On the classifying space of the family of finite and of virtually cyclic subgroups for CAT(0)-groups

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Abstract

Let G be a discrete group which acts properly and isometrically on a complete CAT(0)-space X. Consider an integer d with $d = 1$ or $d \geq 3$ such that the topological dimension of X is bounded by d. We show the existence of a G-CW-model $\underline{E}G$ for the classifying space for proper G-actions with $\dim(\underline{E}G) \leq d$. Provided that the action is also cocompact, we prove the existence of a G-CW-model $\underline{\underline{E}}G$ for the classifying space of the family of virtually cyclic subgroups satisfying $\dim(\underline{\underline{E}}G) \leq d+1$.

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0 Introduction

Given a group G, denote by $\underline{E}G$ a G-CW-model for the classifying space for proper G-actions and by $\underline{\underline{E}}G = E_{\mathcal{VCY}}(G)$ a G-CW-model for the classifying space of the family of virtually cyclic subgroups. Our main theorem which will be proven in Section 3 is

Theorem 0.1. *Let G be a discrete group which acts properly and isometrically on a complete proper* CAT(0)*-space X. Let* top-dim(X) *be the topological dimension of X. Let d be an integer satisfying $d = 1$ or $d \geq 3$ such that* top-dim$(X) \leq d$.

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(i) Then there is G-CW-model $\underline{E}G$ with $\dim(\underline{E}G) \leq d$*;*

(ii) Suppose that G acts by semisimple isometries. (This is the case if we additionally assume that the G-action is cocompact.)

Then there is G-CW-model $\underline{\underline{E}}G$ with $\dim(\underline{\underline{E}}G) \leq d + 1$.

There is the question whether for any group G the inequality

$$\operatorname{hdim}^G(\underline{E}G) - 1 \leq \operatorname{hdim}^G(\underline{\underline{E}}G) \leq \operatorname{hdim}^G(\underline{E}G) + 1 \tag{0.2}$$

holds, where $\operatorname{hdim}^G(\underline{E}G)$ is the minimum of the dimensions of all possible G-CW-models for $\underline{E}G$ and $\operatorname{hdim}^G(\underline{\underline{E}}G)$ is defined analogously (see [11, Introduction]). Since $\operatorname{hdim}(\underline{E}G) \leq 1 + \operatorname{hdim}(\underline{\underline{E}}G)$ holds for all groups G (see [11, Corollary 5.4]), Theorem 0.1 implies

Corollary 0.3. *Let G be a discrete group and let X be complete* CAT(0)*-space X with finite topological dimension* top-dim(X)*. Suppose that G acts properly and isometrically on X. Assume that the G-action is by semisimple isometries. (The last condition is automatically satisfied if we additionally assume that the G-action is cocompact.) Suppose that* top-dim$(X) = \operatorname{hdim}^G(\underline{E}G) \neq 2$.

Then inequality (0.2) *is true.*

We will prove at the end of Section 3

Corollary 0.4. *Suppose that G is virtually torsionfree. Let M be a simply connected complete Riemannian manifold of dimension n with non-negative sectional curvature. Suppose that G acts on M properly, isometrically and cocompactly. Then*

$$\begin{array}{rcccl} & & \operatorname{hdim}(\underline{E}G) & = & n; \\ n-1 & \leq & \operatorname{hdim}(\underline{\underline{E}}G) & \leq & n+1. \end{array}$$

In particular (0.2) *holds.*

If G is the fundamental group of an n-dimensional closed hyperbolic manifold, then $\operatorname{hdim}(\underline{E}G) = \operatorname{hdim}(\underline{\underline{E}}G) = n$ by [11, Example 5.12]. If G is virtually $\mathbb{Z}^n$ for $n \geq 2$, then $\operatorname{hdim}(\underline{E}G) = n$ and $\operatorname{hdim}(\underline{\underline{E}}G) = n+1$ by [11, Example 5.21]. Hence the cases $\operatorname{hdim}(\underline{\underline{E}}G) = \operatorname{hdim}(\underline{E}G)$ and $\operatorname{hdim}(\underline{\underline{E}}G) = \operatorname{hdim}(\underline{E}G)+1$ do occur in the situation of Corollary 0.4. There exists groups G with $\operatorname{hdim}(\underline{\underline{E}}G) = \operatorname{hdim}(\underline{E}G) - 1$ (see [11, Example 5.29]). But we do not believe that this is possible in the situation of Corollary 0.3 or Corollary 0.4.

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1 Classifying Spaces for Families

We briefly recall the notions of a family of subgroups and the associated classifying space. For more information, we refer for instance to the original source [13] or to the survey article [10].

A *family $\mathcal{F}$ of subgroups* of G is a set of subgroups of G which is closed under conjugation and taking subgroups. Examples for $\mathcal{F}$ are

$$\begin{aligned}
\{1\} &= \{\text{trivial subgroup}\};\\
\mathcal{FIN} &= \{\text{finite subgroups}\};\\
\mathcal{VCY} &= \{\text{virtually cyclic subgroups}\};\\
\mathcal{ALL} &= \{\text{all subgroups}\}.
\end{aligned}$$

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying space $E_{\mathcal{F}}(G)$ of the family $\mathcal{F}$* is a G-CW-complex X all of whose isotropy groups belong to $\mathcal{F}$ such that for any G-CW-complex Y with isotropy groups in $\mathcal{F}$ there exists a G-map $Y \to X$ and any two G-maps $Y \to X$ are G-homotopic. In other words, X is a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups belong to $\mathcal{F}$. In particular, two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent.

There exists a model for $E_{\mathcal{F}}(G)$ for any group G and any family $\mathcal{F}$ of subgroups. There is even a functorial construction (see [5, page 223 and Lemma 7.6 (ii)]).

A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if the H-fixed point set X^H is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$.

We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the *universal G-CW-complex for proper G-actions*. We also abbreviate $\underline{\underline{E}}G := E_{\mathcal{VCY}}(G)$.

A model for $E_{\mathcal{ALL}}(G)$ is G/G. A model for $E_{\{1\}}(G)$ is the same as a model for EG, which denotes the total space of the universal G-principal bundle $EG \to BG$.

One can also define a numerable version of the space for proper G-actions to G which is denoted by $\underline{J}G$. It is not necessarily a G-CW-complex. A metric space X on which G acts isometrically and properly is a model for $\underline{J}G$ if and only if the two projections $X \times X \to X$ onto the first and second factor are G-homotopic to one another. If X is a complete CAT(0)-space on which G-acts properly and isometrically, then X is a model for $\underline{J}G$, the desired G-homotopy is constructed using the geodesics joining two points in X (see [3, Proposition 1.4 in II.1 on page 160]).

One motivation for studying the spaces $\underline{E}G$ and $\underline{\underline{E}}G$ comes from the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

2 Topological and CW-dimension

Let X be a topological space. Let $\mathcal{U}$ be an open covering. Its *dimension* $\dim(\mathcal{U}) \in \{0, 1, 2, \ldots\} \amalg \{\infty\}$ is the infimum over all integers $d \geq 0$ such that for any collection $U_0, U_1, \ldots, U_d$ of pairwise distinct elements in $\mathcal{U}$ the intersection $\bigcap_{i=0}^{d} U_i$ is empty. An open covering $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$.

Definition 2.1 (Topological dimension)**.** The *topological dimension* of a topological space X

$$\text{top-dim}(X) \in \{0, 1, 2, \ldots\} \amalg \{\infty\}$$

is the infimum over all integers $d \geq 0$ such that any open covering $\mathcal{U}$ possesses a refinement $\mathcal{V}$ with $\dim(\mathcal{V}) \leq d$.

Let Z be a metric space. We will denote for $z \in Z$ and $r \geq 0$ by $B_r(z)$ and $\overline{B}_r(z)$ respectively the *open ball* and *closed ball* respectively around z with radius r. We call Z *proper* if for each $z \in Z$ and $r \geq 0$ the closed ball $\overline{B}_r(z)$ is compact. A group G acts *properly* on the topological space Z if for any $z \in Z$ there is an open neighborhood U such that the set $\{g \in G \mid g \cdot U \cap U \neq \emptyset\}$ is finite. In particular every isotropy group is finite. If Z is a G-CW-complex, then Z is a proper G-space if and only if the isotropy group of any point in Z is finite (see [9, Theorem 1.23]).

Lemma 2.2. *Let Z be a proper metric space. Suppose that G acts on Z isometrically and properly. Then we get for the topological dimensions of X and $G\backslash X$*

$$\text{top-dim}(G\backslash X) \leq \text{top-dim}(X).$$

Proof. Since G acts properly and isometrically, we can find for every $z \in Z$ a real number $\epsilon(z) > 0$ such that we have for all $g \in G$

$$g \cdot B_{7\epsilon(z)}(z) \cap B_{7\epsilon}(z) \neq \emptyset \Longleftrightarrow g \cdot B_{7\epsilon(z)}(z) = B_{7\epsilon(z)}(z) \Longleftrightarrow g \in G_z.$$

We can arrange that $\epsilon(gz) = \epsilon(z)$ holds for $z \in Z$ and $g \in G$. Consider $G \cdot \overline{B}_\epsilon(z)$. We claim that this set is closed in Z. We have to show for a sequence $(z_n)_{n \geq 0}$ of elements in $\overline{B}_\epsilon(z)$ and $(g_n)_{n \geq 0}$ of elements in G and $x \in Z$ with $\lim_{n\to\infty} g_n z_n = x$ that x belongs to $G \cdot \overline{B}_\epsilon(z)$. Since X is proper, we can find $y \in \overline{B}_\epsilon(z)$ such that $\lim_{n\to\infty} z_n = y$. Choose $N = N(\epsilon)$ such that $d_X(g_n z_n, x) \leq \epsilon$ and $d_X(z_n, y) \leq \epsilon$ holds for $n \geq N$. We conclude for $n \geq N$

$$\begin{aligned} d_x(g_n y, x) &\leq d_X(g_n y, g_n z_n) + d_X(g_n z_n, x) \\ &= d_X(y, z_n) + d_X(g_n z_n, x) \\ &\leq \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned}$$

This implies for $n \geq N$

$$\begin{aligned} d_X(g_n^{-1} g_N z, z) &= d_X(g_N z, g_n z) \\ &\leq d_X(g_N z, g_N y) + d_X(g_N y, x) + d_X(x, g_n y) + d_X(g_n y, g_n z) \\ &= d_X(z, y) + d_X(g_N y, x) + d_X(g_n y, x) + d_X(y, z) \\ &\leq \epsilon + 2\epsilon + 2\epsilon + \epsilon \\ &= 6\epsilon. \end{aligned}$$

Hence $g_n^{-1}g_N \in G_z$ for $n \geq N$. Since G_z is finite, we can arrange by passing to subsequences that $g_0 = g_n$ holds for $n \geq 0$. Hence

$$x = \lim_{n\to\infty} g_n z_n = \lim_{n\to\infty} g_0 z_n = g_0 \cdot \lim_{n\to\infty} z_n = g_0 \cdot y \in G \cdot \overline{B}_\epsilon(z).$$

Choose a set-theoretic section $s\colon G/G_z \to G$ of the projection $G \to G/G_z$. The map

$$G/G_z \times B_{7\epsilon(z)}(z) \xrightarrow{\cong} G \cdot B_{7\epsilon(z)}(z), \quad (gG_z, x) \mapsto s(gG_z) \cdot x$$

is bijective, continuous and open and hence a homeomorphism. It induces a homeomorphism

$$G/G_z \times \overline{B}_{\epsilon(z)}(z) \xrightarrow{\cong} G \cdot \overline{B}_{\epsilon(z)}(z).$$

This implies

$$\operatorname{top-dim}\bigl(\overline{B}_{\epsilon(z)}(z)\bigr) \quad = \quad \operatorname{top-dim}\bigl(G \cdot \overline{B}_{\epsilon(z)}(z)\bigr). \tag{2.3}$$

Let $\operatorname{pr}\colon Z \to G\backslash Z$ be the projection. It induces a bijective continuous map $G_z\backslash\overline{B}_{\epsilon(z)}(z) \xrightarrow{\cong} \operatorname{pr}\bigl(\overline{B}_{\epsilon(z)}(z)\bigr)$ which is a homeomorphism since $\overline{B}_{\epsilon(z)}(z)$ and hence $G_z\backslash\overline{B}_{\epsilon(z)}(z)$ is compact. Hence we get

$$\operatorname{top-dim}\bigl(\operatorname{pr}(\overline{B}_{\epsilon(z)}(z))\bigr) \quad = \quad \operatorname{top-dim}\bigl(G_z\backslash\overline{B}_{\epsilon(z)}(z)\bigr). \tag{2.4}$$

Since the metric space $\overline{B}_{\epsilon(z)}(z)$ is compact and hence contains a countable dense set and G_z is finite, we conclude from [2, Exercise in Chapter II on page 112]

$$\operatorname{top-dim}\bigl(G_z\backslash\overline{B}_{\epsilon(z)}(z)\bigr) \quad \leq \quad \operatorname{top-dim}\bigl(\overline{B}_{\epsilon(z)}(z)\bigr). \tag{2.5}$$

From (2.3), (2.4) and (2.5) we conclude that $G \cdot \overline{B}_{\epsilon(z)}(z) \subseteq Z$ and $\operatorname{pr}\bigl(\overline{B}_{\epsilon(z)}(z)\bigr) \subseteq G\backslash Z$ are closed and satisfy

$$\operatorname{top-dim}\bigl(\operatorname{pr}(\overline{B}_{\epsilon(z)}(z))\bigr) \quad \leq \quad \operatorname{top-dim}\bigl(G \cdot \overline{B}_{\epsilon(z)}(z)\bigr). \tag{2.6}$$

Since Z is proper, it is the countable union of compact subspaces and hence contains a countable dense subset. This is equivalent to the condition that Z has a countable basis for its topology. Obviously the same is true for $G\backslash Z$. We conclude from [12, Theorem 9.1 in in Chapter 7.9 on page 302 and Exercise 9 in Chapter 7.9 on page 315]

$$\operatorname{top-dim}(Z) \quad = \quad \sup\bigl\{\operatorname{top-dim}\bigl(G \cdot \overline{B}_{\epsilon(z)}(z)\bigr)\bigr\}; \tag{2.7}$$

$$\operatorname{top-dim}(G\backslash Z) \quad = \quad \sup\bigl\{\operatorname{top-dim}\bigl(\operatorname{pr}(\overline{B}_{\epsilon(z)}(z))\bigr)\bigr\}. \tag{2.8}$$

Now Lemma 2.2 follows from (2.6), (2.7) and (2.8). □

In the sequel we will equip a simplicial complex with the weak topology, i.e., a subset is closed if and only if its intersection with any simplex σ is a closed subset of σ. With this topology a simplicial complex carries a canonical CW-structure.

Let X be a G-space. We call a subset $U \subseteq X$ a *$\mathcal{FIN}$-set* if we have $gU \cap U \neq \emptyset \implies gU = U$ for every $g \in G$ and $G_U := \{g \in G \mid g \cdot U = U\}$ is finite. Let $\mathcal{U}$ be a covering of X by open $\mathcal{FIN}$-subset. Suppose that $\mathcal{U}$ is G-invariant, i.e., we have $g \cdot U \in \mathcal{U}$ for $g \in G$ and $U \in \mathcal{U}$. Define its *nerve* $\mathcal{N}(\mathcal{U})$ to be the simplicial complex whose vertices are the elements in $\mathcal{U}$ and for which the pairwise distinct vertices $U_0, U_1, \ldots, U_d$ span a d-simplex if and only if $\bigcap_{i=0}^{d} U_i \neq \emptyset$. The action of G on X induces an action on $\mathcal{U}$ and hence a simplicial action on $\mathcal{N}(\mathcal{U})$. The isotropy group of any vertex is finite and hence the isotropy group of any simplex is finite. Let $\mathcal{N}(\mathcal{U})'$ be the barycentric subdivision. It inherits a simplicial G-action from $\mathcal{N}(\mathcal{U})$ such that for any $g \in G$ and any simplex σ whose interior is denoted by σ° and which satisfies $g \cdot \sigma^\circ \cap \sigma^\circ \neq \emptyset$ we have $gx = x$ for all $x \in \sigma^\circ$. In particular $\mathcal{N}(\mathcal{U})'$ is a G-CW-complex and agrees as a G-space with $\mathcal{N}(\mathcal{U})$.

Lemma 2.9. *Let n be an integer with $n \geq 0$. Let X be a proper metric space whose topological dimension satisfies* $\text{top-dim}(X) \leq n$. *Suppose that G acts properly and isometrically on X.*

Then there exists a proper n-dimensional G-CW-complex Y together with a G-map $f\colon X \to Y$.

Proof. Since the G-action is proper we can find for every $x \in X$ an $\epsilon(x) > 0$ such that for every $g \in G$ we have

$$g \cdot \overline{B}_{2\epsilon(x)}(x) \cap \overline{B}_{2\epsilon(x)}(x) \neq \emptyset \Leftrightarrow g \cdot \overline{B}_{2\epsilon(x)}(x) = \overline{B}_{2\epsilon(x)}(x)$$
$$\Leftrightarrow g \cdot B_{2\epsilon(x)}(x) = B_{2\epsilon(x)}(x) \Leftrightarrow g \cdot B_{\epsilon(x)}(x) = B_{\epsilon(x)}(x) \Leftrightarrow g \in G_x.$$

We can arrange that $\epsilon(gx) = \epsilon(x)$ for $g \in G$ and $x \in X$ holds. We obtain a covering of X by open $\mathcal{FIN}$-subsets $\{B_{\epsilon(x)}(x) \mid x \in X\}$. Let $\text{pr}\colon X \to G\backslash X$ be the canonical projection. We obtain an open covering of $G\backslash X$ by $\{\text{pr}\big(B_{\epsilon(x)}(x)\big) \mid x \in X\}$. Since $\text{top-dim}(X) \leq n$ by assumption and G acts properly on X, we get $\text{top-dim}(G\backslash X) \leq n$ from Lemma 2.2. Since G acts properly and isometrically on X, the quotient $G\backslash X$ inherits a metric from X. Hence $G\backslash X$ is paracompact by Stone's theorem (see [12, Theorem 4.3 in Chap. 6.3 on page 256]) and in particular normal. By [6, Theorem 3.5 on page 211] we can find a locally finite open covering $\mathcal{U}$ of $G\backslash X$ such that $\dim(\mathcal{U}) \leq n$ and $\mathcal{U}$ is a refinement of $\{\text{pr}(B_{\epsilon(x)}(x)) \mid x \in X\}$. For each $U \in \mathcal{U}$ choose $x(U) \in X$ with $U \subseteq \text{pr}\big(B_{\epsilon(U)}(x(U)\big)$. Define the index set

$$J = \big\{(U, \overline{g}) \mid U \in \mathcal{U}, \overline{g} \in G/G_{x(U)}\big\}.$$

For $(U, \overline{g}) \in J$ define an open $\mathcal{FIN}$-subset of X by

$$V_{U,\overline{g}} := \text{pr}^{-1}(U) \cap g \cdot B_{2\epsilon(x(U))}\big(x(U)\big).$$

Obviously this is well-defined, i.e., the choice of $g \in \overline{g}$ does not matter, and we have $\text{pr}(V_{U,\overline{g}}) \subseteq U$ and $V_{U,\overline{g}} \subseteq g \cdot B_{2\epsilon(x(U))}\big(x(U)\big)$.

Consider the collection of subsets of X

$$\mathcal{V} = \{V_{U,\overline{g}} \mid (U,\overline{g}) \in J\}.$$

This is a G-invariant covering of X by open $\mathcal{FIN}$-subsets. Its dimension satisfies

$$\dim(\mathcal{V}) \leq \dim(\mathcal{U}) \leq n$$

since for $U \in \mathcal{U}$, $\overline{g_1}, \overline{g_2} \in G/G_{x(U)}$ we have

$$V_{U,\overline{g_1}} \cap V_{U,\overline{g_2}} \neq \emptyset \implies g_1 \cdot B_{2\epsilon(x(U))}\big(x(U)\big) \cap g_2 \cdot B_{2\epsilon(x(U))}\big(x(U)\big) \implies \overline{g_1} = \overline{g_2}.$$

Since $\mathcal{U}$ is locally finite and $G\backslash X$ is paracompact, we can find a locally finite partition of unity $\{e_U \colon G\backslash X \to [0,1] \mid U \in \mathcal{U}\}$ which is subordinate to $\mathcal{U}$, i.e., $\sum_{U\in\mathcal{U}} e_U = 1$ and $\operatorname{supp}(e_U) \subset U$ for every $U \in \mathcal{U}$. Fix a map $\chi\colon [0,\infty) \to [0,1]$ satisfying $\chi^{-1}(0) = [1,\infty)$. Define for $(U,\overline{g}) \in J$ a function

$$\phi_{U,\overline{g}} \colon X \to [0,1], \quad y \mapsto e_U(\operatorname{pr}(y)) \cdot \chi\big(d_X(y, gx(U))/\epsilon(x(U))\big).$$

Consider $y \in X$. Since $\mathcal{U}$ is locally finite and $G\backslash X$ is locally compact, we can find an open neighborhood T of $\operatorname{pr}(y)$ such that $\overline{T}$ meets only finitely many elements of $\mathcal{U}$. Choose an open neighborhood W_0 of y such that $\overline{W_0}$ is compact. Define an open neighborhood of y by

$$W := W_0 \cap \operatorname{pr}^{-1}(T).$$

Since $\overline{W_0}$ is compact, $\overline{W}$ is compact. Since G acts properly, there exists for a given $U \in \mathcal{U}$ only finitely many elements $g \in G$ with $\overline{W} \cap g \cdot B_{\epsilon(x(U))}(x(U)) \neq \emptyset$. Since $\overline{T}$ meets only finitely elements of $\mathcal{U}$, the set

$$J_W := \big\{(U,\overline{g}) \in J \mid \overline{W} \cap g \cdot B_{\epsilon(x(U))}(x(U)) \cap \operatorname{pr}^{-1}(U) \neq \emptyset\big\}$$

is finite. Suppose $\phi_{U,\overline{g}}(z) > 0$ for $(U,\overline{g}) \in J$ and $z \in W$. We conclude $z \in \operatorname{pr}^{-1}(U) \cap g \cdot B_{\epsilon(x(U))}(x(U))$ and hence $(U,\overline{g}) \in J_W$. Thus we have shown that the collection $\{\phi_{U,\overline{g}} \mid (U,\overline{g}) \in J\}$ is locally finite.

We conclude that the map

$$\sum_{(U,\overline{g})\in J} \phi_{U,\overline{g}} \colon X \to [0,1], \quad y \mapsto \sum_{(U,\overline{g})\in J} e_U(\operatorname{pr}(y)) \cdot \chi\big(d_X(y, gx(U))/\epsilon(x(U))\big)$$

is well-defined and continuous. It has always a value greater than zero since for every $y \in X$ there exists $U \in \mathcal{U}$ with $e_U(\operatorname{pr}(y)) > 0$, the set $\operatorname{pr}^{-1}(U)$ is contained in $\bigcup_{g\in G} g \cdot B_{\epsilon(U)}(x(U))$ and $\chi^{-1}(0) = [1,\infty)$. Define for $(U,\overline{g}) \in J$ a map

$$\psi_{U,\overline{g}} \colon X \to [0,1], \quad y \mapsto \frac{\phi_{U,\overline{g}}(y)}{\sum_{(U,\overline{g})\in J} \phi_{U,\overline{g}}(y)}.$$

We conclude that

$$\begin{array}{lcll} \sum_{(U,\overline{g})\in J} \psi_{U,\overline{g}}(y) & = & 1 & \text{for } y \in X; \\ \psi_{U,\overline{g}}(hy) & = & \psi_{U,\overline{h^{-1}g}}(y) & \text{for } h \in G, y \in Y \text{ and } (U,\overline{g}) \in J; \\ \operatorname{supp}(\psi_{U,\overline{g}}) & \subseteq & V_{U,\overline{g}} & \text{for } (U,\overline{g}) \in J, \end{array}$$

and the collection $\{\psi_{U,\overline{g}} \mid (U,\overline{g}) \in J\}$ is locally finite. Define the desired proper n-dimensional G-CW-complex to be the nerve $Y := \mathcal{N}(\mathcal{V})$. Define a map by

$$f\colon X \to \mathcal{N}(\mathcal{V}), \quad y \mapsto \sum_{(U,\overline{g})\in J} \psi_{U,\overline{g}}(y) \cdot V_{U,\overline{g}}.$$

It is well-defined since for $y \in X$ the simplices $V_{U,\overline{g}}$ for which $\psi_{U,\overline{g}}(y) \neq 0$ holds span a simplex because $y \in X$ with $\psi_{U,\overline{g}}(y) \neq 0$ belongs to $V_{U,\overline{g}}$ and hence the intersection of the sets $V_{U,\overline{g}}$ for which $\psi_{U,\overline{g}}(y) \neq 0$ holds contains y and hence is non-empty. The map f is continuous since $\{\psi_{U,\overline{g}} \mid (U,\overline{g}) \in J\}$ is locally finite. It is G-equivariant by the following calculation for $h \in G$ and $y \in Y$:

$$\begin{aligned} f(hy) &= \sum_{(U,\overline{g})\in J} \psi_{U,\overline{g}}(hy) \cdot V_{U,\overline{g}} \\ &= \sum_{(U,\overline{g})\in J} \psi_{U,\overline{hg}}(hy) \cdot V_{U,\overline{hg}} \\ &= \sum_{(U,\overline{g})\in J} \psi_{U,\overline{h^{-1}hg}}(y) \cdot V_{U,\overline{hg}} \\ &= \sum_{(U,\overline{g})\in J} \psi_{U,\overline{g}}(y) \cdot h \cdot V_{U,\overline{g}} \\ &= h \cdot \sum_{(U,\overline{g})\in J} \psi_{U,\overline{g}}(y) \cdot V_{U,\overline{g}} \\ &= h \cdot f(y). \end{aligned}$$

□

Lemma 2.10. *Let X and Y be G-CW-complexes. Let $i\colon X \to Y$ and $r\colon Y \to X$ be G-maps such that $r \circ i$ is G-homotopic to the identity. Consider an integer $d \geq 3$. Suppose that Y has dimension $\leq d$.*

Then X is G-homotopy equivalent to a G-CW-complex Z of dimension $\leq d$.

Proof. By the Equivariant Cellular Approximation Theorem (see [14, Theorem II.2.1 on page 104]) we can assume without loss of generality that i and r are cellular. Let $\operatorname{cyl}(r)$ be the mapping cylinder. Let $k\colon Y \to \operatorname{cyl}(r)$ be the canonical inclusion and $p\colon \operatorname{cyl}(r) \to X$ be the canonical projection. Then p is a G-homotopy equivalence and $p \circ k = r$. Let Z be the union of the 2-skeleton of $\operatorname{cyl}(r)$ and Y. This is a G-CW-subcomplex of $\operatorname{cyl}(r)$ and $\operatorname{cyl}(r)$ is obtained from Z by attaching equivariant cells of dimension ≥ 3. Hence the map $p|_Z\colon Z \to X$ has the property that it induces on every fixed point set a 2-connected map. Let $j\colon X \to Z$ be the composite of $i\colon X \to Y$ with the obvious inclusion $Y \to Z$. Then $p|_Z \circ j = p \circ k \circ i = r \circ i$ is G-homotopy equivalent to the identity and the dimension of Z is still bounded by d since we assume $d \geq 3$. Hence we can assume in the sequel that $r^H\colon Y^H \to X^H$ is 2-connected for all $H \subseteq G$, otherwise replace Y by Z, i by j and r by $p|_Z$.

We want to apply [9, Proposition 14.9 on page 282]. Here the assumption $d \geq 3$ enters. Hence it suffices to show that the cellular $\mathbb{Z}\Pi(G,X)$-chain complex

$C_*^c(X)$ is $\mathbb{Z}\Pi(G,X)$-chain homotopy equivalent to a d-dimensional $\mathbb{Z}\Pi(G,X)$-chain complex. By [9, Proposition 11.10 on page 221] it suffices to show that the cellular $\mathbb{Z}\Pi(G,X)$-chain complex $C_*^c(X)$ is dominated by a d-dimensional $\mathbb{Z}\Pi(G,X)$-chain complex. This follows from the geometric domination (Y,i,r) by passing to the cellular chain complexes over the fundamental categories since r and hence also i induce equivalences between the fundamental categories because $r^H\colon Y^H \to X^H$ is 2-connected for all $H \subseteq G$ and $r \circ i \simeq_G \mathrm{id}_X$.

□

The condition $d \geq 3$ is needed since we want to argue first with the cellular $\mathbb{Z}\mathsf{Or}(G)$-chain complex and then transfer the statement that it is d-dimensional to the statement that the underlying G-CW-complex is d-dimensional. The condition $d \geq 3$ enters for analogous reasons in the classical proof of the theorem that the existence of a d-dimensional $\mathbb{Z}G$-projective resolution for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ implies the existence of a d-dimensional model for BG (see [4, Theorem 7.1 in Chapter VIII.7 on page 205]).

Theorem 2.11. *Let G be a discrete group. Then*

(i) There is a G-homotopy equivalence $\underline{J}G \to \underline{E}G$;

(ii) Suppose that there is a model for $\underline{J}G$ which is a metric space such that the action of G on $\underline{J}G$ is isometric. Consider an integer d with $d = 1$ or $d \geq 3$. Suppose that the topological dimension $\operatorname{top-dim}(\underline{J}G) \leq d$.

Then there is a G-CW-model for $\underline{E}G$ of dimension $\leq d$;

(iii) Let d be an integer $d \geq 0$. Suppose that there is a G-CW-model for $\underline{E}G$ with $\dim(\underline{E}G) \leq d$ such that $\underline{E}G$ after forgetting the group action has countably many cells.

Then there exists a model for $\underline{J}G$ with $\operatorname{top-dim}(\underline{J}G) \leq d$.

Proof. (i) This is proved in [10, Lemma 3.3 on page 278].

(ii) Choose a G-homotopy equivalence $i\colon \underline{E}G \to \underline{J}G$. From Lemma 2.9 we obtain a G-map $f\colon \underline{J}G \to Y$ to a proper G-CW-complex of dimension $\leq d$. By the universal properly of $\underline{E}G$ we can find a G-map $h\colon Y \to \underline{E}G$ and the composite $h \circ f \circ i$ is G-homotopic to the identity on $\underline{E}G$.

Suppose $d \geq 3$. We conclude from Lemma 2.10 that $\underline{E}G$ is G-homotopy equivalent to a G-CW-complex of dimension $\leq d$.

Suppose $d = 1$. By Dunwoody [7, Theorem 1.1] it suffices to show that the rational cohomological dimension of G satisfies $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$. Hence we have to show for any $\mathbb{Q}G$-module M that $\mathrm{Ext}^n_{\mathbb{Q}G}(\mathbb{Q},M\big) = 0$ for $n \geq 2$, where $\mathbb{Q}$ is the trivial $\mathbb{Q}G$-module. Since all isotropy groups of $\underline{E}G$ and Y are finite, their cellular $\mathbb{Q}G$-chain complexes are projective. Since $\underline{E}G$ is contractible, $C_*(\underline{E}G;\mathbb{Q})$ is a projective $\mathbb{Q}G$-resolution and hence

$$\mathrm{Ext}^n_{\mathbb{Q}G}(\mathbb{Q},M\big) \cong H^n\big(\hom_{\mathbb{Q}G}(C_*(\underline{E}G;\mathbb{Q}),M)\big).$$

Since $h \circ f \circ i \simeq_G \mathrm{id}_{\underline{E}G}$, the $\mathbb{Q}$-module $H^n\big(\hom_{\mathbb{Q}G}(C_*(\underline{E}G;\mathbb{Q}),M)\big)$ is a direct summand in the $\mathbb{Q}$-module $H^n\big(\hom_{\mathbb{Q}G}(C_*(Y;\mathbb{Q}),M)\big)$. Since Y is 1-dimensional by assumption, $H^n\big(\hom_{\mathbb{Q}G}(C_*(Y;\mathbb{Q}),M)\big)$ vanishes for $n \geq 2$. This implies that $\mathrm{Ext}^n_{\mathbb{Q}G}(\mathbb{Q},M\big)$ vanishes for $n \geq 2$.

(iii) Using the equivariant version of the simplicial approximation theorem and the fact that changing the G-homotopy class of attaching maps does not change the G-homotopy type, one can find a simplicial complex X with simplicial G-action which is G-homotopy equivalent to $\underline{E}G$, satisfies $\dim(X) = \dim(\underline{E}G)$ and has only countably many simplices. Hence the barycentric subdivision X' is a simplicial complex of dimension $\leq d$ with countably many simplices and carries a G-CW-structure. The latter implies that X' is a G-CW-model for $\underline{E}G$ and hence also a model for $\underline{J}G$. Since the dimension of a simplicial complex with countably many simplices is equal to its topological dimension, we conclude $\text{top-dim}(X') = \dim(X) = \dim(\underline{E}G) \leq d$. ◻

3 The passage from finite to virtually cyclic groups

In [11] it is described how one can construct $\underline{\underline{E}}G$ from $\underline{E}G$. In this section we want to make this description more explicit under the following condition

Condition 3.1. We say that G satisfies condition (C) if for every $g,h \in G$ with $|h| = \infty$ and $k,l \in \mathbb{Z}$ we have

$$gh^kg^{-1} = h^l \implies |k| = |l|.$$

Let $\mathcal{ICY}$ be the set of infinite cyclic subgroup C of G. This is not a family since it does not contain the trivial subgroup. We call $C,D \in \mathcal{ICY}$ *equivalent* if $|C \cap D| = \infty$. One easily checks that this is an equivalence relation on $\mathcal{ICY}$. Denote by $[\mathcal{ICY}]$ the set of equivalence classes and for $C \in \mathcal{ICY}$ by $[C]$ its equivalence class. Denote by

$$N_GC := \{g \in G \mid gCg^{-1} = C\}$$

the *normalizer* of C in G. Define for $[C] \in [\mathcal{ICY}]$ a subgroup of G by

$$N_G[C] := \big\{g \in G \bigm| |gCg^{-1} \cap C| = \infty\big\}.$$

One easily checks that this is independent of the choice of $C \in [C]$. Actually $N_G[C]$ is the isotropy of $[C]$ under the action of G induced on $[\mathcal{ICY}]$ by the conjugation action of G on $\mathcal{ICY}$.

Lemma 3.2. *Suppose that G satisfies Condition (C) (see 3.1). Consider $C \in \mathcal{ICY}$.*

Then obtain a nested sequence of subgroups

$$N_GC \subseteq N_G2!C \subseteq N_G3!C \subseteq N_G4!C \subseteq \cdots$$

where $k!C$ is the subgroup of C given by $\{h^{k!} \mid h \in C\}$, and we have

$$N_G[C] = \bigcup_{k \geq 1} N_G k!C.$$

Proof. Since every subgroup of a cyclic group is characteristic, we obtain the nested sequence of normalizers $N_G C \subseteq N_G 2!C \subseteq N_G 3!C \subseteq N_G 4!C \subseteq \cdots$.

Consider $g \in N_G[C]$. Let h be a generator of C. Then there are $k, l \in \mathbb{Z}$ with $gh^k g^{-1} = h^l$ and $k, l \neq 0$. Condition (C) implies $k = \pm l$. Hence $g \in N_G\langle h^k \rangle \subseteq N_G k!C$. This implies $N_G[C] \subseteq \bigcup_{k \geq 1} N_G k!C$. The other inclusion follows from the fact that for $g \in N_G k!C$ we have $k!C \subseteq gCg^{-1} \cap C$. □

Fix $C \in \mathcal{ICY}$. Define a family of subgroups of $N_G[C]$ by

$$\mathcal{G}_G(C) := \big\{H \subseteq N_G[C] \mid [H : (H \cap C)] < \infty\big\} \cup \big\{H \subseteq N_G[C] \mid |H| < \infty\big\}. \quad (3.3)$$

Notice that $\mathcal{G}_G(C)$ consists of all finite subgroups of $N_G[C]$ and of all virtually cyclic subgroups of $N_G[C]$ which have an infinite intersection with C. Define a quotient group of $N_G C$ by

$$W_G C := N_G C / C.$$

Lemma 3.4. *Let n be an integer. Suppose that G satisfies Condition (C) (see 3.1). Suppose that there exists a G-CW-model for $\underline{E}G$ with $\dim(\underline{E}G) \leq n$ and for every $C \in \mathcal{ICY}$ there exists a $W_G C$-CW-model for $\underline{E}W_G C$ with $\dim(\underline{E}W_G C) \leq n$.*

Then there exists a G-CW-model for $\underline{\underline{E}}G$ with $\dim(\underline{\underline{E}}G) \leq n+1$.

Proof. Because of [11, Theorem 2.3 and Remark 2.5] it suffices to show for every $C \in \mathcal{ICY}$ that there is a $N_G[C]$-model for $E_{\mathcal{G}_G(C)}(N_G[C])$ with

$$\dim(E_{\mathcal{G}_G(C)}(N_G[C])) \quad \leq \quad n+1. \quad (3.5)$$

Because of Lemma 3.2 we have

$$N_G[C] = \operatorname{colim}_{k \to \infty} N_G k!C.$$

We conclude (3.5) from [11, Lemma 4.2 and Theorem 4.3] since every element $H \in \mathcal{G}_G(C)$ is finitely generated and hence lies already in $N_G k!C$ for some $k > 0$, by assumption there exists a $W_G k!C$-CW-model for $\underline{E}W_G k!C$ with $\dim(\underline{E}W_G k!C) \leq n$, and $\operatorname{res}_{N_G k!C \to W_G k!C} \underline{E}W_G k!C$ is $E_{\mathcal{G}_G(C)|_{N_G k!C)}}(N_G k!C)$. □

Now we are ready to prove Theorem 0.1.

Proof of Theorem 0.1. (i) Consider an integer $d \in \mathbb{Z}$ with $d = 1$ or $d \geq 3$ such that $d \geq \text{top-dim}(X)$. The space X is a model for $\underline{J}G$ by [3, Corollary 2.8 in II.2. on page 178]. We conclude from Theorem 2.11 (ii) that there is a d-dimensional model for $\underline{E}G$.

(ii) We will use in the proof some basic facts and notions about isometries of proper complete CAT(0)-spaces which can be found in [3, Chapter II.6].

The group G satisfies condition (C) by the following argument. Suppose that $gh^kg^{-1} = h^l$ for $g, h \in G$ with $|h| = \infty$ and $k, l \in \mathbb{Z}$. The isometry $l_h \colon X \to X$ given by multiplication with h is a hyperbolic isometry since it has no fixed point and is by assumption semisimple. We obtain for the translation length $L(h)$ which is a real number satisfying $L(h) > 0$

$$k \cdot L(h) = L(h^k) = L(gh^kg^{-1}) = L(h^l) = l \cdot L(h).$$

This implies $k = l$.

Let $C \subseteq G$ be any infinite cyclic subgroup. Choose a generator $g \in C$. The isometry $l_g \colon X \to X$ given by multiplication with g is a hyperbolic isometry. Let $\operatorname{Min}(g) \subset X$ be the the union of all axes of g. Then $\operatorname{Min}(g)$ is a closed convex subset of X. There exists a closed convex subset $Y(g) \subseteq X$ and an isometry

$$\alpha\colon \operatorname{Min}(g) \xrightarrow{\cong} Y(g) \times \mathbb{R}.$$

The space $\operatorname{Min}(G)$ is N_GC-invariant since for each $h \in N_GC$ we have $hgh^{-1} = g$ or $hgh^{-1} = g^{-1}$ and hence multiplication with h sends an axis of g to an axis of g. The N_GC-action induces a proper isometric W_GC-action on $Y(g)$. These claims follow from [3, Theorem 6.8 in II.6 on page 231 and Proposition 6.10 in II.6 on page 233]. The space $Y(g)$ inherits from X the structure of a CAT(0)-space and satisfies top-dim$(Y(g)) \leq$ top-dim(X). Hence $Y(g)$ is a model for $\underline{J}W_GC$ with top-dim$(Y(g)) \leq$ top-dim(X) by [3, Corollary 2.8 in II.2. on page 178]. We conclude from Theorem 2.11 (ii) that there is a d-dimensional model for $\underline{E}W_GC$ for every infinite cyclic subgroup $C \subseteq G$. Now Theorem 0.1 follows from Lemma 3.4. □

Finally we prove Corollary 0.4.

Proof of Corollary 0.4. A complete Riemannian manifold M with non-negative sectional curvature is a CAT(0)-space (see [3, Theorem IA.6 on page 173 and Theorem II.4.1 on page 193].) Since G is virtually torsionfree, we can find a subgroup G_0 of finite index in G such that G_0 is torsionfree and acts orientation preserving on M. Hence $G_0\backslash M$ is a closed orientable manifold of dimension n. Hence $H_n(M;\mathbb{Z}) = H_n(BG;\mathbb{Z}) \neq 0$. This implies that every CW-model BG_0 has at least dimension n. Since the restriction of $\underline{E}G$ to G_0 is a G_0-CW-model for EG_0, we conclude $\operatorname{hdim}(\underline{E}G) \geq n$. Since M with the given G_0-action is a G-CW-model for $\underline{E}G$ (see [1, Theorem 4.15]), we conclude

$$\operatorname{hdim}(\underline{E}G) = n = \text{top-dim}(M).$$

If $n \neq 2$, we conclude $\operatorname{hdim}(\underline{\underline{E}}G) \leq n+1$ from Theorem 0.1. Since $\operatorname{hdim}(\underline{E}G) \leq 1 + \operatorname{hdim}(\underline{\underline{E}}G)$ holds for all groups G (see [11, Corollary 5.4]), we get

$$n - 1 \leq \operatorname{hdim}(\underline{\underline{E}}G) \leq n + 1$$

provided that $n \neq 2$.

Suppose $n = 2$. If G_0 is a torsionfree subgroup of finite index in G, then $G_0\backslash X$ is a closed 2-dimensional manifold with non-negative sectional curvature. Hence G_0 is $\mathbb{Z}^2$ or hyperbolic. This implies that G is virtually $\mathbb{Z}^2$ or hyperbolic. Hence $\operatorname{hdim}(\underline{\underline{E}}G) \in \{2, 3\}$ by [11, Example 5.21] in the first case and by [11, Theorem 3.1, Example 3.6, Theorem 5.8 (ii)] or [8, Proposition 6, Remark 7 and Proposition 8] in the second case.

□

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