

CUBIC ALGEBRAS AND IMPLICATION ALGEBRAS

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ABSTRACT. We consider relationships between cubic algebras and implication algebras. We first exhibit a functorial construction of a cubic algebra from an implication algebra. Then we consider an collapse of a cubic algebra to an implication algebra and the connection between these two operations. Finally we use the ideas of the collapse to obtain a Stone-type representation theorem for a large class of cubic algebras.

1. INTRODUCTION

1.1. Cubic Algebras. Cubic algebras first arose in the study of face lattices of n -cubes (see[7]) and in considering the poset of closed intervals of Boolean algebras (see [2]). Both of these families of posets have a partial binary operation Δ – a generalized reflection. Cubic algebras then arise in full generality by taking the variety generated by either of these classes with Δ , join and one.

In this paper we consider another construction of cubic algebras from implication algebras. This construction produces (up to isomorphism) every countable cubic algebra. Cubic algebras also admit a natural collapse to an implication algebra. We show that this collapse operation is a one-sided inverse to this construction.

A consequence of the Stone representation theorem for Boolean algebras is that the set of filters of a Boolean algebra is a Heyting algebra into which the original Boolean algebra embeds naturally. The collapsing process for cubic algebras highlights certain filter-like subimplication algebras of cubic algebras that generate the algebra and let us do a similar construction for cubic algebras. Thus, by looking at the set of all these subobjects we produce a new algebraic structure from which we can pick a subalgebra that is an MR-algebra. And our original cubic algebra embeds into it in a natural way.

Before beginning our study we recall some of the basics of cubic and MR algebras.

Definition 1.1. A cubic algebra is a join semi-lattice with one and a binary operation Δ satisfying the following axioms:

- a. if $x \leq y$ then $\Delta(y, x) \vee x = y$;
- b. if $x \leq y \leq z$ then $\Delta(z, \Delta(y, x)) = \Delta(\Delta(z, y), \Delta(z, x))$;
- c. if $x \leq y$ then $\Delta(y, \Delta(y, x)) = x$;
- d. if $x \leq y \leq z$ then $\Delta(z, x) \leq \Delta(z, y)$;

Let $xy = \Delta(1, \Delta(x \vee y, y)) \vee y$ for any x, y in \mathcal{L} . Then:

- e. $(xy)y = x \vee y$;
- f. $x(yz) = y(xz)$;

Date: 2009, February 4.

1991 Mathematics Subject Classification. 06A12, 06E99.

Key words and phrases. cubes, Boolean algebras, implication algebras.

\mathcal{L} together with $\langle x, y \rangle \mapsto xy$ is an implication algebra. More details on these algebras and some basic representation theory can be found in [2]. A good reference for implication algebras is [1].

1.2. MR-algebras.

Definition 1.2. An MR-algebra is a cubic algebra satisfying the MR-axiom: if $a, b < x$ then

$$\Delta(x, a) \vee b < x \text{ iff } a \wedge b \text{ does not exist.}$$

Example 1.1. Let X be any set, and

$$\mathcal{S}(X) = \{ \langle A, B \rangle \mid A, B \subseteq X \text{ and } A \cap B = \emptyset \}.$$

Elements of $\mathcal{S}(X)$ are called *signed subsets* of X . The operations are defined by

$$\begin{aligned} 1 &= \langle \emptyset, \emptyset \rangle \\ \langle A, B \rangle \vee \langle C, D \rangle &= \langle A \cap C, B \cap D \rangle \\ \Delta(\langle A, B \rangle, \langle C, D \rangle) &= \langle A \cup D \setminus B, B \cup C \setminus A \rangle. \end{aligned}$$

These are all atomic MR-algebras. The face-poset of an n -cube is naturally isomorphic to a signed set algebra.

Example 1.2. Let B be a Boolean algebra, then the *interval algebra* of B is

$$\mathcal{I}(B) = \{ [a, b] \mid a \leq b \text{ in } B \}$$

ordered by inclusion. The operations are defined by

$$\begin{aligned} 1 &= [0, 1] \\ [a, b] \vee [c, d] &= [a \wedge c, b \vee d] \\ \Delta([a, b], [c, d]) &= [a \vee (b \wedge \bar{d}), b \wedge (a \vee \bar{c})]. \end{aligned}$$

These are all atomic MR-algebras. For further details see [2].

We note that $\mathcal{S}(X)$ is isomorphic to $\mathcal{S}(\wp(X))$.

Definition 1.3. Let \mathcal{L} be a cubic algebra. Then for any $x, y \in \mathcal{L}$ we define the (partial) operation $\hat{}$ (caret) by:

$$x \hat{} y = x \wedge \Delta(x \vee y, y)$$

whenever this meet exists.

The operation $\hat{}$ is used as a partial substitute for meets as the next lemma suggests.

Lemma 1.4. If \mathcal{L} is a cubic algebra and $x, y \in \mathcal{L}$ then –

- (a) \mathcal{L} is an MR-algebra iff the caret operation is total.
- (b) if $x \wedge y$ exists then $x \wedge y = x \hat{} \Delta(x \vee y, y)$.

Proof. See [6] lemma 2.4 and theorem 2.6. □

As in any algebra we have subalgebras. If \mathcal{L} is a cubic algebra we denote by $\llbracket X \rrbracket$ the subalgebra generated by X .

1.3. **Enveloping Algebras.** We recall from [4] the existence of *enveloping algebras*.

Theorem 1.5 (Enveloping Algebra). *Let \mathcal{L} be any cubic algebra. Then there is an MR-algebra $\text{env}(\mathcal{L})$ and an embedding $e: \mathcal{L} \rightarrow \text{env}(\mathcal{L})$ such that:*

- (a) *the range of e generates $\text{env}(\mathcal{L})$;*
- (b) *the range of e is an upwards-closed subalgebra;*
- (c) *any cubic homomorphism f from \mathcal{L} into an MR-algebra \mathcal{N} lifts uniquely to a cubic homomorphism \widehat{f} from $\text{env}(\mathcal{L})$ to \mathcal{N} . Furthermore if f is onto or one-one then so is \widehat{f} .*

Definition 1.6. *Let \mathcal{L} be any cubic algebra. Then the MR-algebra $\text{env}(\mathcal{L})$ defined above is called the *enveloping algebra* of \mathcal{L} .*

2. IMPLICATION COLLAPSE

Definition 2.1. *Let \mathcal{L} be a cubic algebra and $a, b \in \mathcal{L}$. Then*

$$\begin{aligned} a \leq b &\text{ iff } \Delta(a \vee b, a) \leq b \\ a \sim b &\text{ iff } \Delta(a \vee b, a) = b. \end{aligned}$$

Lemma 2.2. *Let \mathcal{L} , a, b be as in the definition. Then*

$$a \leq b \text{ iff } b = (b \vee a) \wedge (b \vee \Delta(1, a)).$$

Proof. See [2] lemmas 2.7 and 2.12. □

Lemma 2.3. *Let \mathcal{L} be a cubic algebra and $a \in \mathcal{L}$. If $b, c \geq a$ then*

$$b \sim c \iff b = c.$$

Proof. If $b = \Delta(b \vee c, c)$ then we have $a \leq c$ and $a \leq b = \Delta(b \vee c, c)$ and so $b \vee c = a \vee \Delta(b \vee c, a) \leq c \vee c = c$. Likewise $b \vee c \leq b$ and so $b = c$. □

A small variation of the proof shows that if $a \leq b, c$ then $b \leq c$ iff $b \leq c$.

Remark 2.1. Also from [2] (lemma 2.7c for transitivity) we know that \sim is an equivalence relation. In general it is not a congruence relation, but it does fit well with caret.

It is clear that \leq induces a partial order on \mathcal{L}/\sim . Since $x \leq y$ implies $x \leq y$ we see that $x \mapsto [x]$ is order-preserving.

We will show that the structure \mathcal{L}/\sim is an implication algebra – with $[x] \vee [y] = [x \vee \Delta(x \vee y, y)]$ and $[x] \wedge [y] = [x \wedge \Delta(x \vee y, y)]$ whenever this exists – and is an implication lattice iff \mathcal{L} is an MR-algebra.

Definition 2.4. *The poset \mathcal{L}/\sim is the *implication collapse* (or *just collapse*) of \mathcal{L} .*

The mapping $\eta: \mathcal{L} \rightarrow \mathcal{L}/\sim$ given by

$$\eta(x) = [x]$$

is the collapsing or the collapse mapping. We will often denote this mapping by $\mathcal{L} \mapsto \mathcal{C}(\mathcal{L})$.

2.1. Properties of the collapse. The structure \mathcal{L}/\sim is naturally an implication algebra. To show this we need to show that certain operations cohere with \sim . Before doing so we need to argue that most of our work can be done inside an interval algebra. The crucial tool is the following transfer theorem.

Theorem 2.5 (Transfer). *Let \mathcal{L} be a cubic algebra and $a, b \in \mathcal{L}$. Then*

$$a \sim b \text{ in } \mathcal{L} \iff a \sim b \text{ in } \text{env}(\mathcal{L}).$$

Furthermore, if $a \in \mathcal{L}$, $b \in \text{env}(\mathcal{L})$ and $a \sim b$ then $b \in \mathcal{L}$.

Proof. Since \mathcal{L} is an upwards closed subalgebra of the MR-algebra $\text{env}(\mathcal{L})$. \square

The use of the transfer theorem is to allow us to prove facts about \sim in a cubic algebra by proving them in an MR-algebra. But then we are actually working in a finitely generated sub-algebra of an MR-algebra which is isomorphic to an interval algebra. Thus we can always assume we are in an interval algebra.

In some arbitrary cubic algebra \mathcal{L} there are three operations to consider:

- $a \wedge b$ – will give rise to meets in \mathcal{L}/\sim ;
- $a * b = a \vee \Delta(a \vee b, b)$ – this operation will give rise to joins in \mathcal{L}/\sim ;
- $a \Rightarrow b = \Delta(a \vee b, a) \rightarrow b$ – this operation will give rise to implication in \mathcal{L}/\sim .

We note that $a * b$ and $a \Rightarrow b$ are defined for any two elements in any cubic algebra.

Over any implication algebra the relation \sim simplifies immensely.

Lemma 2.6. *Let $\langle a, b \rangle$ and $\langle c, d \rangle$ be in $\mathcal{I}(I)$. Then*

$$\langle a, b \rangle \sim \langle c, d \rangle \text{ iff } a \wedge b = c \wedge d.$$

Proof. Suppose that $\langle a, b \rangle = \Delta(\langle x, y \rangle, \langle c, d \rangle) = \langle x \wedge (y \rightarrow d), y \wedge (x \rightarrow c) \rangle$. Then $x \wedge (y \rightarrow d) \wedge y \wedge (x \rightarrow c) = [x \wedge (x \rightarrow c)] \wedge [y \wedge (y \rightarrow d)] = c \wedge d$.

Conversely if $a \wedge b = c \wedge d$ we can do all computations in the Boolean algebra $[c \wedge d, 1]$ – so that $\bar{a} \leq b$ and $\bar{c} \leq d$ – to get

$$\begin{aligned} \Delta(\langle a, b \rangle \vee \langle c, d \rangle, \langle c, d \rangle) &= \Delta(\langle a \vee c, b \vee d \rangle) \\ &= \langle (a \vee c) \wedge (\overline{b \vee d} \vee d), (b \vee d) \wedge (\overline{a \vee c} \vee c) \rangle \\ (a \vee c) \wedge (\overline{b \vee d} \vee d) &= (a \vee c) \wedge (\bar{b} \vee d) \\ &= (a \wedge \bar{b}) \vee (a \wedge d) \vee (c \wedge \bar{b}) \vee (c \wedge d) \\ &= \bar{b} \vee (a \wedge d) \vee (c \wedge \bar{b}) \vee (c \wedge d) \\ &= \bar{b} \vee (a \wedge d) \vee (c \wedge d) \\ &= \bar{b} \vee (a \wedge d) \vee (a \wedge b) \\ &= \bar{b} \vee (a \wedge d) \vee a \\ &= a. \\ (b \vee d) \wedge (\overline{a \vee c} \vee c) &= (b \vee d) \wedge (\bar{a} \vee c) \\ &= (b \wedge \bar{a}) \vee (b \wedge c) \vee (d \wedge \bar{a}) \vee (d \wedge c) \\ &= \bar{a} \vee (b \wedge c) \vee (d \wedge \bar{a}) \vee (c \wedge d) \\ &= \bar{a} \vee (b \wedge c) \vee (c \wedge d) \\ &= \bar{a} \vee (b \wedge c) \vee (a \wedge b) \\ &= \bar{a} \vee (b \wedge c) \vee b \\ &= b. \end{aligned}$$

\square

We can restate the lemma by saying that $\iota: \langle a, b \rangle \mapsto a \wedge b$ has the property that

$$(1) \quad \iota(\langle a, b \rangle) = \iota(\langle c, d \rangle) \text{ iff } \langle a, b \rangle \sim \langle c, d \rangle.$$

Thus for all $i \in I$ we have

$$\iota(e_I(i)) = i$$

so that ι is onto and $e_{\mathcal{I}}$ is a right inverse.

Since we will often work in the intuitively clearer setting of Boolean algebras we will restate these results in that context. In this context the relation \sim corresponds to a natural property of intervals – the *length*.

Definition 2.7. *Let $x = [x_0, x_1]$ be any interval in a Boolean algebra B . Then the length of x is $\overline{x_0} \wedge x_1 = \ell(x)$.*

Corollary 2.8. *Let b, c be intervals in a Boolean algebra B . Then*

$$b \sim c \iff \ell(b) = \ell(c).$$

Proof. We recall the isomorphism between the two definitions of $\mathcal{I}(B)$ given by

$$\langle a, b \rangle \mapsto [\overline{a}, b].$$

Then we have

$$\begin{aligned} \iota(\langle a, b \rangle) &= a \wedge b \\ \ell([\overline{a}, b]) &= \overline{\overline{a}} \wedge b \\ &= a \wedge b = \iota(\langle a, b \rangle). \end{aligned}$$

The result is now immediate. \square

The remainder of the proof can be found in [6] wherein we fully establish that \mathcal{L}/\sim is an implication lattice with the following operations:

$$\begin{aligned} \mathbf{1} &= [\mathbf{1}] \\ [a] \wedge [b] &= [a \wedge b] \\ [a] \vee [b] &= [a * b] \\ [a] \rightarrow [b] &= [a \Rightarrow b]; \end{aligned}$$

and that this implication algebra is, locally, exactly the same as \mathcal{L} .

Theorem 2.9. *On each interval $[a, \mathbf{1}]$ in \mathcal{L} the mapping $x \mapsto [x]$ is an implication embedding with upwards-closed range.*

3. IMPLICATION ALGEBRAS TO CUBES

In this section we develop a very general construction of cubic algebras. Although not every cubic algebra is isomorphic to one of this form (see [5]) we will show in the next section that every cubic algebra is very close to one of this form. We leave for later work a detailed analysis of exactly how close.

Let \mathcal{I} be an implication algebra. We define

$$\mathcal{I}(\mathcal{I}) = \{ \langle a, b \rangle \mid a, b \in \mathcal{I}, a \vee b = \mathbf{1} \text{ and } a \wedge b \text{ exists} \}$$

ordered by

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } a \leq c \text{ and } b \leq d.$$

This is a partial order that is an upper semi-lattice with join defined by

$$\langle a, b \rangle \vee \langle c, d \rangle = \langle a \vee c, b \vee d \rangle$$

and a maximum element $\mathbf{1} = \langle 1, 1 \rangle$.

We can also define a Δ function by

$$\text{if } \langle c, d \rangle \leq \langle a, b \rangle \text{ then } \Delta(\langle a, b \rangle, \langle c, d \rangle) = \langle a \wedge (b \rightarrow d), b \wedge (a \rightarrow c) \rangle.$$

We note the natural embedding of \mathcal{I} into $\mathcal{S}(\mathcal{I})$ given by

$$e_{\mathcal{I}}(a) = \langle 1, a \rangle.$$

Note also that in an implication algebra $a \vee b = \mathbf{1}$ iff $a \rightarrow b = b$ iff $b \rightarrow a = a$.

Also $\Delta(\mathbf{1}, \bullet)$ is particularly simply defined as it is exactly $\langle a, b \rangle \mapsto \langle b, a \rangle$.

We wish to show that the structure we have just described is a cubic algebra. We do this by showing that if \mathcal{I} is a Boolean algebra then $\mathcal{S}(\mathcal{I})$ is isomorphic to an interval algebra, and then use the fact that every interval in \mathcal{I} is a Boolean algebra and $\mathcal{S}([a, \mathbf{1}])$ sits naturally inside $\mathcal{S}(\mathcal{I})$.

Lemma 3.1. *Let B be a Boolean algebra. Then $\mathcal{S}(B)$ is isomorphic to the interval algebra of B .*

Proof. Let $\langle a, b \rangle \mapsto [\bar{a}, b]$. Since $a \wedge b$ exists for all $a, b \in B$ this imposes no hardship. The condition $a \rightarrow b = b$ is equivalent to $\bar{a} \leq b$. It is now clear that this mapping is a one-one, onto homomorphism.

We just check how the operations transfer:

$$\begin{aligned} \langle a, b \rangle \vee \langle c, d \rangle &= \langle a \vee c, b \vee d \rangle \\ &\mapsto [\bar{a} \wedge \bar{c}, b \vee d] \\ &= [\bar{a}, b] \vee [\bar{c}, d]. \\ \Delta(\langle a, b \rangle, \langle c, d \rangle) &= \langle a \wedge (b \rightarrow d), b \wedge (a \rightarrow c) \rangle \\ &\mapsto [\bar{a} \vee (b \wedge \bar{d}), b \wedge (\bar{a} \vee c)] \\ &= \Delta([\bar{a}, b], [\bar{c}, d]). \end{aligned}$$

□

Now to check that the axioms of a cubic algebra hold we just need to note that all of the axioms take place in some interval algebra – since working above some $x = [u, v] \in \mathcal{S}(\mathcal{I})$ means that all the computations take place in the interval algebra $\mathcal{S}([u \wedge v, \mathbf{1}])$ – which we already know to be a cubic algebra.

In fact we also have

Lemma 3.2. $[\langle a, b \rangle, \langle 1, 1 \rangle] \sim [a \wedge b, 1]$

Proof. Since $[\langle a, b \rangle, \langle 1, 1 \rangle] \sim [a, 1] \times [b, 1] \sim [a \wedge b, 1]$ by $\langle c, d \rangle \mapsto \langle c, d \rangle \mapsto c \wedge d$. The last is an isomorphism as it is an isomorphism of Boolean algebras and in $[a \wedge b, 1]$ the complement of a is b . □

4. SOME CATEGORY THEORY

The operation \mathcal{S} is a functor where we define $\mathcal{S}(f): \mathcal{S}(\mathcal{I}_1) \rightarrow \mathcal{S}(\mathcal{I}_2)$ by

$$\mathcal{S}(f)(\langle a, b \rangle) = \langle f(a), f(b) \rangle$$

whenever $f: \mathcal{I}_1 \rightarrow \mathcal{I}_2$ is an implication morphism.

Since f preserves all joins, implications and whatever meets exist we easily see that $\mathcal{S}(f)$ is a cubic morphism.

Clearly $\mathcal{S}(fg) = \mathcal{S}(f)\mathcal{S}(g)$. The relation \sim defined above gives rise to a functor \mathcal{C} on cubic algebras. Before defining this we need a lemma.

Lemma 4.1. *Let $\phi: \mathcal{L}_1 \rightarrow \mathcal{L}_2$ be a cubic homomorphism. Let $a, b \in \mathcal{L}_1$. Then*

$$a \sim b \Rightarrow \phi(a) \sim \phi(b).$$

Proof.

$$\begin{aligned}
 a \sim b &\iff \Delta(a \vee b, a) = b \\
 &\implies \phi(\Delta(a \vee b, a)) = \phi(b) \\
 &\iff \Delta(\phi(a) \vee \phi(b), \phi(a)) = \phi(b) \\
 &\iff \phi(a) \sim \phi(b).
 \end{aligned}$$

□

Now \mathcal{C} is defined by

$$\begin{aligned}
 \mathcal{C}(\mathcal{L}) &= \mathcal{L} / \sim \\
 \mathcal{C}(\phi)([x]) &= [\phi(x)].
 \end{aligned}$$

It is easily seen that \mathcal{C} is a functor from the category of cubic algebras to the category of implication algebras.

There are several natural transformations here. The basic ones are $e: \text{ID} \rightarrow \mathcal{I}$ and $\eta: \text{ID} \rightarrow \mathcal{C}$. These two are defined by

$$\begin{aligned}
 e_{\mathcal{I}}(x) &= \langle \mathbf{1}, x \rangle \\
 \eta_{\mathcal{L}}(x) &= [x].
 \end{aligned}$$

The commutativity of the diagram

$$\begin{array}{ccc}
 \mathcal{I}_1 & \xrightarrow{\phi} & \mathcal{I}_2 \\
 e_{\mathcal{I}_1} \downarrow & & \downarrow e_{\mathcal{I}_2} \\
 \mathcal{I}(\mathcal{I}_1) & \xrightarrow{\mathcal{I}(\phi)} & \mathcal{I}(\mathcal{I}_2)
 \end{array}$$

is from – for $x \in \mathcal{I}_1$

$$\begin{aligned}
 e_{\mathcal{I}_2}(\phi(x)) &= \langle \mathbf{1}, \phi(x) \rangle \\
 &= \langle \phi(\mathbf{1}), \phi(x) \rangle \\
 &= \mathcal{I}(\phi)(\langle \mathbf{1}, x \rangle) \\
 &= \mathcal{I}(\phi)e_{\mathcal{I}_1}(x).
 \end{aligned}$$

The commutativity of the diagram

$$\begin{array}{ccc}
 \mathcal{L}_1 & \xrightarrow{\phi} & \mathcal{L}_2 \\
 \eta_{\mathcal{L}_1} \downarrow & & \downarrow \eta_{\mathcal{L}_2} \\
 \mathcal{C}(\mathcal{L}_1) & \xrightarrow{\mathcal{C}(\phi)} & \mathcal{C}(\mathcal{L}_2)
 \end{array}$$

is from – for $x \in \mathcal{L}_1$

$$\begin{aligned}
 \eta_{\mathcal{L}_2}(\phi(x)) &= [\phi(x)] \\
 &= \mathcal{C}(\phi)([x]) \\
 &= \mathcal{C}(\phi)\eta_{\mathcal{L}_1}(x).
 \end{aligned}$$

Then we get the composite transformation $\iota: \text{ID} \rightarrow \mathcal{C}\mathcal{S}$ defined by

$$\iota_I = \eta_{\mathcal{S}(I)} \circ e_I.$$

By standard theory this is a natural transformation. It is easy to see that e_I is an embedding, and that $\eta_{\mathcal{L}}$ is onto. But there's more!

Theorem 4.2. ι_I is an isomorphism.

Proof. Let $x, y \in I$ and suppose that $\iota(x) = \iota(y)$. Then

$$\begin{aligned} \iota(x) &= \eta_{\mathcal{S}(I)}(e_I(x)) \\ &= [\langle \mathbf{1}, x \rangle] \\ &= [\langle \mathbf{1}, y \rangle]. \end{aligned}$$

Thus $\langle \mathbf{1}, x \rangle \sim \langle \mathbf{1}, y \rangle$. Now

$$\begin{aligned} \Delta(\langle \mathbf{1}, x \rangle \vee \langle \mathbf{1}, y \rangle, \langle \mathbf{1}, y \rangle) &= \Delta(\langle \mathbf{1}, x \vee y \rangle, \langle \mathbf{1}, y \rangle) \\ &= \langle (x \vee y) \rightarrow y, x \vee y \rangle. \end{aligned}$$

This equals $\langle \mathbf{1}, x \rangle$ iff $x = x \vee y$ (so that $y \leq x$) and $(x \vee y) \rightarrow y = \mathbf{1}$ so that $y = x \vee y$ and $x \leq y$. Thus $x = y$. Hence ι is one-one.

It is also onto, as if $z \in \mathcal{C}\mathcal{S}(I)$ then we have $z = [w]$ for some $w \in \mathcal{S}(I)$. But we know that $w = \langle x, y \rangle \sim \langle \mathbf{1}, x \wedge y \rangle$ – since $\Delta(\langle \mathbf{1}, y \rangle, \langle \mathbf{1}, x \wedge y \rangle) = \langle x, y \rangle$ – and so $z = [\langle \mathbf{1}, x \wedge y \rangle] = \eta_{\mathcal{S}(I)}(e_I(x \wedge y))$. \square

We note that there is also a natural transformation $\kappa: \text{ID} \rightarrow \mathcal{S}\mathcal{C}$ defined by

$$\kappa_{\mathcal{L}} = e_{\mathcal{C}(\mathcal{L})} \circ \eta_{\mathcal{L}}.$$

In general this is not an isomorphism as there may be an MR-algebra \mathcal{M} which is not a filter algebra, but $\mathcal{S}(\mathcal{C}(\mathcal{M}))$ is always a filter algebra.

We also note that $\iota_{\mathcal{C}(\mathcal{L})} = \mathcal{C}(\kappa_{\mathcal{L}})$ for all cubic algebras \mathcal{L} . The pair \mathcal{S} and \mathcal{C} do not form an adjoint pair.

5. THE RANGE OF \mathcal{S}

In this section we wish to consider the relationship between \mathcal{L} and $\mathcal{S}(\mathcal{L}/\sim)$. In the case of $\mathcal{L} = \mathcal{S}(I)$, we saw in theorem 4.2 that the two structures I and $\mathcal{C}(\mathcal{L})$ are naturally isomorphic and that the set $e_I[I] \subseteq \mathcal{S}(I)$ has a very special place. This leads to the notion of *g-cover*.

Definition 5.1. Let \mathcal{L} be a cubic algebra. Then $J \subseteq \mathcal{L}$ is a *g-cover* iff J is an upwards-closed implication subalgebra and

$$j: J \hookrightarrow \mathcal{L} \xrightarrow{\eta} \mathcal{L}/\sim$$

is an isomorphism.

If J is meet-closed we say that J is a *g-filter*.

We note that $\mathcal{S}(I)$ has a *g-cover* – namely $e_I[I]$. We want to show that this is (essentially) the only way to get *g-covers*, and that having them simplifies the study of such second-order properties as congruences and homomorphisms.

If J is a *g-cover* and $x \in \mathcal{L}$ then we have $x \sim j^{-1}(\eta(x)) \in J$ and so $[[J]] = \mathcal{L}$. We need to be very precise about how J generates \mathcal{L} which leads to the next two lemmas.

Lemma 5.2. Let J be a *g-cover* for \mathcal{L} and $x, y \in J$ with $x \sim y$. Then $x = y$.

Proof. If $x \sim y$ then $\eta(x) = \eta(y)$ and so $j(x) = j(y)$. As j is one-one on J this entails $x = y$. \square

Lemma 5.3. *Let J be a g -cover for \mathcal{L} and $x \in \mathcal{L}$. There exists unique pair α, β in J with $\alpha \geq \beta$ and $\Delta(\alpha, \beta) = x$.*

Proof. Let $x \in \mathcal{L}$. Then $\eta(x) \in \mathcal{L}/\sim = \text{rng}(j)$. Hence there is some $\beta \in J$ with $\eta(\beta) = \eta(x)$ and so $\beta \sim x$. Let $\alpha = \beta \vee x$.

If there is some other α' and β' in J with $\Delta(\alpha', \beta') = x$ then $\beta' \sim x \sim \beta$ and so (by lemma 5.2) $\beta' = \beta$. Then $\alpha' = \beta' \vee x = \beta \vee x = \alpha$. \square

Theorem 5.4. *Suppose that \mathcal{M} is an MR-algebra and J is a g -cover. Then J is a filter – in fact a g -filter by the above remarks.*

Proof. Let $x, y \in J$. Then we have $j(x \vee y) = j(x) \vee j(y) = \eta(x) \vee \eta(y) = \eta(x * y)$ so that $x \vee y \sim x * y = x \vee \Delta(x \vee y, y)$. As $(x \vee y) \wedge (x * y)$ exists this implies $x \vee y = x * y = x \vee \Delta(x \vee y, y)$ and so (by the MR-axiom) $x \wedge y$ exists. Now let $w \in J$ be such that $w \sim (x \wedge y)$. Then there is some $x' \geq w$ with $x' \sim x$ and so $x' = x$ as $x, x' \in J$. Likewise $w \leq y$ and so $w \leq x \wedge y$ i.e. $w = x \wedge y$ is in J . \square

Remark 5.1. The above proof also shows us that if J is a g -cover and $x, y \in J$ are such that $x \wedge y$ exists, then $x \wedge y \in J$.

G-filters were considered in [5] and used to get an understanding of automorphism groups and the lattice of congruences. G-covers generalize the notion of g -filters to a larger class of algebras, but we'll leave applications to second-order properties to another paper.

Now suppose that \mathcal{L} is any cubic algebra with a g -cover J . We want to show that $\mathcal{I}(J) \sim \mathcal{L}$. For each $x \in \mathcal{L}$ there is a unique pair $\alpha(x), \beta(x)$ in J such that $\beta(x) \leq \alpha(x)$ and $x = \Delta(\alpha(x), \beta(x))$. Define

$$\phi: \mathcal{L} \rightarrow \mathcal{I}(J)$$

by

$$\phi(x) = \langle \alpha(x), \alpha(x) \rightarrow \beta(x) \rangle.$$

We need to show that this is one-one, onto and order-preserving.

We first note that $\alpha(x) \rightarrow \beta(x) = \Delta(\mathbf{1}, x) \vee \beta(x)$. Since $x \sim \beta(x)$ we have $\beta(x) = (\Delta(\mathbf{1}, x) \vee \beta(x)) \wedge (x \vee \beta(x))$ and trivially $\mathbf{1} = (\Delta(\mathbf{1}, x) \vee \beta(x)) \vee (x \vee \beta(x))$. Hence the complement of $\alpha(x) = x \vee \beta(x)$ over $\beta(x)$ must be $\alpha(x) \rightarrow \beta(x) = \Delta(\mathbf{1}, x) \vee \beta(x)$.

One-one: Suppose that $\phi(x) = \phi(y)$. Then we have

$$\begin{aligned} \alpha(x) \rightarrow \beta(x) &= \alpha(y) \rightarrow \beta(y) \\ \alpha(x) &= \alpha(y) \end{aligned}$$

Therefore

$$\beta(x) = (\alpha(x) \rightarrow \beta(x)) \wedge \alpha(x) = (\alpha(y) \rightarrow \beta(y)) \wedge \alpha(y) = \beta(y)$$

and so we have

$$x = \Delta(\alpha(x), \beta(x)) = \Delta(\alpha(y), \beta(y)) = y.$$

Onto: Let $\langle a, b \rangle \in \mathcal{I}(J)$. Let $z = \Delta(a, a \wedge b)$. Then we have – by uniqueness – that $\alpha(z) = a$ and $\beta(z) = a \wedge b$ and so $a \rightarrow (a \wedge b) = a \rightarrow b = b$ – by definition of $\mathcal{I}(J)$.

Order-preserving: Suppose that $x \leq y$. Then we have $x \sim \beta(x)$ and so there is some $b \geq \beta(x)$ with $b \sim y$. As $b \in J$ we get $\beta(y) = b$. Hence $\alpha(x) = x \vee \beta(x) \leq y \vee \beta(y) = \alpha(y)$. Also (as $x \leq y$) $\Delta(\mathbf{1}, x) \leq \Delta(\mathbf{1}, y)$ and so $\Delta(\mathbf{1}, x) \vee \beta(x) \leq \Delta(\mathbf{1}, y) \vee \beta(y)$.

Thus we have

Theorem 5.5. *A cubic algebra \mathcal{L} has a g-cover iff \mathcal{L} is isomorphic to $\mathcal{I}(\mathcal{I})$ for some implication algebra \mathcal{I} .*

It follows from the above theorems that not every cubic algebra has a g-cover – as we know that MR-algebras not isomorphic to filter algebras may exist (under certain set-theoretic assumptions) – see [5] section 6.

6. ENV AND G-COVERS

In this section we consider the relationship between g-covers in a cubic algebra and in its envelope. We discover that g-covers go downwards and upwards – ie one has a g-cover iff the other has one.

Theorem 6.1. *Let \mathcal{L} be a cubic algebra and suppose that $\mathcal{I}(\mathcal{F})$ is a filter algebra and $\text{env}(\mathcal{L}) \xrightarrow[\phi]{\sim} \mathcal{I}(\mathcal{F})$ is a cubic homomorphism with upwards-closed range. Then the homomorphism restricts to \mathcal{L} as –*

$$\begin{array}{ccc}
 \mathcal{L} & \xrightarrow[\phi \upharpoonright \mathcal{L}]{\dots\dots\dots} & \mathcal{I}(\mathcal{F} \cap \mathcal{L}) \\
 \downarrow & & \downarrow \mathcal{I}(\text{incl}) \\
 \text{env}(\mathcal{L}) & \xrightarrow[\phi]{} & \mathcal{I}(\mathcal{F})
 \end{array}$$

where $\mathcal{F} \cap \mathcal{L} = \{\phi(l) \mid l \in \mathcal{L} \text{ and } \phi(l) \in \mathcal{F}\}$.

Proof. We first note that $\mathcal{F} \cap \mathcal{L}$ is an implication algebra as $\phi[\mathcal{L}]$ and \mathcal{F} are implication subalgebras of $\mathcal{I}(\mathcal{F})$.

Claim 1: If $l \in \mathcal{L}$ then $\phi(l) \in \mathcal{I}(\mathcal{F} \cap \mathcal{L})$.

\mathcal{L} is upwards closed in $\text{env}(\mathcal{L})$ and so $\phi[\mathcal{L}]$ is upwards closed in $\mathcal{I}(\mathcal{F})$. Thus $\mathcal{F} \cap \mathcal{L}$ is an upper segment of \mathcal{F} .

Let $l \in \mathcal{L}$. Then $\phi(l) \in \mathcal{I}(\mathcal{F})$ and so there is some $l' \in \mathcal{F} \cap \mathcal{L}$ so that $l' \sim \phi(l)$. Then $\phi(l) \vee l' \in \mathcal{F} \cap \mathcal{L}$ and so $\phi(l) = \Delta(\phi(l) \vee l', l') \in \mathcal{I}(\mathcal{F} \cap \mathcal{L})$.

Claim 2: $\phi \upharpoonright \mathcal{L}$ is onto $\mathcal{I}(\mathcal{F} \cap \mathcal{L})$.

If $x \in \mathcal{I}(\mathcal{F} \cap \mathcal{L})$ then we can find some $x' \in \mathcal{F} \cap \mathcal{L}$ so that $x \sim x'$. By definition $x' = \phi(l)$ for some $l \in \mathcal{L}$ and as $x \vee \phi(l) \in \mathcal{F} \cap \mathcal{L}$ there is also some $m \in \mathcal{L}$ with $\phi(m) = x \vee \phi(l)$. Now we have $x = \Delta(x \vee \phi(l), \phi(l)) = \Delta(\phi(m), \phi(l)) = \phi(\Delta(m, l))$ is in the range of $\phi \upharpoonright \mathcal{L}$.

□

Corollary 6.2. *If $\text{env}(\mathcal{L})$ is isomorphic to a filter algebra then \mathcal{L} has a g-cover.*

Proof. Let $\phi: \text{env}(\mathcal{L}) \rightarrow \mathcal{I}(\mathcal{F})$ be the isomorphism. Then $\phi \upharpoonright \mathcal{L}$ is also an isomorphism – it is one-one as it is the restriction of a one-one function, and onto by the theorem. Since $\mathcal{I}(\mathcal{F} \cap \mathcal{L})$ has a g-cover, so does \mathcal{L} . □

The above results show that g-covers go down to certain subalgebras. Now we look at making them go up.

Theorem 6.3. *Let \mathcal{L} be a cubic algebra and suppose that J is a g -cover for \mathcal{L} . Then J has fp in $\text{env}(\mathcal{L})$ and the filter it generates is a g -filter.*

Proof. This is very like the proof to theorem 5.4. Let $x, y \in J$. Then we have $j(x \vee y) = j(x) \vee j(y) = \eta(x) \vee \eta(y) = \eta(x * y)$ so that $x \vee y \sim x * y = x \vee \Delta(x \vee y, y)$. As $(x \vee y) \wedge (x * y)$ exists this implies $x \vee y = x * y = x \vee \Delta(x \vee y, y)$. Thus in $\text{env}(\mathcal{L})$ the meet $x \wedge y$ exists. By earlier work ([5], Lemma 19) this implies J has fp in $\text{env}(\mathcal{L})$.

Let \mathcal{F} be the filter generated by J .

Now if $z \in \text{env}(\mathcal{L})$ we have $x_1, \dots, x_k \in \mathcal{L}$ such that $x_1 \wedge (x_2 \wedge (\dots \wedge x_k)) = z$. Let $y_i \in J$ be such that $x_i \sim y_i$. Then $y_1 \wedge \dots \wedge y_k \leq z$ and so $z \in \llbracket \mathcal{F} \rrbracket$. \square

Corollary 6.4. *Let \mathcal{L} be an upwards-closed cubic subalgebra of a cubic algebra \mathcal{M} with g -cover J . Then \mathcal{L} has a g -cover.*

Proof. Let J be as given and let \hat{J} be the extension to a g -filter for $\text{env}(\mathcal{M})$. Then we have

$$\begin{array}{ccccc}
 \mathcal{L} & \xrightarrow{\phi \upharpoonright \mathcal{L}} & & & \mathcal{I}(\hat{J} \upharpoonright \mathcal{L}) \\
 \downarrow & \searrow & & & \downarrow \\
 & & \mathcal{M} & \xrightarrow{\sim} & \mathcal{I}(J) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{env}(\mathcal{L}) & \hookrightarrow & \text{env}(\mathcal{M}) & \xrightarrow{\sim} & \mathcal{I}(\hat{J})
 \end{array}$$

Since $\phi \upharpoonright \mathcal{L}$ is one-one and onto we have the result. \square

Remark 6.1. By a slightly different argument we can show that if \mathcal{L} is an upwards-closed cubic subalgebra of a cubic algebra \mathcal{M} with g -cover J , then $\mathcal{L} \cap J$ is a g -cover for \mathcal{L} .

Definition 6.5. *A cubic algebra \mathcal{L} is countable presented iff there is a countable set $A \subseteq \mathcal{L}$ such that $\mathcal{L} = \bigcup_{a \in A} \mathcal{L}_a$.*

It is easy to show that if \mathcal{L} is countable presented, then so is $\text{env}(\mathcal{L})$. It then follows from the fact that every countable presented MR-algebra is a filter algebra that every countable presented cubic algebra has a g -cover.

Another interesting consequence for implication algebras is

Theorem 6.6. *Let \mathcal{I} be an implication algebra. Then \mathcal{I} is isomorphic to an upper segment of a filter.*

Proof. Consider

$$p: \mathcal{I} \xrightarrow{e_{\mathcal{I}}} \mathcal{I} \hookrightarrow \text{env}(\mathcal{I}) \xrightarrow{\eta} \text{env}(\mathcal{I}) / \sim.$$

Then p is an implication morphism as each component is one, and it is easy to see that the range of p is upwards closed. We want to see that p is one-one:

$$\begin{aligned}
 p(x) = p(y) &\rightarrow \eta(\text{incl}(e_{\mathcal{I}}(x))) = \eta(\text{incl}(e_{\mathcal{I}}(y))) \\
 &\rightarrow \text{incl}(e_{\mathcal{I}}(x)) \sim \text{incl}(e_{\mathcal{I}}(y)) \\
 &\rightarrow e_{\mathcal{I}}(x) \sim e_{\mathcal{I}}(y).
 \end{aligned}$$

This implies $x = y$ since if $e_{\mathcal{I}}(x) = \langle 1, x \rangle \sim e_{\mathcal{I}}(y) = \langle 1, y \rangle$ then $\langle 1, x \rangle = \Delta(\langle 1, x \vee y \rangle, \langle 1, y \rangle) = \langle (x \vee y) \rightarrow y, x \vee y \rangle$ and so $x \vee y = x$ (and therefore $y \leq x$) and $1 = (x \vee y) \rightarrow y$ (and therefore $x \vee y \leq y$ i.e. $x \leq y$). Thus $x = y$. \square

The filter obtained by this theorem sits over \mathcal{I} in a way similar to the way $\text{env}(\mathcal{L})$ sits over \mathcal{L} . For that reason we will also call this an *enveloping lattice* for an implication algebra and denote it by $\text{env}(\mathcal{I})$. The next theorem is clear.

Theorem 6.7. *Let \mathcal{L} be a cubic algebra with g-cover J . Then $\text{env}(J)$ is isomorphic to $\text{env}(\mathcal{L})/\sim$ and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\sim} & \mathcal{I}(J) \\ \downarrow & & \downarrow \\ \text{env}(\mathcal{L}) & \xrightarrow{\sim} \mathcal{I}(\text{env}(\mathcal{L})/\sim) \xrightarrow{\mathcal{I}(\sim)} & \mathcal{I}(\text{env}(J)) \end{array}$$

Now we consider the last step in the puzzle – the relationship between \mathcal{L} and $\mathcal{I}(\mathcal{L}/\sim)$. Clearly they collapse to the same implication algebra. From corollary 6.4 we know that if \mathcal{L} has no g-cover then we cannot embed \mathcal{L} as an upwards-closed subalgebra of $\mathcal{I}(\mathcal{L}/\sim)$.

Embedding it as a subalgebra seems possible but we have no idea how to do it.

7. AN ALGEBRA OF COVERS

In this section we consider the family of all g-covers of a cubic algebra and deduce an interesting MR-algebra. This section is very like similar material on filters – see [3, 5]. Therein we showed the following results on finite intersection property.

Lemma 7.1. *Let $\mathcal{I}(B)$ be an interval algebra and $A \subseteq \mathcal{I}(B)$. Then A has fip iff for all $x, y \in A$ $x \wedge y$ exists.*

Definition 7.2. *Let \mathcal{L} be a cubic algebra and $A \subseteq \mathcal{L}$. A is compatible iff for all embeddings $e: \mathcal{L} \rightarrow \mathcal{I}(B)$, the set $e[A]$ has fip.*

Corollary 7.3. *Let \mathcal{L} be a cubic algebra and $A \subseteq \mathcal{L}$. Then A is compatible iff for all $x, y \in A$ $x \vee \Delta(\mathbf{1}, y) = \mathbf{1}$.*

For later we have the following useful lemma relating compatibility and the \leq relation.

Lemma 7.4. *If $x \leq y$ and $x \vee \Delta(\mathbf{1}, y) = \mathbf{1}$ then $x \leq y$.*

Proof. $x \leq y$ implies $y = (y \vee x) \wedge (y \vee \Delta(\mathbf{1}, x)) = y \vee x$ as the latter term is $\mathbf{1}$. Thus $x \leq y$. \square

Our interest is in a special class of upwards-closed implication subalgebras.

Definition 7.5. *A special subalgebra of a cubic algebra \mathcal{L} is an upwards-closed implication subalgebra I that is compatible and for all $x, y \in I$, if $x \wedge y$ exists in \mathcal{L} then $x \wedge y \in I$.*

Lemma 7.6. *Every g-cover is special.*

Proof. Let J be a g-cover. As noted in 5.1 the second condition holds.

Compatibility follows from theorem 6.3. \square

Lemma 7.7. *Let \mathbb{I} be a family of special subalgebras. Then $\bigcap \mathbb{I}$ is also special.*

Proof. Immediate. \square

This lemma implies that any compatible set is contained in some smallest special subalgebra.

Now we need to define some operations on special subalgebras.

Lemma 7.8. *Let \mathcal{L} be a cubic algebra and \mathcal{I} and \mathcal{J} be two special subalgebras. Then*

$$\mathcal{I} \cap \mathcal{J} = \{f \vee g \mid f \in \mathcal{I} \text{ and } g \in \mathcal{J}\}.$$

Proof. The RHS set is clearly a subset of both \mathcal{I} and \mathcal{J} .

And if $z \in \mathcal{I} \cap \mathcal{J}$ then $z = z \vee z$ is in the RHS set. \square

Definition 7.9. *Let \mathcal{I}, \mathcal{J} be two special subalgebras of \mathcal{L} . Then $\mathcal{I} \vee \mathcal{J}$ is defined iff $\mathcal{I} \cup \mathcal{J}$ is compatible, in which case it is the special subalgebra generated by $\mathcal{I} \cup \mathcal{J}$.*

Lemma 7.10. *If $\mathcal{I} \vee \mathcal{J}$ exists then it is equal to $\{f \wedge g \mid f \in \mathcal{I} \text{ and } g \in \mathcal{J} \text{ and } f \wedge g \text{ exists}\}$.*

Proof. Let S be this set. It is clearly contained in $\mathcal{I} \vee \mathcal{J}$.

To show the converse we need to show that S is a special subalgebra. Recall that $\mathcal{I} \cup \mathcal{J}$ is assumed to be compatible.

Upwards-closure: if $h \geq f \wedge g$ for $f \in \mathcal{I}$ and $g \in \mathcal{J}$ then $h = (h \vee f) \wedge (h \vee g)$ is also in S .

\rightarrow -closure: follows from upwards-closure.

Compatible: if $a \wedge b \in S$ and $f \wedge g \in S$ with $a, f \in \mathcal{I}$ and $b, g \in \mathcal{J}$ then a is compatible with both f and g so that $\mathbf{1} = a \vee \Delta(\mathbf{1}, f) = a \vee \Delta(\mathbf{1}, g)$, whence $\mathbf{1} = a \vee \Delta(\mathbf{1}, f \wedge g)$. Likewise $\mathbf{1} = b \vee \Delta(\mathbf{1}, f \wedge g)$ so that $\mathbf{1} = (a \wedge b) \vee \Delta(\mathbf{1}, f \wedge g)$.

All available intersections: if $a \wedge b \in S$ and $f \wedge g \in S$ with $a, f \in \mathcal{I}$ and $b, g \in \mathcal{J}$ and $(a \wedge b) \wedge (f \wedge g)$ exists in \mathcal{L} , then $s = a \wedge f \in \mathcal{I}$ and $t = b \wedge g \in \mathcal{J}$ and $s \wedge t$ exists, so that $s \wedge t \in S$. \square

It is easy to show that these operations are commutative, associative, idempotent and satisfy absorption. Distributivity also holds in a weak way.

Lemma 7.11. *Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be special subalgebras of a special subalgebra \mathcal{S} . Then*

$$\mathcal{I} \cap (\mathcal{J} \vee \mathcal{K}) = (\mathcal{I} \vee \mathcal{J}) \cap (\mathcal{I} \vee \mathcal{K}).$$

Proof. As everything sits inside the compatible set \mathcal{S} there are no issues of incompatibility.

Let $x = g \vee (h \wedge k) \in \mathcal{I} \cap (\mathcal{J} \vee \mathcal{K})$. Then $x = (g \vee h) \wedge (g \vee k)$ is in $(\mathcal{I} \vee \mathcal{J}) \cap (\mathcal{I} \vee \mathcal{K})$.

Conversely if $x = (g_1 \vee h) \wedge (g_2 \vee k)$ is in $(\mathcal{I} \vee \mathcal{J}) \cap (\mathcal{I} \vee \mathcal{K})$ then $g_1 \vee k \geq g_1 \in \mathcal{I}$ and $g_2 \vee h \geq g_2 \in \mathcal{I}$ and the meet exists, so $x \in \mathcal{I}$. Also $g_1 \vee k \geq k \in \mathcal{J}$ and $g_2 \vee h \geq h \in \mathcal{K}$ so that $x \in \mathcal{J} \vee \mathcal{K}$. \square

7.1. Near-principal. There is a very special case of special subalgebra that merits attention, as it leads into the general theory so well, *principal subalgebras*. These are of the form $[g, \mathbf{1}]$ for some $g \in \mathcal{L}$. It is easy to verify that these are special.

Also associated with elements of \mathcal{L} is an operation on special subalgebras. Suppose that \mathcal{I} is a special subalgebra.

Lemma 7.12. *The set*

$$\mathcal{I}_g = \{\Delta(g \vee f, f) \mid f \in \mathcal{I}\}$$

is compatible and upwards closed.

Proof. We just need to check this for intervals. Suppose that $g = [g_0, g_1]$, $f_0 = [x, y] \in \mathcal{I}$ and $f_1 = [s, t] \in \mathcal{I}$. Then

$$\begin{aligned}\Delta(g \vee f_0, f_0) &= [(g_0 \wedge x) \vee (g_1 \wedge \bar{y}), (g_0 \vee \bar{x}) \wedge (g_1 \vee y)] \\ \Delta(\mathbf{1}, \Delta(g \vee f_1, f_1)) &= [(\bar{g}_0 \wedge s) \vee (\bar{g}_1 \wedge \bar{t}), (\bar{g}_0 \vee \bar{s}) \wedge (\bar{g}_1 \vee t)]\end{aligned}$$

Thus

$$\begin{aligned}\Delta(g \vee f_0, f_0) \vee \Delta(\mathbf{1}, \Delta(g \vee f_1, f_1)) \\ &= [((g_0 \wedge x) \vee (g_1 \wedge \bar{y})) \wedge ((\bar{g}_0 \wedge s) \vee (\bar{g}_1 \wedge \bar{t})), \\ &\quad ((g_0 \vee \bar{x}) \wedge (g_1 \vee y)) \vee ((\bar{g}_0 \vee \bar{s}) \wedge (\bar{g}_1 \vee t))] \\ &= [(g_0 \wedge x \wedge \bar{g}_0 \wedge s) \vee (g_0 \wedge x \wedge \bar{g}_1 \wedge \bar{t}) \vee (g_1 \wedge \bar{y} \wedge \bar{g}_0 \wedge s) \vee (g_1 \wedge \bar{y} \wedge \bar{g}_1 \wedge \bar{t}), \\ &\quad (g_0 \vee \bar{x} \vee \bar{g}_0 \vee \bar{s}) \wedge (g_0 \vee \bar{x} \vee \bar{g}_1 \vee t) \wedge (g_1 \vee y \vee \bar{g}_0 \vee \bar{s}) \wedge (g_1 \vee y \vee \bar{g}_1 \vee t)] \\ &= [0, 1]\end{aligned}$$

since f_0 and f_1 are compatible and so

$$[0, 1] = f_0 \vee \Delta(\mathbf{1}) = [x \wedge \bar{t}, y \vee \bar{s}].$$

To show upwards closure we note that if $k \geq \Delta(g \vee f, f)$ for some $f \in \mathcal{I}$ then we have $k \in \llbracket \mathcal{I} \rrbracket$ and so there is some $k' \in \mathcal{I}$ with $k \sim k'$. Then we have $\Delta(g \vee k', k') \sim k \geq \Delta(g \vee f, f)$. This implies k and $\Delta(g \vee k', k')$ are compatible, and therefore equal. \square

Lemma 7.13.

$$\mathcal{I} \cap \mathcal{I}_g = [g, \mathbf{1}] \cap \mathcal{I}.$$

Proof. If $f \in \mathcal{I} \cap [g, \mathbf{1}]$ then $g \vee f = f$ and so $\Delta(g \vee f, f) = \Delta(f, f) = f \in \mathcal{I}_g$.

Conversely, if $h \in \mathcal{I} \cap \mathcal{I}_g$ then we have h and $\Delta(g \vee h, h)$ are compatible and so $h = \Delta(g \vee h, h)$. Therefore $g \vee h = h$ and $g \leq h$. \square

Theorem 7.14. *The set*

$$\mathcal{I}_g = \{\Delta(g \vee f, f) \mid f \in \mathcal{I}\}$$

is a special subalgebra and $\llbracket \mathcal{I}_g \rrbracket = \llbracket \mathcal{I} \rrbracket$.

Proof. That \mathcal{I}_g is compatible and upwards-closed follows from the lemma. If $f_1, f_2 \in \mathcal{I}$ and the meet $\Delta(g \vee f_1, f_1) \wedge \Delta(g \vee f_2, f_2)$ exists. Let $h_i = \Delta(g \vee f_i, f_i)$.

In any interval algebra, if $f_1 \wedge f_2$ exists, then $\Delta((g \vee f_1) \wedge (g \vee f_2), f_1 \wedge f_2) = h_1 \wedge h_2$.

In this case, we know that $h_1 \wedge h_2$ exists, and so $(g \vee f_1) \wedge (g \vee f_2)$ exists. This is therefore in \mathcal{I} as both factors are. As it is also in $[g, \mathbf{1}]$ it is in \mathcal{I}_g . From our remark concerning interval algebras we see that $\Delta((g \vee f_1) \wedge (g \vee f_2), h_1 \wedge h_2)$ is below both f_1 and f_2 so that it must equal it in \mathcal{I} . The same formula shows that $h_1 \wedge h_2$ is in \mathcal{I}_g .

By definition, for each $f \in \mathcal{I}$ there is a $f' \in \mathcal{I}_g$ such that $f \sim f'$, and conversely. Thus $\llbracket \mathcal{I}_g \rrbracket = \llbracket \mathcal{I} \rrbracket$. \square

Note that a special case of this is when $g = \mathbf{1}$ and we have $\mathcal{I}_1 = \Delta(\mathbf{1}, \mathcal{I})$ and that for a principal filter $[h, \mathbf{1}]$ we have $[h, \mathbf{1}]_g = [\Delta(g \vee h, h), \mathbf{1}]$.

Corollary 7.15. *The set*

$$g \rightarrow \mathcal{I} = \{g \rightarrow f \mid f \in \mathcal{I}\}$$

is a special subalgebra.

Proof. Recall that $g \rightarrow f = \Delta(\mathbf{1}, \Delta(g \vee f, f)) \vee f$. Hence if $\mathcal{J} = \Delta(\mathbf{1}, \mathcal{I}_g)$ then

$$\begin{aligned} \mathcal{J} \cap \mathcal{J} &= \{f \vee \beta_{\mathcal{J}}(f) \mid f \in \mathcal{J}\} \\ &= \{\Delta(\mathbf{1}, \Delta(g \vee f, f)) \vee f \mid f \in \mathcal{J}\} \\ &= \{g \rightarrow f \mid f \in \mathcal{J}\}. \end{aligned}$$

□

Corollary 7.16. *If $g \in \mathcal{F}$ then*

$$\mathcal{F} \cap \mathcal{F}_g = [g, \mathbf{1}].$$

Proof. Obvious

□

Interestingly enough the converse of lemma 7.13 is also true.

Lemma 7.17. *Suppose that $[[\mathcal{J}]] = [[\mathcal{I}]]$ and $\mathcal{J} \cap \mathcal{I} = [g, \mathbf{1}]$. Then*

$$\mathcal{J} = \mathcal{I}_g.$$

Proof. Clearly $[g, \mathbf{1}] \subseteq \mathcal{J}$.

For arbitrary $h \in \mathcal{J}$ we can find $f \in \mathcal{I}$ and $h' \in \mathcal{I}_g$ with $\Delta(g \vee f, f) = h' \sim h$. Then $h' \vee f = g \vee f$.

Also $h \vee f \in \mathcal{J} \cap \mathcal{I}$ and so $g \leq h \vee f$. Now $h \sim h' \leq g \vee f \in \mathcal{J}$ implies $h \leq g \vee f$ also. Thus $g \vee f = h \vee f = h' \vee f$.

As $f \sim h \sim h'$ we have $h' = \Delta(h' \vee f, f) = \Delta(h \vee f, f) = h$.

Thus $\mathcal{J} \subseteq \mathcal{I}_g$.

The reverse implication follows as $[[\mathcal{J}]] = [[\mathcal{I}]] = [[\mathcal{I}_g]]$ and so if $h \in \mathcal{I}_g$ there is some $h' \in \mathcal{J}$ with $h \sim h'$. As h and h' are compatible (as $\mathcal{J} \subseteq \mathcal{I}_g$) we have $h = h' \in \mathcal{J}$. □

Corollary 7.18. *Let $g, h \in \mathcal{I}$. Then*

- (a) $\mathcal{I} = (\mathcal{I}_g)_g$;
- (b) $(\mathcal{I}_g)_h = (\mathcal{I}_g)_{g \vee h}$.

Proof. (a) Since $\mathcal{I} \cap \mathcal{I}_g = [g, \mathbf{1}]$ and $[[\mathcal{I}]] = [[\mathcal{I}_g]]$ the lemma implies $\mathcal{I} = (\mathcal{I}_g)_g$.

(b)

$$\begin{aligned} \mathcal{I}_g \cap (\mathcal{I}_g)_h &= [h, \mathbf{1}] \cap \mathcal{I}_g \\ &= [h, \mathbf{1}] \cap \mathcal{I} \cap \mathcal{I}_g \\ &= [h, \mathbf{1}] \cap [g, \mathbf{1}] \\ &= [h \vee g, \mathbf{1}]. \end{aligned}$$

The lemma now implies $(\mathcal{I}_g)_h = (\mathcal{I}_g)_{g \vee h}$.

□

7.2. Relative Complements. Let $\mathcal{J} \subseteq \mathcal{I}$ be two special subalgebras. There are several ways to define the relative complement of \mathcal{J} in \mathcal{I} .

Definition 7.19. *Let $\mathcal{J} \subseteq \mathcal{I}$ be two special subalgebras. Then*

- (a) $\mathcal{J} \supset \mathcal{I} = \bigcap \{\mathcal{H} \mid \mathcal{H} \vee \mathcal{J} = \mathcal{I}\}$;
- (b) $\mathcal{J} \Rightarrow \mathcal{I} = \bigvee \{\mathcal{H} \mid \mathcal{H} \subseteq \mathcal{I} \text{ and } \mathcal{H} \cap \mathcal{J} = \{\mathbf{1}\}\}$;
- (c) $\mathcal{J} \rightarrow \mathcal{I} = \{h \in \mathcal{I} \mid \forall g \in \mathcal{J} \ h \vee g = \mathbf{1}\}$.

We will now show that these all define the same set.

Lemma 7.20. $\mathcal{J} \rightarrow \mathcal{I} = \mathcal{J} \Rightarrow \mathcal{I}$.

Proof. Let $h \in (\mathcal{J} \rightarrow \mathcal{I}) \cap \mathcal{J}$. Then $\mathbf{1} = h \vee h = h$. Thus $\mathcal{J} \rightarrow \mathcal{I} \subseteq \mathcal{J} \Rightarrow \mathcal{I}$.

Suppose that $\mathcal{H} \subseteq \mathcal{I}$ and $\mathcal{H} \cap \mathcal{J} = \{\mathbf{1}\}$. Let $h \in \mathcal{H}$ and $g \in \mathcal{J}$. Then $h \vee g \in \mathcal{H} \cap \mathcal{J} = \{\mathbf{1}\}$ so that $h \vee g = \mathbf{1}$. Hence $\mathcal{H} \subseteq (\mathcal{J} \rightarrow \mathcal{I})$ and so $\mathcal{J} \Rightarrow \mathcal{I} \subseteq \mathcal{J} \rightarrow \mathcal{I}$. \square

Lemma 7.21. *Let $h \in \mathcal{I}$ and $g \in \mathcal{J}$ be such that $g \vee h < \mathbf{1}$. Then $h \notin g \rightarrow \mathcal{I}$.*

Proof. This is clear as $h = g \rightarrow f$ implies $h \vee g = \mathbf{1}$. \square

Theorem 7.22. $\mathcal{J} \supset \mathcal{I} = \mathcal{J} \rightarrow \mathcal{I}$.

Proof. Suppose that $h \notin \mathcal{J} \rightarrow \mathcal{I}$ so that there is some $g \in \mathcal{J}$ with $h \vee g < \mathbf{1}$. Then $h \notin g \rightarrow \mathcal{I}$ and clearly $\mathcal{I} = [g, \mathbf{1}] \vee (g \rightarrow \mathcal{I})$ so that $\mathcal{G} \supset \mathcal{I} \subseteq g \rightarrow \mathcal{I}$ does not contain h . Thus $\mathcal{J} \supset \mathcal{I} \subseteq \mathcal{J} \rightarrow \mathcal{I}$.

Conversely if $\mathcal{H} \vee \mathcal{J} = \mathcal{I}$ and $k \in \mathcal{J} \rightarrow \mathcal{I}$ then there is some $h \in \mathcal{H}$ and $g \in \mathcal{J}$ with $k = h \wedge g$. But then

$$\begin{aligned} k &= k \vee (h \wedge g) \\ &= (k \vee h) \wedge (k \vee g) \\ &= k \vee h \end{aligned} \qquad \text{as } k \vee g = \mathbf{1}$$

and so $k \geq h$ must be in \mathcal{H} . Thus $\mathcal{J} \rightarrow \mathcal{I} \subseteq \mathcal{J} \supset \mathcal{I}$. \square

We earlier defined a filter $g \rightarrow \mathcal{I}$. We now show that this new definition of \rightarrow extends this earlier definition.

Lemma 7.23. *Let $g \in \mathcal{I}$. Then*

$$g \rightarrow \mathcal{I} = [g, \mathbf{1}] \rightarrow \mathcal{I}.$$

Proof. Let $g \rightarrow f \in g \rightarrow \mathcal{I}$ and $k \in [g, \mathbf{1}]$. Then $k \vee (g \rightarrow f) \geq g \vee (g \rightarrow f) = \mathbf{1}$. Thus $g \rightarrow f \in [g, \mathbf{1}] \rightarrow \mathcal{I}$ and so $g \rightarrow \mathcal{I} \subseteq [g, \mathbf{1}] \rightarrow \mathcal{I}$.

Conversely, if $h \in [g, \mathbf{1}] \rightarrow \mathcal{I}$ then $h \vee g = \mathbf{1}$ and so h is the complement of g in $[h, \mathbf{1}]$. Thus $h = g \rightarrow h \in g \rightarrow \mathcal{I}$ and so $[g, \mathbf{1}] \rightarrow \mathcal{I} \subseteq g \rightarrow \mathcal{I}$. \square

Lemma 7.24. *Let $\mathcal{J} \subseteq \mathcal{H} \subseteq \mathcal{I}$. Then*

$$\mathcal{J} \rightarrow \mathcal{H} \subseteq \mathcal{J} \rightarrow \mathcal{I}.$$

Proof. If $h \in \mathcal{H}$ and $h \vee g = \mathbf{1}$ for all $g \in \mathcal{J}$ then $h \in \mathcal{J} \rightarrow \mathcal{I}$. \square

Corollary 7.25. *Let $\mathcal{J} \subseteq \mathcal{H} \subseteq \mathcal{I}$ and $\mathcal{J} \rightarrow \mathcal{I} \subseteq \mathcal{H}$. Then*

$$\mathcal{J} \rightarrow \mathcal{H} = \mathcal{J} \rightarrow \mathcal{I}.$$

Proof. LHS \subseteq RHS by the lemma. Conversely if $h \in \mathcal{J} \rightarrow \mathcal{I}$ then $h \in \mathcal{H}$ has the defining property for $\mathcal{J} \rightarrow \mathcal{H}$ and so is in $\mathcal{J} \rightarrow \mathcal{H}$. \square

Corollary 7.26.

$$\mathcal{J} \rightarrow (\mathcal{J} \vee (\mathcal{J} \rightarrow \mathcal{I})) = \mathcal{J} \rightarrow \mathcal{I}.$$

Lemma 7.27. *Let $\mathcal{J} \subseteq \mathcal{H} \subseteq \mathcal{I}$. Then*

$$\mathcal{H} \rightarrow \mathcal{I} \subseteq \mathcal{J} \rightarrow \mathcal{I}.$$

Proof. This is clear as $k \vee h = \mathbf{1}$ for all $h \in \mathcal{H}$ implies $k \vee g = \mathbf{1}$ for all $g \in \mathcal{J}$. \square

7.3. Delta on Filters. Now the critical lemma in defining our new Δ operation.

Lemma 7.28. $(\mathcal{J} \rightarrow \mathcal{I}) \cup \Delta(\mathbf{1}, \mathcal{J})$ is compatible.

Proof. If $x \in \mathcal{J} \rightarrow \mathcal{I}$ and $y \in \Delta(\mathbf{1}, \mathcal{J})$ then $\Delta(\mathbf{1}, y) \in \mathcal{J}$ and so $x \vee \Delta(\mathbf{1}, y) = \mathbf{1}$. \square

Definition 7.29. Let $\mathcal{J} \subseteq \mathcal{I}$. Then

$$\Delta(\mathcal{J}, \mathcal{I}) = \Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{I}) \vee \mathcal{J}.$$

The simplest special algebras in \mathcal{I} are the principal ones. In this case we obtain the following result.

Lemma 7.30. Let $g \in \mathcal{I}$. Then $\Delta([g, \mathbf{1}], \mathcal{I}) = \mathcal{I}_g$.

Proof. From lemma 7.23 we have $[g, \mathbf{1}] \rightarrow \mathcal{I} = g \rightarrow \mathcal{I}$ and we know from corollary 7.15 that $\Delta(\mathbf{1}, \mathcal{I}_g) \cap \mathcal{I} = g \rightarrow \mathcal{I}$. Thus $\Delta(\mathbf{1}, g \rightarrow \mathcal{I}) \subseteq \mathcal{I}_g$. Also $g \in \mathcal{I}_g$ so we have $\Delta([g, \mathbf{1}], \mathcal{I}) \subseteq \mathcal{I}_g$.

Conversely, if $f \in \mathcal{I}$ then $\Delta(g \vee f, f) = (g \vee f) \wedge \Delta(\mathbf{1}, g \rightarrow f)$ is in $\Delta(\mathbf{1}, g \rightarrow \mathcal{I}) \vee [g, \mathbf{1}] = \Delta([g, \mathbf{1}], \mathcal{I})$. \square

Corollary 7.31. Let $g \geq h$ in \mathcal{I} . Then

$$\Delta([g, \mathbf{1}], [h, \mathbf{1}]) = [\Delta(g, h), \mathbf{1}].$$

Proof. As $\Delta([g, \mathbf{1}], [h, \mathbf{1}]) = [h, \mathbf{1}]_g = [\Delta(g, h), \mathbf{1}]$. \square

For further properties of the Δ operation we need some facts about the interaction between \rightarrow and Δ . Here is the first.

Lemma 7.32.

$$\mathcal{J} \rightarrow \Delta(\mathcal{J}, \mathcal{I}) = \Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{I}).$$

Proof. Let $k \in \mathcal{J} \rightarrow \mathcal{I}$, $h = \Delta(\mathbf{1}, k)$ and $g \in \mathcal{J}$. Then $k, g \in \mathcal{I}$ implies they are compatible and so $\Delta(\mathbf{1}, k) \vee g = h \vee g = \mathbf{1}$. Thus $h \in \mathcal{J} \rightarrow \Delta(\mathcal{J}, \mathcal{I})$ and we get $\Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{I}) \subseteq \mathcal{J} \rightarrow \Delta(\mathcal{J}, \mathcal{I})$.

Conversely, suppose that $h \in \Delta(\mathcal{J}, \mathcal{I})$ and for all $g \in \mathcal{J}$ we have $h \vee g = \mathbf{1}$. Then there is some $k \in \mathcal{J} \rightarrow \mathcal{H}$ and $g' \in \mathcal{J}$ such that $h = \Delta(\mathbf{1}, k) \wedge g'$. Therefore $\mathbf{1} = h \vee g' = (\Delta(\mathbf{1}, k) \wedge g') \vee g' = g'$ and so $h = \Delta(\mathbf{1}, k) \in \Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{I})$. \square

Corollary 7.33.

$$\Delta(\mathcal{J}, \Delta(\mathcal{J}, \mathcal{I})) = \mathcal{J} \vee (\mathcal{J} \rightarrow \mathcal{I}).$$

Proof.

$$\begin{aligned} \Delta(\mathcal{J}, \Delta(\mathcal{J}, \mathcal{I})) &= \Delta(\mathbf{1}, \mathcal{J} \rightarrow \Delta(\mathcal{J}, \mathcal{I})) \vee \mathcal{J} \\ &= \Delta(\mathbf{1}, \Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{I})) \vee \mathcal{J} \\ &= (\mathcal{J} \rightarrow \mathcal{I}) \vee \mathcal{J}. \end{aligned}$$

\square

Lemma 7.34. Let $\mathcal{J} \subseteq \mathcal{H} \subseteq \mathcal{I}$. Then

$$\Delta(\mathcal{J}, \mathcal{H}) \subseteq \Delta(\mathcal{J}, \mathcal{I}).$$

Proof. As $\Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{H}) \vee \mathcal{J} \subseteq \Delta(\mathbf{1}, \mathcal{J} \rightarrow \mathcal{I}) \vee \mathcal{J}$. \square

Lemma 7.35. $\mathcal{I} \cap \Delta(\mathcal{J}, \mathcal{I}) = \mathcal{J}$.

Proof. Clearly $\mathcal{J} \subseteq \mathcal{I} \cap \Delta(\mathcal{J}, \mathcal{I})$.

Let $g \in \mathcal{I}$ and $k \in \mathcal{J} \rightarrow \mathcal{I}$ be such that $f = g \wedge \Delta(\mathbf{1}, k) \in \mathcal{I} \cap \Delta(\mathcal{J}, \mathcal{I})$. Then $k \in \mathcal{I}$ so k and $\Delta(\mathbf{1}, k)$ are compatible. Thus $k = \mathbf{1}$ and so $f = g \in \mathcal{J}$. \square

7.4. Boolean elements. Corollary 7.33 shows us what happens to $\Delta(\mathcal{Q}, \Delta(\mathcal{Q}, \mathcal{P}))$. We are interested in knowing when this produces \mathcal{P} .

Definition 7.36. *Let \mathcal{P} and \mathcal{Q} be special subalgebras. Then*

- (a) \mathcal{Q} is weakly \mathcal{P} -Boolean iff $\mathcal{Q} \subseteq \mathcal{P}$ and $(\mathcal{Q} \rightarrow \mathcal{P}) \rightarrow \mathcal{P} = \mathcal{Q}$.
- (b) \mathcal{Q} is \mathcal{P} -Boolean iff $\mathcal{Q} \subseteq \mathcal{P}$ and $\mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{P}) = \mathcal{P}$.

Before continuing however we show that “weak” really is weaker.

Lemma 7.37. *Suppose that \mathcal{Q} is \mathcal{P} -Boolean. Then \mathcal{Q} is weakly \mathcal{P} -Boolean.*

Proof. We know that $\mathcal{Q} \subseteq (\mathcal{Q} \rightarrow \mathcal{P}) \rightarrow \mathcal{P}$.

Since $\mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{P}) = \mathcal{P}$ we also have that $(\mathcal{Q} \rightarrow \mathcal{P}) \supset \mathcal{P} \subseteq \mathcal{Q}$. □

And now the simplest examples of \mathcal{P} -Boolean subalgebras.

Lemma 7.38. *Let $g \in \mathcal{P}$. Then $[g, \mathbf{1}]$ is \mathcal{P} -Boolean.*

Proof. We know that

$$\Delta([g, \mathbf{1}], \mathcal{P}_g) = (\mathcal{P}_g)_g = \mathcal{P}$$

and so

$$\begin{aligned} \mathcal{P} &= [g, \mathbf{1}] \vee \Delta(\mathbf{1}, [g, \mathbf{1}] \rightarrow \mathcal{P}_g) \\ &= [g, \mathbf{1}] \vee \Delta(\mathbf{1}, [g, \mathbf{1}] \rightarrow \Delta([g, \mathbf{1}], \mathcal{P})) \\ &= [g, \mathbf{1}] \vee \Delta(\mathbf{1}, \Delta(\mathbf{1}, [g, \mathbf{1}] \rightarrow \mathcal{P})) \\ &= [g, \mathbf{1}] \vee (g \rightarrow \mathcal{P}). \end{aligned}$$

□

Essentially because we have so many internal automorphisms we can show that Boolean is not a local concept – that is if \mathcal{Q} is \mathcal{P} -Boolean somewhere then it is Boolean in all special subalgebras equivalent to \mathcal{P} . And similarly for weakly Boolean.

Lemma 7.39. *Let $\mathcal{P} \sim \mathcal{H}$ and $\mathcal{Q} \subseteq \mathcal{P} \cap \mathcal{H}$ be special subalgebras. Let $\beta = \beta_{\mathcal{P}\mathcal{H}}$ (and so $\beta^{-1} = \beta_{\mathcal{H}\mathcal{P}}$). Then $\beta[\mathcal{Q} \rightarrow \mathcal{P}] = \beta[\mathcal{Q}] \rightarrow \mathcal{H}$.*

Proof. Indeed if $g \in \mathcal{Q}$ and $h \in \mathcal{Q} \rightarrow \mathcal{P}$ then we have

$$\begin{aligned} \mathbf{1} &= \beta(h \vee g) \\ &= \beta(h) \vee \beta(g) \end{aligned}$$

and so $\beta(h) \in \beta[\mathcal{Q}] \rightarrow \mathcal{H}$.

Likewise, if $h \in \beta[\mathcal{Q}] \rightarrow \mathcal{H}$ and $g \in \mathcal{Q}$ then $\mathbf{1} = h \vee \beta(g) = \beta(\beta^{-1}(h) \vee g)$ so that $\beta^{-1}(h) \vee g = \mathbf{1}$. Thus $\beta^{-1}(h) \in \mathcal{Q} \rightarrow \mathcal{P}$ whence $h = \beta(\beta^{-1}(h)) \in \beta[\mathcal{Q} \rightarrow \mathcal{P}]$. □

Theorem 7.40. *Let \mathcal{Q} be \mathcal{P} -Boolean, and $\mathcal{P} \sim \mathcal{R}$ with $\mathcal{Q} \subseteq \mathcal{R}$. Then \mathcal{Q} is \mathcal{R} -Boolean.*

Proof. We have $\mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{P}) = \mathcal{P}$ and $\mathcal{Q} \subseteq \mathcal{R}$. Let $\beta = \beta_{\mathcal{P}\mathcal{R}}$, $h \in \mathcal{R}$ and find $g \in \mathcal{Q}$, $k \in \mathcal{Q} \rightarrow \mathcal{P}$ with $\beta^{-1}(h) = g \wedge k$. Then $h = \beta(\beta^{-1}(h)) = \beta(g \wedge k) = \beta(g) \wedge \beta(k) = g \wedge \beta(k)$ as $g \in \mathcal{H}$ implies $\beta(g) = g$. As $\beta(k) \in \beta[\mathcal{Q} \rightarrow \mathcal{P}] = \beta[\mathcal{Q}] \rightarrow \mathcal{H} = \mathcal{Q} \rightarrow \mathcal{H}$ we have $h \in \mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{H})$. □

Theorem 7.41. *Let \mathcal{Q} be weakly \mathcal{P} -Boolean for some special subalgebra \mathcal{P} , and $\mathcal{P} \sim \mathcal{R}$ with $\mathcal{Q} \subseteq \mathcal{R}$. Then \mathcal{Q} is weakly \mathcal{R} -Boolean.*

Proof. Claim 1: $\beta[\mathcal{Q}] = \mathcal{Q}$ – since $\mathcal{Q} \subseteq \mathcal{R}$ implies $\beta \upharpoonright \mathcal{Q}$ is the identity.

Claim 2: Now suppose that \mathcal{Q} is weakly \mathcal{P} -Boolean. Then

$$\begin{aligned}
\mathcal{Q} &= \beta[\mathcal{Q}] \\
&= \beta[(\mathcal{Q} \rightarrow \mathcal{P}) \rightarrow \mathcal{P}] \\
&= \beta[\mathcal{Q} \rightarrow \mathcal{P}] \rightarrow \mathcal{R} \\
&= (\beta[\mathcal{Q}] \rightarrow \mathcal{R}) \rightarrow \mathcal{R} \\
&= (\mathcal{Q} \rightarrow \mathcal{R}) \rightarrow \mathcal{R}.
\end{aligned}$$

□

We need to know certain persistence properties of Boolean-ness.

Lemma 7.42. *Let $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$ be \mathcal{P} -Boolean. Then \mathcal{Q} is \mathcal{R} -Boolean and $\mathcal{Q} \rightarrow \mathcal{R} = (\mathcal{Q} \rightarrow \mathcal{P}) \cap \mathcal{R}$.*

Proof. First we note that $\mathcal{Q} \rightarrow \mathcal{R} = (\mathcal{Q} \rightarrow \mathcal{P}) \cap \mathcal{R}$ as $x \in \text{LHS}$ iff $x \in \mathcal{R}$ and for all $g \in \mathcal{Q}$ $x \vee g = \mathbf{1}$ iff $x \in \text{RHS}$.

Thus we have

$$\begin{aligned}
\mathcal{R} &= \mathcal{P} \cap \mathcal{R} \\
&= (\mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{P})) \cap \mathcal{R} \\
&= (\mathcal{Q} \cap \mathcal{R}) \vee ((\mathcal{Q} \rightarrow \mathcal{P}) \cap \mathcal{R}) \\
&= \mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{R}).
\end{aligned}$$

□

Lemma 7.43. *Let \mathcal{Q} be \mathcal{R} -Boolean, \mathcal{R} be \mathcal{P} -Boolean. Then \mathcal{Q} is \mathcal{P} -Boolean.*

Proof. Let $f \in \mathcal{P}$. Then there is some $h \in \mathcal{R}$ and $k \in \mathcal{H} \rightarrow \mathcal{P}$ such that $h \wedge k = f$. Also there is some $g \in \mathcal{Q}$ and $l \in \mathcal{Q} \rightarrow \mathcal{R}$ such that $h = g \wedge l$. Thus $g \wedge l \wedge k = f$ – so it suffices to show that $l \wedge k \in \mathcal{Q} \rightarrow \mathcal{P}$.

Clearly $k \wedge l \in \mathcal{P}$. So let $p \in \mathcal{Q}$. Then $\mathcal{Q} \subseteq \mathcal{R}$ and $k \in \mathcal{R} \rightarrow \mathcal{P}$ implies $p \vee k = \mathbf{1}$. $l \in \mathcal{Q} \rightarrow \mathcal{R}$ implies $p \vee l = \mathbf{1}$. Therefore $p \vee (k \wedge l) = (p \vee k) \wedge (p \vee l) = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}$. □

So far we have few examples of Boolean special subalgebras. The next lemma produces many more.

Lemma 7.44. *Let $\mathcal{P} \sim \mathcal{R}$. Then $\mathcal{P} \cap \mathcal{R}$ is \mathcal{P} -Boolean and*

$$(\mathcal{P} \cap \mathcal{R}) \rightarrow \mathcal{P} = \Delta(\mathbf{1}, \mathcal{R}) \cap \mathcal{P}.$$

Proof. First we show that $(\mathcal{P} \cap \mathcal{R}) \rightarrow \mathcal{P} = \Delta(\mathbf{1}, \mathcal{R}) \cap \mathcal{P}$.

Let $f \in \mathcal{P} \cap \mathcal{R}$ and $k \in \Delta(\mathbf{1}, \mathcal{R}) \cap \mathcal{P}$. Then $\Delta(\mathbf{1}, k) \in \mathcal{R}$ so $\Delta(\mathbf{1}, k)$ and f are compatible, ie $k \vee f = \mathbf{1}$. Hence $\Delta(\mathbf{1}, \mathcal{R}) \cap \mathcal{P} \subseteq (\mathcal{P} \cap \mathcal{R}) \rightarrow \mathcal{P}$.

Conversely suppose that $k \in (\mathcal{P} \cap \mathcal{R}) \rightarrow \mathcal{P}$. Let $h \in \mathcal{R}$. Then $h \vee k \in \mathcal{P} \cap \mathcal{R}$ and so $h \vee k = (h \vee k) \vee k = \mathbf{1}$. As there is some $k' \sim k$ in \mathcal{R} this implies $k' \vee k = \mathbf{1}$ and (as $k \sim k'$) we have $k = \Delta(\mathbf{1}, k')$. Thus $k \in \Delta(\mathbf{1}, \mathcal{R}) \cap \mathcal{P}$.

Now let $f \in \mathcal{P}$. Then let $f' \in \mathcal{R}$ with $f' \sim f$. Then $(f \vee f') \rightarrow f = f \wedge \Delta(\mathbf{1}, \Delta(f' \vee f, f)) = f \wedge \Delta(\mathbf{1}, f') \in \mathcal{P} \cap \Delta(\mathbf{1}, \mathcal{R})$. Also $f \vee f' \in \mathcal{P} \cap \mathcal{R}$ and $(f \vee f') \wedge ((f \vee f') \rightarrow f) = f$ so $f \in (\mathcal{P} \cap \mathcal{R}) \vee (\mathcal{P} \cap \Delta(\mathbf{1}, \mathcal{R}))$. □

Corollary 7.45. *Let $\mathcal{P} \sim \mathcal{R}$. Then*

$$\Delta(\mathcal{P} \cap \mathcal{R}, \mathcal{P}) = \mathcal{R}.$$

Proof.

$$\begin{aligned}
\Delta(\mathcal{P} \cap \mathcal{R}, \mathcal{P}) &= (\mathcal{P} \cap \mathcal{R}) \vee \Delta(\mathbf{1}, (\mathcal{P} \cap \mathcal{R}) \rightarrow \mathcal{P}) \\
&= (\mathcal{P} \cap \mathcal{R}) \vee \Delta(\mathbf{1}, \Delta(\mathbf{1}, \mathcal{R}) \cap \mathcal{P}) \\
&= (\mathcal{P} \cap \mathcal{R}) \vee (\mathcal{R} \cap \Delta(\mathbf{1}, \mathcal{P})) \\
&= (\mathcal{P} \cap \mathcal{R}) \vee ((\mathcal{P} \cap \mathcal{R}) \rightarrow \mathcal{R}) \\
&= \mathcal{R}
\end{aligned}$$

since $\mathcal{P} \cap \mathcal{R}$ is also \mathcal{R} -Boolean. □

Lemma 7.46. *Let g, h in \mathcal{L} be such that $g \wedge h$ exists and $g \vee h = \mathbf{1}$. Then $\Delta(g, g \wedge h) = g \wedge \Delta(\mathbf{1}, h)$.*

Proof.

$$\begin{aligned}
\Delta(g, g \wedge h) &= g \wedge \Delta(\mathbf{1}, g \rightarrow (g \wedge h)) \\
&= g \wedge \Delta(\mathbf{1}, (g \vee h) \rightarrow h) && \text{by modularity in } [g \wedge h, \mathbf{1}] \\
&= g \wedge \Delta(\mathbf{1}, \mathbf{1} \rightarrow h) \\
&= g \wedge \Delta(\mathbf{1}, h)
\end{aligned}$$

□

Theorem 7.47. *$\mathcal{R} \sim \mathcal{P}$ iff there is an \mathcal{P} -Boolean subalgebra \mathcal{Q} such that $\mathcal{R} = \Delta(\mathcal{Q}, \mathcal{P})$.*

Proof. The right to left direction is the last corollary.

So we want to prove that $\Delta(\mathcal{Q}, \mathcal{P}) \sim \mathcal{P}$ whenever \mathcal{Q} is \mathcal{P} -Boolean.

Let $f \in \mathcal{P}$. We will show that there is some $f' \in \Delta(\mathcal{Q}, \mathcal{P})$ with $f \sim f'$. As $\mathcal{Q} \vee (\mathcal{Q} \rightarrow \mathcal{P}) = \mathcal{P}$ we can find $g \in \mathcal{Q}$ and $h \in \mathcal{Q} \rightarrow \mathcal{P}$ with $f = g \wedge h$. As $g \vee h = \mathbf{1}$ we know that $\Delta(g, g \wedge h) = g \wedge \Delta(\mathbf{1}, h)$. But $g \wedge \Delta(\mathbf{1}, h) \in \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{P}) = \Delta(\mathcal{Q}, \mathcal{P})$ and $f = g \wedge h \sim \Delta(g, g \wedge h) = g \wedge \Delta(\mathbf{1}, h)$. □

The Boolean elements have nice properties with respect to Δ . We want to show more – that the set of \mathcal{P} -Boolean elements is a Boolean subalgebra of $[\mathcal{P}, \{\mathbf{1}\}]$ with the reverse order.

It suffices to show closure under \cap and \vee – closure under \rightarrow follows from lemma 7.37.

Lemma 7.48. *Let \mathcal{Q}_1 and \mathcal{Q}_2 be \mathcal{P} -Boolean. Then $(\mathcal{Q}_1 \rightarrow \mathcal{P}) \vee (\mathcal{Q}_2 \rightarrow \mathcal{P}) = (\mathcal{Q}_1 \cap \mathcal{Q}_2) \rightarrow \mathcal{P}$.*

Proof. Suppose that $h_i \in \mathcal{Q}_i \rightarrow \mathcal{P}$ and $g \in \mathcal{Q}_1 \cap \mathcal{Q}_2$. Then $(h_1 \wedge h_2) \vee g = (h_1 \vee g) \wedge (h_2 \vee g) = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}$ and so $(h_1 \wedge h_2) \in (\mathcal{Q}_1 \cap \mathcal{Q}_2) \rightarrow \mathcal{P}$.

Conversely, let $h \vee g = \mathbf{1}$ for all $g \in \mathcal{Q}_1 \cap \mathcal{Q}_2$. As \mathcal{Q}_i are both \mathcal{P} -Boolean there exists $h_i \in \mathcal{Q}_i \rightarrow \mathcal{P}$ and $g_i \in \mathcal{Q}_i$ with $h = h_1 \wedge g_1 = h_2 \wedge g_2$. Then

$$\begin{aligned}
h_1 \wedge h_2 \wedge (g_1 \vee g_2) &= (h_1 \wedge h_2 \wedge g_1) \vee (h_1 \wedge h_2 \wedge g_2) \\
&= (h_2 \wedge h) \vee (h_1 \wedge h) \\
&= h \wedge h = h.
\end{aligned}$$

As $h_1 \wedge h_2 \in (\mathcal{Q}_1 \rightarrow \mathcal{P}) \vee (\mathcal{Q}_2 \rightarrow \mathcal{P})$ and $g_1 \vee g_2 \in \mathcal{Q}_1 \cap \mathcal{Q}_2$ we then have $h = [h \vee (h_1 \wedge h_2)] \wedge (h \vee g_1 \vee g_2) = h \vee (h_1 \wedge h_2)$ and so $h = h_1 \wedge h_2$ is in $(\mathcal{Q}_1 \rightarrow \mathcal{P}) \vee (\mathcal{Q}_2 \rightarrow \mathcal{P})$. □

Corollary 7.49. *Let \mathcal{Q}_1 and \mathcal{Q}_2 be \mathcal{P} -Boolean. Then so is $\mathcal{Q}_1 \cap \mathcal{Q}_2$.*

Proof. Let $f \in \mathcal{P}$. As \mathcal{Q}_i are both \mathcal{P} -Boolean there exists $h_i \in \mathcal{G}_i \rightarrow \mathcal{P}$ and $g_i \in \mathcal{Q}_i$ with $f = h_1 \wedge g_1 = h_2 \wedge g_2$. Then as above $f = h_1 \wedge h_2 \wedge (g_1 \vee g_2)$ and $g_1 \vee g_2 \in \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $h_1 \wedge h_2 \in (\mathcal{Q}_1 \rightarrow \mathcal{P}) \vee (\mathcal{Q}_2 \rightarrow \mathcal{P}) = (\mathcal{Q}_1 \cap \mathcal{Q}_2) \rightarrow \mathcal{P}$. \square

Corollary 7.50. *Let \mathcal{Q}_1 and \mathcal{Q}_2 be \mathcal{P} -Boolean. Then so is $\mathcal{Q}_1 \vee \mathcal{Q}_2$.*

Proof. Since we have $(\mathcal{Q} \rightarrow \mathcal{P}) \rightarrow \mathcal{P} = \mathcal{Q}$ for \mathcal{P} -Booleans we know that $\mathcal{Q}_i \rightarrow \mathcal{P}$ are also \mathcal{P} -Boolean and so

$$\begin{aligned} \mathcal{Q}_1 \vee \mathcal{Q}_2 &= ((\mathcal{Q}_1 \rightarrow \mathcal{P}) \rightarrow \mathcal{P}) \vee ((\mathcal{Q}_2 \rightarrow \mathcal{P}) \rightarrow \mathcal{P}) \\ &= ((\mathcal{Q}_1 \rightarrow \mathcal{P}) \cap (\mathcal{Q}_2 \rightarrow \mathcal{P})) \rightarrow \mathcal{P} \end{aligned}$$

Therefore

$$\begin{aligned} (\mathcal{Q}_1 \vee \mathcal{Q}_2) \rightarrow \mathcal{P} &= (((\mathcal{Q}_1 \rightarrow \mathcal{P}) \cap (\mathcal{Q}_2 \rightarrow \mathcal{P})) \rightarrow \mathcal{P}) \rightarrow \mathcal{P} \\ &= (\mathcal{Q}_1 \rightarrow \mathcal{P}) \cap (\mathcal{Q}_2 \rightarrow \mathcal{P}). \end{aligned}$$

Thus we have

$$(\mathcal{Q}_1 \vee \mathcal{Q}_2) \vee ((\mathcal{Q}_1 \vee \mathcal{Q}_2) \rightarrow \mathcal{P}) = (\mathcal{Q}_1 \vee \mathcal{Q}_2) \vee ((\mathcal{Q}_1 \rightarrow \mathcal{P}) \cap (\mathcal{Q}_2 \rightarrow \mathcal{P})).$$

Let $f \in \mathcal{P}$ and $g_i \in \mathcal{Q}_i$, $h_i \in \mathcal{Q}_i \rightarrow \mathcal{P}$ be such that $f = g_i \wedge h_i$. Then $f \leq g_1, g_2$ so that $g_1 \wedge g_2 \in \mathcal{Q}_1 \vee \mathcal{Q}_2$, $h_1 \vee h_2 \in (\mathcal{Q}_1 \rightarrow \mathcal{P}) \cap (\mathcal{Q}_2 \rightarrow \mathcal{P})$ and

$$\begin{aligned} g_1 \wedge g_2 \wedge (h_1 \vee h_2) &= (g_1 \wedge g_2 \wedge h_1) \vee (g_1 \wedge g_2 \wedge h_2) \\ &= (g_2 \wedge f) \vee (g_1 \wedge f) \\ &= f \wedge f && \text{as } f \leq g_i \\ &= f. \end{aligned}$$

\square

Thus we have

Theorem 7.51. *Let \mathcal{P} be any special subalgebra. Then $\{\mathcal{Q} \mid \mathcal{Q} \text{ is } \mathcal{P}\text{-Boolean}\}$ ordered by reverse inclusion is a Boolean algebra with $\wedge = \vee$, $\vee = \cap$, $1 = \{\mathbf{1}\}$, $0 = \mathcal{P}$ and $\overline{\mathcal{Q}} = \mathcal{Q} \rightarrow \mathcal{P}$.*

Proof. This is immediate from lemma 7.11 and preceding remarks, and from lemma 7.37. \square

We need a stronger closure property for Boolean filters under intersection.

Lemma 7.52. *Let $\mathcal{P} \sim \mathcal{R}$, \mathcal{Q} be \mathcal{P} -Boolean and \mathcal{K} be \mathcal{R} -Boolean. Then $\mathcal{Q} \cap \mathcal{K}$ is $\mathcal{P} \cap \mathcal{R}$ -Boolean.*

Proof. Let $p \in \mathcal{P} \cap \mathcal{R}$ be arbitrary. Choose $g \in \mathcal{Q}$, $g' \in \mathcal{Q} \rightarrow \mathcal{P}$ with $g \wedge g' = p$ and choose $k \in \mathcal{K}$, $k' \in \mathcal{K} \rightarrow \mathcal{R}$ with $k \wedge k' = p$.

Then g' and k' are both above p so $g' \wedge k'$ exists and is in $\mathcal{P} \cap \mathcal{R}$. Also $(g \vee k) \wedge (g' \wedge k') = p$. $g \vee k \in \mathcal{Q} \cap \mathcal{K}$ so we need to show that $g' \wedge k'$ is in $(\mathcal{Q} \cap \mathcal{K}) \rightarrow (\mathcal{P} \cap \mathcal{R})$. Let $q \in \mathcal{Q} \cap \mathcal{K}$. Then $q \vee g' = \mathbf{1} = q \vee k'$ so that $q \vee (g' \wedge k') = (q \vee g') \wedge (q \vee k') = \mathbf{1}$. \square

Corollary 7.53. *Let \mathcal{Q} be \mathcal{P} -Boolean, \mathcal{K} be \mathcal{R} -Boolean and $\mathcal{P} \sim \mathcal{R}$. Then $\mathcal{Q} \cap \mathcal{K}$ is \mathcal{P} -Boolean.*

Proof. The lemma tells us that $\mathcal{Q} \cap \mathcal{K}$ is $\mathcal{P} \cap \mathcal{R}$ -Boolean. Theorem 7.47 tells us that $\mathcal{P} \cap \mathcal{R}$ is \mathcal{P} -Boolean. And from lemma 7.43 we have $\mathcal{Q} \cap \mathcal{K}$ to be \mathcal{P} -Boolean. \square

The last closure property we need is with respect to Δ .

Lemma 7.54. *Let $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$ be \mathcal{P} -Boolean subalgebras. Then*

$$\Delta(\mathcal{Q}, \mathcal{R}) \rightarrow \Delta(\mathcal{Q}, \mathcal{P}) = \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P}).$$

Proof. As $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$ in a Boolean algebra we have

$$(\mathcal{Q} \rightarrow \mathcal{R}) \rightarrow (\mathcal{Q} \rightarrow \mathcal{P}) = \mathcal{R} \rightarrow \mathcal{P}.$$

Also we have

$$\begin{aligned} \Delta(\mathcal{Q}, \mathcal{R}) &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{R}) \\ \Delta(\mathcal{Q}, \mathcal{P}) &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{P}). \end{aligned}$$

Let $x \in \Delta(\mathcal{Q}, \mathcal{R})$ and $g \in \mathcal{Q}$, $h \in \mathcal{Q} \rightarrow \mathcal{R}$ with $x = g \wedge \Delta(\mathbf{1}, h)$. Let $y \in \Delta(\mathcal{Q}, \mathcal{P})$ and $g' \in \mathcal{Q}$, $f \in \mathcal{Q} \rightarrow \mathcal{P}$ with $y = g' \wedge \Delta(\mathbf{1}, f)$ and suppose that $x \vee y = \mathbf{1}$ for all such x . Then

$$\begin{aligned} y \vee x &= (g' \wedge \Delta(\mathbf{1}, f)) \vee (g \wedge \Delta(\mathbf{1}, h)) \\ &= (g' \vee g) \wedge (g' \vee \Delta(\mathbf{1}, h)) \wedge (\Delta(\mathbf{1}, f) \vee g) \wedge \Delta(\mathbf{1}, f \vee h) \\ &= (g' \vee g) \wedge \Delta(\mathbf{1}, f \vee h) \end{aligned}$$

since g and f are compatible, as are g' and h .

Thus $g' \vee g = \mathbf{1}$ and $f \vee h = \mathbf{1}$ for all $g \in \mathcal{Q}$ and all $h \in \mathcal{Q} \rightarrow \mathcal{R}$. Choosing $g = g'$ implies $g' = \mathbf{1}$ and so $f \in (\mathcal{Q} \rightarrow \mathcal{R}) \rightarrow (\mathcal{Q} \rightarrow \mathcal{P}) = \mathcal{R} \rightarrow \mathcal{P}$. Hence $y = \Delta(\mathbf{1}, f) \in \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P})$.

Conversely if $f \in \mathcal{R} \rightarrow \mathcal{P}$ then $g \vee \Delta(\mathbf{1}, f) = \mathbf{1}$ for all $g \in \mathcal{Q}$. And $f \in (\mathcal{Q} \rightarrow \mathcal{R}) \rightarrow (\mathcal{Q} \rightarrow \mathcal{P})$ implies $h \vee f = \mathbf{1}$ for all $h \in \mathcal{Q} \rightarrow \mathcal{R}$. Hence $(g \wedge \Delta(\mathbf{1}, h)) \vee \Delta(\mathbf{1}, f) = \mathbf{1}$ and so $\Delta(\mathbf{1}, f)$ is in $\Delta(\mathcal{Q}, \mathcal{R}) \rightarrow \Delta(\mathcal{Q}, \mathcal{P})$. \square

Lemma 7.55. *Let $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$ be \mathcal{P} -Boolean subalgebras. Then $\Delta(\mathcal{Q}, \mathcal{R})$ is $\Delta(\mathcal{Q}, \mathcal{P})$ -Boolean.*

Proof. Since

$$\begin{aligned} \Delta(\mathcal{Q}, \mathcal{R}) \vee (\Delta(\mathcal{Q}, \mathcal{R}) \rightarrow \Delta(\mathcal{Q}, \mathcal{P})) &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{R}) \vee \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P}) \\ &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{R}) \vee \Delta(\mathbf{1}, (\mathcal{Q} \rightarrow \mathcal{R}) \rightarrow (\mathcal{Q} \rightarrow \mathcal{P})) \\ &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{R}) \vee ((\mathcal{Q} \rightarrow \mathcal{R}) \rightarrow (\mathcal{Q} \rightarrow \mathcal{P})) \\ &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{P}) \\ &= \Delta(\mathcal{Q}, \mathcal{P}). \end{aligned}$$

\square

From this lemma we can derive another property of Δ .

Lemma 7.56. *Let $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$ be \mathcal{P} -Boolean subalgebras. Then*

$$\mathcal{Q} \rightarrow \Delta(\mathcal{R}, \mathcal{P}) = (\mathcal{Q} \rightarrow \mathcal{R}) \vee \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P}).$$

Proof. The RHS is clearly a subset of $\Delta(\mathcal{R}, \mathcal{P})$. Let $g \in \mathcal{Q}$. If $h \in \mathcal{Q} \rightarrow \mathcal{R}$ then $h \vee g = \mathbf{1}$. If $k \in \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P})$ then $\Delta(\mathbf{1}, k) \in \mathcal{R} \rightarrow \mathcal{P} \subseteq \mathcal{Q} \rightarrow \mathcal{P}$ so that $g \vee k = \mathbf{1}$. Thus the RHS is a subset of the LHS.

Conversely suppose that $h = h_1 \wedge h_2$ is in $\mathcal{R} \vee \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P}) = \Delta(\mathcal{R}, \mathcal{P})$ and $g \vee h = \mathbf{1}$ for all $g \in \mathcal{Q}$. Then $g \vee h_1 = \mathbf{1}$ for all $g \in \mathcal{Q}$ and so $h_1 \in \mathcal{Q} \rightarrow \mathcal{R}$. Thus the LHS is a subset of the RHS. \square

Corollary 7.57. *Let $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$ be \mathcal{P} -Boolean subalgebras. Then*

$$\Delta(\mathcal{Q}, \Delta(\mathcal{R}, \mathcal{P})) = \Delta(\Delta(\mathcal{Q}, \mathcal{R}), \Delta(\mathcal{Q}, \mathcal{P})).$$

Proof.

$$\begin{aligned} \Delta(\mathcal{Q}, \Delta(\mathcal{R}, \mathcal{P})) &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \Delta(\mathcal{R}, \mathcal{P})) \\ &= \mathcal{Q} \vee \Delta(\mathbf{1}, (\mathcal{Q} \rightarrow \mathcal{R}) \vee \Delta(\mathbf{1}, \mathcal{R} \rightarrow \mathcal{P})) \\ &= \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{R}) \vee (\mathcal{R} \rightarrow \mathcal{P}) \\ &= \Delta(\mathcal{Q}, \mathcal{R}) \vee \Delta(\mathbf{1}, \Delta(\mathcal{Q}, \mathcal{R}) \rightarrow \Delta(\mathcal{Q}, \mathcal{P})) \\ &= \Delta(\Delta(\mathcal{Q}, \mathcal{R}), \Delta(\mathcal{Q}, \mathcal{P})). \end{aligned}$$

□

7.5. An MR-algebra. The results of the last section show us that there is a natural MR-algebra sitting over the top of any cubic algebra. The first theorem describes the case for cubic algebras with g-covers.

Theorem 7.58. *Let \mathcal{L} be a cubic algebra with a g-cover. Let \mathcal{L}_{sB} be the set of all special subalgebras that are \mathcal{P} -Boolean for some g-cover \mathcal{P} . Order these by reverse inclusion. Then*

- (a) \mathcal{L}_{sB} contains $\{\mathbf{1}\}$ and is closed under the operations \vee and Δ .
- (b) $\langle \mathcal{L}_{sB}, \{\mathbf{1}\}, \vee, \Delta \rangle$ is an atomic MR-algebra.
- (c) The mapping $e: \mathcal{L} \rightarrow \mathcal{L}_{sB}$ given by $g \mapsto [g, \mathbf{1}]$ is a full embedding.
- (d) The atoms of \mathcal{L}_{sB} are exactly the g-covers of \mathcal{L} .

Proof. (a) It is easy to see that $\mathbf{1} \rightarrow = \mathcal{P}$ for all filters \mathcal{P} . Corollary 7.53 and lemma 7.55 give the closure under join and Delta respectively.

(b) We will proceed sequentially through the axioms.

- i. if $x \leq y$ then $\Delta(y, x) \vee x = y$ – this is lemma 7.35.
- ii. if $x \leq y \leq z$ then $\Delta(z, \Delta(y, x)) = \Delta(\Delta(z, y), \Delta(z, x))$ – this is corollary 7.57.
- iii. if $x \leq y$ then $\Delta(y, \Delta(y, x)) = x$ – this is corollary 7.33 and the definition of \mathcal{F} -Boolean.
- iv. if $x \leq y \leq z$ then $\Delta(z, x) \leq \Delta(z, y)$ – this is lemma 7.34.

Let $xy = \Delta(\mathbf{1}, \Delta(x \vee y, y)) \vee y$ for any x, y in \mathcal{L} .

First we note that if $\mathcal{Q} \subseteq \mathcal{P}$ then

$$\begin{aligned} \Delta(\mathbf{1}, \Delta(\mathcal{Q}, \mathcal{P})) \cap \mathcal{P} &= \Delta(\mathbf{1}, \mathcal{Q} \vee \Delta(\mathbf{1}, \mathcal{Q} \rightarrow \mathcal{P})) \cap \mathcal{P} \\ &= (\Delta(\mathbf{1}, \mathcal{Q}) \vee (\mathcal{Q} \rightarrow \mathcal{P})) \cap \mathcal{P}. \end{aligned}$$

If $g \in \mathcal{Q}$ and $h \in \mathcal{P}$ is such that $\Delta(\mathbf{1}, g) \wedge h \in \mathcal{P}$ then $g = \Delta(\mathbf{1}, g)$ (since $g \simeq \Delta(\mathbf{1}, g)$ and $g \wedge \Delta(\mathbf{1}, g)$ exists). Thus $(\Delta(\mathbf{1}, \mathcal{Q}) \vee (\mathcal{Q} \rightarrow \mathcal{P})) \cap \mathcal{P} = \mathcal{Q} \rightarrow \mathcal{P}$.

- v. $(xy)y = x \vee y$ and
- vi. $x(yz) = y(xz)$. These last two properties hold as \mathcal{L}_{sB} is locally Boolean and hence an implication algebra.

To see that \mathcal{L}_{sB} is an MR-algebra it suffices to note that if \mathcal{Q}_1 and \mathcal{Q}_2 are in \mathcal{L}_{sB} and we have g-covers $\mathcal{P}_1, \mathcal{P}_2$ with $\mathcal{Q}_i \subseteq \mathcal{P}_i$ then $\Delta(\mathcal{P}_1 \cap \mathcal{P}_2, \mathcal{P}_2) = \mathcal{P}_1 \supseteq \mathcal{Q}_1$ so that $\mathcal{P}_2 \preceq \mathcal{Q}_1$. It is clear that $\mathcal{P}_2 \preceq \mathcal{Q}_2$.

(c) It is clear that this mapping preserves order and join. Preservation of Δ is corollary 7.31.

It is full because $[g, \mathbf{1}] \subseteq \mathcal{Q}$ whenever $g \in \mathcal{G}$.

(d) This is theorem 7.47. □

The structure \mathcal{L}_{sB} is another notion of envelope for cubic algebras. The existence of such an envelope – it is an MR-algebra with a g-filter into which \mathcal{L} embeds as a full subalgebra – implies that \mathcal{L} has a g-cover, so this result cannot be directly extended to all cubic algebras.

We note that if \mathcal{L} is finite then \mathcal{L}_{sB} is the same as the enveloping algebra given by theorem 1.5.

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