MINIMAL NUMBER OF SELF-INTERSECTIONS OF THE BOUNDARY OF AN IMMERSED SURFACE IN THE PLANE

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Abstract. We find the minimal number of self-intersections of the boundary of a surface of genus g generically immersed in $\mathbb{R}^2$.

Let Σ be an oriented surface of genus $g \geq 1$ with one boundary component. We consider the class of all immersions $I : \Sigma \to \mathbb{R}^2$ so that $I(\partial\Sigma)$ intersects itself transversely. Among this class of immersions, we determine the minimal number of self-intersections of $I(\partial\Sigma)$.

Proposition 1. *If I is an immersion $I : \Sigma \to \mathbb{R}^2$ and $I(\partial\Sigma)$ intersects itself transversely, then $I(\partial\Sigma)$ has at least $2g+2$ self-intersections. For each g, there is such an immersion so $I(\partial\Sigma)$ has exactly $2g+2$ self-intersections.*

This proposition answers a very simple case of a question that Gromov studied in the recent paper [1]. Gromov gave estimates for the number of self-intersections of the critical set of a generic map from one manifold to another. We can rewrite Proposition 1 in that language as follows. Suppose that Σ' is a closed surface of genus $2g$ without boundary. Let S be an embedded curve in Σ' which divides Σ' into two surfaces each with genus g. It is possible to find a map F from Σ' to $\mathbb{R}^2$ folded along the curve S and with no other singularities. The curve S is the singular set of the map F, and $F(S) \subset \mathbb{R}^2$ is the critical set of F. Gromov observed that as the topological complexity of Σ increases, then the topological complexity of the critical set $F(S)$ must also increase. As a corollary of Proposition 1, we see that for a generic F folded along S, the critical set must have at least $2g+2$ self-intersections, and this estimate is sharp.

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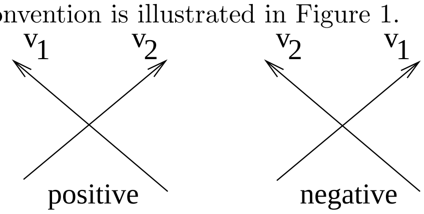
Proof. First we prove the lower bound. The main ingredient of the proof is the Whitney index formula, which relates the index of an immersed curve with its self-intersections. Whitney's formula appears in his famous paper on immersed curves [2]. (The more famous result of that paper is that any two immersed curves with equal index are regular homotopic.)

Let C be an oriented immersed curve given by an immersion $\phi : S^1 \to \mathbb{R}^2$. At any point θ of S^1, the derivative of ϕ is a non-vanishing vector in $\mathbb{R}^2$. Therefore, the derivative of ϕ defines a map from S^1 to $\mathbb{R}^2 - \{0\}$. The winding number of this map is called the index of the immersed curve.

We give $\mathbb{R}^2$ its standard orientation, and we orient Σ so that I is orientation preserving. We let C be the image $I(\partial\Sigma)$, with the boundary orientation. The first step of our proof is to show that the index of C is $1-2g$. This step follows from the Euler-Poincare formula.

Let V be the pullback $I^*(\partial/\partial x)$. The vector field V is a non-vanishing vector field on the surface Σ. We trivialize the bundle $T\Sigma$ restricted to $\partial\Sigma$ so that the tangent vector to the boundary is constant. With respect to that trivialization, we let $W(V)$ be the winding number of the vector field V along the boundary $\partial\Sigma$. According to the Euler-Poincare formula, since V is nowhere vanishing, $-W(V) = \chi(\Sigma) = 1-2g$. The immersion I induces a trivialization of $T\Sigma$. In particular, it gives a second trivialization of $T\Sigma$ over $\partial\Sigma$. The index of C is the winding number of the tangent vector to $\partial\Sigma$ in this second trivialization. The second trivialization has the same orientation as the first, and in the second trivialization, the vector V is constant. Therefore, the winding number of the tangent vector to $\partial\Sigma$ is $-W(V) = 1-2g$.

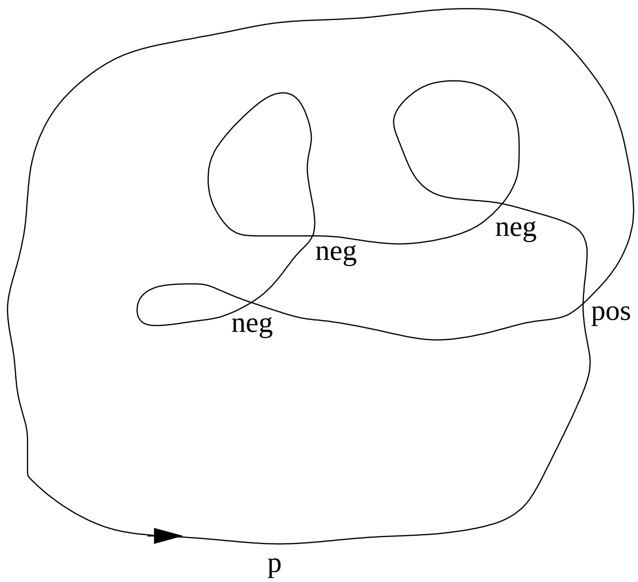
If C is an oriented immersed curve with transverse self-intersections, then its index and its self-intersections are related by the Whitney index formula. Let p be a point of C where the coordinate function y achieves its minimum. With respect to p, we can give each self-intersection a sign ± 1. If x is a self-intersection, then at x there are two distinct unit tangent vectors tangent to the curve C with the correct orientation. We call them v_1 and v_2. We let v_1 be the tangent vector that occurs first if one follows the immersed curve from the point p until one reaches x. Finally, we say that x is positive if v_1 is a positive rotation from v_2. The sign convention is illustrated in Figure 1.



We define N^+ to be the number of positive self-intersections and N^- to be the number of negative self-intersections. Finally, we define a number $\mu = \pm 1$ which depends on the tangent vector of C at p. Because the function y achieves its minimum value at p, the tangent vector to C at p must be $\pm\partial/\partial x$. If the tangent vector is $\partial/\partial x$, then $\mu = 1$, and if the tangent vector is $-\partial/\partial x$, then $\mu = -1$. In terms of these conventions, the Whitney index formula reads as follows.

Theorem. *(Whitney)* $ind(C) = \mu + N^+ - N^-$.

Figure 2 gives an example to illustrate the conventions and the formula. (Incidentally, this example bounds a surface of genus 1.) For the curve in the figure, we have $ind(C) = -1$, $\mu = 1$, $N^+ = 1$, and $N^- = 3$.



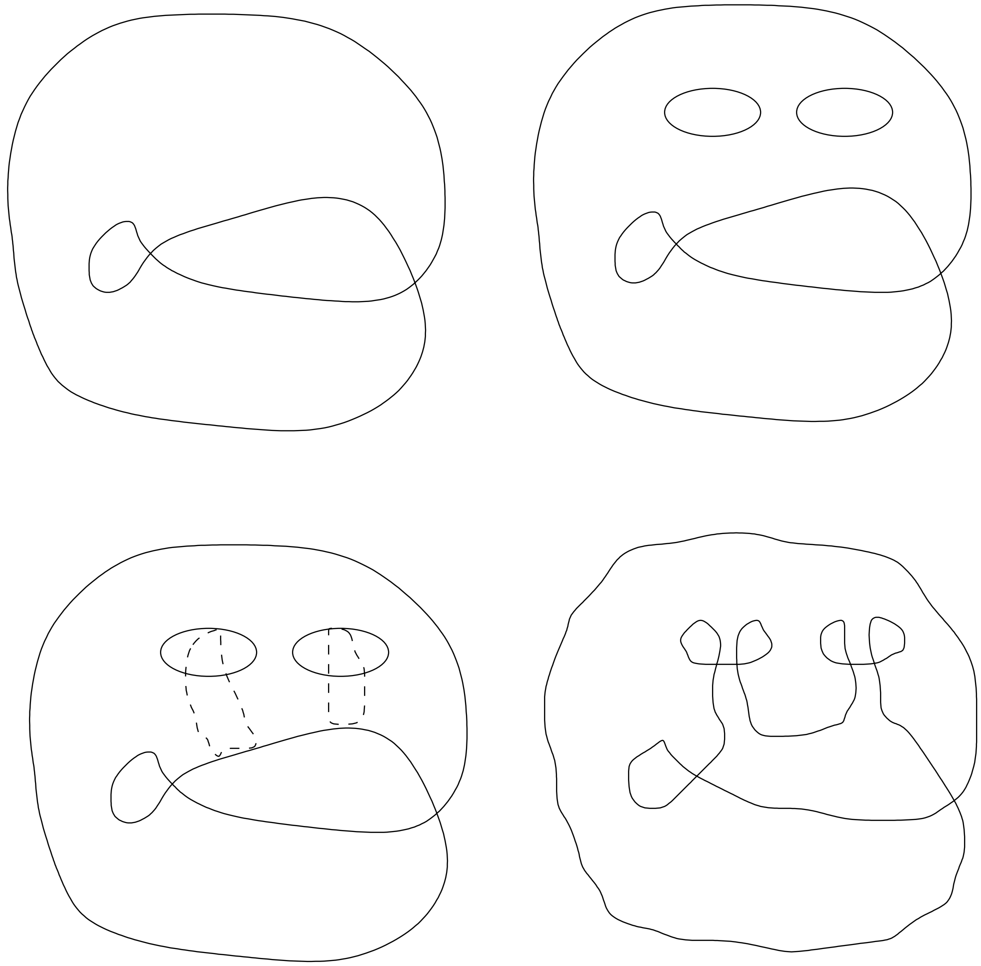
For a general curve C of index $1-2g$, the Whitney index formula shows that the total number of self-intersections, $N^- + N^+$, is at least $2g-2$. Knowing only the index of C, this estimate is the best possible, but using the immersed surface, we can improve it in two places.

By translation, we can assume that the minimal value of y on C is 0. Since I is an immersion, the minimal value of y on $I(\Sigma)$ occurs on $I(\partial\Sigma)$, and therefore the image $I(\Sigma)$ lies above the line $y=0$. Therefore, the inward normal vector to $I(\Sigma)$ at p must point in the positive y-direction, and this implies that $\mu = +1$. This is the first improvement.

Because the surface Σ has genus $g \geq 1$, the curve $I(\partial\Sigma)$ must have at least one self-intersection. Let x be the first point of self-intersection that one reaches following the curve $I(\partial\Sigma)$ from p. We claim that the self-intersection at x is positive. This is the second improvement. Let C_1 denote the arc from p to x, and let C_2 denote a short piece of the other arc of C through x. The positivity of the self-intersection is equivalent to knowing that the inward normal vector of $I(\Sigma)$ along C_2 points on the opposite side of C_2 from C_1. But if the inward normal vector lay on the same side as C_1, there would be a second sheet of $I(\Sigma)$ under C_1, which would run down to p and then down past the line $y=0$, giving a contradiction. Therefore $N^+ \geq 1$.

According to the Whitney index formula, $N^- = \mu + N^+ - ind(C) \geq 1 + 1 + (2g-1) = 2g+1$. Since we already showed that $N^+ \geq 1$, the total number of self-intersections, $N^- + N^+$, is at least $2g+2$. This finishes the proof of the lower bound.

Next we construct an immersion with $2g+2$ self-intersections, for any g. Our construction involves a few steps, illustrated in Figure 3 in the case $g=2$. We start with an immersion of the disk with 2 self-intersections, illustrated in Figure 3. Next, we cut g disjoint disks out of this immersed disk, removing them from the multiplicity 1 region. The result is an immersed curve with $g+1$ components, bounding an immersed surface with genus 0 and $g+1$ boundary components. The last step is to do surgery on this immersed surface. We glue in g strips, as shown in the figure. Each strip connects one of the new circles to the boundary of the original immersed disk. The result is an immersed circle with $2g+2$ transverse self-intersections, bounding an immersed surface of genus g.



References

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[2] Whitney, H.; On regular closed curves in the plane, Comp. Math. 4 (1937) 276-284.

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