

EMBEDDED CMC HYPERSURFACES ON HYPERBOLIC SPACES

OSCAR M. PERDOMO

ABSTRACT. In this paper we will prove that for every integer $n > 1$, there exists a real number $H_0 < -1$ such that every $H \in (-\infty, H_0)$ can be realized as the mean curvature of a embedding of $H^{n-1} \times S^1$ in the $n + 1$ -dimensional spaces H^{n+1} . For $n = 2$ we explicitly compute the value H_0 . For a general value n , we provide function ξ_n defined on $(-\infty, -1)$, which is easy to compute numerically, such that, if $\xi_n(H) > -2\pi$, then, H can be realized as the mean curvature of a embedding of $H^{n-1} \times S^1$ in the $n + 1$ -dimensional spaces H^{n+1} .

1. INTRODUCTION AND PRELIMINARIES

Here we will be considering the following model of the hyperbolic space,

$$H^{n+1} = \{x \in \mathbf{R}^{n+2} : x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = -1\}$$

where the space \mathbf{R}^{n+2} is endowed with the following inner product

$$\langle v, w \rangle = v_1 w_1 + \cdots + v_{n+1} w_{n+1} - v_{n+2} w_{n+2} \quad \text{for } v = (v_1, \dots, v_{n+2}) \text{ and } w = (w_1, \dots, w_{n+2})$$

In [2] we proved the following theorem that shows that $S^{n-1} \times \mathbf{R}$ can be embedded in the hyperbolic space with constant mean curvature.

Theorem 1.1. *Let $g_{C,H} : \mathbf{R} \rightarrow \mathbf{R}$ be a positive solution of the equation*

$$(1.1) \quad (g')^2 + g^{2-2n} + (H^2 - 1)g^2 + 2Hg^{2-n} = C$$

associated with a non negative H and a positive constant C . If $\mu, \lambda, r, \theta : \mathbf{R} \rightarrow \mathbf{R}$ are defined by

$$r = \frac{g_{C,H}}{\sqrt{C}}, \quad \lambda = H + g_{C,H}^{-n}, \quad \mu = nH - (n-1)\lambda = H - (n-1)g_{C,H}^{-n} \quad \text{and} \quad \theta(u) = \int_0^u \frac{r(s)\lambda(s)}{1+r^2(s)} ds$$

then, the map $\phi : S^{n-1} \times \mathbf{R} \rightarrow H^{n+1}$ given by

$$(1.2) \quad \phi(y, u) = (r(u)y, \sqrt{1+r(u)^2} \sinh(\theta(u)), \sqrt{1+r(u)^2} \cosh(\theta(u)))$$

defines an embedded hypersurface in H^{n+1} with constant mean curvature H . Moreover, if $H^2 > 1$, the embedded manifold defined by (1.2) admits the group $O(n) \times Z$ in its group of isometries, where Z is the group of integers.

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The existence of the previous examples just as immersions were studied in [5] as Delaunay-type hypersurfaces of the hyperbolic space and also in [4] as rotational hypersurfaces of spherical type. In this paper we will prove that a subfamily of the family of immersions named as *rotational hypersurfaces of hyperbolic type* in [4] provides different ways to embed the manifold $H^{n-1} \times S^1$ in the $n + 1$ -dimensional hyperbolic space.

2. EMBEDDED HYPERBOLIC TYPE ROTATIONAL SURFACES IN H^3

It is not difficult to show that the function

$$\xi : (-\infty, -1) \rightarrow \mathbf{R} \quad \text{given by} \quad \xi(H) = \int_0^\pi \frac{\sqrt{2} H dt}{\sqrt{2H^2 + \sin(2t) - 1}}$$

is decreasing, $\lim_{H \rightarrow -\infty} \xi(H) = -\pi$ and $\xi(H) < -2\pi$ for values of H close to -1 . The previous observations guarantee the existence of a unique H_0 such that $\xi(H_0) = -2\pi$. A numerical computation shows that,

$$H_0 \simeq -1.0158136657178574$$

In this section we will show that every $H < H_0$ can be realized as the mean curvature of a hyperbolic type rotational embedded constant mean curvature surface in the hyperbolic three dimensional space. Let us state and prove the main and only theorem in this section.

Theorem 2.1. *For any $H < -1$ and $C \in (C_1, 0)$ where $C_1 = 2(H + \sqrt{-1 + H^2})$, let us define $f : \mathbf{R} \rightarrow \mathbf{R}$ by*

$$f(t) = \sqrt{\frac{C - 2H + \sqrt{4 + C^2 - 4CH} \sin(2\sqrt{H^2 - 1} t)}{2H^2 - 2}}$$

If we define,

$$r(t) = \frac{f(t)}{\sqrt{-C}} \quad \text{and} \quad \lambda(t) = H + (f(t))^{-2}$$

then, the function $\frac{\lambda(t)r(t)}{r^2(t)-1}$ is a smooth function everywhere and if we define

$$\theta(t) = \int_0^t \frac{\lambda(s)r(s)}{r^2(s)-1} ds$$

then, the map

$$(2.1) \quad \phi(y, u) = (\sqrt{r(u)^2 - 1} \cos(\theta(u)), \sqrt{r(u)^2 - 1} \sin(\theta(u)), r(u) \sinh(v), r(u) \cosh(v))$$

defines an immersion from \mathbf{R}^2 to H^3 . We also have that for every $H < -1$ there exist infinitely many choices of C such that the immersion ϕ is periodic in the variable u and therefore it defines immersions from $\mathbf{R} \times S^1$ to H^3 . Moreover, we have that for every $H < H_0$, there exists a value C such that ϕ defines an embedding from $\mathbf{R} \times S^1$ to H^3 .

Proof. Since $H < -1$ and $C \in (C_1, 0)$, we have that the function f is a real-value T -periodic function, that oscillates from t_1 to t_2 where,

$$t_1, t_2 = \sqrt{\frac{C - 2H \pm \sqrt{4 + C^2 - 4CH}}{2H^2 - 2}} \quad \text{and} \quad T = \frac{\pi}{\sqrt{H^2 - 1}}$$

A direct computation shows that

$$(f')^2 + f^{-2} + (H^2 - 1)f^2 + 2H = C$$

The equation above shows that the function $r(t)$ satisfies the following identity

$$(2.2) \quad (r')^2 + \lambda^2 r^2 = r^2 - 1$$

This equation shows that $r(t) \geq 1$, moreover, it shows that $r(t^*) = 1$, if and only if $\lambda(t^*) = 0$ and $r'(t^*) = 0$. These last two conditions imply that t^* is a zero with multiplicity 2 of the function λ . We can easily see that t^* is a zero with multiplicity 2 of the function $r^2 - 1$. Since the function r is analytic, we get that the function $\frac{\lambda(s)r(s)}{r^2(s)-1}$ is smooth near t^* , therefore it is smooth everywhere. A direct computation shows that

$$\frac{\partial \phi}{\partial u} = \frac{r r'}{\sqrt{r^2 - 1}} (\cos(\theta), \sin(\theta), 0, 0) + \frac{r \lambda}{\sqrt{r^2 - 1}} (-\sin(\theta), \cos(\theta), 0, 0) + r'(0, 0, \sinh(v), \cosh(v))$$

and

$$\frac{\partial \phi}{\partial v} = r(u) (0, 0, \cosh(v), \sinh(v))$$

It is not difficult to prove that the map

$$\nu = -r\lambda (0, 0, \sinh(v), \cosh(v)) - \frac{r^2 \lambda}{\sqrt{r^2 - 1}} (\cos(\theta), \sin(\theta), 0, 0) + \frac{r'}{\sqrt{r^2 - 1}} (-\sin(\theta), \cos(\theta), 0, 0)$$

is a Gauss map of the immersion ϕ . It follows that the immersion ϕ has constant mean curvature H by noticing that

$$\frac{\partial \nu}{\partial v} = -\lambda \frac{\partial \phi}{\partial v} \quad \text{and} \quad \frac{\partial \nu}{\partial u} = -(2H - \lambda) \frac{\partial \phi}{\partial u}$$

Let us define the function K that depends on H and C , by

$$K(C, H) = \int_0^T \frac{\lambda(s)r(s)}{r^2(s)-1} ds$$

A direct computation shows that for every fixed H we have,

$$(2.3) \quad \lim_{C \rightarrow C_1} K(C, H) = -\pi \sqrt{2 - \frac{2H}{\sqrt{H^2 - 1}}} = b_2(H) \quad \text{and} \quad \lim_{C \rightarrow 0} K(C, H) = 0$$

Since $H < -1$ we have that $b_2(H) < -2\pi$. Using the limits in (2.3) we get that for any fixed value $H < -1$ and for every positive integer m , there exists a real number C^* between C_1 and 0 such that $K(C^*, H) = -\frac{2\pi}{m}$. Since the function θ satisfies that

$$\text{For any integer } j \text{ and } u \in [jT, (j+1)T] \text{ we have that } \theta(u) = jK + \theta(u - jT)$$

we get that if we choose the value C^* , we get that $\theta(mT) = -2\pi$ and therefore the immersion $\phi(u, v)$ will be mT -periodic in the variable u and it will define an immersion from $\mathbf{R} \times S^1$ to H^3 . Let us prove that for every $H < H_0$ there exists an embedding from $\mathbf{R} \times S^1$ to H^3 . By using the definition of the function λ and the expression for the bounds t_1 and t_2 of the function f , we have that for a given H , the function $\lambda < 0$ if and only if $C_1 < C < \frac{1}{H}$. Notice that if λ is always negative, then the function θ is strictly decreasing, and in particular it is one to one. A direct computation shows that

$$K\left(\frac{1}{H}, H\right) = \int_0^T \frac{H \sqrt{2H^2 - 2}}{\sqrt{2H^2 - 1 + \sin(2\sqrt{H^2 - 1} s)}} ds = \int_0^\pi \frac{H \sqrt{2}}{\sqrt{2H^2 - 1 + \sin(2t)}} dt$$

As pointed out at the beginning of this section, the function $\xi(H) = K(\frac{1}{H}, H)$ is decreasing and the limit when $H \rightarrow -\infty$ is $-\pi$. Therefore for any $H < H_0$ there exists a C^* between C_1 and $\frac{1}{H}$ such that $K(C^*, H) = -2\pi$. By the way we picked C^* we get that the function θ is strictly decreasing and $\theta(T) = -2\pi$, these two conditions guarantee that the immersion $\phi(u, v)$ is T -periodic and injective in $\mathbf{R} \times (0, T)$, therefore ϕ defines an embedding from $\mathbf{R} \times S^1$ to H^3 . This completes the proof of the theorem. \square

2.1. Graph of some profile curves. The examples described above are obtained by doing a hyperbolic rotation of the profile curve

$$\alpha(t) = (\sqrt{r^2(t) - 1} \cos(\theta(t)), \sqrt{r^2(t) - 1} \sin(\theta(t)))$$

We will show the graphs of a profile curve that corresponds to an embedded example and two profile curves corresponding to immersed examples, all of them represent examples with constant mean curvature $H = -1.1$. To finish the section we will show one of the numerical difficulties to do the graph. This difficulty is the fact that the angle function θ moves a lot in a small variation of the parameter t , during this small variation of parameter t , the radius function $\sqrt{r^2(t) - 1}$ is very close to zero. We will show this fact by graphing the function $\theta'(t)$, first by limiting the codomain to some values close to zero, and then by showing the whole graph of θ' .

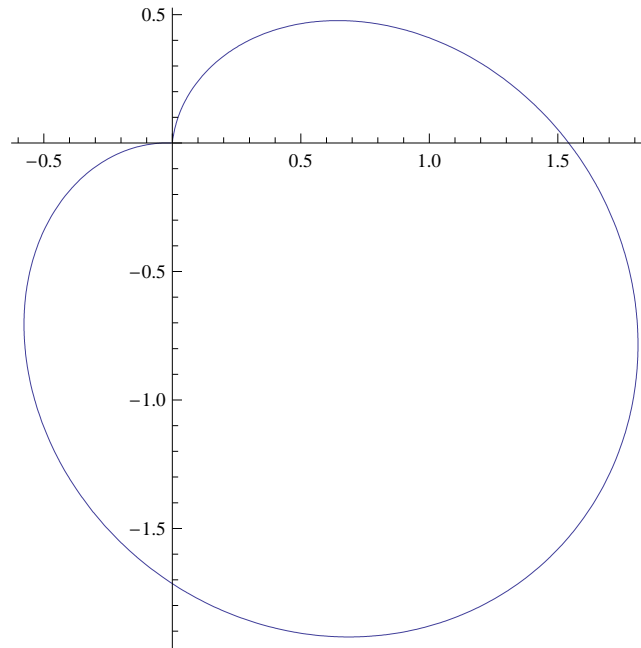


FIGURE 2.1. Profile curve for a surface with CMC $H = -1.1$, in this case the surface is embedded and $C = -0.9091743461769703$ and $K = -2\pi$

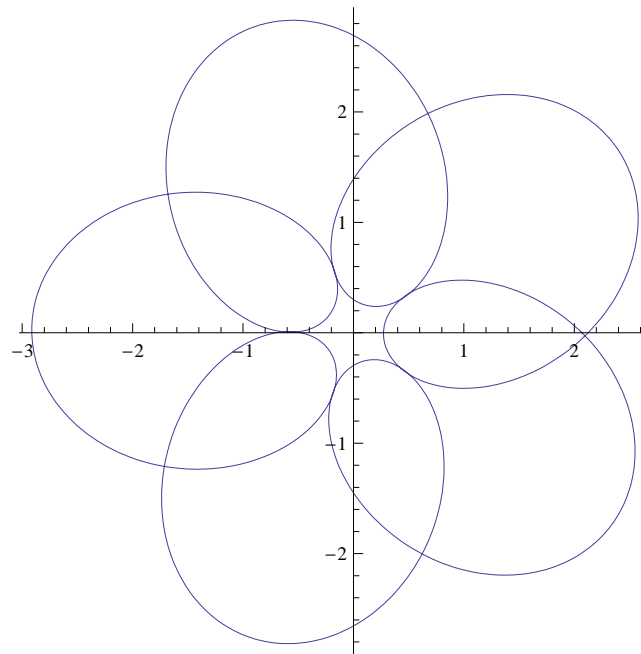


FIGURE 2.2. Profile curve for a surface with CMC $H = -1.1$, in this case $C = -0.6835660909345689$ and $K = -\frac{2\pi}{5}$

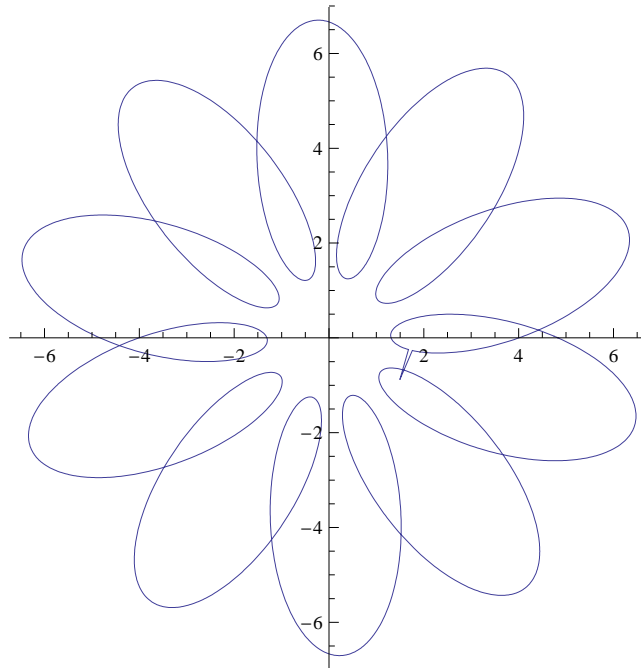


FIGURE 2.3. Profile curve for a surface with CMC $H = -1.1$, in this case $C = -0.19607165524075582$ and $K = -\frac{2\pi}{10}$

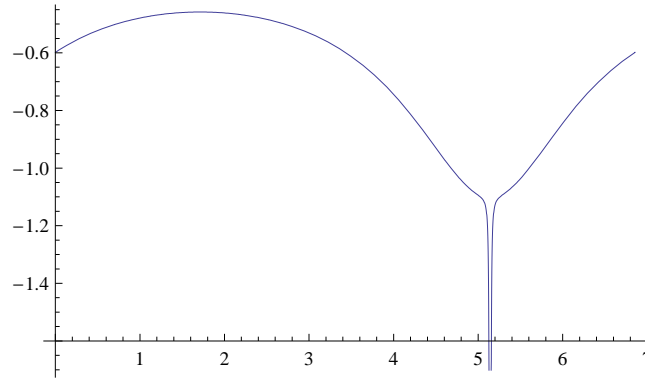


FIGURE 2.4. Graph of the function θ' associated with the embedded example which profile curve is shown above, in this case just part of the graph is shown

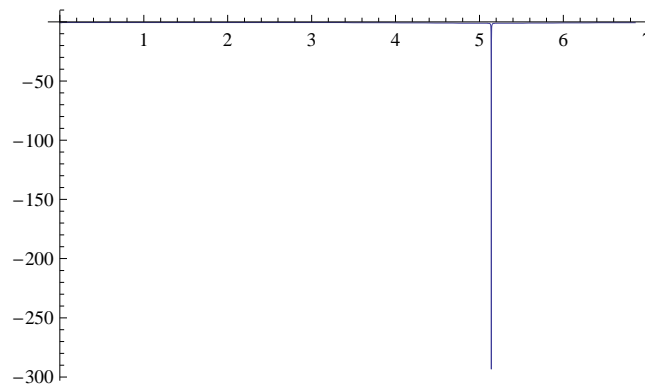


FIGURE 2.5. Graph of the function θ' example associated with the embedded which profile curve is shown above.

3. EMBEDDED SOLUTIONS IN HYPERBOLIC SPACES.

It is well known that the existence of CMC hypersurfaces in hyperbolic spaces relies on the existence of solutions of the following differential equation,

$$(g')^2 + g^{2-2n} + (H^2 - 1)g^2 + 2Hg^{2-n} = C$$

It is not difficult to check that, when $H < -1$, it is possible to obtain solutions of this equation associated with negative values of C . This CMC examples produced by these solutions when $C < 0$ correspond to those named as rotational hyperbolic type in [4]. Similar arguments as those shown in [2] will give us explicit immersions for such a choice of the constant C . The following sequences of statements tell us how to pick the negative values of C to obtain solutions in the case that $H < -1$ and several other properties that will be useful in the proof of the main theorem in this paper.

Remark 3.1. *The function $q : (0, \infty) \rightarrow \mathbf{R}$ defined by $q(v) = C - v^{2-2n} + (1 - H^2)v^2 - 2Hv^{2-n}$, where $H < -1$ and $C < 0$, has the following properties:*

- (1) *The positive real number v_0 given by*

$$v_0 = \left(\frac{H(n-2) + \sqrt{4-4n+H^2n^2}}{2H^2-2} \right)^{\frac{1}{n}} = \left(\frac{2(n-1)}{\sqrt{4-4n+H^2n^2} - H(n-2)} \right)^{\frac{1}{n}}$$

is the only positive critical point of q .

- (2) *$p(v) = v^{2n-2}q(v)$ is a polynomial with even degree, negative leading coefficient and $p(0) = -1$.*

- (3) *Since $q'(v) > 0$ if $v < v_0$, $q'(v) < 0$ if $v > v_0$, and $q(v_0) = C - C_0$ where*

$$(3.1) \quad C_0 = n \frac{H^2n - 2 + H\sqrt{4-4n+H^2n^2}}{(H(n-2) + \sqrt{4-4n+H^2n^2})^{\frac{2n-2}{n}}} (2H^2 - 2)^{\frac{n-2}{n}}$$

then, q has exactly 2 roots whenever $0 > C > C_0$.

- (4) *The functions $t_1, t_2 : (C_0, 0) \times (-\infty, -1) \rightarrow (0, \infty)$ defined by the equations*

$$(3.2) \quad q(t_1(C, H)) = 0 \quad q(t_2(C, H)) = 0 \quad \text{with} \quad t_1(C, H) < t_2(C, H)$$

are smooth, $t_1(C, H)$ is decreasing with respect to C , $t_2(C, H)$ is increasing with respect to C and the limit of both functions when $C \rightarrow C_0$ is v_0 .

- (5) *Since the roots of q when $C = 0$ are $v_1 = \frac{1}{(1-h)^{\frac{1}{n}}}$ and $v_2 = \frac{1}{(-1-h)^{\frac{1}{n}}}$ then for any fixed H the derivative of the functions t_1 and t_2 defined on $(C_0, 0)$ never vanish and*

$$\lim_{C \rightarrow 0} t_1(C) = v_1, \quad \lim_{C \rightarrow 0} t_2(C) = v_2 \quad \text{and} \quad \lim_{C \rightarrow C_0} t_1(C) = \lim_{C \rightarrow C_0} t_2(C) = v_0$$

- (6) *The following identities are true,*

$$\lambda_1 = H + v_0^{-n} = \frac{nH + \sqrt{H^2 n^2 - 4(n-1)}}{2(n-1)} < 0 \quad \text{and} \quad \lambda_2 = H + v_1^{-n} = 1$$

- (7) For a fixed $H < -1$, the previous two items guarantee the existence of a unique $\tilde{C}(H) \in (C_0, 0)$ such that $t_1^*(H) = t_1(\tilde{C}(H), H)$ satisfies that

$$H + (t_1^*(H))^{-n} = 0$$

The equality above defines a smooth function $\tilde{C} : (-\infty, -1) \rightarrow (C_0, 0)$

- (8) We can explicitly compute the function \tilde{C} by noticing first that for that special value of C , the number $t_1 = (-H)^{-\frac{1}{n}}$ must be a root of the function q , therefore $q((-H)^{-\frac{1}{n}})$ must be zero, i.e.,

$$q((-H)^{-\frac{1}{n}}) = C + (-H)^{-\frac{2}{n}} = 0$$

$$\text{Therefore, } \tilde{C}(H) = -(-H)^{-\frac{2}{n}}$$

- (9) The function $\tilde{q}(v) = -\frac{1}{C}q(\sqrt{-C}v)$ has the following expression

$$\tilde{q}(v) = -1 - (-C)^{-n}v^{2-2n} + v^2(1 - H^2 - 2H(\sqrt{-C}v)^{-n})$$

Moreover, by the definition of \tilde{q} and the properties of the function q we have that \tilde{q} , for any $C \in (C_0, C)$, the only 2 positive roots of \tilde{q} are

$$\tilde{t}_1(C, H) = \frac{t_1(C, H)}{\sqrt{-C}} \quad \text{and} \quad \tilde{t}_2(C, H) = \frac{t_2(C, H)}{\sqrt{-C}}$$

$$\text{Therefore, we have that } \tilde{t}_1(\tilde{C}, H) = \frac{(-H)^{-\frac{1}{n}}}{\sqrt{(-H)^{-\frac{2}{n}}}} = 1$$

- (10) A direct computation shows that when $C = \tilde{C}$, the polynomial \tilde{q} , reduces to the polynomial Q given by,

$$Q = -1 + v^2 - H^2v^2 - H^2v^{2-2n} + 2H^2v^{2-n}$$

It is not difficult to check that, when $n > 2$, for any positive ϵ , $\lim_{H \rightarrow -\infty} Q(1 + \epsilon) = -\infty$, therefore we have that

$$(3.3) \quad \tilde{t}_1(\tilde{C}, H) = 1 \quad \text{and} \quad \lim_{H \rightarrow -\infty} \tilde{t}_2(\tilde{C}, H) = 1$$

- (11) Let us define the function $h : (0, \infty) \rightarrow \mathbf{R}$ by

$$h(v) = \frac{2Hv^{1-n}(-1 + v^n)}{v^2 - 1} = 2Hv^{1-n} \frac{1 + v + \dots + v^{n-1}}{1 + v}$$

and the function $\xi_n : (-\infty, -1) \rightarrow \mathbf{R}$

$$\xi_n(H) = \int_1^{\tilde{t}_2(\tilde{C}, H)} \frac{h_n(v)}{\sqrt{Q(v)}} dv$$

(12) A direct computation shows that

$$(3.4) \quad \tilde{a} = -\frac{1}{2}Q''(1) = n^2H^2 - 1$$

Therefore using a small modification of lemma 5.1 and its corollary in [2] we get that

$$(3.5) \quad \lim_{H \rightarrow -\infty} \xi_n(H) = -\pi$$

Theorem 3.2. Let $g : \mathbf{R} \rightarrow \mathbf{R}$ be a positive solution of the equation

$$(3.6) \quad (g')^2 + g^{2-2n} + (H^2 - 1)g^2 + 2Hg^{2-n} = C$$

associated with a negative constant C . If $\mu, \lambda, r, \theta : \mathbf{R} \rightarrow \mathbf{R}$ are defined by

$$r = \frac{g}{\sqrt{-C}}, \quad \lambda = H + g^{-n}, \quad \mu = nH - (n-1)\lambda = H - (n-1)g^{-n} \quad \text{and} \quad \theta(u) = \int_0^u \frac{r(s)\lambda(s)}{r^2(s)-1} ds$$

then, the map $\phi_{C,H} : H^{n-1} \times \mathbf{R} \rightarrow H^{n+1}$ given by

$$(3.7) \quad \phi_{C,H}(y, u) = (\sqrt{r(u)^2 - 1} \cos(\theta(u)), \sqrt{r(u)^2 - 1} \sin(\theta(u)), r(u)y)$$

defines an immersed hypersurface in H^{n+1} with constant mean curvature H . We also have that when $H < -1$, the function g is periodic and if we denote its period by T , then, $\phi_{C,H}$ defines an immersion from $H^{n-1} \times S^1$ to H^n whenever

$$(3.8) \quad K(C, H) = \int_0^T \frac{r(s)\lambda(s)}{r^2(s)-1} ds = -\frac{2k\pi}{m} \quad \text{for some pair of positive integers } k \text{ and } m$$

Moreover, we have that anytime $\xi_n(H_1) > -2\pi$, where ξ_n is the function defined in item (11) in Remark (3.1), then, there exists a constant C such that the immersion ϕ_{C,H_1} defines an embedding from $H^{n-1} \times S^1$ to H^{n+1} .

Proof. A direct computation shows the following identities,

$$(r')^2 + r^2\lambda^2 = r^2 - 1, \quad \text{and} \quad \lambda r' + r\lambda' = \mu r'$$

Let us define

$$B_2(u) = (\cos(\theta(u)), \sin(\theta(u)), 0, \dots, 0) \quad \text{and} \quad B_3(u) = (-\sin(\theta(u)), \cos(\theta(u)), 0, \dots, 0)$$

Notice that $\langle B_2, B_2 \rangle = 1$, $\langle B_3, B_3 \rangle = 1$, $\langle B_2, B_3 \rangle = 0$, $B_2' = \frac{r\lambda}{r^2-1}B_3$ and $B_3' = -\frac{r\lambda}{r^2-1}B_2$, moreover, we have that the map $\phi = \phi_{C,H}$ can be written as

$$\phi = r(0, 0, y) + \sqrt{r^2 - 1}B_2$$

A direct verification shows that $\langle \phi, \phi \rangle = -1$ and that

$$\frac{\partial \phi}{\partial u} = r'(0, 0, y) + \frac{rr'}{\sqrt{r^2 - 1}} B_2 + \frac{r\lambda}{\sqrt{r^2 - 1}} B_3$$

is a unit vector, i.e., $\langle \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial u} \rangle = 1$. We have that the tangent space of the immersion at (y, u) is given by

$$T_{\phi(y,u)} = \{(v, 0, 0) + s \frac{\partial \phi}{\partial u} : \langle v, y \rangle = 0 \quad \text{and} \quad s \in \mathbf{R}\}$$

A direct verification shows that the map

$$\nu = -r\lambda(0, 0, y) - \frac{r^2\lambda}{\sqrt{r^2 - 1}} B_2 + \frac{r'}{\sqrt{r^2 - 1}} B_3$$

satisfies that $\langle \nu, \nu \rangle = 1$, $\langle \nu, \frac{\partial \phi}{\partial u} \rangle = 0$ and, for any $v \in \mathbf{R}^n$ with $\langle v, y \rangle = 0$, we have that $\langle \nu, (v, 0, 0) \rangle = 0$. It then follows that ν is a Gauss map of the immersion ϕ . The fact that the immersion ϕ has constant mean curvature H follows because, for any unit vector v in \mathbf{R}^n perpendicular to y , we have that

$$\beta(t) = (0, 0, r \cosh(t) y + r \sinh(t) v) + \sqrt{r^2 - 1} B_2 = \phi(\cosh(t)y + \sinh(t)v, u)$$

satisfies that $\beta(0) = \phi(y, u)$, $\beta'(0) = rv$ and

$$\left. \frac{d\nu(\beta(t))}{dt} \right|_{t=0} = d\nu(rv) = -r\lambda v$$

Therefore, λ is a principal curvature with multiplicity $n - 1$. Now, since $\langle \frac{\partial \nu}{\partial u}, (v, 0, 0) \rangle = 0$ for every $(v, 0, 0) \in T_{\phi(y,u)}$, we have that $\frac{\partial \phi}{\partial u}$ defines a principal direction, i.e. we must have that $\frac{\partial \phi}{\partial u}$ is a multiple of $\frac{\partial \phi}{\partial u}$. A direct verification shows that,

$$\left\langle \frac{\partial \nu}{\partial u}, y \right\rangle = -\lambda' r - \lambda r' = -\mu r' = -(nH - (n - 1)\lambda)r'$$

We also have that $\langle \frac{\partial \phi}{\partial u}, y \rangle = r'$, therefore,

$$\frac{\partial \nu}{\partial u} = d\nu\left(\frac{\partial \phi}{\partial u}\right) = -\mu \frac{\partial \phi}{\partial u} = -(nH - (n - 1)\lambda) \frac{\partial \phi}{\partial u}$$

It follows that the other principal curvature is $nH - (n - 1)\lambda$. Therefore ϕ defines an immersion with constant mean curvature H , this proves the first item in the Theorem. The fact that the map defines an immersion from $H^{n-1} \times S^1$ whenever $K(C, H) = -\frac{2k\pi}{m}$, follows from the following property

$$\text{For any integer } j \text{ and } u \in [jT, (j + 1)T] \text{ we have that } \theta(u) = jK + \theta(u - jT)$$

which implies that the map ϕ is periodic in the variable u , with period mT . Let us prove the embedding part of the theorem. In this part of the proof we will be using the functions and constants

$$q, \tilde{q}, Q, \xi_n, t_1, t_2, \tilde{t}_1, \tilde{t}_2, \tilde{C}, C_0, \quad \text{and} \quad v_0$$

defined in Remark (3.1). Let us start by noticing that the differential equations for the functions g and r can be written as

$$(g') = q(g) \quad \text{and} \quad (r')^2 = \tilde{q}(r)$$

It follows that, in order to obtain a solution g of this differential equation, we need that $C > C_0$ and, once we have the solution g associated with the number C and H , this solution g varies from $t_1(C, H)$ to $t_2(C, H)$. Since we know the maximum and the minimum of the function g in terms of C and H , we can verify that anytime $C < \tilde{C} = -(-H)^{-\frac{2}{n}}$, the function λ is negative, we also have that when $C = \tilde{C}$, 0 is the maximum of the function λ . The previous affirmation guarantees that anytime $C \in (C_0, \tilde{C})$, the function θ is one to one. By doing the substitution $v = g(s)$ in the integral $K(C, H)$, we get that

$$K(C, H) = \int_{t_1(C, H)}^{t_2(C, H)} \frac{2\sqrt{-C} (1 + Hv^n) v^{1-n}}{(C + v^2) \sqrt{q(v)}} dv$$

In the previous expression we have used the symmetry of the function g , and therefore the symmetries of the functions r and λ , to express K as

$$K = 2 \int_0^{\frac{\pi}{2}} \frac{r(s)\lambda(s)}{r^2(s) - 1} ds$$

When $C = C_0$, we have that $q(v_0) = 0 = q'(v_0)$, then, we can apply the lemma 5.1 in [2] and its corollary, to obtain that

$$\lim_{C \rightarrow C_0} K(C, H) = -\sqrt{2} \sqrt{1 - \frac{nH}{\sqrt{n^2 H^2 - 4(n-1)}}} \quad \pi = lb$$

Notice that for any $n \geq 2$ and any $H < -1$, the bound $lb < -2\pi$. By doing the substitution $v = r(s)$ in the integral $K(C, H)$, we get that

$$K(C, H) = \int_{\tilde{t}_1(C, H)}^{\tilde{t}_2(C, H)} \frac{2v (H + (\sqrt{-C}v)^{-n})}{(v^2 - 1) \sqrt{\tilde{q}(v)}} dv$$

When we replace C by \tilde{C} the integral above reduces to,

$$K(\tilde{C}, H) = \xi_n(H)$$

Using the intermediate value theorem we conclude the theorem because anytime $\xi_n(H) > -2\pi$ there exists a $C^* \in (C_0, \tilde{C})$ such that $K(C^*, H) = -2\pi$, therefore the map $\phi_{C^*, H}$ is periodic in the u variable, and since $C < \tilde{C}$ the function θ is injective and therefore the map $\phi_{C^*, H}$ is an embedding. \square

Corollary 3.3. *For any integer $n > 1$ there exists an $H_0 \leq -1$ such that for any $H < H_0$ there exists an embedding with constant mean curvature H from $H^{n-1} \times S^1$ to H^{n+1} .*

Proof. The corollary follows from the fact that $\lim_{H \rightarrow -\infty} \xi_n(H) = -\pi$. See item (12) in Remark (3.1). \square

Remark 3.4. *The integral ξ_n is easy to evaluate numerically, for example*

$$\xi_3(-1) = -5.97106763713693 \quad \xi_4(-1) = -4.599155062889069 \quad \xi_5(-1) = -4.13016242612799$$

The following graphs suggest that for $n = 3, 4, 5$, there exist embeddings for all $H < -1$.

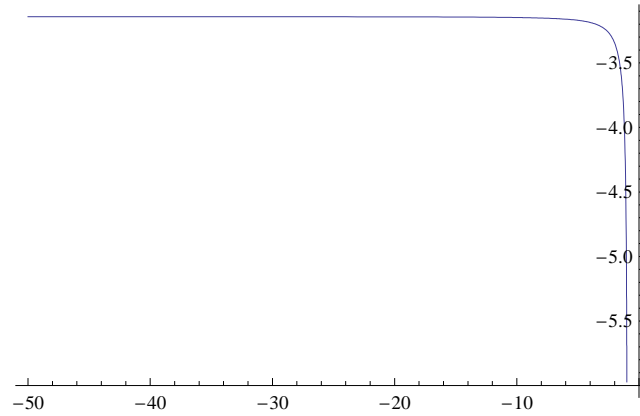
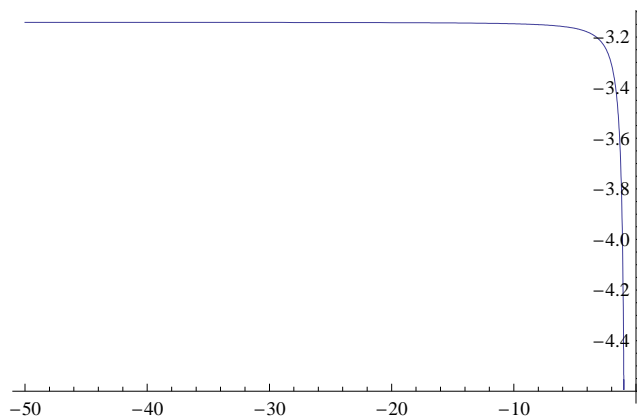
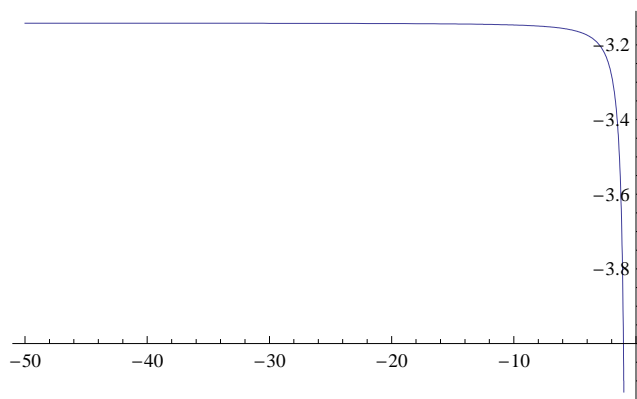


FIGURE 3.1. Graph of the function ξ_3 on $[-50, -1]$

FIGURE 3.2. Graph of the function ξ_4 on $[-50, -1]$ FIGURE 3.3. Graph of the function ξ_5 on $[-50, -1]$

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Current address: Department of Mathematics, Central Connecticut State University, New Britain, CT 06050,

E-mail address: perdomoosm@ccsu.edu