$N-$BUNDLES FOR N AN EXTENSION OF A FINITE GROUP BY AN ABELIAN GROUP

YASHONIDHI PANDEY

Abstract. Let W be a finite group and T be an abelian group. Consider an extension $0 \to T \to N \to W \to 0$. For a smooth projective curve X, we give a precise description of the fiber of the quotient by T map $q_T : \mathcal{M}_X(N) \to \mathcal{M}_X(W)$ as a torsor over an abelian variety. We also prove a result on Mumford groups.

1. Introduction

Let T be an arbitrary abelian group and W be an arbitrary finite group. We fix an action $\sigma : W \to Aut(T)$.

Let $\pi : Z \to X$ be an étale Galois cover of smooth projective curves of Galois group a finite group W. Let E_T be a $T-$bundle on Z. Combining the pull-back action of W on Z and the action σ on the fibers, we have the twisted action of W on $T-$bundles on Z defined as $(w, E) \mapsto w^*E \times_w T$ where $\sigma(w) : T \to T$ is the understood extension of structure group from T to itself. The action of W on Z does not in general lift to E_T, however on $W-$invariant principal $T-$bundles on Z, we can define a Mumford group parametrising the 2-uples (w, γ) where γ is an isomorphism between E and wE of $T-$bundles of Z.

The second main result of this paper is the Proposition 3.1 on Mumford groups.

Proposition 1.1. *Let $\pi : Z \to X$ be an étale Galois cover of smooth projective curves with Galois group a finite group W. We have a homomorphism of groups*

$$\begin{array}{cccc} c: & H^1(Z,\underline{T})^W & \to & H^2(W,T) \\ & E_T & \mapsto & [0 \to T \to \mathcal{G}^\sigma(E_T) \to W \to 0] \end{array}$$

Let η denote the class of an extension $[0 \to T \to N \to W \to 0] \in H^2(W, T)$.

The main theorem 4.7 of this paper can be described as follows.

Theorem 1.2. *The fiber of the quotient by T map $q_T : \mathcal{M}_X(N) \to \mathcal{M}_X(W)$ over $\pi : Z \to X$ is $H^1(Z, \underline{T})^W_\eta$.*

We give applications of these results by taking N to be the dihedral group, the Weyl group of type B_n, C_n and D_n to describe the set of $N-$bundles $\mathcal{M}_X(N)$ on a curve X.

Much of these results were obtained to study the Weyl group, the torus and the Lie group in the context of abelianisation. We thus use suggestively the notation W for an arbitrary finite group and T for a finite abelian group.

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2. Principal $N-$bundles: notation and known results

Let T be an arbitrary abelian group and W be an arbitrary finite group. We fix an action $\sigma : W \to Aut(T)$. Let N be an arbitrary extension

$$0 \to T \to N \to W \to 0$$

inducing the action σ.

Definition 2.1. *For $w \in W$ and a $T-$bundle E we denote by $E \times_w T$ the quotient of $E \times T$ by the relation*

$$(et, t') \ (e, w(t)t').$$

and call it the extension of structure group of E by w. We call the following map the evaluation morphism

$$\begin{array}{rccc} \nu_w : & E_T \times_w T & \to & E_T \\ & \overline{(e,t)} & \mapsto & ew^{-1}(t) \end{array}$$

Definition 2.2. *We define the twisted action of W on $T-$bundles on Z*

$$\begin{array}{ccc} W \times H^1(Z,T) & \to & H^1(Z,T) \\ (w, E) & \mapsto & w^*E \times_w T \end{array}$$

Notice that pull-back by w and extension of structure group by w commute with eachother. We shall denote the twisted action of w on E as $w.E$.

For the rest of this section let E_T denote a $T-$bundle on Z invariant under the twisted action of W.

Definition 2.3. *We define for E_T the Mumford group as follows*

$$\mathcal{G}^\sigma(E) = \{(w, \gamma) | \gamma : E \to w.E\}$$

We define the composition $(w_1, \gamma_1) \circ (w_2, \gamma_2)$ as $(w_1 \circ w_2, w_1(\gamma_2)\gamma_1)$ where $w_1(\gamma_2) = w_1^*\gamma_2 \times_{w_1} \circ \gamma_1$.

Since E and $w.E$ are principal $T-$bundles and T is abelian, so the set of isomorphisms between them is a torsor on T and we have $\mathrm{Aut}_T(E) = T$. Thus we can talk of a Mumford group (instead of a sheaf of groups).

Definition 2.4. *For $E_T \in H^1(Z,T)^W$, we define a right action of $\mathcal{G}^\sigma(E)$ on E_T as follows: for $(w,\gamma) = n \in \mathcal{G}^\sigma(E)$, we define μ_n as the composition of morphisms*

$$
\begin{array}{ccccccc}
E_T & \xrightarrow{\gamma} & w^*E_T \times_w T & \longrightarrow & E_T \times_w T & \xrightarrow{\nu_w} & E_T \\
 & & \downarrow & & \downarrow & & \\
 & & Z & \xrightarrow{\quad w \quad} & Z & &
\end{array} \tag{1}
$$

For ease of notation we will denote the Mumford group $\mathcal{G}^\sigma(E)$ of E_T by N.

Remark 2.5. *It follows directly (cf.* [1] *[Prop 6.3]) that this action extends the action of T on E_T and lifts the action of W on Z over X.*

Consider the following action:

$$
\begin{array}{rccc}
\Psi_{E_T}: & (E \times_T N) \times N & \longrightarrow & E \times_T N \\
 & (\overline{(e,n')}, n) & \mapsto & \overline{(\mu_n(e), n^{-1}n')}
\end{array}
$$

Remark 2.6. *It is proved in* [1] *Prop 6.6 that $E_N := E_T \times^T N$ admits a canonical $W-$linearisation induced from E_T for T a torus. But the same proof works for T an arbitrary abelian group.*

We denote the linearisation by Φ_{E_T}.

3. A result on Mumford group

Proposition 3.1. *Let $\pi : Z \to X$ be an étale Galois cover of smooth projective curves with Galois group a finite group W. We have a homomorphism of groups*

$$
\begin{array}{rccc}
c: & H^1(Z,\underline{T})^W & \to & H^2(W,T) \\
 & E_T & \mapsto & [0 \to T \to \mathcal{G}^\sigma(E_T) \to W \to 0]
\end{array}
$$

This Proposition is the Proposition 7.1 in [1].

Proof. We have two left exact functors

(1) Γ_Z : {Sheaves of W-modules on Z} $\to$ {W-modules}

$$\underline{A} \to H^0(Z, \underline{A})$$

(2) Γ^W : {W-modules} $\to$ {abelian groups } $M \to M^W$

We denote the composite functor as $\Gamma_Z^W = \Gamma^W \circ \Gamma_Z$ whose $n-$th derived fonctor is calculated by the Grothendieck spectral sequence of composite of two functors

$$E_2^{p,q} = H^p(W, H^q(Z, \underline{A})) \Rightarrow E^{p,q} = H^{p+q}(Z; W; \underline{A})$$

We have the following short exact sequence of low degree terms

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{c} E_2^{2,0} \to E^2$$

In the case when the group W acts upon T, this sequence becomes

$$0 \to H^1(W,T) \to H^1(Z;W;\underline{T}) \to H^1(Z,\underline{T})^W \xrightarrow{c} H^2(W,T) \to H^2(Z;W;\underline{T})$$

Let us fix a principal T-bundle $E_T \in H^1(Z,\underline{T})^W$. We associate to the extension

$$0 \to T \to \mathcal{G}^\sigma(E_T) \xrightarrow{p} W \to 0 \tag{2}$$

an element of $H^2(W,T)$ by taking a set theoretic section α of p.

For each $w \in W$, we get an isomorphism $\alpha(w) : E_T \to wE_T$. We identify $Aut_T(E_T)$ and T as T is abelian. To α we associate the 2-cocyle

$$\begin{array}{rccc} f_\alpha : & W \times W & \to & T \\ & (w_1, w_2) & \mapsto & \alpha(w_2 w_1)^{-1} \circ w_1 \alpha(w_2) \circ \alpha(w_1) \end{array} \tag{3}$$

If we change α by β, we have $f_\beta - f_\alpha = d\theta$, where θ is the map

$$\begin{array}{rccc} \theta : & W & \to & T \\ & w & \mapsto & \beta(w)^{-1} \circ \alpha(w). \end{array}$$

Thus the class of f_α in $H^2(W,T)$ is well defined. It is equal to the classs of (2) by the natural correspondance between extensions of groups by abelian groups and $2-$cocycles.

Let us explicit the map c. Let $\mathfrak{U}$ be a W-invariant covering of Z by affine open sets ,that is for $U \in \mathfrak{U}$, $w.U \in \mathfrak{U}$ also. The Cech complexes gives a resolution $\underline{T} \to \mathcal{C}^\cdot(\mathfrak{U}, \underline{T})$ of the sheaf $\underline{T}$ on Z. We deduce from the resolution two short exact sequences

$$0 \to Z^0(\mathcal{C}^\cdot(\mathfrak{U},\underline{T})) \to C^0(\mathfrak{U},\underline{T}) \to B^1(\mathcal{C}^\cdot(\mathfrak{U},\underline{T})) \to 0 \tag{4}$$

$$0 \to B^1(\mathcal{C}^\cdot(\mathfrak{U},\underline{T})) \to Z^1(\mathcal{C}^\cdot(\mathfrak{U},\underline{T})) \to H^1(\mathcal{C}^\cdot(\mathfrak{U},\underline{T})) \to 0 \tag{5}$$

Let us abbreviate the above groups by Z^0, C^0, B^1, Z^1 et H^1 respectively.

By the twisted action

$$\begin{array}{ccc} W \times C^i(\mathfrak{U},\underline{T}) & \to & C^i(\mathfrak{U},\underline{T}) \\ (w, a_{k,..,l}) & \mapsto & w(a_{w^{-1}k,..,w^{-1}l})_{k,..,l} \end{array}$$

we take the resolution by the complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z^0 & \longrightarrow & C^0 & \longrightarrow & B^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{\cdot}(W, Z^0) & \longrightarrow & C^{\cdot}(W, C^0) & \longrightarrow & C^{\cdot}(W, B^1) & \longrightarrow & 0
\end{array}
\tag{6}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B^1 & \longrightarrow & Z^1 & \longrightarrow & H^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{\cdot}(W, B^1) & \longrightarrow & C^{\cdot}(W, Z^1) & \longrightarrow & C^{\cdot}(W, H^1) & \longrightarrow & 0
\end{array}
\tag{7}
$$

By the definition of the spectral sequence, the map c is the composition of

$$H^0(C^{\cdot}(W, H^1)) \to H^1(C^{\cdot}(W, B^1)) \to H^2(C^{\cdot}(W, Z^0)),$$

where the morphisms between the groups are connection morphisms.

For $E_T \in H^1(Z, \underline{T})^W \subset C^0(W, H^1)$, let $a := (a_{i,j}) \in Z^1(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))$ be a 1-cocycle antecedent of E_T. Associated to a, there is a *canonical* choice of a 1-cocycle wa antecedent of the principal wE_T bundle for the twisted action of W on E_T. Recall that by definition $(wa)_{i,j} := w(a_{w^{-1}i, w^{-1}j})_{i,j}$. We consider the first terms of 7 and denote the vertical arrows by d

(8)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))^W & \longrightarrow & Z^1(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))^W & \longrightarrow & H^1(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))^W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(W, B^1) & \longrightarrow & C^0(W, Z^1) & \longrightarrow & C^0(W, H^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1(W, B^1) & \longrightarrow & C^1(W, Z^1) & \longrightarrow & C^1(W, H^1) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & C^2(W, B^1) & & & & & &
\end{array}
$$

Let us denote $d(a)$ by h. Since $d(E_T) = 0$, we have $h \in C^1(W, B^1)$. Explicitly h is given by

$$h = d(a) : \begin{array}{ccc} W & \to & Z^1(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T})) \\ w & \mapsto & w(a) \circ a^{-1} \end{array} \tag{9}$$

where over $u \in U_i \cap U_j$, we define $w(a) \circ a^{-1}(i,j)(u) := w(a)(i,j)(u)a(i,j)(u)^{-1}$. We consider the first terms of (6) and let us denote the vertical arrows by d

(10)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^0(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))^W & \longrightarrow & C^0(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))^W & \longrightarrow & B^1(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T}))^W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(W, Z^0) & \longrightarrow & C^0(W, C^0) & \longrightarrow & C^0(W, B^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1(W, Z^0) & \longrightarrow & C^1(W, C^0) & \longrightarrow & C^1(W, B^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^2(W, Z^0) & \longrightarrow & C^2(W, C^0) & \longrightarrow & C^2(W, B^1) & \longrightarrow & 0
\end{array}$$

We denote by $g \in C^1(W, C^0)$ an antecedent of h. Then for all $w \in W$, we have

$$g(w)(i)g(w)(j)^{-1} = h(w)(i,j) \tag{11}$$

Since $Aut_T(E_T) = T$ is commutative, by (11) and (9) we deduce the equality

$$w(a)(i,j) = g(w)(i)a(i,j)g(w)(j)^{-1}$$

which means that g explicitly gives an isomorphism $g(w) : E_T \to wE_T$ in local coordinates for all $w \in W$. Thus g is a section of p of the short exact sequence (2). Since $0 = d(d(a)) = d(h)$ so $k := d(g) \in C^2(W, Z^0)$ is a 2-cocycle. Now since $H^0(\mathcal{C}^{\cdot}(\mathfrak{U}, \underline{T})) = \underline{T}$, we have

$$\begin{array}{ccc} k : W \times W & \to & T \\ (w_1, w_2) & \mapsto & w_1(g(w_2))g(w_1 w_2)^{-1}g(w_1) \end{array}$$

The map c sends E_T to the class of k in $H^2(W, T)$, by the definition of the spectral sequence. This definition is independant of the choice of the antecedant a and g of E_T and h respectively. Since $g(w) : E_T \to wE_T$ is an isomorphism, f_g (ref 3) is equal to k because the passage from g to k and to f_g is the same. Thus a fortiori the image of k and of f_g- the class of Mumford group in $H^2(W,T)$ coincide. $\square$

4. The fiber of $\mathcal{M}_X(N) \to \mathcal{M}_X(W)$

Fix an extension class $\eta = [0 \to T \to N \to W] \in H^2(W,T)$.

Definition 4.1. *One calls the abelianisation map, the map that for a $T-$bundle $E_T \in H^1(Z, \underline{T})^W_\eta$ associates the quotient F_N of E_N by Φ_{E_T}.*

Lemma 4.2. *The principal bundle quotient of F_N by T is canonically isomorphic to Z.*

Proof. To show that F_N/T is canonically isomorphic to Z it suffices to show that there is a canonical isomorphism of principal W-bundles on Z between $\pi^*(F_N)$ and $\pi^*(Z)$ which, moreover, respects the W-linearisations induced by pull-back. To verify this condition, we shall firstly describe a canonical isomorphism between each principal bundle and $Z \times W$, and then show that, by transport of structure via these isomorphisms, we get the same W-linearisation on $Z \times W$, namely, the canonical W-linearisation.

Since $\pi : Z \to X$ is a Galois covering of Galois group W, we have canonical isomorphisms

$$\pi^* Z \simeq Z \times_X Z \simeq Z \times W.$$

The W-linearisation on $\pi^* Z$ induces the canonical W-linearisation on $Z \times W$, that is

$$\begin{array}{rccc} \phi : & W \times Z \times W & \to & Z \times W \\ & (w,(z,w')) & \mapsto & (zw, w^{-1}w'). \end{array}$$

We have the natural isomorphisms

$$\pi^*(F_N/T) \to \pi^*(F_N)/T \to (E_T \times^T N)/T.$$

By the isomorphism $E_T \times^T_w T/T \simeq Z$ we deduce the following commutatif diagram where all arrows are isomorphisms

$$\begin{array}{ccccc} (E_T \times^T N)/T & \longrightarrow & E_T/T \times N/T & \longrightarrow & Z \times W \\ \uparrow {\scriptstyle (\nu_w \times^T N)/T} & & \uparrow {\scriptstyle \nu_w/T \times \mathrm{Id}} & & \uparrow {\scriptstyle \mathrm{Id}} \\ (E_T \times^T_w T \times^T N)/T & \longrightarrow & (E_T \times^T_w T)/T \times N/T & \longrightarrow & Z \times W \end{array} \tag{12}$$

If we extend the upper sequence of (1) by $\times^T N$ and then we quotient by the group T, by the isomorphism

$$E \times_w T/T \simeq Z$$

and by (12) we obtain the following action of W on $Z \times W$: for $(w, \gamma') \in \mathcal{G}^\sigma(Z \times W)$

$$\begin{array}{ccccccc} Z \times W & \xrightarrow{\gamma'} & (w^* Z \times W) & \longrightarrow & Z \times W & \xrightarrow[\text{Id}]{\simeq} & Z \times W \\ & & \downarrow & & \downarrow & & \\ & & Z & \xrightarrow{w} & Z & & \end{array}$$

where $\gamma' : Z \to w^* Z$ is an isomorphism. We remark that this action is constant on the second factor W of $Z \times W$. Thus the action in propositon 2.6 by passing to quotient by T becomes

$$\overline{\phi(E_T)}(w)(z, w') = (zw, w^{-1}w') :$$

which is the canonical W-linearisation.

□

Proposition 4.3. *Let $\pi : Z \to X$ be a Galois cover with Galois group W. Let E be a principal $T-$bundle on Z. Then E is a principal $\mathcal{G}^\sigma(E)-$bundle on X.*

Proof. Let us denote the Mumford group $\mathcal{G}^\sigma(E)$ by N.

Consider the natural map of sheaves on Z

$$\begin{array}{cccc} \phi : & E & \to & E \times^T N \\ & e & \mapsto & \overline{(e, 1)} \end{array}$$

The bundle $E \times^T N$ admits a canonical $W-$linearisation by the proposition 2.6 and we denote by $p : E \times^T N \to E \times^T N/W =: F_N$ the quotient by W. The lemma 4.2 implies that F_N/T is isomorphic to Z. Thus F_N seen as a sheaf on Z is a principal $T-$bundle. Thus the composition of arrows $p \circ \phi : E \to F_N$ is a map of sheaves between principal $T-$bundles on Z. Now $p \circ \phi$ is also $T-$linear, thus it is also a map of principal $T-$bundles. So it is an isomorphism since in the category of principal group bundles, morphisms are isomorphisms. Thus E is a fortiori isomorphic to F_N on X. Now F_N is a principal$N-$bundle, and therefore so is E. □

Lemma 4.4. *Let N be an arbitrary extension of W by T and $\eta \in H^2(W, T)$ denote its class. Let $p : E_N \to X$ be a principal $N-$bundle on X and we denote by $\pi : Z \to X$ its quotient by T. There exists a canonical isomorphism between N and the Mumford group of the principal bundle E_N seen as a principal $T-$bundle on Z. Moreover via these isomorphisms the natural actions of these groups on E_N can also be identified.*

Proof. For $n \in N$, let $\overline{n}$ denote the class of n in W. For all $n \in N$, for the action of N on the princiapl $N-$bundle E_N and of W on the principal $W-$bundle Z, we have the following cartesian diagram preserving the fibers of $E_N \to Z \to X$

$$\begin{array}{ccc} E_N & \xrightarrow{n} & E_N \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\overline{n}} & Z. \end{array}$$

Now E_N is a principal $T-$bundle on Z. The commutation of the above diagram is equivalent to the fact that E_N is a $W-$invariant principal $T-$bundle for the twisted action of the group W. Thus, we conclude that the Mumford group of E_N, seen as a principal $T-$bundle on Z, is N. From this it also follows that the action of the Mumford group upon E_N gets identified with the usual multiplication action by N. □

Lemma 4.5. *Let $p : E_N \to X$ be a principal $N-$bundle on X. We denote by $\pi : Z \to X$ the quotient by T. The bundle E_N seen as a principal $T-$bundle on Z is sent by the abelianisation map to E_N seen as a principal $N-$bundle on X.*

Proof. By the lemma 4.4 we conclude that $E_N \in H^1(Z, \underline{T})^W$ and since its Mumford group is N so $E_N \in H^1(Z, \underline{T})^W_\eta$. Since E_N is a principal $N-$bundle on X, we have a canonical isomorphism

$$\begin{array}{ccc} E_N \times N & \to & E_N \times_X E_N \\ (e, n) & \mapsto & (e, en). \end{array}$$

When we quotient $E_N \times N$ by T by putting the relation $(et, n) = (e, tn)$, the last isomorphism implies the relation $(et, etn) = (e, etn)$ on $E_N \times_X E_N$ and we obtain

$$\begin{array}{cccc} \alpha : & E_N \times_T N & \to & Z \times_X E_N \\ & \overline{(e, n)} & \mapsto & (\overline{e}, en) \end{array}$$

where $\overline{e}$ is the image of $e \in E_N$ in Z. Recall the action of N on $E_N \times_T N$ of abelianisation

$$\begin{array}{ccc} N \times E_N \times_T N & \to & E_N \times_T N \\ (n, (e, n')) & \mapsto & (e.n, n^{-1}n'), \end{array}$$

and consider the action of N on $Z \times_X E_N$

$$\begin{array}{ccc} N \times Z \times_X E_N & \to & Z \times_X E_N \\ (n, (z, e)) & \to & (z\overline{n}, e). \end{array}$$

Now for these actions for all $n \in N$, we have the following commutative diagram

$$\begin{array}{ccc} E_N \times_T N & \xrightarrow{n} & E_N \times_T N \\ \downarrow \alpha & & \downarrow \alpha \\ Z \times_X E_N & \xrightarrow{n} & Z \times_W E_N. \end{array}$$

Thus, E_N - the quotient of $Z \times_X E_N$ by W is isomorphic on X to the quotient of $E_N \times_T N$ by W for the canonical $W-$linearisation of abelianisation. This last quotient is the image of E_N seen as a principal $T-$bundle on Z by Δ_θ. Now the assertion follows. □

Corollary 4.6. *Let $\pi : Z \to X$ be a Galois étale covering. Let $p : E \to Z$ be a principal $T-$bundle. The abelianisation map*

$$\Delta_\theta : H^1(Z, \underline{T})^W_\eta \to \mathcal{M}_X(N)$$

is injective.

Proof. By the proposition 4.3, E is a principal $N-$bundle on X, where N is the Mumford group. By the lemma 4.5, we have $\triangle_\theta(E) = E$ seen as a principal $N-$bundle on X . If E and F are principal $T-$bundles on Z such that $\triangle_\theta(E) = \triangle_\theta(F)$ then $E = F$ as principal $N-$bundles on X. By the lemma 4.2 we have E/T (resp. F/T) is isomorphic to Z. Thus E and F are isomorphic as principal $T-$bundles on Z. □

Theorem 4.7. *The fiber of the quotient by T map $q_T : \mathcal{M}_X(N) \to \mathcal{M}_X(W)$ over $\pi : Z \to X$ is $H^1(Z, \underline{T})^W_\eta$.*

Proof. By the lemma 4.2 the map quotient by T maps $H^1(Z, \underline{T})^W_\eta$ to the fiber of q_T. By the lemma 4.6, this map to the fiber of q_T is injective and by 4.5 it is surjective. □

5. Applications

Example 1. *Let $\pi : Z \to X$ be a covering Galois group D_{2n} the dihedral group of order $2n$. When we quotient Z by $\mathbb{Z}/n\mathbb{Z} \subset D_{2n}$, we obtain a double covering $p : Y \to X$. We have the following exact sequence*

$$0 \to \mathbb{Z}/n\mathbb{Z} \to D_{2n} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

with the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/n\mathbb{Z}$ by $e(\overline{k}) = \overline{k}$ and $\sigma(\overline{k}) = -\overline{k}$. The curve Z determines a primitive element, denoted as Z again, in $\mathrm{Jac}(Y)[n]$ *invariant under the action of $\mathbb{Z}/2\mathbb{Z}$ switching the two sheets.*

Let us suppose that n is odd. We denote by $\mathrm{Prym}(Y/X)$ *the Prym variety associated to the étale double cover $p : Y \to X$. We have the isogeny* $\mathrm{Jac}(X) \times \mathrm{Prym}(Y/X) \to \mathrm{Jac}(Y)$. *Let us consider the points* *of $n-$torsion in* $\mathrm{Jac}(Y)$*, denoted* $\mathrm{Jac}(Y)[n]$*. Thus the isogeny gives an isomorphism between abelian groups*

$$\mathrm{Jac}(X)[n] \times \mathrm{Prym}(Y/X)[n] \to \mathrm{Jac}(Y)[n],$$

since the kernel of the isogeny is consists of points of $2-$torsion in $\mathrm{Jac}(X)$*. Now the involution $\sigma : Y \to Y$ above X operates as* $+\mathrm{Id}$ *on* $\mathrm{Jac}(X)[n]$ *and* $-\mathrm{Id}$ *on* $\mathrm{Prym}(Y/X)[n]$*. Let $\alpha = (\beta, \gamma) \in$* $\mathrm{Jac}(X)[n] \times \mathrm{Prym}(Y/X)[n]$.

(1) *If $\sigma(\alpha) = \alpha$, then $(\beta, -\gamma) = (\beta, \gamma)$, that is $\gamma = 0$. Thus $\alpha \in$* $\mathrm{Jac}(X)[n]$*. Thus the étale covering α is the pull-back on Y of a cyclic étale covering of degree n on X and the associated total covering α is Galois of Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.*

(2) *If $\sigma(\alpha) = -\alpha$, then $(\beta, -\gamma) = (-\beta, -\gamma)$, that is $\beta = 0$. Thus $\alpha \in$* $\mathrm{Prym}(Y/X)[n]$*. Thus α is a cyclic étale covering of order n on Y. The total covering*

$$Z \overset{n:1}{\to} Y \overset{2:1}{\to} X$$

gives an étale Galois covering of Galois group D_{2n}.

Proposition 5.1. *The Weyl group W acts transitively on the fibers of the map*

$$\mathcal{M}_X(T) \to \mathcal{M}_X(N)$$

of extension of structure group from T to $N(T)$. The action is generically without fixed points.

Proof. For $E \in H^1(X, \underline{T})$, let us denote $E \times^T N$ by E_N. We consider the short exact sequence of group schemes

$$0 \to \mathrm{Aut}_T(E) \to \mathrm{Aut}_N(E_N) \to \underline{W} \to 0 \tag{13}$$

We have the associated long exact sequence (we omit the curve X in the notation)

$$\begin{array}{ccccccc} 0 & \to & H^0(\mathrm{Aut}_T(E)) & \to & H^0(\mathrm{Aut}_N(E_N)) & \to & H^0(\underline{W}) \\ & \overset{\delta}{\to} & H^1(\mathrm{Aut}_T(E)) & \to & H^1(\mathrm{Aut}_N(E_N)) & \to & H^1(\underline{W}) \end{array}$$

The distinguished elements of the sets $H^1(\mathrm{Aut}_T(E))$ and $H^1(\mathrm{Aut}_N(E_N))$ are E and E_N respectively. Let $\sigma : X \to E_N/T$ be the section corresponding to E. The group $H^0(\underline{W}) = W$ acts on E in the following way: $\delta(w)E$ is the principal T-bundle obtained by the section $(\sigma, w) : X \to E_N \times^N N/T \simeq E_N/T$. Thus we have

$$\delta(w)E = E \times^w T. \tag{14}$$

As the sequence gives the fiber on the distinguished element, we deduce that the group W acts transitively on the fiber of E_N, from which the first assertion follows. For a generic $E \in H^1(X,T)$ we have $Stab(E) = \{e\}$ which implies the second assertion.

□

Proposition 5.2. *Let T be a torus of a Lie group. Let $0 \to T \to N \to W \to 0$ be an extension of W by T. The map extension of structure group from N to W*

$$\mathcal{M}_X(N(T)) \to \mathcal{M}_X(W)$$

is surjective.

Proof. An element $(\pi : Z \to X) \in \mathcal{M}_X(W)$ corresponds to an étale Galois covering of curves on X. Since X is smooth and π is étale, so Z is smooth. Since X is projective and π is finite, so Z is projective also. By the lemma 7.2 of Lange-Pauly [1], we have that the set $H^1(Z,\underline{T})^W_\eta$ is non-empty where η denotes the extension class of $[0 \to T \to N \to W \to 0] = \eta \in H^2(W,T)$. By the lemma 4.2 $\pi : Z \to X$ is the image of the composition of arrows

$$\mathrm{Prym}(\pi,\Lambda)_\eta = H^1(Z,\underline{T})^W_\eta \to H^1(X,\underline{N}) \to H^1(X,\underline{W}).$$

□

Example 2. *Let us consider a Weyl group W of type B_n or C_n. It is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \ltimes \Sigma_n$. By the theorem 4.7, we have*

$$\mathcal{M}_X(W) = \bigsqcup_{(\pi:Z\to X)\in\mathcal{M}_X(\Sigma_n)} H^1(Z,\underline{T})^{\Sigma_n}_\eta$$

where $T = (\mathbb{Z}/2\mathbb{Z})^n$ and η is the extension class

$$[0 \to (\mathbb{Z}/2\mathbb{Z})^n \to W \to \Sigma_n \to 0] = \eta \in H^2(\Sigma_n,(\mathbb{Z}/2\mathbb{Z})^n).$$

Now a principal $T-$bundle on Z is a $n-$uple $(L_1,\cdots,L_n)$ where $L_i \in \mathrm{Jac}(Z)[2]$ are line bundles of order 2.By the equation 14 of the proposition 5.1, we deduce that Σ_n operates by permuting factors. Thus, the L_i are isomorphic to eachother. Thus we have

$$H^1(Z,\underline{T})^W_\eta = \{L \in \mathrm{Jac}(Z)[2] | [\mathcal{G}^\sigma((L,\cdots,L))] = \eta\}$$

Example 3. *The Weyl group D_n is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1} \ltimes \Sigma_n$ where Σ_n acts by permuting the factors on the subgroup of $(\mathbb{Z}/2\mathbb{Z})^n$ having an even number of ones. Reasoning as before in the example 2* *we find that*

$$\mathcal{M}_X(W) = \begin{cases} \mathcal{M}_X(\Sigma_n) & n \quad odd \\ \bigsqcup_{(\pi:Z\to X)\in\mathcal{M}_X(\Sigma_n)} \{L \in \mathrm{Jac}(Z)[2] | [\mathcal{G}^\sigma((L,\cdots,L))] = \eta\} & n \quad even \end{cases}$$

References

1. Herbert LANGE and Christian PAULY, *Polarizations of Prym varieties for Weyl groups via Abelianization*, to appear in Journal of the European Mathematical Society

Chennai Mathematical Institute, Plot No. H1, Sipcot IT Park, Padur Post Office, Siruseri 603103, India Email: ypandey@cmi.ac.in