1-DIMENSIONAL REPRESENTATIONS AND PARABOLIC INDUCTION FOR W-ALGEBRAS

IVAN LOSEV

Abstract. A W-algebra is an associative algebra constructed from a reductive Lie algebra and its nilpotent element. This paper concentrates on the study of 1-dimensional representations of W-algebras. Under some conditions on a nilpotent element (satisfied by all rigid elements) we obtain a criterium for a finite dimensional module to have dimension 1. It is stated in terms of the Brundan-Goodwin-Kleshchev highest weight theory. This criterium allows to compute highest weights for certain completely prime primitive ideals in universal enveloping algebras. We make an explicit computation in a special case in type E_8. Our second principal result is a version of a parabolic induction for W-algebras. In this case, the parabolic induction is an exact functor between the categories of finite dimensional modules for two different W-algebras. The most important feature of the functor is that it preserves dimensions. In particular, it preserves one-dimensional representations. A closely related result was obtained previously by Premet. We also establish some other properties of the parabolic induction functor.

1. Introduction

1.1. **W-algebras and their 1-dimensional representations.** Our base field is an algebraically closed field $\mathbb{K}$ of characteristic 0. A W-algebra $U(\mathfrak{g}, e)$ (of finite type) is a certain finitely generated associative algebra constructed from a reductive Lie algebra $\mathfrak{g}$ and its nilpotent element e. There are several definitions of W-algebras. The definitions introduced in [Pr2] and [Lo2] will be given in Subsection 4.1.

During the last decade W-algebras were extensively studied starting from Premet's paper [Pr2][1], see, for instance, [BrKl],[BGK],[GG],[Gi],[Lo2]-[Lo4],[Pr3]-[Pr5]. One of the reasons why W-algebras are interesting is that they are closely related to the universal enveloping algebra $U(\mathfrak{g})$. For instance, there is a map $\mathcal{I} \mapsto \mathcal{I}^\dagger$ from the set of two-sided ideals of $U(\mathfrak{g}, e)$ to the set of two-sided ideals of $U(\mathfrak{g})$ having many nice properties, see [Gi],[Lo2],[Lo3],[Pr3],[Pr4], for details.

The central topic in this paper is the study of 1-dimensional representations of W-algebras. A key conjecture here was stated by Premet in [Pr3].

Conjecture 1.1.1. For any e the algebra $U(\mathfrak{g}, e)$ has at least one 1-dimensional representation.

Key words and phrases: W-algebras, 1-dimensional representations, parabolic induction, deformation quantization, category $\mathcal{O}$, Hamiltonian reduction.

2000 Mathematics Subject Classification. 17B35, 53D55.

Address: MIT, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, MA 02139, United States.

e-mail: ivanlosev@math.mit.edu.

[1]We should mention here that there are affine counterparts of finite W-algebras and they were studied even before finite ones. The general definition of affine W-algebras was sketched in Kac's ICM talk [Kac1], and fully developed in [KRW].

There are two major results towards this conjecture obtained previously.

First, the author proved Conjecture 1.1.1 in [Lo2] provided the algebra $\mathfrak{g}$ is classical. Actually, the proof given there can be easily deduced from earlier results: by Moeglin [Mo2], and Brylinski [Bry].

Second, in [Pr5] Premet reduced Conjecture 1.1.1 to the case when the nilpotent orbit is rigid, that is, cannot be induced (in the sense of Lusztig and Spaltenstein) from a proper Levi subalgebra. Namely, let $\underline{\mathfrak{g}}$ be a Levi subalgebra in $\mathfrak{g}$. To a nilpotent orbit $\underline{\mathbb{O}} \subset \underline{\mathfrak{g}}$ Lusztig and Spaltenstein assigned a nilpotent orbit $\mathbb{O} = \operatorname{Ind}_{\underline{\mathfrak{g}}}^{\mathfrak{g}}(\underline{\mathbb{O}})$ in $\mathfrak{g}$. See Subsection 6.1 for a precise definition. Premet checked in [Pr5, Theorem 1.1], that $U(\mathfrak{g}, e)$ has a one-dimensional representation provided $U(\underline{\mathfrak{g}}, \underline{e})$ does, where $\underline{e} \in \underline{\mathbb{O}}$. His proof used a connection with modular representations of semisimple Lie algebras.

There are several reasons to be interested in one-dimensional $U(\mathfrak{g}, e)$-modules. They give rise to completely prime primitive ideals in $U(\mathfrak{g})$. This was first proved by Moeglin in [Mo1] (her Whittaker models correspond to one-dimensional $U(\mathfrak{g}, e)$-modules via the Skryabin equivalence from [S]). Moreover, as Moeglin proved in [Mo2], any one-dimensional $U(\mathfrak{g}, e)$-module (=Whittaker model) leads to an (appropriately understood) *quantization* of a covering of Ge, see Introduction to [Mo2].

On the other hand, in [Pr5] Premet proved that the existence of a one-dimensional $U(\mathfrak{g}, e)$-module implies the Humphreys conjecture on the existence of a *small* non-restricted representation in characteristic $p \gg 0$.[2]

1.2. **Main results.** One of the main results of this paper strengthens Theorem 1.1 from [Pr5].

Theorem 1.2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\underline{\mathfrak{g}}$ its Levi subalgebra, $\underline{e}$ an element of a nilpotent orbit $\underline{\mathbb{O}}$ in $\underline{\mathfrak{g}}$, and e an element from the induced nilpotent orbit $\operatorname{Ind}_{\underline{\mathfrak{g}}}^{\mathfrak{g}}(\underline{\mathbb{O}})$. Then there is a dimension preserving exact functor ρ from the category of finite dimensional $U(\underline{\mathfrak{g}}, \underline{e})$-modules to the category of finite dimensional $U(\mathfrak{g}, e)$-modules.*

The functor ρ mentioned in the theorem will be called the *parabolic induction* functor. The proof of this theorem given in Subsection 6.3 follows from a stronger result: we show that $U(\mathfrak{g}, e)$ can be embedded into a certain completion of $U(\underline{\mathfrak{g}}, \underline{e})$. This completion has the property that any finite dimensional representation of $U(\underline{\mathfrak{g}}, \underline{e})$ extends to it. Now our functor is just the pull-back. To get an embedding we use techniques of quantum Hamiltonian reduction.

In Subsection 6.4 we will relate our parabolic induction functor to the parabolic induction map between the sets of ideals of $U(\underline{\mathfrak{g}})$ and $U(\mathfrak{g})$.

So to complete the proof of Conjecture 1.1.1 it remains to deal with rigid orbits in exceptional Lie algebras. The first result here is also due to Premet. In [Pr3] he proved that $U(\mathfrak{g}, e)$ has a one-dimensional representation provided e is a minimal nilpotent element (outside of type A such an element is always rigid)[3].

[2]Actually, this conjecture that appeared in Humphreys review [H] was also stated before (in one form or another). It appeared in the same form in Kac's review, [Kac2], to [Pr1] and in [Pr1] itself but in a slightly different form.

[3]When the present paper was almost ready to be made public, another result towards Premet's conjecture appeared: in [GRU] Goodwin, Röhrle and Ubly checked that $U(\mathfrak{g}, e)$ has a one-dimensional representation provided $\mathfrak{g}$ is G_2, F_4, E_6 or E_7. They use Premet's approach based on the analysis of relations in $U(\mathfrak{g}, e)$, [Pr3], together with GAP computations.

We make a further step to the proof of Conjecture 1.1.1. Namely, under some condition on a nilpotent element $e \in \mathfrak{g}$ we find a criterium for a finite dimensional module over $U(\mathfrak{g}, e)$ to be one-dimensional in terms of the highest weight theory for W-algebras established in [BGK] and further studied in [Lo4]. The precise statement of this criterium, Theorem 5.2.1, will be given in Subsection 5.2. The condition on e is that the reductive part of the centralizer $\mathfrak{z}_{\mathfrak{g}}(e)$ is semisimple. In particular, this is the case for all rigid nilpotent elements. Moreover, our criterium makes it possible to find explicitly highest weights of some completely prime primitive ideals in $U(\mathfrak{g})$, see Subsection 5.3.

1.3. **Content of this paper.** The paper consists of sections that are divided into subsections. Definitions, theorems, etc., are numbered within each subsection. Equations are numbered within sections.

Let us describe the content of this paper in more detail. In Section 2 we gather some standard notation used in the paper. In Section 3 we recall different more or less standard facts regarding deformation quantization, classical and quantum Hamiltonian actions, and the Hamiltonian reduction. Essentially, it does not contain new results, with possible exception of some technicalities. In Section 4 different facts about W-algebras are gathered. In Subsection 4.1 we recall two definitions of W-algebras (from [Lo2] and [Pr2]). In Subsection 4.2 we recall a crucial technical result about W-algebras, the decomposition theorem from [Lo2], as well as two maps between the sets of two-sided ideals of $U(\mathfrak{g})$ and of $U(\mathfrak{g}, e)$ also defined in [Lo2]. Finally in Subsection 4.3 we recall the notion of the categories $\mathcal{O}$ for W-algebras introduced in [BGK] and a category equivalence theorem from [Lo4] (Theorem 4.3.2). This theorem asserts that there is an equivalence between the category $\mathcal{O}$ for $U(\mathfrak{g}, e)$ and a certain category of generalized Whittaker modules for $U(\mathfrak{g})$.

In Section 5 we apply Theorem 4.3.2 to the study of irreducible finite dimensional and, in particular, one-dimensional $U(\mathfrak{g}, e)$-modules. Recall that, according to [BGK], one can consider an irreducible finite dimensional $U(\mathfrak{g}, e)$-module N as the irreducible highest weight module $L(N^0)$ (below we use the notation $L^\theta(N^0)$), where the "highest weight" N^0 is a module over the W-algebra $U(\mathfrak{g}_0, e)$, $\mathfrak{g}_0$ being a Levi subalgebra of $\mathfrak{g}$ containing e.

In Subsection 5.1 we prove a criterium, Theorem 5.1.1, for $L(N^0)$ to be finite dimensional. This criterium generalizes [BGK, Conjecture 5.2] and is stated in terms of primitive ideals in $U(\mathfrak{g})$. In Subsection 5.2, under the condition on e mentioned in the previous subsection, we prove a criterium for $L^\theta(N^0)$ to be one-dimensional, Theorem 5.2.1. Roughly speaking, this criterium asserts that a primitive ideal $J(\lambda)$ corresponds to a one-dimensional representation of $U(\mathfrak{g}, e)$ if λ satisfies certain four conditions, the most implicit (as well as the most difficult to check) one being that the associated variety of $J(\lambda)$ is $\overline{\mathbb{O}}$. In Subsection 5.3 we provide some technical statements that allow to check whether the last condition holds. Also we verify the four conditions for an explicit highest weight in the case of the nilpotent element $A_5 + A_1$ in E_8.

Section 6 is devoted to the study of a parabolic induction for W-algebras. In Subsection 6.1 we recall the definition and some properties of the Lusztig-Spaltenstein induction for nilpotent orbits. Subsection 6.2 is very technical. There we prove some results about certain classical Hamiltonian actions to be used in the next two subsections. The first of them, Subsection 6.3, contains the proof of Theorem 1.2.1. Subsection 6.4 relates the parabolic induction for W-algebras with the parabolic induction for ideals in universal enveloping algebras (Corollary 6.4.2). In Subsection 6.5 we study the morphism of representation schemes induced by the parabolic induction. Our main result is that this morphism is finite. In

Subsection 6.6 we show that the parabolic induction functor has both left and right adjoint functors.

Finally, in Section 7 we establish two results, which are not directly related to the main content of this paper but seem to be of interest. In Subsection 7.1 we will relate the Fedosov quantization of the cotangent bundle to the quantization by twisted differential operator. The material of this subsection seems to be a folklore knowledge that was never written down explicitly. In Subsection 7.2 we prove a general result on filtered algebras that is used in Subsection 6.5.

Acknowledgements. This paper would never appear without numerous fruitful discussions with Alexander Premet and David Vogan. Their precious help is gratefully acknowledged. I also wish to thank Roman Bezrukavnikov, Jonathan Brundan, Alexander Elashvili, Pavel Etingof, Anthony Joseph, Alexander Kleshchev, George Lusztig, and Roman Travkin for stimulating discussions. Finally, I wish to thank Simon Goodwin and Alexander Premet for pointing out several mistakes and gaps in the previous versions of this text.

2. Notation and conventions

Algebraic groups and their Lie algebras. If an algebraic group is denoted by a capital Latin letter, e.g., G or N^-, then its Lie algebra is denoted by the corresponding small German letter, e.g., $\mathfrak{g},\mathfrak{n}^-$.

Locally finite parts. Let $\mathfrak{g}$ be some Lie algebra and let M be a module over $\mathfrak{g}$. By the locally finite (shortly, l.f.) part of M we mean the sum of all finite dimensional $\mathfrak{g}$-submodules of M. The similar definition can be given for algebraic group actions.

The main algebras. Below G is a connected reductive algebraic group and $\mathfrak{g}$ is its Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ will be mostly denoted by $\mathcal{U}$. Let e be a nilpotent element in $\mathfrak{g}$. The W-algebra $U(\mathfrak{g},e)$ will be denoted shortly by $\mathcal{W}$. In Section 5 we will consider the situation when e lies in a certain Levi subalgebra $\mathfrak{g}_0\subset\mathfrak{g}$. In this case we set $\mathcal{U}^0:=U(\mathfrak{g}_0),\mathcal{W}^0:=U(\mathfrak{g}_0,e)$. In Section 6 the element e will be induced from a nilpotent element $\underline{e}$ of some Levi subalgebra $\underline{\mathfrak{g}}$ of $\mathfrak{g}$. There we will write $\underline{\mathcal{U}}:=U(\underline{\mathfrak{g}}),\underline{\mathcal{W}}:=U(\underline{\mathfrak{g}},\underline{e})$.

$\mathbf{A}_V$	the Weyl algebra of a symplectic vector space V.
$\operatorname{Ann}_{\mathcal{A}}(M)$	the annihilator of an $\mathcal{A}$-module M in an algebra $\mathcal{A}$.
$\operatorname{gr}\mathcal{A}$	the associated graded algebra of a filtered algebra $\mathcal{A}$.
G_x	the stabilizer of a point x under an action of a group G.
(G,G)	the derived subgroup of a group G.
$H^i_{DR}(X)$	i-th De Rham cohomology of a smooth algebraic variety (or of a formal scheme) X.
$\mathfrak{Id}(\mathcal{A})$	the set of all (two-sided) ideals of an algebra $\mathcal{A}$.
$\sqrt{\mathcal{J}}$	the radical of an ideal $\mathcal{J}$.
$\mathbb{K}[X]^\wedge_Y$	the completion of the algebra $\mathbb{K}[X]$ of regular functions on X with respect to a subvariety $Y\subset X$.
$\operatorname{mult}_{\mathbb{O}}(\mathcal{M})$	the multiplicity of a Harish-Chandra $U(\mathfrak{g})$-bimodule $\mathcal{M}$ on an open orbit $\mathbb{O}\subset\operatorname{V}(\mathcal{M})$.
$N\rtimes L$	the semidirect product of groups N and L (N is normal).
$R_\hbar(\mathcal{A})$	the Rees algebra of a filtered algebra $\mathcal{A}$.
$\mathrm{R}_u(H)$	the unipotent radical of an algebraic group H.
$\operatorname{Span}_A(X)$	the A-linear span of a subset X in some A-module.

$U^\angle$	the skew-orthogonal complement to a subspace U in a symplectic vector space.
$\mathrm{V}(\mathcal{M})$	the associated variety (in $\mathfrak{g}^*$) of a Harish-Chandra $U(\mathfrak{g})$-bimodule $\mathcal{M}$.
X^G	the fixed-point set for an action $G:X$.
$\mathfrak{z}_\mathfrak{g}(x)$	the centralizer of x in $\mathfrak{g}$.
$Z_G(x)$	the centralizer of $x\in\mathfrak{g}$ in G.
$\Omega^i(X)$	the space of i-forms on a variety X.
$\widehat{\otimes}$	the completed tensor product (of complete topological vector spaces or modules).

3. Deformation quantization, Hamiltonian actions and Hamiltonian reduction

3.1. **Deformation quantization.** Let A be a commutative associative algebra with unit equipped with a Poisson bracket.

Definition 3.1.1. A $\mathbb{K}[[\hbar]]$-bilinear map $*: A[[\hbar]]\times A[[\hbar]]\rightarrow A[[\hbar]]$ is called a *star-product* if it satisfies the following conditions:

(*1) $*$ is associative, equivalently, $(f*g)*h=f*(g*h)$ for all $f,g,h\in A$, and $1\in A$ is a unit for $*$.
(*2) $f*g-fg\in \hbar^2A[[\hbar]], f*g-g*f-\hbar^2\{f,g\}\in\hbar^4A[[\hbar]]$ for all $f,g\in A$.
(*3) $A*A\subset A[[\hbar^2]]$

Clearly, a star-product is uniquely determined by its restriction to A. One may write $f*g=\sum_{i=0}^\infty D_i(f,g)\hbar^{2i}, f,g\in A, D_i: A\otimes A\rightarrow A$. If all D_i are bidifferential operators, then the star-product $*$ is called *differential*. In this case we can extend the star-product to $B[[\hbar]]$ for any localization or completion B of A. When we consider $A[[\hbar]]$ as an algebra with respect to the star-product, we call it a *quantum algebra*.

Also we remark that usually in condition (*2) one has $f*g-fg\in\hbar A[[\hbar]], f*g-g*f-\hbar\{f,g\}\in\hbar^2A[[\hbar]]$ (so we get our definition from the usual one replacing $\hbar$ with $\hbar^2$).

Let G be an algebraic group acting on A by automorphisms. It makes sense to speak about G-invariant star-products ($\hbar$ is supposed to be G-invariant). Now let $\mathbb{K}^\times$ act on $A, (t,a)\mapsto t.a$, by automorphisms. Consider the action of $\mathbb{K}^\times$ on $A[[\hbar]]$ given by

$$t.\sum_{i=0}^\infty a_j\hbar^j=\sum_{j=0}^\infty t^j(t.a_j)\hbar^j.$$

If $\mathbb{K}^\times$ acts by automorphisms of $*$, then we say that $*$ is *homogeneous*. Clearly, $*$ is homogeneous if and only if the map $D_l: A\otimes A\rightarrow A$ is homogeneous of degree $-2l$.

Let X be a smooth affine variety equipped with a symplectic form ω. Let $A:=\mathbb{K}[X]$ be its algebra of regular functions. There is a construction of a differential star-product due to Fedosov. According to this construction, see [F, Section 5.3] one needs to fix a symplectic connection (=a torsion-free affine connection annihilating the symplectic form), say ∇, and a $K[[\hbar^2]]$-valued 2-form Ω on X. Then from ω,∇ and Ω one canonically constructs differential operators D_i defining a star-product.

Now suppose $G\times\mathbb{K}^\times$ acts on X. If ω,∇,Ω are G-invariant, then $*$ is G-invariant. If ∇,Ω are $\mathbb{K}^\times$-invariant and $t.\omega=t^2\omega$ for any $t\in\mathbb{K}^\times$ (recall, that $t.\hbar=t\hbar$), then $*$ is homogeneous. Note also that if G is reductive, then there is always a $G\times\mathbb{K}^\times$-invariant symplectic connection on X, [Lo2, Proposition 2.2.2].

Let us now consider the question of the classification of star-products on $\mathbb{K}[X][[\hbar]]$. We say that two G-invariant homogeneous star-products $*,*'$ are *equivalent* if there is a $G\times\mathbb{K}^\times$-equivariant map (equivalence) $T := \mathrm{id}+\sum_{i=1}^\infty T_i\hbar^{2i} : \mathbb{K}[X][[\hbar]] \to \mathbb{K}[X][[\hbar]]$, where $T_i : \mathbb{K}[X]\to\mathbb{K}[X]$ is a linear map, such that $(Tf)*'(Tg) = T(f*g)$. We say that the equivalence T is *differential* when all T_i are differential operators.

Theorem 3.1.2. (1) *Let $*,*'$ be Fedosov star-products constructed from the pairs (∇,Ω), (∇',Ω') such that $\nabla,\nabla',\Omega,\Omega'$ are $G\times\mathbb{K}^\times$-invariant. Then $*,*'$ are equivalent if and only if $\Omega'-\Omega$ is exact. In this case one can choose a differential equivalence.*
(2) *Any G-invariant homogeneous star-product is $G\times\mathbb{K}^\times$-equivariantly equivalent to a Fedosov one.*

The first part of the theorem is, essentially, due to Fedosov, see [F, Section 5.5]. The second assertion is an easy special case of [BeKa, Theorem 1.8]. The fact that an equivalence can be chosen $G\times\mathbb{K}^\times$-equivariant follows from the proof of that theorem, compare with [BeKa, Subsection 6.1].

Remark 3.1.3. Although Fedosov worked in the C^∞-category, his results remain valid for smooth affine varieties too. Also proofs of parts of Theorem 3.1.2 in [F] and [BeKa] of the results that we need can be generalized to the case of smooth formal schemes in a straightforward way.

3.2. **Classical Hamiltonian actions.** In this subsection G is an algebraic group and X is a smooth affine variety equipped with a regular symplectic form ω and an action of G by symplectomorphisms. Let $\{\cdot,\cdot\}$ denote the Poisson bracket on X induced by ω.

To any element $\xi\in\mathfrak{g}$ one assigns in a standard way the velocity vector field ξ_X on X. Suppose there is a linear map $\mathfrak{g}\to\mathbb{K}[X],\xi\mapsto H_\xi$, satisfying the following two conditions:

(H1) The map $\xi\mapsto H_\xi$ is G-equivariant.
(H2) $\{H_\xi,f\}=\xi_X f$ for any $f\in\mathbb{K}[X]$.

Definition 3.2.1. The action $G:X$ equipped with a linear map $\xi\mapsto H_\xi$ satisfying (H1),(H2) is said to be *Hamiltonian* and X is called a Hamiltonian G-variety. The functions H_ξ are said to be the hamiltonians of the action.

For a Hamiltonian action $G:X$ we define a morphism $\mu: X\to\mathfrak{g}^*$ (called a *moment map*) by the formula

$$\langle\mu(x),\xi\rangle = H_\xi(x), \xi\in\mathfrak{g}, x\in X.$$

The map $\xi\mapsto H_\xi$ is often referred to as a *comoment map*.

Let X_1,X_2 be two G-varieties with Hamiltonian G-actions and φ be a G-equivariant morphism $X_1\to X_2$. We say that φ is *Hamiltonian* if φ intertwines the symplectic forms and the moment maps.

Now let us recall a local description of Hamiltonian actions obtained in [Lo1].

Till the end of the subsection we assume that G is reductive. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using an invariant symmetric form $(\cdot,\cdot)$, whose restriction to the rational part of a Cartan subalgebra in $\mathfrak{g}$ is positively definite. Then the restriction of $(\cdot,\cdot)$ to the Lie algebra of any reductive subgroup of G is non-degenerate. So for any such subalgebra $\mathfrak{h}\subset\mathfrak{g}$ (in particular, for $\mathfrak{h}=\mathfrak{g}$) we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form. For a point $x\in X$ let $\mathfrak{g}_*x$ denote the tangent space to the orbit Gx in x.

Let $x\in X$ be a point with closed G-orbit. Set $H=G_x,\eta=\mu_G(x)$ and $V:=\mathfrak{g}_*x^\angle/(\mathfrak{g}_*x\cap\mathfrak{g}_*x^\angle)$ (in other words, V is the symplectic part of the normal space to Gx in X). Then V is a symplectic H-module.

Proposition 3.2.2 ([Lo1]). *Let X_1, X_2 be two Hamiltonian G-varieties and $x_1 \in X_1, x_2 \in X_2$ be two points with closed G-orbits. Suppose that $G_{x_1} = G_{x_2}, \mu_G(x_1) = \mu_G(x_2)$ and the G_{x_i}-modules $V_i := \mathfrak{g}_* x_i^{\angle}/(\mathfrak{g}_* x_i \cap \mathfrak{g}_* x_i^{\angle})$ are isomorphic. Then there is a G-equivariant Hamiltonian morphism $\varphi : (X_1)^{\wedge}_{Gx_1} \xrightarrow{\sim} (X_2)^{\wedge}_{Gx_2}$ mapping x_1 to x_2.*

Suppose, in addition, that we have $\mathbb{K}^\times$-actions on X_1, X_2 such that

(A) *they stabilize Gx_i and the stabilizers of x_1, x_2 in $G \times \mathbb{K}^\times$ are the same.*
(B) *$t \in \mathbb{K}^\times$ multiplies the symplectic forms and the functions H_ξ by t^2.*
(C) *V_1 and V_2 are isomorphic as $(G \times \mathbb{K}^\times)_{x_i}$-modules.*

Then an isomorphism φ can be made, in addition, $\mathbb{K}^\times$-equivariant.

Proof. We remark that if two symplectic modules are equivariantly isomorphic, then they are actually equivariantly symplectomorphic. Now the part without a $\mathbb{K}^\times$-action basically follows from the uniqueness result in [Lo1] (which was stated in the complex-analytic category). See also the preprint [Kn, Theorem 5.1] for the proof in the formal scheme setting.

The proof in [Lo1] can be adjusted to the $\mathbb{K}^\times$-equivariant situation too, compare with the proof of [Lo2, Theorem 3.1.3]. □

Let us construct explicitly a Hamiltonian G-variety corresponding to a triple (H, η, V), a so called *model variety* $M_G(H, \eta, V)$ introduced in [Lo1]. For simplicity we will assume that $\eta = e$ is nilpotent. In this case there is also a $\mathbb{K}^\times$-action satisfying the conditions (A),(B),(C) of the previous proposition.

Let V be a symplectic H-module with symplectic form ω_V. Define a subspace $U \subset \mathfrak{g}$ as follows. If $e = 0$, then $U := \mathfrak{h}^\perp$. Otherwise, consider an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}^H$ and set $U := \mathfrak{z}_\mathfrak{g}(f) \cap \mathfrak{h}^\perp$.

Consider the homogeneous vector bundle $X := G *_H (U \oplus V)$ and embed $U \oplus V$ to X by $(u, v) \mapsto [1, (u, v)]$. Define the map $\mu_G : U \oplus V \to \mathfrak{g}$ by

$$\mu_G([1, (u, v)]) = e + u + \mu_H(v),$$

where $\mu_H : V \to \mathfrak{h}$ is the moment map for the action of H on V given by $(\mu_H(v), \xi) = \frac{1}{2}\omega(\xi v, v)$. Since this map is H-equivariant, we can extend it uniquely to a G-equivariant map $\mu_G : X \to \mathfrak{g}$.

Also define the 2-form $\omega_x \in \bigwedge^2 T_x^* X, x = [1, (u, v)]$, by

$$\omega_x(\xi_1 + u_1 + v_1, \xi_2 + u_2 + v_2) = \langle \mu_G(x), [\xi_1, \xi_2] \rangle + \langle \xi_1, u_2 \rangle - \langle \xi_2, u_1 \rangle + \omega_V(v_1, v_2),$$
$$\xi_1, \xi_2 \in \mathfrak{h}^\perp, u_1, u_2 \in U, v_1, v_2 \in V.$$

Note that the section $\omega : x \mapsto \omega_x$ of $\bigwedge^2 T^*X|_{U \oplus V}$ is H-invariant, so we can extend it to X by G-invariance. It turns out that the form ω is symplectic and $\mu_G : X \to \mathfrak{g}$ is a moment map for this form, see [Lo1].

Define the $\mathbb{K}^\times$-action on X as follows. Let $\gamma : \mathbb{K}^\times \to G$ be the composition of the homomorphism $\mathrm{SL}_2 \to G$ induced by the $\mathfrak{sl}_2$-triple and of the embedding $\mathbb{K}^\times \to \mathrm{SL}_2$ given by $t \mapsto \mathrm{diag}(t, t^{-1})$. Define a $\mathbb{K}^\times$-action on $M_G(H, \eta, V)$ as follows:

$$t.(g, u, v) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)u, t^{-1}\beta(t)v),$$

where β is a group homomorphism $\mathbb{K}^\times \to \mathrm{Sp}(V)^H$. The action of $t \in \mathbb{K}^\times$ multiplies ω and μ_G by t^2.

In the sequel we will need a technical result concerning Hamiltonian actions on formal schemes.

Lemma 3.2.3. *Let G be a reductive group, and X be an affine Hamiltonian G-variety with symplectic form ω and moment map μ_G. Let $x \in X$ be a point with closed G-orbit. Suppose that $\mu_G(x)$ is nilpotent. Further, let $\mathfrak{z}$ be an algebraic Lie subalgebra in $\mathfrak{z}(\mathfrak{g})$ satisfying $\mathfrak{z} \oplus (\mathfrak{g}_x + [\mathfrak{g},\mathfrak{g}]) = \mathfrak{g}$. Finally, let ζ be a G-invariant symplectic vector field on $X^{\wedge}_{Gx}$ such that $\zeta H_\xi = 0$ for all $\xi \in \mathfrak{z}$. Then $\zeta = v_f$ for a G-invariant element $f \in \mathbb{K}[X]^{\wedge}_{Gx}$, where v_f is the Hamiltonian vector field associated with f.*

Actually, the proof generalizes directly to the case when η is not necessarily nilpotent, but we will not need this more general result.

Proof. Note that the $v_f H_\xi = 0$ for any $f \in (\mathbb{K}[X]^{\wedge}_{Gx})^G$.

Replacing G with a covering we may assume that $G = Z \times G_0$, where $Z \subset Z(G)$ is the torus with Lie algebra $\mathfrak{z}$ and G_0 is the product of a torus and of a simply connected semisimple group with $G_0 = H^\circ(G,G)$. Thanks to Proposition 3.2.2, we can replace X with a model variety $M_G(H,\eta,V)$.

Let us consider the model variety $\widetilde{X} := M_G(H^\circ,\eta,V)$. From the construction of model varieties recalled above, we have the action of the finite group $\Gamma := H/H^\circ$ on $\widetilde{X}$ by Hamiltonian automorphisms, and $X = \widetilde{X}/\Gamma$. We have the decomposition $\widetilde{X} = T^*Z \times \widetilde{X}_0$, where $\widetilde{X}_0 := M_{G_0}(H^\circ,\eta,V)$. The homogeneous space G_0/H° is simply connected. Note also that the action of Γ on $\widetilde{X}$ preserves the decomposition $\widetilde{X} = T^*Z \times \widetilde{X}_0$. Pick a point $\widetilde{x} \in \widetilde{X}$ mapping to $x = [1,(0,0)] \in X = M_G(H,\eta,V)$. Lift ω and ζ to $\widetilde{X}^{\wedge}_{G\widetilde{x}}$. We will denote the liftings by the same letters.

Let us check that the contraction $\iota_\zeta\omega$ is an exact form.

The projection $\widetilde{X}^{\wedge}_{G\widetilde{x}} \twoheadrightarrow Z$ induces an isomorphism $H^1_{DR}(\widetilde{X}^{\wedge}_{Gx}) \to H^1_{DR}(Z)$. Fix an identification $Z \cong (\mathbb{K}^\times)^m$ and let $z_i, i = 1,\ldots,m$, be the coordinate on the i-th copy of $\mathbb{K}^\times$. The forms $\frac{dz_i}{z_i}$ form a basis in $H^1_{DR}(Z)$. Choose the basis $p_1,\ldots,p_m$ in $\mathfrak{z}^*$ corresponding to $z_1,\ldots,z_m$. Then the symplectic form on T^*Z is written as $\sum_{i=1}^m dp_i \wedge dz_i$. Consider the vector field $\zeta_i := \frac{1}{z_i}\frac{\partial}{\partial p_i}$, then $\iota_{\zeta_i}\omega = \frac{dz_i}{z_i}$. The classes of the forms $\iota_{\zeta_i}\omega$ form a basis in $H^1_{DR}(Z)$. Finally, let $\xi_1,\ldots,\xi_m$ be the basis of $\mathfrak{z}$ corresponding to the choice of $z_1,\ldots,z_m$. Then $H_{\xi_i} = p_i z_i$ and $\zeta_j H_{\xi_i} = \delta_{ij}$.

Now let ζ be an arbitrary G-invariant symplectic vector field on $\widetilde{X}^{\wedge}_{Gx}$. There are scalars $a_1,\ldots,a_m$ such that $\iota_\zeta\omega - \sum_{i=1}^m a_i\iota_{\zeta_i}\omega = df$ for some $f \in \mathbb{K}[\widetilde{X}]^{\wedge}_{Gx}$. Since the $G\times\Gamma$-module $\mathbb{K}[\widetilde{X}]^{\wedge}_{Gx}$ is pro-finite, we may assume that f is $G\times\Gamma$-invariant. It follows that $0 = \zeta H_{\xi_i} = a_i$. Thus $\zeta = v_f$. Since $f \in (\mathbb{K}[\widetilde{X}]^{\wedge}_{Gx})^\Gamma = \mathbb{K}[X]^{\wedge}_{Gx}$, we are done. □

3.3. **Quantum Hamiltonian actions.** In this subsection we consider a quantum version of Hamiltonian actions. We preserve the notation of the previous subsection.

Let $*$ be a star-product on $\mathbb{K}[X][[\hbar]]$. A *quantum comoment map* for the action $G : X$ is, by definition, a G-equivariant linear map $\mathfrak{g} \to \mathbb{K}[X][[\hbar]], \xi \mapsto \widehat{H}_\xi$, satisfying the equality

$$[\widehat{H}_\xi, f] = \hbar^2\xi_X f, \forall \xi \in \mathfrak{g}, f \in \mathbb{K}[X][[\hbar]], \tag{3.1}$$

where $[\widehat{H}_\xi, f] := \widehat{H}_\xi * f - f * \widehat{H}_\xi$. The elements $\widehat{H}_\xi$ are said to be the quantum hamiltonians of the action. A G-equivariant homomorphism of quantum algebras is called *Hamiltonian* if it intertwines the quantum comoment maps.

The following theorem gives a criterium for the existence of a quantum comoment map in the case when G is reductive.

Theorem 3.3.1 ([GR], Theorem 6.2)**.** *Let X be an affine symplectic Hamiltonian G-variety, $\xi \mapsto H_\xi$ being the comoment map, and $*$ be the star-product on $\mathbb{K}[X][[\hbar]]$ obtained by the Fedosov construction with a G-invariant connection ∇ and $\Omega \in \Omega^2(X)[[\hbar^2]]^G$. Then the following conditions are equivalent:*

1. *The G-variety X has a quantum comoment map $\xi \mapsto \widehat{H}_\xi$ with $\widehat{H}_\xi \equiv H_\xi \mod \hbar^2$.*
2. *The 1-form $i_{\xi_X}\Omega$ is exact for each ξ, where i_{ξ_X} stands for the contraction with ξ_X.*

Moreover, if (2) holds then for $\widehat{H}_\xi$ we can take $H_\xi + \hbar^2 a_\xi$, where $\xi \mapsto a_\xi$ is a G-equivariant map $\mathfrak{g} \to \mathbb{K}[X][[\hbar]]$ such that $da_\xi = \iota_{\xi_X}\Omega$ for any $\xi \in \mathfrak{g}$.

Remark 3.3.2. Although Gutt and Rawnsley worked with C^∞-manifolds, their techniques can be carried over to the algebraic setting without any noticeable modifications. Also Theorem 3.3.1 can be directly generalized to the case when X is a smooth affine formal scheme (say, the completion of a closed G-orbit in an affine variety).

Remark 3.3.3. We preserve the notation of Theorem 3.3.1. Suppose $\mathbb{K}^\times$ acts on X such that ∇, Ω are $\mathbb{K}^\times$-invariant and ω, H_ξ have degree 2 for all $\xi \in \mathfrak{g}$. Then the last assertion of Theorem 3.3.1 insures that there are $\widehat{H}_\xi$ of degree 2.

In particular, Theorem 3.3.1 implies that if $\Omega = 0$, then we can take 0 for a_ξ and so H_ξ for $\widehat{H}_\xi$.

We will need a quantum version of Proposition 3.2.2.

Theorem 3.3.4. *Let X_1, X_2, x_1, x_2 be such as in Proposition 3.2.2. Suppose there are actions of $\mathbb{K}^\times$ as in that proposition. Equip X_1, X_2 with homogeneous G-invariant star-products corresponding to $\Omega = 0$ and let the quantum hamiltonians coincide with the classical ones. Then there is a $G \times \mathbb{K}^\times$-equivariant differential Hamiltonian isomorphism $\mathbb{K}[X_1]^\wedge_{Gx_1}[[\hbar]] \to \mathbb{K}[X_2]^\wedge_{Gx_2}[[\hbar]]$ lifting $(\varphi^*)^{-1}$, where φ is as in Proposition 3.2.2.*

Proof. Let $X, x, Z, G_0, \widetilde{X}, \widetilde{x}, \widetilde{X}_0, \Gamma$ be such as in the proof of Lemma 3.2.3. Let us equip $\mathbb{K}[\widetilde{X}][[\hbar]]$ with a star-product as follows. Consider the Fedosov star-product on T^*Z constructed from the *trivial* connection and $\Omega = 0$. This star-product is invariant w.r.t the action of Γ on T^*Z. Then consider some $G \times \mathbb{K}^\times \times \Gamma$-invariant symplectic connection on $\widetilde{X}_0$ and construct the star-product from this connection and $\Omega = 0$. Taking the tensor product of the two star-products we get a star-product $*$ on $\widetilde{X}$ (that can be extended to $\widetilde{X}^\wedge_{G\widetilde{x}}$), again, with the quantum comoment map $\xi \mapsto H_\xi$. This star-product is Γ-invariant so it descends to $\mathbb{K}[X][[\hbar]]$.

Lemma 3.3.5. *Let $\psi \in \mathfrak{z}^*$. Then there is a derivation $\widetilde{\zeta}$ of the quantum algebra $\mathbb{K}[X][[\hbar]]$ such that $\widetilde{\zeta}(\hbar) = 0, \widetilde{\zeta}(H_\xi) = \langle \psi, \xi \rangle$.*

Proof of Lemma 3.3.5. Again, as in the proof of Lemma 3.2.3 choose coordinates $z_1, \ldots, z_m$ on Z and let p_i, ξ_i have the same meaning as in that proof. The algebra $\mathbb{K}[T^*Z][[\hbar]]$ is generated (as a $\mathbb{K}[[\hbar]]$-algebra) by the elements $z_i, z_i^{-1}, p_i, i = 1, \ldots, m$, subject to the relations $[z_i, p_j] = \hbar^2\delta_{ij}, [z_i, z_j] = 0, [p_i, p_j] = 0$.

Set $a_i = \langle \psi, \xi_i \rangle$. It is clear that the map $\widetilde{\zeta} : z_i \mapsto 0, p_i \mapsto a_i/z_i$ can be uniquely extended to a $\mathbb{K}[\hbar]$-linear derivation of the quantum algebra $\mathbb{K}[T^*Z][\hbar]$ also denoted by $\widetilde{\zeta}$. Extend $\widetilde{\zeta}$ to the derivation of $\mathbb{K}[\widetilde{X}][\hbar]$ by making it act trivially on $\mathbb{K}[\widetilde{X}_0][\hbar]$. Note that $\widetilde{\zeta}(H_\xi) = \langle \psi, \xi \rangle$ for $\xi \in \mathfrak{z}$. By the construction, $\widetilde{\zeta}$ is Γ-invariant. So it descends to $\mathbb{K}[X][[\hbar]]$. $\square$

By Proposition 3.2.2, we have $G\times\mathbb{K}^\times$-equivariant Hamiltonian isomorphisms $\mathbb{K}[X_1]^\wedge_{Gx_1}\cong\mathbb{K}[X]^\wedge_{Gx}\cong\mathbb{K}[X_2]^\wedge_{Gx_2}$. Transfer the star-products and the quantum (=classical) comoment maps from $(X_1)^\wedge_{Gx_1},(X_2)^\wedge_{Gx_2}$ to $X^\wedge_{Gx}$. So we get two star-products $*^1,*^2$ on $\mathbb{K}[X]^\wedge_{Gx}[[\hbar]]$ such that $\xi\mapsto H_\xi$ is a quantum comoment map for both of them.

It remains to check that there is a $G\times\mathbb{K}^\times$-equivariant differential equivalence $T:=\mathrm{id}+\sum_{i=1}^\infty T_i\hbar^{2i}:\mathbb{K}[X]^\wedge_{Gx}\rightarrow\mathbb{K}[X]^\wedge_{Gx}$ such that $T(f*g)=(Tf)*^1(Tg)$ and $T(H_\xi)=H_\xi$. In general, $T(H_\xi)-H_\xi\in\mathbb{K}\hbar^2$ (compare with the proof of [Lo3, Theorem 2.3.1]).

As we have seen in the proof of [Lo3, Theorem 2.3.1], T_1 is a Poisson derivation of $\mathbb{K}[X]^\wedge_{Gx}$. Let v denote the corresponding vector field. Let ζ_i have the same meaning as in Lemma 3.2.3. Then, as we have seen in the proof of that lemma, $v=v_f+\sum_{i=1}^m a_i\zeta_i$ for uniquely determined $a_i\in\mathbb{K}$ and some $f\in\mathbb{K}[X]^\wedge_{Gx}[[\hbar]]^G$.

Let $\widetilde{\zeta}$ be a derivation from Lemma 3.3.5 constructed for the linear function $\psi\in\mathfrak{z}^*$ given by $\langle\psi,\xi_i\rangle=a_i$. Set $T':=T\circ\exp(-\widetilde{\zeta})$. This is an isomorphism intertwining the star-products $*,*^1$. Moreover, $T'=\mathrm{id}+\sum_{i=1}^\infty T'_i\hbar^{2i}$, where T'_1 now corresponds to a Hamiltonian vector field. So $T'(H_\xi)-H_\xi$ lies in $\hbar^4\mathbb{K}[[\hbar]]$ whence is zero. □

We will also need the following corollary of Theorems 3.3.1,3.3.4.

Corollary 3.3.6. *Let X be an affine Hamiltonian G-variety with symplectic form ω and comoment map $\xi\mapsto H_\xi$. Let $x\in X$ be a point with closed G-orbit and $G_x=\{1\}$. Next, suppose that X is equipped with a $\mathbb{K}^\times$-action and also with an action of a reductive group Q such that*

- *Q, $\mathbb{K}^\times$ and G pairwise commute.*
- *Q acts by Hamiltonian automorphisms.*
- *$t.\omega=t^2\omega, t.H_\xi=t^2H_\xi$ for any $t\in\mathbb{K}^\times$.*
- *$Q\times\mathbb{K}^\times$ preserves Gx.*

Let $*$ be a star-product on $X^\wedge_{Gx}$ obtained by the Fedosov construction with a $G\times Q\times\mathbb{K}^\times$-invariant connection and $\Omega=0$. Further, let $*'$ be some other $G\times Q$-equivariant homogeneous star-product on $X^\wedge_{Gx}$ and $\mathfrak{g}\rightarrow\mathbb{K}[X]^\wedge_{Gx}[[\hbar]],\xi\mapsto\widehat{H}'_\xi$, be a quantum comoment map for $*'$. We suppose that $\widehat{H}'_\xi\equiv H_\xi \mod \hbar^2$ and that $\widehat{H}'_\xi$ is Q-invariant and of degree 2 with respect to $\mathbb{K}^\times$ for any $\xi\in\mathfrak{g}$. Then there exists a $G\times Q\times\mathbb{K}^\times$-equivariant linear map $T:\mathbb{K}[X]^\wedge_{Gx}[[\hbar]]\rightarrow\mathbb{K}[X]^\wedge_{Gx}[[\hbar]]$ intertwining the star-products and the quantum comoment maps.

Proof. We may assume that $*'$ is obtained by the Fedosov construction using a $Q\times\mathbb{K}^\times$-invariant two-form Ω'. Let us show that Ω' is exact.

From Theorem 3.3.1 (applied to formal schemes rather than to varieties) it follows that $\iota_{\xi_*}\Omega'$ is exact for any $\xi\in\mathfrak{z}(\mathfrak{g})$. The inclusion $G=Gx\hookrightarrow X^\wedge_{Gx}$ and the projection $G\twoheadrightarrow G/(G,G)$ induce isomorphisms of the second cohomology groups. So we need to check that if θ is a closed (left) invariant 2-form on a torus $G/(G,G)$, then θ is exact whenever the forms $i_{\xi_*}\theta$ are exact for all $\xi\in\mathfrak{z}(\mathfrak{g})$. Let $z_i, i=1,\ldots,m$, have the same meaning as before. Then any second cohomology class of $(\mathbb{K}^\times)^m$ is uniquely represented in the form $\sum a_{ij}\frac{dz_i\wedge dz_j}{z_iz_j}, a_{ij}=-a_{ji}$. The condition of the exactness of all forms $\iota_{\xi_*}\theta$ means that the forms $\sum_i a_{ij}\frac{dz_i}{z_i}$ are exact for all j. So $a_{ij}=0$ and Ω' is exact.

Applying Theorem 3.3.4 (as the proof shows an equivalence there can be chosen to be, in addition, Q-equivariant), we complete the proof of the corollary. □

3.4. **Hamiltonian reduction.** In this subsection $\widetilde{G}$ is an algebraic group and X an affine Hamiltonian $\widetilde{G}$-variety equipped with a symplectic form ω and with a moment map $\mu_{\widetilde{G}}$. We fix a normal subgroup $G \subset \widetilde{G}$ such that the G-action on X is free. We suppose that there exists a categorical quotient X/G and that the quotient morphism $X \to X/G$ is affine.

Let μ_G be the moment map for the action $G : X$, in other words, μ_G is the composition of $\mu_{\widetilde{G}}$ and the natural projection $\widetilde{\mathfrak{g}}^* \twoheadrightarrow \mathfrak{g}^*$. By the standard properties of the moment map, compare, for example, with [GS, Section 26] $\mu_G^{-1}(0)$ is a smooth complete intersection in X. Further, $\mu_G^{-1}(0)/G \subset X/G$ is a smooth subvariety and there is a unique symplectic form $\underline{\omega}$ on $\mu_G^{-1}(0)/G$ whose pullback to $\mu_G^{-1}(0)$ coincides with the restriction of ω. So $\underline{\omega}$ is $\widetilde{G}/G$-invariant and the action of $\widetilde{G}/G$ on $\mu_G^{-1}(0)/G$ is Hamiltonian, the moment map sends an orbit $Gx, x \in \mu_G^{-1}(0)$, to $\mu_{\widetilde{G}}(x)$. The variety $\mu_G^{-1}(0)/G$ is said to be the *Hamiltonian reduction* of X under the action of G and is denoted by $X///G$. Note that the algebra of functions on $X///G$ is nothing else but $(\mathbb{K}[X]/I)^G$, where I is the ideal in $\mathbb{K}[X]$ generated by $H_\xi, \xi \in \mathfrak{g}$.

Suppose there is a $\mathbb{K}^\times$-action on X such that $t.\omega = t^2\omega$ and $t.\mu_{\widetilde{G}} = t^2\mu_{\widetilde{G}}$. Then this action descends to $X///G$ and has analogous properties.

Now let us consider a quantum version of this construction. Suppose that X is equipped with a $\widetilde{G}$-invariant homogeneous star-product $*$ and $\xi \mapsto \widehat{H}_\xi$ is a quantum comoment map for this action. Set $\mathcal{A}_\hbar := (\mathbb{K}[X][[\hbar]]/\mathcal{I}_\hbar)^G$, where $\mathcal{I}_\hbar$ is the left ideal in $\mathbb{K}[X][[\hbar]]$ generated by $\widehat{H}_\xi, \xi \in \mathfrak{g}$.

Proposition 3.4.1. *$\mathcal{A}_\hbar$ is a complete flat $\mathbb{K}[[\hbar]]$-algebra, $\mathcal{I}_\hbar/(\mathcal{I}_\hbar \cap \hbar\mathbb{K}[X][[\hbar]]) = I$, and the natural homomorphism $\mathcal{A}_\hbar/(\hbar) \to (\mathbb{K}[X]/I)^G$ is an isomorphism. Furthermore, if $\widetilde{G}/G$ is reductive, then there is a $(\widetilde{G}/G)\times\mathbb{K}^\times$-equivariant isomorphism $\mathcal{A}_\hbar \cong \mathbb{K}[X///G][[\hbar]]$ of $\mathbb{K}[[\hbar]]$-modules, so we get a $\widetilde{G}/G$-invariant homogeneous star-product on $\mathbb{K}[X///G][[\hbar]]$. Finally, a map $(\widetilde{\mathfrak{g}}/\mathfrak{g})^* \to \mathcal{A}_\hbar$ sending ξ to the image of $\widehat{H}_\xi$ in $\mathbb{K}[X][[\hbar]]/\mathcal{I}_\hbar$ (the image is easily seen to be G-invariant) is a quantum comoment map.*

Proof. Clearly, $\mathcal{A}_\hbar$ is complete and $\mathcal{I}_\hbar/(\mathcal{I}_\hbar \cap \hbar\mathbb{K}[X][[\hbar]]) = I$. Flatness of $\mathcal{A}_\hbar$ stems from the equality $\hbar\mathcal{I}_\hbar = \mathcal{I}_\hbar \cap \hbar\mathbb{K}[X][[\hbar]]$. This equality follows from the fact that $H_{\xi_1}, \ldots, H_{\xi_n}$ form a regular sequence in $\mathbb{K}[X]$ (here $\xi_1, \ldots, \xi_n$ is a basis in $\mathfrak{g}$), compare, for example, with the proof of [Lo2, Lemma 3.6.1].

Let us prove that $\mathcal{A}_\hbar/(\hbar) = (\mathbb{K}[X]/I)^G$. Set $\mathcal{M}_\hbar := \mathbb{K}[X][[\hbar]]/\mathcal{I}_\hbar, \mathcal{M}_{\hbar,k} := \mathcal{M}_\hbar/(\hbar^k)$. We need to prove that the natural map $(\mathcal{M}_{\hbar,k})^G \to (\mathcal{M}_{\hbar,k-1})^G$ is surjective for any k. We can rewrite $(\mathcal{M}_{\hbar,k})^G$ as $H^0(\mathfrak{n}, M_{\hbar,k})^L$, where $G = N \rtimes L$ is a Levi decomposition. The group L is reductive and all L-modules in consideration are locally finite. From the exact sequence $0 \to \mathbb{K}[X]/I \to \mathcal{M}_{\hbar,k} \to \mathcal{M}_{\hbar,k-1} \to 0$ we see that the following sequence

$$0 \to H^0(\mathfrak{n}, \mathbb{K}[X]/I)^L \to H^0(\mathfrak{n}, \mathcal{M}_{\hbar,k})^L \to H^0(\mathfrak{n}, \mathcal{M}_{\hbar,k-1})^L \to H^1(\mathfrak{n}, \mathbb{K}[X]/I)^L$$

is exact.

Set, for brevity, $X_0 := \mu^{-1}(0)$, so that $\mathbb{K}[X]/I = \mathbb{K}[X_0]$. We claim that $H^1(\mathfrak{n}, \mathbb{K}[X_0])^L = 0$. Assume first that the quotient map $\pi : X_0 \to X_0/G$ is a trivial principal G-bundle. In particular, $\mathbb{K}[X_0] \cong \mathbb{K}[G] \otimes \mathbb{K}[X_0]^G$ as a G-module and hence $\mathbb{K}[X_0] \cong \mathbb{K}[N] \otimes \mathbb{K}[X_0]^N$ as an N-module. Since the group N is unipotent, the first cohomology $H^1(\mathfrak{n}, \mathbb{K}[N])$ vanishes, compare, for example, with [GG, 5.3]. So $H^1(\mathfrak{n}, \mathbb{K}[X_0]) = \{0\}$.

Proceed to the general case. There is a faithfully flat étale morphism $\varphi : Y \to X_0/G$ such that the natural morphism $Y \times_{X_0/G} X_0 \to Y$ is a trivial G-bundle. Then $H^1(\mathfrak{n}, \mathbb{K}[Y \times_{X_0/G} X_0])^L = \mathbb{K}[Y] \otimes_{\mathbb{K}[X_0/G]} H^1(\mathfrak{n}, \mathbb{K}[X_0])^L$. As we have seen above, the left hand side of the previous equality vanishes. Since the morphism $Y \to X_0/G$ is faithfully flat, we see that $H^1(\mathfrak{n}, \mathbb{K}[X_0])^L = \{0\}$.

So we have proved that $\mathcal{A}_\hbar/(\hbar) = \mathbb{K}[X///G]$. Since the algebra $\mathcal{A}_\hbar$ is complete, from here it follows that $\mathcal{A}_\hbar \cong \mathbb{K}[X///G][[\hbar]]$ as $\mathbb{K}[[\hbar]]$-modules and an isomorphism can be made $\widetilde{G}/G$-equivariant if $\widetilde{G}/G$ is reductive (because we can average an isomorphism over $\widetilde{G}/G$). The claim about a quantum comoment map is straightforward to check. □

4. W-algebras

4.1. **Definition of W-algebras.** In this subsection we are going to recall the definition of a W-algebra given in [Lo2, Subsection 3.1] and also a variant of Premet's definition, [Pr2].

Recall that G denotes a connected reductive algebraic group, and $\mathfrak{g}$ is the Lie algebra of G. Fix a nilpotent element $e \in \mathfrak{g}$. Set $\mathbb{O} := Ge$. Choose an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ and set $Q := Z_G(e, h, f)$. Let T denote a maximal torus in Q. Fix a G-invariant symmetric form $(\cdot, \cdot)$ on $\mathfrak{g}$ and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using it.

Define the Slodowy slice $S := e + \mathfrak{z}_\mathfrak{g}(f)$. It will be convenient for us to consider S as a subvariety in $\mathfrak{g}^*$. Let $\gamma : \mathbb{K}^\times \to G$ have the same meaning as in Subsection 3.2.

Consider the cotangent bundle T^*G of G. Identify T^*G with $G \times \mathfrak{g}^*$ assuming that $\mathfrak{g}^*$ consists of left-invariant 1-forms. The variety T^*G is equipped with a $G \times Q \times \mathbb{K}^\times$-action, where G acts by left translations: $g.(g_1, \alpha) = (gg_1, \alpha)$, Q acts by right translations: $q.(g_1, \alpha) = (g_1 q^{-1}, q.\alpha)$, while $\mathbb{K}^\times$ acts by

$$t.(g_1, \alpha) = (g_1 \gamma^{-1}(t), t^{-2}\gamma(t)\alpha),$$

where $t \in \mathbb{K}^\times, q \in Q, g, g_1 \in G, \alpha \in \mathfrak{g}^*$.

Recall the canonical symplectic form $\widetilde{\omega}$ on T^*G. This form is $G \times Q$-invariant and $t.\widetilde{\omega} = t^2\widetilde{\omega}$. Both G and Q-actions are Hamiltonian with moment maps $\mu_G(g, \alpha) = g\alpha, \mu_Q(g, \alpha) = \alpha|_\mathfrak{q}$.

By the *equivariant Slodowy slice* we mean the subvariety $X := G \times S \hookrightarrow G \times \mathfrak{g}^* = T^*G$. It turns out that X is a $G \times Q \times \mathbb{K}^\times$-stable symplectic subvariety of T^*G, see [Lo2, Subsection 3.1]. Let ω denote the restriction of $\widetilde{\omega}$ to X. It is easy to see that the symplectic variety X is nothing else but the model variety $M_G(\{1\}, e, \{0\})$ introduced in [Lo1].

Choose a $G \times Q \times \mathbb{K}^\times$-invariant symplectic connection ∇ on X and construct a star-product on $\mathbb{K}[X][[\hbar]]$ from ∇ and $\Omega = 0$. It turns out, see [Lo2, Subsection 3.1], that $\widetilde{\mathcal{W}_\hbar} := \mathbb{K}[X][\hbar]$ is a subalgebra in the quantum algebra $\mathbb{K}[X][[\hbar]]$. Set $\mathcal{W}_\hbar := \widetilde{\mathcal{W}}_\hbar^G$. This $\mathbb{K}[\hbar]$-algebra is called a *homogeneous W-algebra*. We define a *W-algebra* $\mathcal{W}$ by $\mathcal{W} := \mathcal{W}_\hbar/(\hbar - 1)$. This is a filtered associative algebra equipped with a Q-action and a quantum comoment map $\mathfrak{q} \to \mathcal{W}$. Also from the quantum comoment map $\mathfrak{g} \to \widetilde{\mathcal{W}_\hbar}$ we get a homomorphism $\mathcal{Z} \to \mathcal{W}$, where $\mathcal{Z}$ stands for the center of the universal enveloping algebra $\mathcal{U} := U(\mathfrak{g})$.

There is another (earlier) definition of $\mathcal{W}$ due to Premet [Pr2]. Let us recall it. Introduce a grading on $\mathfrak{g}$ by eigenvalues of $\operatorname{ad} h$: $\mathfrak{g} := \bigoplus \mathfrak{g}(i), \mathfrak{g}(i) := \{\xi \in \mathfrak{g} | [h, \xi] = i\xi\}$ so that $\gamma(t)\xi = t^i\xi$ for $\xi \in \mathfrak{g}(i)$. Define the element $\chi \in \mathfrak{g}^*$ by $\chi = (e, \cdot)$ and the skew-symmetric form ω_χ on $\mathfrak{g}(-1)$ by $\omega_\chi(\xi, \eta) = \langle \chi, [\xi, \eta] \rangle$. It turns out that this form is symplectic. Pick a $\mathfrak{t}$-stable lagrangian subspace $l \subset \mathfrak{g}(-1)$ and define the subalgebra $\mathfrak{m} := l \oplus \bigoplus_{i \leqslant -2} \mathfrak{g}(i)$. Then χ is a character of $\mathfrak{m}$. Define the shift $\mathfrak{m}_\chi = \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{m}\} \subset \mathfrak{g} \oplus \mathbb{K}$. Essentially, in [Pr2] the W-algebra was defined as the quantum Hamiltonian reduction $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\operatorname{ad} \mathfrak{m}}$ (this variant of a Hamiltonian reduction is slightly different from the one recalled above).

We checked in [Lo2], see also [Lo3, Theorem 2.2.1], and the discussion after it, that both definitions agree and also that the homomorphism $\mathcal{Z} \to \mathcal{W}$ coincides with one from [Pr2] and so is an isomorphism of $\mathcal{Z}$ with the center of $\mathcal{W}$, [Pr3, footnote to Question 5.1].

A useful feature of Premet's construction is that it allows to construct functors between the categories of $\mathcal{U}$- and $\mathcal{W}$-modules. We say that a left $\mathcal{U}$-module M is a *Whittaker* module if $\mathfrak{m}_\chi$ acts on M by locally nilpotent endomorphisms. In this case $M^{\mathfrak{m}_\chi} = \{m \in M | \xi m = \langle\chi, \xi\rangle m, \forall \xi \in \mathfrak{m}\}$ is a nonzero $\mathcal{W}$-module. As Skryabin proved in the appendix to [Pr2], the functor $M \mapsto M^{\mathfrak{m}_\chi}$ is an equivalence between the category of Whittaker $\mathcal{U}$-modules and the category $\mathcal{W}$-Mod of $\mathcal{W}$-modules. A quasiinverse equivalence is given by $N \mapsto \mathcal{S}(N) := (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi) \otimes_{\mathcal{W}} N$, where $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ is equipped with a natural structure of a $\mathcal{U}$-$\mathcal{W}$-bimodule. In the sequel we will call $\mathcal{S}$ the *Skryabin functor*.

4.2. **Decomposition theorem and the correspondence between ideals.** The decomposition theorem, roughly, says that, up to suitably understood completions, the universal enveloping algebra is decomposed into the tensor product of the W-algebra and of a Weyl algebra. We start with the equivariant version of this theorem, [Lo2, Subsection 3.3], because it is similar in spirit to some constructions of the present paper.

First of all, let us note that we can apply the Fedosov construction (with the trivial 2-form Ω) to get a $G \times G$-equivariant homogeneous (with respect to the action of $\mathbb{K}^\times$ defined by $t.(g, \alpha) = (g, t^{-2}\alpha)$) star-product on $\mathbb{K}[T^*G][[\hbar]]$. Automatically, $\mathbb{K}[T^*G][\hbar]$ is a subalgebra in $\mathbb{K}[T^*G][[\hbar]]$. The algebra $\mathcal{U}_\hbar := \mathbb{K}[T^*G][\hbar]^G$ of G-invariants (for the action by left translations) is said to be the homogeneous universal enveloping algebra of $\mathfrak{g}$. The reason for the terminology is that the quotient $\mathcal{U}_\hbar/(\hbar - 1)$ is isomorphic to the universal enveloping algebra $\mathcal{U}$, see [Lo2, Example 3.2.4].

Set $V := [\mathfrak{g}, f]$. Equip V with the symplectic form $\omega(\xi, \eta) = \langle\chi, [\xi, \eta]\rangle$, the action of $\mathbb{K}^\times, t.v := \gamma(t)^{-1}v$, and the action of Q restricted from $\mathfrak{g}$. Then we can equip the space $\mathbf{A}_{V,\hbar} := S(V)[\hbar]$ with the Moyal-Weyl star-product, compare with [Lo2, Example 3.2.3]. The algebra $\mathbf{A}_{V,\hbar}$ is called the *homogeneous Weyl algebra* of the vector space V. The quotient $\mathbf{A}_V := \mathbf{A}_{V,\hbar}/(\hbar - 1)$ is the usual Weyl algebra of V.

Set $x := (1, \chi) \in X$. Since the star-products on both $\mathbb{K}[X][[\hbar]]$ and $\mathbb{K}[T^*G][[\hbar]]$ are differential, we can extend them to the completions $\mathbb{K}[X]^\wedge_{Gx}[[\hbar]], \mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]]$ along the G-orbit Gx. Next, we can form the completion $\mathbf{A}^\wedge_{V,\hbar} := \mathbb{K}[V^*]^\wedge_0[[\hbar]]$ of the Weyl algebra. This space also has a natural star-product. We remark that the algebras $\mathbb{K}[X]^\wedge_{Gx}[[\hbar]], \mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]]$, $\mathbf{A}^\wedge_{V,\hbar}$ have natural topologies (of completions).

The following theorem was proved in [Lo3, Theorem 2.3.1, Remark 2.3.2].

Theorem 4.2.1. *There is a $G \times Q \times \mathbb{K}^\times$-equivariant $\mathbb{K}[[\hbar]]$-linear Hamiltonian isomorphism $\Phi_\hbar : \mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]] \to \mathbf{A}^\wedge_{V,\hbar}\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[X]^\wedge_{Gx}[[\hbar]]$ of topological algebras.*

Taking the G-invariants in the algebras from Theorem 4.2.1, we get a non-equivariant decomposition theorem. Namely, set $\mathcal{W}^\wedge_\hbar := \mathbb{K}[S]^\wedge_\chi[[\hbar]], \mathcal{U}^\wedge_\hbar := \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]]$.

Corollary 4.2.2. *There is a $\mathbb{K}[[\hbar]]$-linear $Q \times \mathbb{K}^\times$-equivariant Hamiltonian isomorphism $\Phi_\hbar : \mathcal{U}^\wedge_\hbar \to \mathbf{A}^\wedge_{V,\hbar}\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathcal{W}^\wedge_\hbar$ of topological algebras.*

This proposition allows to define a map from the set $\mathfrak{Id}(\mathcal{W})$ of two-sided ideals of $\mathcal{W}$ to the analogous set $\mathfrak{Id}(\mathcal{U})$ for $\mathcal{U}$. Namely, take a two-sided ideal $\mathcal{I} \subset \mathcal{W}$. As we noted in [Lo2, Subsection 3.2], there is a unique ideal $\mathcal{I}_\hbar \subset \mathcal{W}_\hbar$ with the following properties:

- $\mathcal{I} = \mathcal{I}_\hbar/(\hbar - 1)$,
- $\mathcal{I}_\hbar$ is $\mathbb{K}^\times$-stable,
- $\mathcal{I}_\hbar$ is $\hbar$-*saturated*, i.e., $\mathcal{I}_\hbar \cap \hbar\mathcal{W}_\hbar = \hbar\mathcal{I}_\hbar$.

Let $\mathcal{I}_\hbar^\wedge$ denote the closure of $\mathcal{I}_\hbar$ in $\mathcal{W}_\hbar^\wedge$. Set $\mathcal{J}_\hbar := \Phi_\hbar^{-1}(\mathbf{A}_{V,\hbar}^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathcal{I}_\hbar^\wedge) \cap \mathcal{U}_\hbar$. Finally, set $\mathcal{I}^\dagger := \mathcal{J}_\hbar/(\hbar-1) \subset \mathcal{U}$.

Reversing the procedure, we can construct a map $\mathcal{J} \mapsto \mathcal{J}_\dagger : \mathfrak{Id}(\mathcal{U}) \to \mathfrak{Id}(\mathcal{W})$, see [Lo3, Subsection 3.1]. More precisely, we first construct the homogeneous ideal $\mathcal{J}_\hbar \subset \mathcal{U}_\hbar$, then complete it to get the ideal $\mathcal{J}_\hbar^\wedge \subset \mathcal{U}_\hbar^\wedge$. This ideal has the form $\mathbf{A}_\hbar^\wedge(\mathcal{I}_\hbar^\wedge) := \mathbf{A}_\hbar^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathcal{I}_\hbar^\wedge$ for a unique ideal $\mathcal{I}_\hbar^\wedge \subset \mathcal{W}_\hbar^\wedge$. The ideal $\mathcal{I}_\hbar^\wedge$ is the completion of a unique ideal $\mathcal{I}_\hbar \subset \mathcal{W}_\hbar$. Then for $\mathcal{J}_\dagger$ we take the image of $\mathcal{I}_\hbar$ in $\mathcal{W}$. This map restricts to a surjection:

- from the set of all $\mathcal{J} \in \mathfrak{Id}(\mathcal{U})$ such that the associated variety $\mathrm{V}(\mathcal{U}/\mathcal{J})$ equals $\overline{\mathbb{O}}$
- to the set of all Q-stable ideals of finite codimension in $\mathcal{W}$.

The map given by $\mathcal{I} \mapsto \mathcal{I}^\dagger$ is a section of $\mathcal{J} \mapsto \mathcal{J}_\dagger$, see [Lo3, Theorem 1.2.2]. Moreover, $\operatorname{codim}_{\mathcal{W}} \mathcal{J}_\dagger$ coincides with the multiplicity $\operatorname{mult}_{\mathbb{O}} \mathcal{U}/\mathcal{J}$. This follows from [Lo2, Proposition 3.4.2]. In particular, $\mathcal{J}$ has multiplicity 1 on $\mathbb{O}$ if and only if $\mathcal{J}_\dagger$ is the annihilator of a one-dimensional module. Moreover, if V is a $\mathcal{W}$-module with $\operatorname{Ann}_{\mathcal{W}}(V)^\dagger = \mathcal{J}$, then $\mathcal{J}_\dagger \subset \operatorname{Ann}_{\mathcal{W}}(V)$, see [Lo2, Theorem 1.2.2,(ii), Proposition 3.4.4]. So if $\operatorname{mult}_{\overline{\mathbb{O}}}\mathcal{U}/\mathcal{J} = 1$, then $\dim V = 1$ and V is stable with respect to the action of Q on the set of irreducible $\mathcal{W}$-modules.

4.3. **Category $\mathcal{O}$.** Recall the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}$. Consider the centralizer $\mathfrak{g}_0 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{g}$. This is a minimal Levi subalgebra of $\mathfrak{g}$ containing e. Fix an element θ lying in the cocharacter lattice $\operatorname{Hom}(\mathbb{K}^\times, T) \hookrightarrow \mathfrak{t}$ with $\mathfrak{z}_{\mathfrak{g}}(\theta) = \mathfrak{g}_0$. Let $\mathcal{W} := \bigoplus_{\alpha\in\mathbb{Z}} \mathcal{W}_\alpha$ be the decomposition into eigenspaces of $\operatorname{ad}\theta$. Set

$$\mathcal{W}_{\geqslant 0} := \bigoplus_{\alpha\geqslant 0} \mathcal{W}_\alpha, \quad \mathcal{W}_{>0} := \bigoplus_{\alpha>0} \mathcal{W}_\alpha, \quad \mathcal{W}_{\geqslant 0}^+ := \mathcal{W}_{\geqslant 0} \cap \mathcal{W}\mathcal{W}_{>0}.$$

Clearly, $\mathcal{W}_{\geqslant 0}$ is a subalgebra of $\mathcal{W}$, while $\mathcal{W}_{>0}$ and $\mathcal{W}_{\geqslant 0}^+$ are two-sided ideals of $\mathcal{W}_{\geqslant 0}$. Note that we have an embedding $\mathfrak{t} \hookrightarrow \mathfrak{q} \hookrightarrow \mathcal{W}$. The image of $\mathfrak{t}$ lies in $\mathcal{W}_{\geqslant 0}$. Hence we get a homomorphism $\mathfrak{t} \to \mathcal{W}_{\geqslant 0}/\mathcal{W}_{\geqslant 0}^+$ of Lie algebras.

Consider also the W-algebra $\mathcal{W}^0$ associated with the pair $(\mathfrak{g}_0, e)$ (we remark that this notation is different from [Lo4], where $\mathcal{W}^0$ denoted the quotient $\mathcal{W}_{\geqslant 0}/\mathcal{W}_{\geqslant 0}^+$, while the W-algebra of $(\mathfrak{g}_0, e)$ was denoted by $\underline{\mathcal{W}}$). Again, we have a natural embedding $\mathfrak{t} \hookrightarrow \mathcal{W}^0$. We are going to describe a relation between $\mathcal{W}_{\geqslant 0}/\mathcal{W}_{\geqslant 0}^+$ and $\mathcal{W}^0$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ containing h. Let Δ be the corresponding root system of $\mathfrak{g}$. Fix a system $\Pi \subset \Delta$ of simple roots such that θ is dominant (in particular, Π contains a system of simple roots for $\mathfrak{g}_0$). Set $\Delta^+ := \{\alpha \in \Delta | \langle\alpha, \theta\rangle > 0\}, \Delta^- = -\Delta^+$. Following [BGK, Subsection 4.1], define an element $\delta \in \mathfrak{h}^*$ by

$$\delta := \sum_{\alpha\in\Delta^-, \langle\alpha,h\rangle=-1} \alpha/2 + \sum_{\alpha\in\Delta^-, \langle\alpha,h\rangle\leqslant-2} \alpha. \tag{4.1}$$

Now we can state the following proposition, see [Lo4, Remark 5.4]. A similar result was obtained by Brundan, Goodwin and Kleshchev in [BGK, Theorem 4.3].

Proposition 4.3.1. *There is a T-equivariant isomorphism $\Psi : \mathcal{W}_{\geqslant 0}/\mathcal{W}_{\geqslant 0}^+ \to \mathcal{W}^0$ making the following diagram commutative.*

$$\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathcal{W}_{\geqslant 0}/\mathcal{W}^+_{\geqslant 0} \\ \Big\downarrow {\scriptstyle x\mapsto x-\langle\delta,x\rangle} & & \Big\downarrow {\scriptstyle \Psi} \\ \mathfrak{t} & \longrightarrow & \mathcal{W}^0 \end{array}$$

Below we consider $\mathcal{W}_{\geqslant 0}/\mathcal{W}^+_{\geqslant 0}$-modules as $\mathcal{W}^0$-modules and vice-versa using the isomorphism Ψ.

By definition, the category $\widetilde{\mathcal{O}}(\theta)$ for the pair $(\mathcal{W},\theta)$ consists of all $\mathcal{W}$-modules N satisfying the following conditions:

- N is finitely generated.
- $\mathfrak{t}$ acts on N by diagonalizable endomorphisms.
- $\mathcal{W}_{>0}$ acts on N by locally nilpotent endomorphisms.

In [Lo4] this category was denoted by $\widetilde{\mathcal{O}}^{\mathfrak{t}}(\theta)$.

There are analogs of Verma modules in $\widetilde{\mathcal{O}}^{\mathfrak{t}}(\theta)$. Take a finitely generated $\mathcal{W}^0$-module N^0 with diagonalizable action of $\mathfrak{t}$. The module $\mathcal{M}^\theta(N^0):=\mathcal{W}\otimes_{\mathcal{W}_{\geqslant 0}}N^0$ lies in $\widetilde{\mathcal{O}}(\theta)$; here we consider N^0 as a $\mathcal{W}_{\geqslant 0}$-module via the natural projection $\mathcal{W}_{\geqslant 0}\twoheadrightarrow \mathcal{W}_{\geqslant 0}/\mathcal{W}^+_{\geqslant 0}\cong\mathcal{W}^0$. Also we have a functor between $\mathcal{W}$-Mod and $\mathcal{W}^0$-Mod defined by $N\mapsto N^{\mathcal{W}_{>0}}:=\{v\in N|xv=0,\forall x\in\mathcal{W}_{>0}\}$.

Suppose now that N^0 is irreducible. Then $\mathcal{M}^\theta(N^0)$ has a unique irreducible quotient $L^\theta(N^0)$, see [BGK, Theorem 4.5]. Moreover, the modules $L^\theta(N^0)$ form a complete set of pairwise non-isomorphic simple objects in $\widetilde{\mathcal{O}}(\theta)$. In particular, any irreducible finite dimensional $\mathcal{W}$-module is of the form $L^\theta(N^0)$, where N^0 is finite dimensional and irreducible. However, $L^\theta(N^0)$ may be infinite dimensional even if N^0 is finite dimensional. In Subsection 5.1 we will obtain a criterium for $L^\theta(N^0)$ to be finite dimensional. For this we will need to recall the main result of [Lo4].

Set $\mathcal{U}^0:=U(\mathfrak{g}_0)$. Recall the subalgebra $\mathfrak{m}\subset\mathfrak{g}$ defined in Subsection 4.1 and $\chi:=(e,\cdot)$. Let $\mathfrak{m}_0:=\mathfrak{m}\cap\mathfrak{g}_0$. Then $\mathfrak{m}_0$ plays the same role for $(\mathfrak{g}_0,e)$ as $\mathfrak{m}$ played for $(\mathfrak{g},e)$. Let $\mathcal{S}_0:\mathcal{W}^0\text{-Mod}\to\mathcal{U}^0\text{-Mod}$ be the Skryabin functor for $\mathfrak{g}_0,e$.

Let $\mathfrak{g}:=\bigoplus_{\alpha\in\mathbb{Z}}\mathfrak{g}_\alpha$ be the grading by eigenspaces of $\operatorname{ad}\theta$. Set

$$\mathfrak{n}_+:=\bigoplus_{\alpha>0}\mathfrak{g}_\alpha,\quad \mathfrak{p}:=\bigoplus_{\alpha\geqslant 0}\mathfrak{g}_\alpha=\mathfrak{g}_0\oplus\mathfrak{n}_+,$$
$$\widetilde{\mathfrak{m}}:=\mathfrak{n}_+\oplus\mathfrak{m}_0,\quad \widetilde{\mathfrak{m}}_\chi:=\{\xi-\langle\chi,\xi\rangle|\xi\in\widetilde{\mathfrak{m}}\}.$$

Let M be a $\mathcal{U}$-module. We say that M is a *generalized Whittaker module* (for e and θ) if:

(1) M is finitely generated.
(2) $\mathfrak{t}$ acts on M by diagonalizable endomorphisms.
(3) $\widetilde{\mathfrak{m}}_\chi$ acts by locally nilpotent endomorphisms.

For example, let N^0 be a $\mathcal{W}^0$-module with diagonalizable action of $\mathfrak{t}$. Set $\mathcal{M}^{e,\theta}(N^0):=\mathcal{U}\otimes_{U(\mathfrak{p})}\mathcal{S}_0(N^0)$. Here $U(\mathfrak{p})$ acts on $\mathcal{S}_0(N^0)$ via the natural projection $U(\mathfrak{p})\twoheadrightarrow\mathcal{U}^0$. Then $\mathcal{M}^{e,\theta}(N^0)$ is a generalized Whittaker module. We denote the category of generalized Whittaker modules by $\widetilde{\mathrm{Wh}}(e,\theta)$.

We have a functor from $\widetilde{\mathrm{Wh}}(e,\theta)$ to $\mathcal{W}^0$-Mod constructed as follows. For a $\mathfrak{g}$-module M the space $M^{\mathfrak{n}_+}$ is a Whittaker $\mathfrak{g}_0$-module. Now $\mathcal{S}_0^{-1}(M^{\mathfrak{n}_+})=M^{\widetilde{\mathfrak{m}}_\chi}$ is a $\mathcal{W}^0$-module.

The following theorem follows from [Lo4, Theorem 4.1].

Theorem 4.3.2. *There is an equivalence of abelian categories $\mathcal{K}: \widetilde{\mathcal{O}}(\theta) \to \widetilde{\mathrm{Wh}}(e,\theta)$ having the following properties:*

1. *The functors $N \mapsto N^{\mathcal{W}>0}$ and $N \mapsto \mathcal{K}(N)^{\widetilde{\mathfrak{m}}_\chi}$ from $\mathcal{W}$-Mod to $\mathcal{W}^0$-Mod are isomorphic.*
2. *The functors $N^0 \mapsto \mathcal{M}^\theta(N^0)$ and $N^0 \mapsto \mathcal{K}^{-1}(\mathcal{M}^{e,\theta}(N^0))$ from $\mathcal{W}^0$-Mod to $\mathcal{W}$-Mod are isomorphic.*
3. *For any $M \in \widetilde{\mathcal{O}}(\theta)$ we have $\mathrm{Ann}_{\mathcal{U}} \mathcal{K}(M) = (\mathrm{Ann}_{\mathcal{W}}(M))^\dagger$.*

Corollary 4.3.3. *The functor $\mathcal{K}$ induces a bijection between the following two sets:*

- *The set of finite dimensional irreducible $\mathcal{W}$-modules.*
- *The set of irreducible modules M in $\widetilde{\mathrm{Wh}}(e,\theta)$ with $\mathrm{V}(\mathcal{U}/\mathrm{Ann}_{\mathcal{U}}(M)) = \overline{\mathbb{O}}$.*

This bijection sends $L^\theta(N^0)$ to a unique irreducible quotient $L^{e,\theta}(N^0)$ of $\mathcal{M}^{e,\theta}(N^0)$ and, moreover, $\mathrm{Ann}_{\mathcal{W}}(L^\theta(N^0))^\dagger = \mathrm{Ann}_{\mathcal{U}}(L^{e,\theta}(N^0))$.

5. 1-dimensional representations via category $\mathcal{O}$

5.1. **Highest weights of finite dimensional representations.** We use the notation of Subsection 4.3. In particular, let $\mathfrak{g}_0 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ and let $\mathcal{W}^0$ be the W-algebra constructed from the pair $(\mathfrak{g}_0, e)$.

Recall the element $\theta \in \mathrm{Hom}(\mathbb{K}^\times, T) \hookrightarrow \mathfrak{t}$, the subalgebras $\mathfrak{n}_+, \mathfrak{p} \subset \mathfrak{g}$, the category $\widetilde{\mathcal{O}}(\theta)$ and also the Verma modules $\mathcal{M}^\theta(N^0)$ and the irreducible modules $L^\theta(N^0)$ in this category introduced in Subsection 4.3.

Let $\mathcal{I} \mapsto \mathcal{I}^{\dagger_0}$ denote the map $\mathfrak{Id}(\mathcal{W}^0) \to \mathfrak{Id}(\mathcal{U}^0)$ defined analogously to $\bullet^\dagger$, where $\mathcal{U}^0 := U(\mathfrak{g}_0)$. Let $L(\lambda)$ (resp., $L_0(\lambda)$) denote the irreducible highest weight module for $\mathfrak{g}$ (resp., $\mathfrak{g}_0$) with highest weight λ. Set $J(\lambda) := \mathrm{Ann}_{\mathcal{U}}(L(\lambda)), J_0(\lambda) = \mathrm{Ann}_{\mathcal{U}^0}(L_0(\lambda))$.

Theorem 5.1.1. *Let N^0 be an irreducible $\mathcal{W}^0$-module. Suppose that $(\mathrm{Ann}_{\mathcal{W}^0}(N^0))^{\dagger_0} = J_0(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. Then $\big(\mathrm{Ann}_{\mathcal{W}}(L^\theta(N^0))\big)^\dagger = J(\lambda)$.*

This theorem has the following immediate corollary.

Corollary 5.1.2. *In the notation of Theorem 5.1.1 the following conditions are equivalent:*

1. $\dim L^\theta(N^0) < \infty$.
2. $\dim N^0 < \infty$ *and* $\mathrm{V}(\mathcal{U}/J(\lambda)) = \overline{\mathbb{O}}$.

In the proof of the theorem a crucial role is played by the following lemma.

Lemma 5.1.3. *Let M^0 be an irreducible $\mathcal{U}^0$-module. Then the induced $\mathcal{U}$-module $\mathcal{U} \otimes_{U(\mathfrak{p})} M^0$ has a unique irreducible quotient M. The annihilator of M depends only on the annihilator of M^0.*

The first claim of this lemma is completely standard. The idea of the proof of the second one was communicated to me by David Vogan.

Proof. Let α be the eigenvalue of θ on M^0. For any other eigenvalue β of θ on M the difference $\alpha - \beta$ is a positive integer. There is the largest submodule of $\mathcal{U} \otimes_{U(\mathfrak{p})} M^0$ contained in $\sum_{\beta<\alpha} M_\beta$, say $R(M^0)$. Since $\mathcal{U} \otimes_{U(\mathfrak{p})} M^0$ is generated by M^0, we see that $R(M^0)$ is the largest proper submodule in $\mathcal{U} \otimes_{U(\mathfrak{p})} M^0$, hence the first claim.

To prove the second claim let us define a certain map $\pi : \mathcal{U} \twoheadrightarrow \mathcal{U}^0$. Namely, let π be a unique T-equivariant linear map such that π is the identity on $\mathcal{U}^0 \subset \mathcal{U}$ and $\ker \pi \cap \mathcal{U}_0 = \mathcal{U}_0 \cap \mathcal{U}\mathfrak{p}$ (recall that $\mathcal{U}_\beta$ denotes the eigenspace of $\operatorname{ad}\theta$ in $\mathcal{U}$ with eigenvalue β). We claim that $\operatorname{Ann}_{\mathcal{U}} M$ coincides with

$$\mathcal{I} := \{u \in \mathcal{U} | \pi(aub) \in \operatorname{Ann}_{\mathcal{U}^0} M^0, \forall a, b \in \mathcal{U}\} \tag{5.1}$$

Let us note that $u \in \mathcal{U}_0$ acts on M^0 by $\pi(u)$. Let $u \in \operatorname{Ann}_{\mathcal{U}} M \cap \mathcal{U}_\beta$. Then for $a \in \mathcal{U}_\gamma, b \in \mathcal{U}_{-\beta-\gamma}$ the element aub lies in $\operatorname{Ann}_{\mathcal{U}} M \cap \mathcal{U}_0$ and acts trivially on M and, in particular, on M^0. So $aub \in \mathcal{I}$. It follows that $\operatorname{Ann}_{\mathcal{U}} M \subset \mathcal{I}$.

Let us show that $\mathcal{I} \subset \operatorname{Ann}_{\mathcal{U}}(M)$, i.e., any element $u \in \mathcal{I}$ acts trivially on M. We may assume that $u \in U_\beta$ for some β. Then aub acts trivially on M^0 for any $a \in \mathcal{U}_\gamma, b \in \mathcal{U}_{-\beta-\gamma}$. It follows that $\mathcal{U}u\mathcal{U}M^0$ has zero intersection with $M^0 = M_\alpha$. Since M is irreducible, we see that $\mathcal{U}u\mathcal{U}M^0 = \{0\}$ whence $u \in \operatorname{Ann}_{\mathcal{U}}(M)$. □

Proof of Theorem 5.1.1. By Theorem 4.3.2, $\left(\operatorname{Ann}_{\mathcal{W}}(L^\theta(N^0))\right)^\dagger = \operatorname{Ann}_{\mathcal{U}} L^{e,\theta}(N^0)$. Note that $L^{e,\theta}(N^0)$ is obtained from $M^0 := \mathcal{S}_0(N^0)$ as described in Lemma 5.1.3. The $\mathcal{U}$-module $L(\lambda)$ is obtained from the $\mathcal{U}^0$-module $L_0(\lambda)$ by the same construction. Applying Lemma 5.1.3, we complete the proof. □

5.2. **Highest weights for 1-dimensional representations.** Recall that the group Q acts on $\mathcal{W}$ by automorphisms and so acts also on the set of isomorphism classes of irreducible $\mathcal{W}$-modules. The action of Q° on the latter is trivial.

Recall that in Subsection 4.3 we chose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the set of simple roots $\Pi \subset \mathfrak{h}^*$ and then defined the element $\delta \in \mathfrak{h}^*$ by (4.1).

Theorem 5.2.1. *Let N^0 be an irreducible $\mathcal{W}^0$-module such that $\dim L^\theta(N^0) < \infty$. Suppose that the stabilizer of $L^\theta(N^0)$ in $N_Q(\mathfrak{t})$ acts on $\mathfrak{t}$ without nonzero fixed points. Then the following conditions are equivalent:*

(1) $\dim N^0 = 1$ *and $\mathfrak{t}$ acts on N^0 by $\delta|_{\mathfrak{t}}$.*
(2) $\dim L^\theta(N^0) = 1$.

Proof. Consider the action of $\mathfrak{t}$ on $L^\theta(N^0)$ via the embedding $\mathfrak{t} \hookrightarrow \mathfrak{q} \hookrightarrow \mathcal{W}$. By the construction of $L^\theta(N^0)$, $\mathfrak{t}$ acts on $N^0 \subset L^\theta(N^0)$ by a single character, say α. For any other $\mathfrak{t}$-weight β of $L^\theta(N^0)$ we have $\langle\beta,\theta\rangle < \langle\alpha,\theta\rangle$. By Proposition 4.3.1, the second condition of (1) is equivalent to $\alpha = 0$.

Let Γ denote the stabilizer of $L^\theta(N^0)$ in $N_Q(\mathfrak{t})$. Some central extension of Γ acts on $L^\theta(N^0)$. So the set of $\mathfrak{t}$-weights in $L^\theta(N^0)$ is Γ-stable.

Suppose (1) holds. So $\langle\beta,\theta\rangle < 0$ for all other weights β of $L^\theta(N^0)$. On the other hand, $\sum_{\gamma\in\Gamma/T} \gamma.\theta = 0$ for Γ has no fixed points in $\mathfrak{t}$. So 0 is the only weight of $\mathfrak{t}$ in $L^\theta(N^0)$ whence $L^\theta(N^0) = N^0$.

Now let $\dim L^\theta(N^0) = 1$. Then α is Γ-stable. So α is zero. □

If $\mathfrak{g} = \mathfrak{sl}_n$, then $Q/Z(G)$ is a connected group with a nontrivial center (unless e is principal). So the theorem does not work in this situation. However, in this case the classification of one-dimensional $\mathcal{W}$-modules should follow from results of Brundan and Kleshchev, [BrKl], where an explicit presentation of $\mathcal{W}$ in terms of generators and relations is found, see [Pr5, Subsection 3.8].

For the types B and C we have the following corollary.

Corollary 5.2.2. *Let $\mathfrak{g}$ be of type B or C and let $\lambda \in \mathfrak{h}^*$ be such that $\mathrm{V}(\mathcal{U}/J(\lambda)) = \overline{\mathbb{O}}$ and $\mathrm{V}(\mathcal{U}^0/J_0(\lambda)) = \overline{\mathbb{O}_0}$, where $\mathbb{O}_0$ is the orbit of e in $\mathfrak{g}_0$. Then the following conditions (1) and (2) are equivalent.*

(1) $\mathrm{mult}_{\mathbb{O}}\, \mathcal{U}/J(\lambda) = 1$.
(2) (A) $\lambda - \delta$ *vanishes on* $\mathfrak{t}$,
   (B) $\mathrm{mult}_{\mathbb{O}_0} \mathcal{U}^0/J_0(\lambda) = 1$,
   (C) *and for any* $\lambda' \in \mathfrak{h}^*$ *the conditions* $\mathrm{V}(\mathcal{U}^0/J_0(\lambda')) = \overline{\mathbb{O}_0}$ *and* $J(\lambda) = J(\lambda')$ *imply* $J_0(\lambda) = J_0(\lambda')$.

Proof. First of all, let us check that $N_Q(\mathfrak{t})$ acts on $\mathfrak{t}$ without nonzero fixed points. Let e correspond to the partition $(n_1^{r_1}, \ldots, n_k^{r_k}), n_1 > n_2 > \ldots > n_k$ (the superscripts r_i denote the multiplicities). We may assume that G is SO_n (n is odd) or Sp_n (n is even). Then, as a linear group, Q is the subgroup in $G_1 \times \ldots \times G_k$ consisting of all matrices with determinant 1. Here $G_i = \mathrm{O}_{r_i}$ if either n_i is even and $G = \mathrm{Sp}_n$ or n_i is odd and $G = \mathrm{SO}_n$. Otherwise, $G_i = \mathrm{Sp}_{r_i}$. This description implies that $N_Q(\mathfrak{t})$ acts on $\mathfrak{t}$ without nonzero fixed vectors.

Let us prove the implication (1)$\Rightarrow$(2). Let N^0 be a finite dimensional irreducible $\mathcal{W}^0$-module with $\mathrm{Ann}_{\mathcal{W}^0}(N^0)^{\dagger_0} = J_0(\lambda)$. Then, by Theorem 5.1.1, $\mathrm{Ann}_{\mathcal{W}}(L^\theta(N^0))^\dagger = J(\lambda)$. Since $\mathrm{mult}_{\mathbb{O}}\mathcal{U}/J(\lambda) = 1$, the discussion at the end of Subsection 4.2 implies that $\dim L^\theta(N^0) = 1$ and $L^\theta(N^0)$ is Q-stable. By the choice of N^0, $\mathfrak{t}$ acts on N^0 by λ, hence $\lambda - \delta$ vanishes on $\mathfrak{t}$. Since $L^\theta(N^0)$ is Q-stable, we see that N^0 is $Q \cap G_0$-stable (where G_0 stands for the Levi subgroup of G with Lie algebra $\mathfrak{g}_0$). Also $\dim N^0 = 1$. This implies $\mathrm{mult}_{\mathbb{O}_0}(\mathcal{U}^0/J_0(\lambda)) = 1$. Now let λ' be such that $J(\lambda) = J(\lambda')$, $\mathrm{V}(\mathcal{U}^0/J_0(\lambda')) = \overline{\mathbb{O}}_0$. Let N'^0 be an irreducible $\mathcal{W}^0$-module with $\mathrm{Ann}_{\mathcal{W}^0}(N'^0)^{\dagger_0} = J_0(\lambda')$. Then $\mathrm{Ann}_{\mathcal{W}}(L^\theta(N'^0))^\dagger = J(\lambda') = J(\lambda) = \mathrm{Ann}_{\mathcal{W}}(L^\theta(N^0))^\dagger$. Therefore $\mathrm{Ann}_{\mathcal{W}}(L^\theta(N'^0))$ is Q-conjugate to $\mathrm{Ann}_{\mathcal{W}}(L^\theta(N^0))$. But the last ideal coincides with $J(\lambda)_\dagger$ and hence is Q-stable. So $L^\theta(N'^0) \cong L^\theta(N^0)$ hence $N'^0 \cong N^0$. We conclude that $J_0(\lambda) = \mathrm{Ann}_{\mathcal{W}^0}(N^0)^\dagger = \mathrm{Ann}_{\mathcal{W}^0}(N'^0)^\dagger = J_0(\lambda')$.

To prove (2)$\Rightarrow$(1) one reverses the argument of the previous paragraph. Namely, pick N^0 with $\mathrm{Ann}_{\mathcal{W}^0}(N^0) = J_0(\lambda)$. Using (B) one sees that $\dim N^0 = 1$ and $\mathrm{Ann}_{\mathcal{W}^0}(N^0)$ is $G_0 \cap Q$-stable. From (C) one deduces that $\mathrm{Ann}_{\mathcal{W}} L^\theta(N^0)$ is Q-stable. Now one uses (A) and Theorem 6.5 to show that $\dim L^\theta(N^0) = 1$. (1) follows. ☐

For many orbits in type D the group $N_Q(\mathfrak{t})$ still acts on $\mathfrak{t}$ without nonzero fixed points (and for these orbits both claims of the corollary hold). However, this is not always the case (for instance, recall that $\mathfrak{so}_6 \cong \mathfrak{sl}_4$). The partitions for which $N_Q(\mathfrak{t})$ has a nonzero fixed vector in $\mathfrak{t}$ are precisely those where there are exactly two odd parts and they are equal.

On the other hand Theorem 5.2.1 works (for any representation $L^\theta(N^0)$) whenever $\mathfrak{q}$ is semisimple. This is the case when e is a rigid nilpotent element. For classical $\mathfrak{g}$ this follows from the combinatorial description of rigid elements, see, for example, [CM, Corollary 7.3.5]. For the case of an exceptional Lie algebra $\mathfrak{g}$ see the following table.

Table 5.1: Rigid elements in exceptional algebras

N	$\mathfrak{g}$	e	$\mathfrak{q}$	$\dim \mathfrak{z}_{\mathfrak{g}}(e)$
1	G_2	A_1	A_1	8
2	G_2	$\widetilde{A}_1$	A_1	6
3	F_4	A_1	C_3	36
4	F_4	$\widetilde{A}_1$	A_3	30

N	$\mathfrak{g}$	e	$\mathfrak{q}$	$\dim \mathfrak{z}_{\mathfrak{g}}(e)$
5	F_4	$A_1+\widetilde{A}_1$	A_1+A_1	24
6	F_4	$A_2+\widetilde{A}_1$	A_1	18
7	F_4	$\widetilde{A}_2+A_1$	A_1	16
8	E_6	A_1	A_5	56
9	E_6	$3A_1$	A_2+A_1	38
10	E_6	$2A_2+A_1$	A_1	24
11	E_7	A_1	D_6	99
12	E_7	$2A_1$	B_4+A_1	81
13	E_7	$(3A_1)'$	C_3+A_1	69
14	E_7	$4A_1$	C_3	63
15	E_7	A_2+2A_1	$3A_1$	51
16	E_7	$2A_2+A_1$	$2A_1$	43
17	E_7	$(A_3+A_1)'$	$3A_1$	41
18	E_8	A_1	E_7	190
19	E_8	$2A_1$	B_6	156
20	E_8	$3A_1$	F_4+A_1	136
21	E_8	$4A_1$	C_4	120
22	E_8	A_2+A_1	A_5	112
23	E_8	A_2+2A_1	B_3+A_1	102
24	E_8	A_2+3A_1	G_2+A_1	94
25	E_8	$2A_2+A_1$	G_2+A_1	86
26	E_8	A_3+A_1	B_3+A_1	84
27	E_8	$2A_2+2A_1$	B_2	80
28	E_8	A_3+2A_1	B_2+A_1	76
29	E_8	$D_4(a_1)+A_1$	$3A_1$	72
30	E_8	$A_3+A_2+A_1$	$2A_1$	66
31	E_8	$2A_3$	B_2	60
32	E_8	A_4+A_3	A_1	48
33	E_8	A_5+A_1	$2A_1$	46
34	E_8	$D_5(a_1)+A_2$	A_1	46

The information in this table is taken from [McG, Subsection 5.7], and [C, Subsection 13.1]. A nilpotent element is given by its Bala-Carter label. This label also indicates a minimal Levi subalgebra containing the element. In all cases but NN29,34 the nilpotent element is regular in the Levi subalgebra. In the remaining two cases its D_l-component, $l=4,5$, is subregular.

5.3. **Toolkit.** Recall that we have fixed a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and a system of simple roots Π. We assume that $\mathfrak{g}$ is an exceptional Lie algebra and $\mathbb{O}$ is its rigid nilpotent orbit. To prove that the corresponding algebra $\mathcal{W}$ has a one-dimensional representation we need to find

- A Levi subalgebra $\mathfrak{g}_0$, whose simple root system Π_0 is contained in Π and a nilpotent element $e\in\mathbb{O}\cap\mathfrak{g}_0$ that is not contained in a proper Levi subalgebra of $\mathfrak{g}_0$. These data are needed to construct the category $\mathcal{O}$, see Subsection 4.3. Let $\mathbb{O}_0$ denote the orbit of e in $\mathfrak{g}_0$.

- $\lambda \in \mathfrak{h}^*$ such that
  (A) $\mathrm{V}(\mathcal{U}^0/J_0(\lambda)) = \overline{\mathbb{O}}_0$ so that there is an irreducible finite dimensional $\mathcal{W}^0$-module N^0 with $\mathrm{Ann}_{\mathcal{W}^0}(N^0) = J_0(\lambda)$. This yields $\mathbb{O} \subset \mathrm{V}(\mathcal{U}/J(\lambda))$.
  (B) $\mathrm{V}(\mathcal{U}/J(\lambda)) = \overline{\mathbb{O}}$ or, equivalently, modulo (A), $\dim \mathrm{V}(\mathcal{U}/J(\lambda)) \leqslant \dim \mathbb{O}$. By Corollary 5.1.2, this means $\dim L^\theta(N^0) < \infty$.
  (C) $\lambda - \delta \in \mathrm{Span}_{\mathbb{K}} \Pi_0$, where δ is defined by (4.1). By the discussion in Subsection 5.2, this means that $L(N^0) = N^0$.
  (D) $J_0(\lambda) = \mathcal{I}_0^{\dagger_0}$ for some ideal $\mathcal{I}_0$ of $\mathcal{W}^0$ of codimension 1 (this condition always holds when e is regular in $\mathfrak{g}_0$, because in this case $\mathcal{W}^0$ is commutative, and so all irreducible $\mathcal{W}^0$-modules are 1-dimensional). This means that $\dim N^0 = 1$.

Recall the notion of a *special* nilpotent orbit due to Lusztig. One of their characterizations is that an orbit $\mathbb{O}$ is special if and only if there is an integral weight $\lambda \in \mathfrak{h}^*$ such that $\overline{\mathbb{O}} = \mathrm{V}(\mathcal{U}/J(\lambda))$.

Consider the Langlands dual algebra $\mathfrak{g}^\vee$ with the Cartan subalgebra $\mathfrak{h}^\vee := \mathfrak{h}^*$ and the root system $\Delta^\vee$, the dual root system of $\mathfrak{g}$. Let $\Pi^\vee \subset \Delta^\vee$ be the simple root system corresponding to Π. Note that $\mathfrak{g} \cong \mathfrak{g}^\vee$ provided $\mathfrak{g}$ is exceptional. Spaltenstein constructed an order-reversing bijection $\mathbb{O} \mapsto \mathbb{O}^\vee$ between the sets of special nilpotent orbits.

Suppose e is a special element. Take $e^\vee \in \mathbb{O}^\vee$ and let $(e^\vee, h^\vee, f^\vee)$ be the corresponding $\mathfrak{sl}_2$-triple. We may assume that $h^\vee$ lies in $\mathfrak{h}^*$ and is dominant. This specifies $h^\vee$ uniquely. As Barbash and Vogan checked in [BV, Proposition 5.1], the element $e^\vee$ is even (i.e., all eigenvalues of $\mathrm{ad}\, h^\vee$ are even) provided e is rigid.

Let, as usual, ρ denote half the sum of all positive roots of $\mathfrak{g}$.

Proposition 5.3.1 ([BV], Proposition 5.10). *One has the equality* $\mathrm{V}(\mathcal{U}/J(h^\vee - \rho)) = \overline{\mathbb{O}}$.

The list of special nilpotent orbits in exceptional algebras and the description of the Spaltenstein duality are given in [C, Subsection 13.4]. In Table 5.2 we list the special *rigid* elements e together with their duals $e^\vee$. Here numbers in the first column are those from Table 5.1.

Table 5.2: Non-minimal special rigid elements in exceptional algebras

N	$\mathfrak{g}$	e	$e^\vee$
4	F_4	$\widetilde{A}_1$	$F_4(a_1)$
5	F_4	$A_1 + \widetilde{A}_1$	$F_4(a_2)$
12	E_7	$2A_1$	$E_7(a_2)$
15	E_7	$A_2 + 2A_1$	$E_7(a_4)$
19	E_8	$2A_1$	$E_8(a_2)$
22	E_8	$A_2 + A_1$	$E_8(a_4)$
23	E_8	$A_2 + 2A_1$	$E_8(b_4)$
29	E_8	$D_4(a_1) + A_1$	$E_8(a_6)$

It turns out that for 5 special elements (with exception of $(F_4, \widetilde{A}_1), (E_8, A_2{+}A_1), (E_8, D_4(a_1){+}A_1)$) the weight $\lambda = h^\vee - \rho$ satisfies conditions (A) and (C) (under a suitable choice of $\mathfrak{g}_0$). (B) follows from Proposition 5.3.1, while (D) holds because the five elements are principal in $\mathfrak{g}_0$. Hence in these 5 cases $J(h^\vee - \rho)$ has the form $\mathcal{I}^\dagger$ with $\dim \mathcal{W}/\mathcal{I} = 1$.

Now suppose $\mathbb{O}$ is not special. Pick some $\lambda \in \mathfrak{h}^*$. Let $\Delta(\lambda)^\vee$ be the subset of all coroots $\alpha^\vee \in \Delta^\vee$ such that $\langle \alpha^\vee, \lambda\rangle \in \mathbb{Z}$. Let $\Delta(\lambda)$ be the dual root system of $\Delta^\vee(\lambda)$. Note that for $\mathfrak{g} = E_6, E_7, E_8$ the root system $\Delta(\lambda)$ is a subsystem in Δ.

Let $\Pi^\vee(\lambda)$ be a unique system of simple roots in $\Delta^\vee(\lambda)$ such that the corresponding system of positive coroots in $\Delta^\vee(\lambda)$ consists of positive coroots for $\Delta^\vee$. Let $W(\lambda)$ denote the Weyl group of $\Delta(\lambda)$ (or of $\Delta^\vee(\lambda)$). Let $w \in W(\lambda)$ be a unique element of minimal length such that $w(\lambda+\rho)$ is antidominant, i.e. $\langle w(\lambda+\rho), \alpha^\vee\rangle \leqslant 0$ for all $\alpha^\vee \in \Pi^\vee(\lambda)$.

Recall that the group $W(\lambda)$ is decomposed into equivalence classes called *two-sided* cells and there is a bijection between the set of two-sided cells in $W(\lambda)$ and the set of special nilpotent orbits in the Lie algebra $\mathfrak{g}(\lambda)$ associated with $\mathfrak{h}, \Delta(\lambda)$. We will denote the nilpotent orbit corresponding to $w \in W(\lambda)$ by $\mathbb{O}_w$.

Proposition 5.3.2. *We have* $\dim \mathrm{V}(\mathcal{U}/J(\lambda)) = \dim \mathfrak{g} - \dim \mathfrak{g}(\lambda) + \dim \mathbb{O}_w$.

This result is well known to specialists. However, as far as we know, the proof was never written explicitly, so it is written down below. I wish to thank A. Joseph and D. Vogan for explaining the details.

Proof. Using an appropriate translation functor, see, for example, [Ja, 4.12,4.13], we may assume that λ is regular. Let w_0 denote the longest element in $W(\lambda)$. Then $\lambda = w_0 w \cdot \mu$, where μ is dominant. Here for $x \in W(\lambda)$ we set $x \cdot \mu := x(\mu + \rho(\lambda)) - \rho(\lambda)$, where $\rho(\lambda)$ is half of the sum of all positive roots in $\Delta(\lambda)$.

For $x, y \in W(\lambda)$ let $a(x,y)$ denote the coefficient of the character of the Verma module $\mathcal{M}(y\cdot\mu)$ in the character of the irreducible module $L(x\cdot\mu)$. Set $n = \frac{1}{2}(\#\Delta - \dim \mathrm{V}(\mathcal{U}/J(\lambda)))$. Joseph proved in [Jo2] that n coincides with the smallest non-negative integer m such that

$$\sum_{y\in W(\lambda)} a(x,y) y^{-1}\rho(\lambda)^m \neq 0.$$

But the Kazhdan-Lusztig conjecture implies that $a(x,y)$ coincides with the coefficient of the character of the Verma module $\mathcal{M}_{(\lambda)}(y\cdot\mu)$ in the character of the irreducible module $L_{(\lambda)}(x\cdot\mu)$ for the algebra $\mathfrak{g}(\lambda)$. It follows that $n = \#\Delta(\lambda) - \dim \mathbb{O}_w$. Since $\#\Delta - \#\Delta(\lambda) = \frac{1}{2}(\dim\mathfrak{g} - \dim\mathfrak{g}(\lambda))$, we are done. $\square$

The easiest case here is when w is the longest element in $W(\lambda)$. Then $\mathbb{O}_w$ is just the zero orbit. So we have the following corollary originally suggested to us by A. Premet.

Corollary 5.3.3 ([Jo1], Corollary 3.5)**.** *If* $\langle\lambda+\rho, \alpha^\vee\rangle > 0$ *for all* $\alpha^\vee \in \Pi(\lambda)^\vee$*, then* $\dim \mathrm{V}(\mathcal{U}/J(\lambda)) = \dim\mathfrak{g} - \dim\mathfrak{g}(\lambda)$.

To finish the section let us provide an example of computation.

As we have seen, it is more convenient to work with the element $\lambda+\rho$ rather than with λ. Also from the point of view of computations it is better to replace δ with the element

$$\delta' := \frac{1}{2}\sum_{\alpha\in\Delta^+, \langle\alpha,h\rangle=0,1} \alpha.$$

We claim that $\delta' - \delta - \rho \in \mathrm{Span}_{\mathbb{Q}}\Pi_0$. Indeed, since $e,h,f \in \mathfrak{g}_0$, we see that the $\mathfrak{sl}_2$-triple (e,h,f) preserves all weight subspaces of $\mathfrak{t}$. In particular, by the representation theory of $\mathfrak{sl}_2$, for any k, l we have

$$\sum_{\alpha,\langle\alpha,\theta\rangle=k,\langle\alpha,h\rangle=l} \alpha|_{\mathfrak{t}} = \sum_{\alpha,\langle\alpha,\theta\rangle=k,\langle\alpha,h\rangle=-l} \alpha|_{\mathfrak{t}}$$

or equivalently,

$$\sum_{\alpha,\langle\alpha,\theta\rangle=k,\langle\alpha,h\rangle=l}\alpha\equiv\sum_{\alpha,\langle\alpha,\theta\rangle=k,\langle\alpha,h\rangle=-l}\alpha \mod \operatorname{Span}_{\mathbb{Q}}(\Pi_0). \tag{5.2}$$

Applying (5.2), we get

$$\delta\equiv\frac{1}{2}\sum_{\langle\alpha,\theta\rangle<0,\langle\alpha,h\rangle\neq 0,1}\alpha \mod \operatorname{Span}_{\mathbb{Q}}(\Pi_0).$$

It follows that

$$\delta+\rho\equiv-\frac{1}{2}\sum_{\langle\alpha,\theta\rangle<0,\langle\alpha,h\rangle=0,1}\alpha\equiv\frac{1}{2}\sum_{\langle\alpha,\theta\rangle>0,\langle\alpha,h\rangle=0,-1}\alpha \mod \operatorname{Span}_{\mathbb{Q}}(\Pi_0).$$

Applying (5.2) once more, we have

$$\sum_{\langle\alpha,\theta\rangle>0,\langle\alpha,h\rangle=-1}\alpha\equiv\sum_{\langle\alpha,\theta\rangle>0,\langle\alpha,h\rangle=1}\alpha \mod \operatorname{Span}_{\mathbb{Q}}(\Pi_0).$$

It follows that $\delta'\equiv\delta+\rho \mod \operatorname{Span}_{\mathbb{Q}}(\Pi_0)$.

We will check the existence of $\lambda\in\mathfrak{h}^*$ with properties (A)-(D) for the non-special orbit A_5+A_1 in E_8. We use the notation for roots from [OV]. Namely, fix an orthonormal basis $\varepsilon_i, i=1,\ldots,9$, in the Euclidean space $\mathbb{R}^9$. Then we represent the real form $\mathfrak{h}(\mathbb{R})^*$ of $\mathfrak{h}^*$ as the quotient of $\mathbb{R}^9$ by the diagonal. The image of ε_i in $\mathfrak{h}(\mathbb{R})^*$ is again denoted by ε_i. Now simple roots are $\alpha_i:=\varepsilon_i-\varepsilon_{i+1}, i=1,\ldots,7, \alpha_8=\varepsilon_6+\varepsilon_7+\varepsilon_8$. The element ρ equals $\sum_{i=1}^7(8-i)\varepsilon_i-22\varepsilon_9$. Let $\pi_i, i=1,\ldots,8$, denote the fundamental weights. They are given by $\pi_i=\sum_{i=1}^7\varepsilon_i-\min(i,15-2i)\varepsilon_9, i=1,2,\ldots,7, \pi_8=3\varepsilon_9$. We identify $\mathfrak{h}(\mathbb{R})^*$ with $\mathfrak{h}(\mathbb{R})$ by means of the scalar product on $\mathfrak{h}(\mathbb{R})^*$ (induced from $\mathbb{R}^9$).

First of all, we choose $\mathfrak{g}_0$ such that $\Pi_0=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_7\}$. Then for h we can take $5\varepsilon_1+3\varepsilon_2+\varepsilon_3-\varepsilon_4-3\varepsilon_5-5\varepsilon_6+\varepsilon_7-\varepsilon_8$. We get

$$\delta'=\frac{3}{2}\varepsilon_1+\varepsilon_2+2\varepsilon_3+\varepsilon_4+\frac{3}{2}\varepsilon_5+\frac{1}{2}\varepsilon_6+\varepsilon_7-4\varepsilon_9.$$

Set

$$\lambda':=\varepsilon_1+\frac{7}{6}\varepsilon_2+\frac{1}{3}\varepsilon_3+\frac{1}{2}\varepsilon_4+\frac{2}{3}\varepsilon_5+\frac{5}{6}\varepsilon_6+\frac{1}{6}\varepsilon_7-\frac{1}{6}\varepsilon_8-\frac{9}{2}\varepsilon_9, \lambda:=\lambda'-\rho.$$

One can check directly that $\lambda'-\delta'\in\mathbb{Q}\Pi_0$, which is equivalent to (C). The restriction of λ to $\mathfrak{g}_0$ is antidominant, so (A) holds. The system $\Delta(\lambda)$ is isomorphic to $A_5+A_2+A_1$ and the set of simple roots in $\Delta(\lambda)$ is $\varepsilon_7+\varepsilon_4+\varepsilon_3, \varepsilon_2-\varepsilon_7, \varepsilon_8+\varepsilon_7+\varepsilon_1, \varepsilon_6-\varepsilon_8, \varepsilon_8+\varepsilon_5+\varepsilon_4, \varepsilon_8+\varepsilon_6+\varepsilon_3, \varepsilon_7+\varepsilon_5+\varepsilon_2, \varepsilon_1+\varepsilon_3+\varepsilon_5$. The element λ' is positive on all these roots. By Corollary 5.3.3, $\dim \operatorname{V}(\mathcal{U}/J(\lambda))=248-46=202=\dim\mathbb{O}$, which is (B). So we have checked (A),(B),(C). The condition (D) is vacuous because e is of principal Levi type.

6. Parabolic induction for W-algebras

6.1. **Lusztig-Spaltenstein induction.** Let $\underline{G}$ be a Levi subgroup of G, $\underline{\mathfrak{g}}$ its Levi subalgebra, and $\underline{e}\in\underline{\mathfrak{g}}$ be a nilpotent element. Let $\underline{\mathbb{O}}\subset\underline{\mathfrak{g}}\cong\underline{\mathfrak{g}}^*$ be the orbit of $\underline{e}$.

Definition 6.1.1 ([LS], Introduction, Theorem 2.2)**.** Let P be a parabolic subgroup of G with Levi subgroup $\underline{G}$ and let $\mathfrak{p}$ be the Lie algebra of P. There is a unique nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$ such that $(\underline{\mathbb{O}}+\operatorname{R}_u(\mathfrak{p}))\cap\mathbb{O}$ is dense in $\underline{\mathbb{O}}+\operatorname{R}_u(\mathfrak{p})$. The orbit $\mathbb{O}$ does not depend on the choice of P and is said to be induced from $\underline{\mathbb{O}}$ and is denoted by $\operatorname{Ind}_{\underline{\mathfrak{g}}}^{\mathfrak{g}}(\underline{\mathbb{O}})$.

We remark that, although Lusztig and Spaltenstein considered unipotent orbits in $G, \underline{G}$, their results are transferred to the case of nilpotent orbits in a straightforward way, see [McG] for an exposition.

Here are some properties of $\mathrm{Ind}_{\underline{\mathfrak{g}}}^{\mathfrak{g}}(\underline{\mathbb{O}})$ due to Lusztig and Spaltenstein.

Proposition 6.1.2 ([LS], Theorem 1.3)**.** *Let $e \in (\underline{\mathbb{O}} + \mathrm{R}_u(\mathfrak{p})) \cap \mathrm{Ind}_{\underline{\mathfrak{g}}}^{\mathfrak{g}}(\underline{\mathbb{O}})$. Then the following assertions hold:*

1. $\dim Z_G(e) = \dim Z_{\underline{G}}(\underline{e})$.
2. $Ge \cap (\underline{\mathbb{O}} + \mathrm{R}_u(\mathfrak{p})) = Pe$.
3. $Z_G(e)^\circ \subset P$.
4. *If $g \in G$ is such that $\mathrm{Ad}(g)e \in \underline{\mathbb{O}} + \mathrm{R}_u(\mathfrak{p})$, then there is $z \in Z_G(e)$ such that $z^{-1}g \in P$.*

In the sequel we will also need the following lemma.

Lemma 6.1.3. *Let $h_0 \in \mathfrak{g}$ be such that $[h_0, e] = 2e$. Then $h_0 \in \mathfrak{p}$.*

Proof. Since $\mathfrak{z}_{\mathfrak{g}}(e) \subset \mathfrak{p}$, we may assume that $h_0 = h$ is the semisimple element of some $\mathfrak{sl}_2$-triple (e, h, f). Let $\gamma : \mathbb{K}^\times \to G$ be the one-parameter subgroup corresponding to h, i.e. such that $\frac{d}{dt}|_{t=0}\gamma = h$. By Proposition 6.1.2, (4), for any $t \in \mathbb{K}^\times$ there is $z_t \in Z_G(e)$ such that $z_t^{-1}\gamma(t) \in P$. Since $Z_G(e)^\circ \subset P$, we see that $\{\gamma(t)P, t \in \mathbb{K}^\times\}$ is a finite subset of G/P. Therefore $\gamma(t) \in P$ for all $t \in \mathbb{K}^\times$, equivalently, $h \in \mathfrak{p}$. □

6.2. **Construction on the classical level.** Let $\underline{G}, \underline{e}, P, \mathbb{O}$ be such as in the previous subsection. Set $N := \mathrm{R}_u(P)$. Choose $e \in \mathbb{O} \cap (\underline{\mathbb{O}} + \mathfrak{n})$. Let (e, h, f) be an $\mathfrak{sl}_2$-triple. Also recall the group $Q := Z_G(e, h, f)$. By Proposition 6.1.2 and Lemma 6.1.3, we have $h \in \mathfrak{p}, \mathfrak{q} \subset \mathfrak{p}$. The subalgebra $\mathbb{K}h \oplus \mathfrak{q} \subset \mathfrak{p}$ is reductive, so replacing $\underline{G}$ by a P-conjugate subgroup, we may assume $h \in \underline{\mathfrak{g}}, \mathfrak{q} \subset \underline{\mathfrak{g}}$. Hence $Q^\circ \subset \underline{G}$.

Our goal in this subsection is to study the Hamiltonian reductions of some open subsets in X, T^*G.

Let P^- denote the opposite parabolic to P containing $\underline{G}$ and $N^- := \mathrm{R}_u(P^-)$. Set $Z = T^*(PN^-) = PN^- \times \mathfrak{g}^* \subset G \times \mathfrak{g}^* = T^*G$. This is an affine P and $\mathbb{K}^\times$-stable open subvariety in T^*G containing $(1, \chi)$. Note that $Z = P \times N^- \times \mathfrak{g}^*$. Set $\widetilde{Y} := Z///N, Y := (Z \cap X)///N = \widetilde{Y} \cap (X/N)$.

Since $\underline{G}$ normalizes N, we have Hamiltonian actions of $\underline{G}$ on $Y, \widetilde{Y}$. The Kazhdan actions of $\mathbb{K}^\times$ on X, T^*G descend to $\mathbb{K}^\times$-actions on $Y, \widetilde{Y}$, because Z is $\mathbb{K}^\times$-stable and $\mu_G(t.x) = t^2\mu_G(x)$ for all $x \in X$. The reduction $\widetilde{Y}$ can be naturally identified with the Hamiltonian $\underline{G}$-variety T^*P^-. The latter, in its turn, is naturally isomorphic to $T^*\underline{G} \times T^*N^-$, where $\underline{G}$ acts only on the first factor (however, $\mathbb{K}^\times$ acts non-trivially on both factors). Set $\underline{Q} := Q \cap \underline{G}$. Clearly, $Z \subset T^*G$ is stable with respect to the right $\underline{Q}$-action.

Let y denote the image of $(1, \chi)$ in $Y \subset \widetilde{Y}$ (the inclusion $(1, \chi) \in \mu^{-1}(\mathfrak{p})$ holds because χ vanishes on $\mathfrak{n}$). Note that the orbit $\underline{G}y$ is stable under the $\mathbb{K}^\times$-action because $\gamma(t) \in \underline{G}$ for any $t \in \mathbb{K}^\times$.

According to Proposition 3.2.2, there is a $G \times Q \times \mathbb{K}^\times$-equivariant Hamiltonian isomorphism $\varphi : (T^*G)^\wedge_{Gx} \to (X \times V^*)^\wedge_{Gx}$, which is identical on Gx. Completing further, we get a $P \times \underline{Q} \times \mathbb{K}^\times$-equivariant symplectomorphism $(T^*G)^\wedge_{Px} \to (X \times V^*)^\wedge_{Px}$ and, since $Px \subset Z$, also a symplectomorphism $Z^\wedge_{Px} \to [(X \cap Z) \times V^*]^\wedge_{Px}$, which we also denote by φ. Taking the reduction, we get a $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$-equivariant Hamiltonian isomorphism $\underline{\varphi} : \widetilde{Y}^\wedge_{\underline{G}y} \to (Y \times V^*)^\wedge_{\underline{G}y} = Y^\wedge_{\underline{G}y} \times (V^*)^\wedge_0$.

Pick an $\mathfrak{sl}_2$-triple $(\underline{e}, \underline{h}, \underline{f})$ in $\underline{\mathfrak{g}}^Q$ such that $[h, \underline{h}] = 0, [h, \underline{f}] = -2\underline{f}$ (if $\underline{e} = 0$, we set $\underline{h} = \underline{f} = 0$).

Now let $\underline{X}$ denote the equivariant Slodowy slice constructed for $\underline{G}$ and $\underline{e}$. In other words, as a $\underline{G}$-variety $\underline{X}$ is nothing else but $\underline{G} \times \underline{S}$, where $\underline{S}$ is the Slodowy slice for $\underline{e}$ in $\underline{\mathfrak{g}}$. Since $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \dim \mathfrak{z}_{\underline{\mathfrak{g}}}(\underline{e})$, we see that $\dim \underline{S} = \dim S$. Let $\underline{\chi} \in \underline{\mathfrak{g}}^*$ be such that $\langle \underline{\chi}, \xi \rangle = (\underline{e}, \xi)$ for $\xi \in \underline{\mathfrak{g}}$. The group $\underline{Q}$ acts on $\underline{X}$ and its action commutes with the actions of $\underline{G}$ and $\mathbb{K}^\times$.

Set $\underline{x} = (1, \underline{\chi})$. We have the projection $\widetilde{Y} = T^*P^- = T^*\underline{G} \times T^*N^- \twoheadrightarrow T^*\underline{G}$ mapping y to $\underline{x}$. So it gives rise to the projection $\widetilde{Y}^\wedge_{\underline{G}y} \twoheadrightarrow (T^*\underline{G})^\wedge_{\underline{G}\underline{x}}$.

Lemma 6.2.1. *There is a $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$-equivariant Hamiltonian isomorphism $\underline{X}^\wedge_{\underline{G}\underline{x}} \to Y^\wedge_{\underline{G}y}$.*

Proof. According to Proposition 3.2.2, we need to check that

1. The stabilizers of $\underline{x}, y$ in $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$ coincide.
2. $\mu_{\underline{G}}(\underline{x}) = \mu_{\underline{G}}(y), \mu_{\underline{Q}}(\underline{x}) = \mu_{\underline{Q}}(y)$, where $\mu_{\underline{G}}, \mu_{\underline{Q}}$ are the moment maps for the $\underline{G}$- and $\underline{Q}$-actions.
3. The normal spaces to $\underline{G}\underline{x}$ in $\underline{X}$ and to $\underline{G}y$ in Y are isomorphic as $(\underline{G} \times \underline{Q} \times \mathbb{K}^\times)_{\underline{x}}$-modules.

The stabilizer of x in $G \times \underline{Q} \times \mathbb{K}^\times$ is $\{(q\gamma(t), q, t), q \in \underline{Q}, t \in \mathbb{K}^\times\}$. Since the latter is contained in $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$, we see that it coincides with the stabilizer of y in $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$. Also, this subgroup coincides with the stabilizer of $\underline{x}$. Hence (1). Moreover, it is easy to see from the definition of the moment map for Y that $\mu_{\underline{G}}(y) = \mu_{\underline{G}}(\underline{x}) = \underline{e}, \mu_{\underline{Q}}(y) = \mu_{\underline{Q}}(\underline{x}) = 0$. Hence (2). Since the orbit $\underline{G}\underline{x}$ is a coisotropic subvariety in $\underline{X}$, (3) reduces to checking that $\dim Y = \dim \underline{X}$.

From the definition of Y, we have $\dim Y = \dim X - 2\dim N$. But $\dim X = \dim G + \dim S = \dim \underline{G} + 2\dim N + \dim S$ and hence $\dim Y = \dim S + \dim \underline{G}$. On the other hand, $\dim \underline{X} = \dim \underline{S} + \dim \underline{G}$. As we have seen above, $\dim S = \dim \underline{S}$. Hence $\dim Y = \dim S + \dim \underline{G} = \dim \underline{X}$. □

Now we will discuss some constructions to be used in the proof of Theorem 6.4.1. Set $\underline{V} := [\underline{\mathfrak{g}}, \underline{f}]$. Consider the action of $\mathbb{K}^\times$ on $\underline{V}$ given by $t \mapsto \gamma(t)^{-1}$. Also $\underline{V}$ has a natural $\underline{Q}$-action. Similarly to φ, we have a $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$-equivariant Hamiltonian morphism $\varphi_0 : (T^*\underline{G})^\wedge_{\underline{G}\underline{x}} \to \underline{X}^\wedge_{\underline{G}\underline{x}} \times (\underline{V}^*)^\wedge_0$.

Lemma 6.2.2. *There is a $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$-equivariant Hamiltonian isomorphism*

$$\underline{\varphi}' : \widetilde{Y}^\wedge_{\underline{G}y} \to Y^\wedge_{\underline{G}y} \times (V^*)^\wedge_0$$

making the following diagram commutative.

$$\begin{array}{ccccc}
Y^\wedge_{\underline{G}y} \times (V^*)^\wedge_0 & & \xleftarrow{\quad \varphi' \quad} & & \widetilde{Y}^\wedge_{\underline{G}y} \\
\downarrow & & & & \downarrow \\
 & & \underline{X}^\wedge_{\underline{G}\underline{x}} \times (\underline{V}^*)^\wedge_0 & \xleftarrow{\varphi_0} & (T^*\underline{G})^\wedge_{\underline{G}\underline{x}} \\
\downarrow & & \downarrow & & \\
Y^\wedge_{\underline{G}y} & \longleftarrow & \underline{X}^\wedge_{\underline{G}\underline{x}} & &
\end{array}$$

Here all vertical arrows are projections and the arrow $\underline{X}^\wedge_{\underline{G}x} \to Y^\wedge_{\underline{G}y}$ *is an isomorphism constructed in Lemma 6.2.1.*

Proof. Set $\widetilde{V} := T_y(\underline{G}y)^\angle/[T_y(\underline{G}y)^\angle \cap T_y(\underline{G}y)]$. This is a symplectic vector space acted on by $(\underline{G}\times\underline{Q}\times\mathbb{K}^\times)_y$. Recall that the stabilizer $(\underline{G}\times\underline{Q}\times\mathbb{K}^\times)_y$ consists of all elements of the form $(q\gamma(t), q, t), q\in\underline{Q}, t\in\mathbb{K}^\times$, so we can identify it with $\underline{Q}\times\mathbb{K}^\times$. The action of $\underline{Q}\times\mathbb{K}^\times$ on $\widetilde{Y} = T^*(P^-)$ is given by

$$q.(p,\alpha) := (qpq^{-1}, q\alpha),\ t.(p,\alpha) := (\gamma(t)p\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha),$$
$$q\in\underline{Q}, t\in\mathbb{K}^\times, p\in P^-, \alpha\in\mathfrak{p}^{-*}.$$

So we see that as a $\underline{Q}\times\mathbb{K}^\times$-module, $\widetilde{V}$ is isomorphic to $\underline{V}\oplus\mathfrak{n}^-\oplus\mathfrak{n}^{-*}$, where the action of $\underline{Q}\times\mathbb{K}^\times$ on $\underline{V}$ was described previously, the action on $\mathfrak{n}^-$ factors through the adjoint action of $\underline{G}$ via the homomorphism $\underline{Q}\times\mathbb{K}^\times\to\underline{G}, (q,t)\mapsto q\gamma(t)$, and, finally, the action on $\mathfrak{n}^{-*}\cong\mathfrak{n}$ is given by $(q,t)\mapsto t^{-2}\operatorname{Ad}(q\gamma(t))$.

Proposition 3.2.2 implies that

$$\widetilde{Y}^\wedge_{\underline{G}y} \cong \underline{X}^\wedge_{\underline{G}x}\times(\widetilde{V}^*)^\wedge_0.$$

So to complete the proof we need to check that V is isomorphic to $\widetilde{V}$ as a $\underline{Q}\times\mathbb{K}^\times$-module (and then an isomorphism can be chosen to be symplectic). But we have already constructed a $\underline{G}\times\underline{Q}\times\mathbb{K}^\times$-equivariant isomorphism $\underline{\varphi}: \widetilde{Y}^\wedge_{\underline{G}y}\to Y^\wedge_{\underline{G}y}\times(V^*)^\wedge_0$ mapping y to y. The existence of $\underline{\varphi}$ implies $V\cong\widetilde{V}$. □

Unfortunately, we do not know whether $\underline{\varphi}=\underline{\varphi}'$. So let us consider $\lambda := \underline{\varphi}'\circ\underline{\varphi}^{-1}$. This is an $\underline{G}\times\underline{Q}\times\mathbb{K}^\times$-equivariant Hamiltonian automorphism of $Y^\wedge_{\underline{G}y}\times(V^*)^\wedge_0$, which is trivial on $\underline{G}y$.

We are going to describe the structure of λ, more precisely, to prove that λ is, in a sense, an inner automorphism.

Lemma 6.2.3. *The isomorphism λ can be written as a composition $A\exp(v)$, where $A\in\operatorname{Sp}(V^*)$ (when we consider A as an automorphism of $Y^\wedge_{\underline{G}y}\times(V^*)^\wedge_0$ we mean that A acts trivially on the first factor) and v is the Hamiltonian vector field v_f of a $\underline{G}\times\underline{Q}$-invariant element $f\in\mathbb{K}[Y\times V^*]^\wedge_{\underline{G}y}$ of degree 2 with respect to the $\mathbb{K}^\times$-action.*

We will see in the proof that v will be constructed in such a way that $\exp(v)$ converges.

Proof. Since λ is the identity on $\underline{G}y$, the subspace $(T_y\underline{G}y)^\angle$ is stable w.r.t $d_y\lambda$. Let A be the operator induced by $d_y\lambda$ on $V^* = (T_y\underline{G}y)^\angle/([T_y\underline{G}y)^\angle\cap T_y\underline{G}y]$. Set $\lambda_0 := A^{-1}\circ\lambda$. Note that $d_y\lambda_0$ is a unipotent operator on $T_y(Y\times V^*)$. Therefore λ_0^* induces a unipotent operator on all quotients $\mathbb{K}[Y\times V^*]/I^k$, where I stands for the ideal of $\underline{G}y$. This means that the operator $\ln\lambda_0^*:\mathbb{K}[Y\times V^*]^\wedge_{\underline{G}y}\to\mathbb{K}[Y\times V^*]^\wedge_{\underline{G}y}$ is well-defined. This operator is a derivation, and the corresponding vector field v is symplectic and annihilates all hamiltonians $H_\xi, \xi\in\underline{\mathfrak{g}}$. It remains to apply Lemma 3.2.3 to conclude that v has the required form. □

6.3. **Proof of the main theorem.** Consider the formal scheme $X^\wedge_{Px}$. Its algebra of functions is equipped with the differential star-product induced from $\mathbb{K}[X]$. Since $Y^\wedge_{\underline{G}y}$ is naturally identified with $X^\wedge_{Px}///N$, we can apply the construction of Subsection 3.4 to get a star-product on $\mathbb{K}[Y]^\wedge_{\underline{G}y}$. Note that we have a natural map

$$\mathcal{W}^\wedge_\hbar = \mathbb{K}[X]^\wedge_{Gx}[[\hbar]]^G \hookrightarrow \mathbb{K}[X]^\wedge_{Px}[[\hbar]]^P \to \mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}. \tag{6.1}$$

The following proposition is a quantum analog of Lemma 6.2.1.

Proposition 6.3.1. *There is a $\underline{G}\times\underline{Q}\times\mathbb{K}^\times$-equivariant Hamiltonian isomorphism of quantum algebras*

$$\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]\to\mathbb{K}[\underline{X}]^\wedge_{\underline{G}x}[[\hbar]].$$

Proof. By Lemma 6.2.1, we have a $\underline{G}\times\underline{Q}\times\mathbb{K}^\times$-equivariant Hamiltonian isomorphism $Y^\wedge_{\underline{G}y}\to\underline{X}^\wedge_{\underline{G}x}$. It follows from Corollary 3.3.6 that the corresponding isomorphism $\mathbb{K}[Y]^\wedge_{\underline{G}y}\to\mathbb{K}[\underline{X}]^\wedge_{\underline{G}x}$ can be lifted to an $\underline{G}\times\underline{Q}\times\mathbb{K}^\times$-equivariant Hamiltonian isomorphism $\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]\to\mathbb{K}[\underline{X}]^\wedge_{\underline{G}x}[[\hbar]]$. □

Set $h_0:=h-\underline{h}$, then $h_0\in\mathfrak{z}_\mathfrak{g}(\underline{e},\underline{h},\underline{f})$. There is an embedding $\mathfrak{z}_\mathfrak{g}(\underline{e},\underline{h},\underline{f})\hookrightarrow\underline{\mathcal{W}}$ coming from the quantum comoment map. So consider h_0 as an element in $\underline{\mathcal{W}}$. Form the completion $\underline{\mathcal{W}}'$ of $\underline{\mathcal{W}}$ consisting of all infinite sums $\sum_{i\leqslant k}f_i$, where $[h_0,f_i]=if_i$. The algebra $\underline{\mathcal{W}}'$ has a topology, the sets $O_k:=\{\sum_{i\leqslant k}f_i\}$ are fundamental neighborhoods of 0. Clearly, $\underline{\mathcal{W}}'$ is complete and separated with respect to this topology.

Theorem 6.3.2. *There is an embedding $\Xi:\mathcal{W}\hookrightarrow\underline{\mathcal{W}}'$.*

Proof. The isomorphism $\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]\to\mathbb{K}[\underline{X}]^\wedge_{\underline{G}x}[[\hbar]]$ restricts to a $\mathbb{K}^\times$-equivariant isomorphism $\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}\to\mathbb{K}[\underline{X}]^\wedge_{\underline{G}x}[[\hbar]]^{\underline{G}}$. The latter isomorphism intertwines the embeddings of $\mathfrak{q}$. Let us note that $\mathbb{K}[\underline{X}]^\wedge_{\underline{G}x}[[\hbar]]^{\underline{G}}$ is nothing else but $\mathbb{K}[\underline{S}]^\wedge_{\underline{\chi}}[[\hbar]]=\underline{\mathcal{W}}_\hbar^\wedge$. On the other hand, from the construction of the star-product on $\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]$ we have a $\mathbb{K}^\times\times\underline{Q}$-equivariant embedding $\Xi_\hbar:\mathcal{W}_\hbar=\mathbb{K}[X][\hbar]^G\hookrightarrow\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}$. Therefore we get an embedding $\Xi_\hbar:\mathcal{W}_\hbar\to(\underline{\mathcal{W}}_\hbar^\wedge)_{\mathbb{K}^\times-l.f}$, where the subscript "$\mathbb{K}^\times-l.f$" means the subalgebra of locally finite vectors, and also homomorphisms

$$\Xi_1:\mathcal{W}=\mathcal{W}_\hbar/(\hbar-1)\to\underline{\mathcal{W}}^\heartsuit:=(\underline{\mathcal{W}}_\hbar^\wedge)_{\mathbb{K}^\times-l.f}/(\hbar-1).\tag{6.2}$$

$$\Xi_0:\mathbb{K}[S]=\mathcal{W}_\hbar/(\hbar)\to(\underline{\mathcal{W}}_\hbar^\wedge)_{\mathbb{K}^\times-l.f}/(\hbar)=(\mathbb{K}[\underline{S}]^\wedge_{\underline{\chi}})_{\mathbb{K}^\times-l.f}.\tag{6.3}$$

Let us note that both algebras in (6.2) are filtered and Ξ_1 preserves the filtrations. The algebras and the homomorphism in (6.3) are associated graded of those in (6.2). Also let us note that the homomorphism in (6.3) is injective. To show this it is enough to check that the morphism $Y/\underline{G}\to X/G$ is dominant, that is, that any G-orbit in X intersects $\mu_G^{-1}(\mathfrak{p})$. The latter is clear, because any adjoint orbit intersects $\mathfrak{p}$.

So the homomorphism in (6.2) is also injective. It remains to embed the algebra $\underline{\mathcal{W}}^\heartsuit$ into $\underline{\mathcal{W}}'$. The algebra $\mathbb{K}[\underline{S}]$ is the symmetric algebra $A:=S(\mathfrak{v})$, where $\mathfrak{v}$ is $\underline{S}$ considered as a vector space (with $\underline{\chi}$ as an origin). The space $\mathfrak{v}$ is equipped with two $\mathbb{K}^\times$-actions. The first one is the Kazhdan action: $t.\underline{s}=t^{-2}\underline{\gamma}(t)\underline{s}$, where $\underline{\gamma}$ is the one-parameter subgroup of $\underline{G}$ corresponding to $\underline{h}$. The second action is given by the one-parameter subgroup $\gamma_0(t)=\gamma(t)\underline{\gamma}(t)^{-1}$, whose differential at $t=0$ is h_0. So we have a bi-grading $A=\bigoplus_{i,j\in\mathbb{Z}}A(i,j)$, where $A(i,j)=\{a\in A|t.a=t^ia,\gamma_0(t).a=t^ja\}$. It follows that $A(0,0)=\mathbb{K}$, $A(i,j)=0$ for $i<0$ and all j, and $A(0,k)=0$ provided $k\neq0$. Analogously to [Lo2, Subsection 3.2], $\underline{\mathcal{W}}^\heartsuit$ consists of all infinite sums $a=\sum_{i,j\in\mathbb{Z}}a_{ij}$, $a_{ij}\in A(i,j)$, such that there is $n\in\mathbb{Z}$ (depending on a) with $a_{ij}=0$ provided $i+j\geqslant n$. On the other hand, $\underline{\mathcal{W}}'$ consists of all sums $\sum_{i,j\in\mathbb{Z}}a_{ij}$, such that

(1) there is n with $a_{ij}=0$ for $j>n$,
(2) and for any j only finitely many elements a_{ij} are nonzero.

The products on both algebras can be written as

$$(\sum_{i,j} a_{ij})(\sum_{i',j'} a'_{i'j'}) = \sum_{i,j,i',j'} a_{ij} *_1 a'_{i'j'}. \tag{6.4}$$

Here for $f, g \in \mathbb{K}[\underline{S}]$ we write $f *_1 g = \sum_{i=0}^{\infty} D_i(f,g)$, where $f * g = \sum_{i=0}^{\infty} D_i(f,g)\hbar^{2i}$ is the star-product on $\mathbb{K}[\underline{S}][\hbar]$.

Since $A(i,j) = 0$ for $i < 0$ and $\dim \bigoplus_j A(i,j) < \infty$ for any i, we see that $\underline{\mathcal{W}}^{\heartsuit} \subset \underline{\mathcal{W}}'$. By (6.4), this inclusion is a homomorphism of algebras. $\square$

Proof of Theorem 1.2.1. Any finite dimensional representation of $\underline{\mathcal{W}}$ has only finitely many eigenvalues of h_0. So it uniquely extends to a continuous (with respect to the discrete topology) representation of $\underline{\mathcal{W}}'$. Now the functor ρ we need is just the pull-back from $\underline{\mathcal{W}}'$ to $\mathcal{W}$. $\square$

We would like to deduce a corollary from the proof of Theorem 6.3.2 concerning the behavior of the centers. Let $\mathcal{Z}, \underline{\mathcal{Z}}$ denote the centers of $\mathcal{U}, \underline{\mathcal{U}}$, respectively. We have identifications of $\mathcal{Z}$ (resp., $\underline{\mathcal{Z}}$) with the center of $\mathcal{W}$ (resp., $\underline{\mathcal{W}}$).

Note that the choice of the parabolic subgroup P with Levi subgroup $\underline{G}$ gives rise to an embedding $\iota : \mathcal{Z} \to \underline{\mathcal{Z}}$. More precisely, ι is the restriction of the map $\mathcal{U} \mapsto \underline{\mathcal{U}}$ defined analogously to the map from the proof of Lemma 5.1.3.

Corollary 6.3.3. *The image of $\mathcal{Z} \subset \mathcal{W}$ under Ξ lies in $\underline{\mathcal{Z}} \subset \underline{\mathcal{W}} \hookrightarrow \underline{\mathcal{W}}'$. The corresponding homomorphism $\mathcal{Z} \to \underline{\mathcal{Z}}$ coincides with the one described in the previous paragraph.*

Proof. The action of G on X is free, so the homomorphism $\mathcal{U}_\hbar \hookrightarrow \widetilde{\mathcal{W}}_\hbar$ induced by the quantum comoment map is an embedding. Set $\mathcal{Z}_\hbar := \mathcal{U}_\hbar^G \subset \widetilde{\mathcal{W}}_\hbar^G = \mathcal{W}_\hbar$. It is clear that $\mathcal{Z}_\hbar/(\hbar - 1) = \mathcal{Z}$.

The homomorphism $\mathcal{W}_\hbar \to \mathbb{K}[Y]^{\wedge}_{\underline{G}y}[[\hbar]]$ factors through $\mathcal{W}_\hbar \to (\widetilde{\mathcal{W}}_\hbar/\widetilde{\mathcal{W}}_\hbar\mathfrak{n})^{\operatorname{ad}\mathfrak{n}} \to \mathbb{K}[Y]^{\wedge}_{\underline{G}y}[[\hbar]]$. Therefore the homomorphism $\mathcal{Z}_\hbar = \mathcal{U}_\hbar^G \hookrightarrow \mathcal{W}_\hbar \to \mathbb{K}[Y]^{\wedge}_{\underline{G}y}[[\hbar]]$ factors through $(\mathcal{U}_\hbar/\mathcal{U}_\hbar\mathfrak{n})^{\mathfrak{n}}$. For any element $u \in \mathcal{Z}_\hbar$ there are unique $\underline{u} \in \underline{\mathcal{Z}}_\hbar := \underline{\mathcal{U}}_\hbar^{\underline{G}}$ and $u_+ \in \mathcal{U}_\hbar\mathfrak{n}$ such that $u = \underline{u} + u_+$. The image of u in $\mathbb{K}[Y]^{\wedge}_{\underline{G}y}[[\hbar]]$ coincides with the image of $\underline{u}$ under the quantum comoment map. This image lies in the center of $\mathbb{K}[Y]^{\wedge}_{\underline{G}y}[[\hbar]]^{\underline{G}}$. Now the claim of the corollary follows from the construction of Ξ. $\square$

Remark 6.3.4. In some special cases we can embed $\mathcal{W}$ into $\underline{\mathcal{W}}$ itself not just into a completion of $\underline{\mathcal{W}}$. Namely, suppose that the element h is even. In this case we can take $\underline{\mathfrak{g}} = \mathfrak{z}_{\mathfrak{g}}(h)$ and $\underline{e} = 0$. Then $\underline{\mathcal{W}} = \underline{\mathcal{U}}$, and $h_0 = h$. The algebra $\underline{\mathcal{W}}'$ coincides with $\underline{\mathcal{W}}$. So we get an embedding $\mathcal{W} \hookrightarrow \underline{\mathcal{W}}$. Note that, at least, some embedding $\mathcal{W} \hookrightarrow \underline{\mathcal{W}}$ in this case was known previously: this is a so called *generalized Miura transform* first discovered in [Ly]. Using techniques developed in [Lo2, Subsections 3.2,3.3] it should not be very difficult to show that these two embeddings coincide (at least, up to an automorphism of $\underline{\mathcal{U}}$). However, to save space we are not going to study this question.

We do not know if it is always possible to embed $\mathcal{W}$ into $\underline{\mathcal{W}}$. Neither it is clear for us how to define the map Θ using Premet's construction of W-algebras (the generalized Miura transform can be easily seen from that construction).

6.4. **Behavior of ideals.** Recall the procedure of the parabolic induction for ideals in universal enveloping algebras. To a two-sided ideal $\underline{\mathcal{J}} \subset \underline{\mathcal{U}}$ we can assign a two-sided ideal $\mathcal{J}$ in $\mathcal{U}$ as follows. Let $\underline{\mathcal{J}}_{\mathfrak{p}}$ denote the inverse image of $\underline{\mathcal{J}}$ in $U(\mathfrak{p})$ under the projection

$U(\mathfrak{p}) \twoheadrightarrow \underline{\mathcal{U}}$. Then for $\mathcal{J}$ we take the largest (with respect to inclusion) two-sided ideal of $\mathcal{U}$ contained in $\mathcal{U}\underline{\mathcal{J}}_{\mathfrak{p}}$. It is well known that if $\underline{M}$ is a $\underline{\mathcal{U}}$-module with $\operatorname{Ann}_{\underline{\mathcal{U}}} \underline{M} = \underline{\mathcal{J}}$, then $\mathcal{J} := \operatorname{Ann}_{\mathcal{U}}(\mathcal{U} \otimes_{U(\mathfrak{p})} \underline{M})$. We say that $\mathcal{J}$ is obtained from $\underline{\mathcal{J}}$ by parabolic induction.

The goal of this subsection is to relate the inclusion $\mathcal{W} \hookrightarrow \underline{\mathcal{W}}'$ to parabolic induction for ideals.

Theorem 6.4.1. *Let $\underline{\mathcal{I}}$ be a two-sided ideal in $\underline{\mathcal{W}}$ and let $\underline{\mathcal{I}}'$ denote the closure of $\underline{\mathcal{I}}$ in $\underline{\mathcal{W}}'$. Set $\mathcal{I} := \Xi'^{-1}(\underline{\mathcal{I}}') \subset \mathcal{W}, \underline{\mathcal{J}} := \underline{\mathcal{I}}^{\dagger} \subset \underline{\mathcal{U}}$. Finally, let $\mathcal{J} \subset \mathcal{U}$ be the ideal obtained from $\underline{\mathcal{J}}$ by parabolic induction. Then $\mathcal{I}^{\dagger} = \mathcal{J}$.*

Here is an immediate corollary of this theorem.

Corollary 6.4.2. *Let $\underline{N}$ be a finite dimensional $\underline{\mathcal{W}}$-module. Then the ideal $\operatorname{Ann}_{\mathcal{W}}(\rho(\underline{N}))^{\dagger} \subset \mathcal{U}$ is obtained from $\operatorname{Ann}_{\underline{\mathcal{W}}}(\underline{N})^{\dagger} \subset \underline{\mathcal{U}}$ by the parabolic induction.*

Proof of Theorem 6.4.1. Let us explain the scheme of the proof. In Step 1 we will give an alternative construction of the parabolic induction map $\underline{\mathcal{J}} \mapsto \mathcal{J}$. The main result of Step 3 is commutative diagram (6.8), whose vertices are various sets of two-sided ideals. The sink of this diagram is $\mathfrak{Id}(\underline{\mathcal{W}})$, and the source is $\mathfrak{Id}(\mathcal{U})$. Diagram (6.8) is obtained from diagram (6.6) of algebra homomorphisms to be established on Step 2. It is more or less easy to see that (6.8) implements the composition $\underline{\mathcal{I}} \mapsto \mathcal{I} \mapsto \mathcal{I}^{\dagger}$. The analogous claim for the composition $\underline{\mathcal{I}} \mapsto \underline{\mathcal{J}} \mapsto \mathcal{J}$ will be verified on Step 4 using the description of Step 1.

Step 1. We can extend the star-product from $\mathbb{K}[T^*G]^{\wedge}_{Gx}$ to $\mathbb{K}[T^*G]^{\wedge}_{Px}$. Applying the quantum Hamiltonian reduction described in Subsection 3.4 to the action of N on $(T^*G)^{\wedge}_{Px}$, we get the structure of a quantum algebra on $\mathbb{K}[\widetilde{Y}]^{\wedge}_{\underline{G}y}[[\hbar]]$. Similarly to (6.1), we have a homomorphism $\mathcal{U}_{\hbar}^{\wedge} \hookrightarrow \mathbb{K}[\widetilde{Y}]^{\wedge}_{\underline{G}y}[[\hbar]]^{\underline{G}}$.

A remarkable feature here, however, is that, unlike in the case of W-algebras, we can get a non-completed version of this homomorphism. The quantum algebra $\mathbb{K}[T^*G][\hbar]$ is $G \times G \times \mathbb{K}^{\times}$-equivariantly isomorphic to the "homogeneous version" of the algebra of differential operators $\mathcal{D}_{\hbar}(G)$, for a rigorous proof see, e.g., Subsection 7.1. This isomorphism gives rise to a $P \times P^- \times \mathbb{K}^{\times}$-equivariant identification $\mathbb{K}[Z][\hbar] \xrightarrow{\sim} \mathcal{D}_{\hbar}(NP^-)$ (recall that $Z = P^- \times \mathfrak{g}^* = T^*(NP^-)$). The quantum Hamiltonian reduction of $\mathcal{D}_{\hbar}(NP^-) = \mathcal{D}_{\hbar}(N) \otimes_{\mathbb{K}[\hbar]} \mathcal{D}_{\hbar}(P^-)$ under the action of N is naturally identified with $\mathcal{D}_{\hbar}(P^-)$. So we get a $\underline{G} \times P^- \times \mathbb{K}^{\times}$-equivariant identification $\mathbb{K}[\widetilde{Y}][\hbar] \xrightarrow{\sim} \mathcal{D}_{\hbar}(P^-)$.

Embed P^- into G/N via $p \mapsto Np$. Consider a homomorphism $\mathcal{U}_{\hbar} \to \mathcal{D}_{\hbar}(P^-)$ sending $\xi \in \mathfrak{g}$ to the velocity vector $\xi_{G/N}$ on $P^- \hookrightarrow G/N$. The image of this homomorphism consists of $\underline{G}$-invariants, so we have a homomorphism $\mathcal{U}_{\hbar} \to \mathbb{K}[\widetilde{Y}][\hbar]^{G} \xrightarrow{\sim} \mathcal{D}_{\hbar}(P^-)^{\underline{G}} = \underline{\mathcal{U}}_{\hbar} \otimes_{\mathbb{K}[\hbar]} \mathcal{D}_{\hbar}(N^-)$. The latter homomorphism is $\underline{G}$-equivariant, where we consider the adjoint action of $\underline{G}$ on $\mathcal{U}$, and the $\underline{G}$-action on $\mathcal{D}_{\hbar}(P^-)^{\underline{G}}$ coming from the $\underline{G}$-action on P^- by right translations. We can also take the quotients by $\hbar - 1$ and get the $\underline{G}$-equivariant homomorphism $\widetilde{\Xi} : \mathcal{U} \to \underline{\mathcal{U}} \otimes \mathcal{D}(N^-)$.

The reason why we need this construction is that we can give an alternative definition of the parabolic induction for ideals in universal enveloping algebras.

Lemma 6.4.3. *Let $\underline{\mathcal{J}}$ be a two-sided ideal in $\underline{\mathcal{U}}$ and $\mathcal{J}$ be the ideal in $\mathcal{U}$ obtained from $\underline{\mathcal{J}}$ by parabolic induction. Then $\mathcal{J} = \widetilde{\Xi}^{-1}(\underline{\mathcal{J}} \otimes \mathcal{D}(N^-))$.*

Proof of Lemma 6.4.3. Set, for brevity, $\mathcal{B} := \underline{\mathcal{U}} \otimes \mathcal{D}(N^-)$. As a vector space, $\mathcal{D}(N^-) = \mathcal{B}_+ \otimes \mathcal{B}_-$, where $\mathcal{B}_+ = \mathbb{K}[N^-], \mathcal{B}_- = U(\mathfrak{n}^-)$. Note that the construction of the homomorphism $\widetilde{\Xi} : \mathcal{U} \to \mathcal{B}$ implies that the restrictions of $\widetilde{\Xi}$ to $\underline{\mathcal{U}}, \mathcal{B}_- \subset \mathcal{U}$ are the identity maps.

Pick a rational element $\vartheta \in \mathfrak{z}(\underline{\mathfrak{g}})$ such that all eigenvalues of $\operatorname{ad}\vartheta$ on $\mathfrak{n}$ are positive (and then the eigenvalues of $\operatorname{ad}\vartheta$ on $\mathfrak{n}^-$ are automatically negative). Recall that $\underline{G}$ acts on $\mathcal{B}$. Let ϑ_* denote the image of ϑ in $\operatorname{Der}(\mathcal{B})$. All eigenvalues of ϑ_* on $\underline{\mathcal{U}}$, (resp., $\mathcal{B}_-, \mathcal{B}_+$) are zero (resp., nonpositive, nonnegative). Moreover, $\mathcal{B}_+ = \mathbb{K} \oplus \mathcal{B}_{++}$, where all eigenvalues of ϑ_* on $\mathcal{B}_{++}$ are strictly positive. Clearly, $\mathcal{B}_{++}$ is an ideal in $\mathcal{B}_+$.

Pick a $\underline{\mathcal{U}}$-module $\underline{M}$ with $\operatorname{Ann}_{\underline{\mathcal{U}}}(\underline{M}) = \underline{\mathcal{J}}$. Consider the induced module $M = \mathcal{B} \otimes_{\underline{\mathcal{U}} \otimes \mathcal{B}_+} \underline{M}$. Here $\mathcal{B}_+$ acts on $\underline{M}$ via the projection $\mathcal{B}_+ \twoheadrightarrow \mathcal{B}_+/\mathcal{B}_{++} = \mathbb{K}$.

We claim that, as a $\mathcal{U}$-module, $M = \mathcal{U} \otimes_{U(\mathfrak{p})} \underline{M}$. Since $\mathcal{B} = \mathcal{B}_- \otimes \underline{\mathcal{U}} \otimes \mathcal{B}_+$ we see that $M = U(\mathfrak{n}^-) \otimes \underline{M}$ as a $U(\mathfrak{p}^-)$-module, where we consider $U(\mathfrak{p}_-)$ as a subalgebra of $\mathcal{B}$. All eigenvectors of ϑ_* with positive eigenvalues act by zero on $\underline{M} \subset M$. Since the map $\widetilde{\Xi}$ is $\underline{G}$-equivariant, we have $\widetilde{\Xi}([\vartheta, u]) = \vartheta_* u$ for all $u \in U$. Therefore $\mathfrak{n}$ acts trivially on $\underline{M}$. The identity map $\underline{M} \to \underline{M}$ extends to a homomorphism $\mathcal{U} \otimes_{U(\mathfrak{p})} \underline{M} \to M$ of $\mathcal{U}$-modules. Since both $\mathcal{U} \otimes_{U(\mathfrak{p})} \underline{M}$ and M are naturally identified with $U(\mathfrak{n}^-) \otimes \underline{M}$, we see that this homomorphism is, in fact, an isomorphism.

It is easy to see the annihilator of M in $\mathcal{B}$ coincides with $\underline{\mathcal{J}} \otimes \mathcal{D}(N^-)$. So $\operatorname{Ann}_{\mathcal{U}}(M) = \widetilde{\Xi}^{-1}(\underline{\mathcal{J}} \otimes \mathcal{D}(N^-))$. On the other hand, from the previous paragraph it follows that $\operatorname{Ann}_{\mathcal{U}}(M) = \mathcal{J}$. $\square$

Step 2. Recall the isomorphism

$$\Phi_\hbar : \mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]] \xrightarrow{\sim} \mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[X]^\wedge_{Gx}[[\hbar]]) (:= \mathbf{A}^\wedge_{V,\hbar} \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathbb{K}[X]^\wedge_{Gx}[[\hbar]])$$

from Theorem 4.2.1. We have a similar isomorphism $\Phi_{0\hbar} : \mathbb{K}[T^*\underline{G}]^\wedge_{\underline{Gx}} \xrightarrow{\sim} \mathbf{A}^\wedge_{\underline{V},\hbar}(\mathbb{K}[\underline{X}]^\wedge_{\underline{Gx}}[[\hbar]])$.

We extend $\Phi_\hbar$ to the isomorphism $\mathbb{K}[T^*G]^\wedge_{Px}[[\hbar]] \xrightarrow{\sim} \mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[X]^\wedge_{Px}[[\hbar]])$ and then, performing the quantum Hamiltonian reduction, we get an isomorphism $\underline{\Phi}_\hbar : \mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]] \xrightarrow{\sim} \mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]])$.

The following lemma is a quantum analog of Lemmas 6.2.1,6.2.2,6.2.3.

Lemma 6.4.4. *There is a commutative diagram*

$$\begin{array}{ccccc}
\mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]) & \xrightarrow{\Lambda_\hbar} & \mathbf{A}_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]) & \xrightarrow{\underline{\Phi}_\hbar^{-1}} & \mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]] \\
\uparrow & & & & \uparrow \\
 & & \mathbf{A}^\wedge_{\underline{V},\hbar}(\mathbb{K}[\underline{X}]^\wedge_{\underline{Gx}}[[\hbar]]) & \xrightarrow{\Phi_{0\hbar}^{-1}} & \mathbb{K}[T^*\underline{G}]^\wedge_{\underline{Gx}}[[\hbar]] \\
 & & \uparrow & & \\
\mathbb{K}[Y^\wedge_{\underline{G}y}][[\hbar]] & \longrightarrow & \mathbb{K}[\underline{X}]^\wedge_{\underline{Gx}}[[\hbar]] & &
\end{array} \tag{6.5}$$

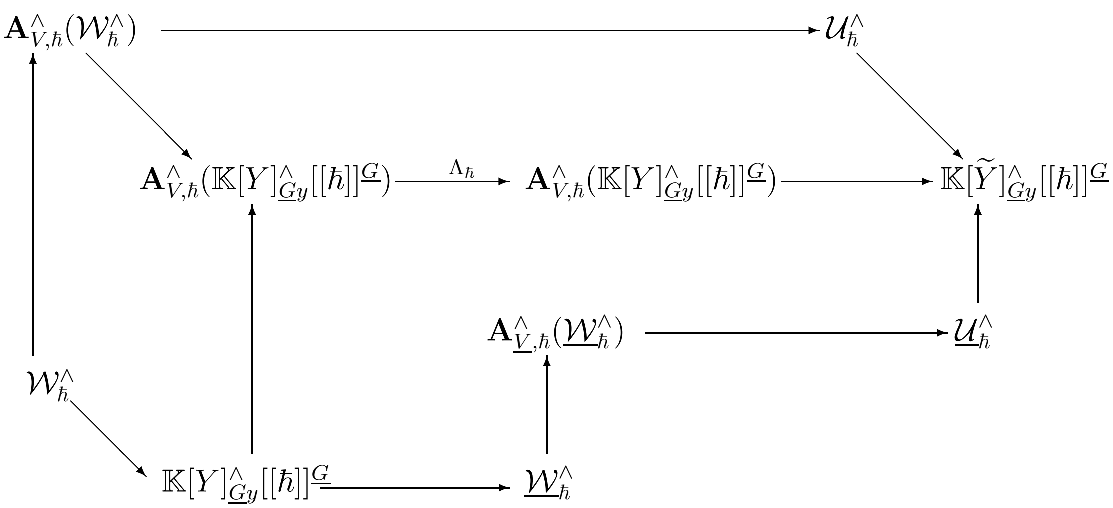
Here all vertical maps are natural embeddings, while $\Lambda_\hbar$ is an automorphism of the form $A\exp(\frac{1}{\hbar^2}\operatorname{ad}(\hat{f}))$ for some $\underline{G} \times \underline{Q}$-invariant element $\hat{f} \in \mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]])$ of degree 2 with respect to $\mathbb{K}^\times$. The bottom horizontal arrow is an isomorphism from Proposition 6.3.1.

Proof. Similarly to Lemma 6.2.2, we get a $\underline{G} \times \underline{Q} \times \mathbb{K}^\times$-equivariant automorphism $\Lambda_\hbar$ of $\mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]])$ making the diagram commutative and preserving the quantum comoment map. In addition, we may assume that modulo $\hbar^2$ the automorphism $\Lambda_\hbar$ equals $A\exp(\frac{1}{\hbar^2}\operatorname{ad}(f))$, where f, A are as in Lemma 6.2.3. In particular, $\exp(\frac{1}{\hbar^2}\operatorname{ad}(f))$ converges.

So $\Lambda_{\hbar,1} := \Lambda_\hbar \exp(-\frac{1}{\hbar^2}\operatorname{ad}(f))A^{-1}$ has the form $\mathrm{id} + \sum_{i=1}^{\infty} T_i\hbar^{2i}$. Again, $\Lambda_{\hbar,1}$ is a $\underline{G}\times\underline{Q}\times\mathbb{K}^\times$-equivariant Hamiltonian automorphism. As in the proof of [Lo3, Theorem 2.3.1], we see that T_1 is a $\underline{G}\times\underline{Q}$-equivariant Poisson derivation of $\mathbb{K}[Y\times V^*]^\wedge_{\underline{G}y}$. Also let us note that T_1 annihilates the image of the classical comoment map and therefore, by Lemma 3.2.3, $T_1 = v_{f_1}$ for some $\underline{G}\times\underline{Q}$-invariant element $f_1\in\mathbb{K}[Y\times V^*]^\wedge_{\underline{G}y}$ that has degree 0 w.r.t $\mathbb{K}^\times$. Replace $\Lambda_{\hbar,1}$ with $\Lambda_{\hbar,2} := \Lambda_{\hbar,1}\exp(-\operatorname{ad}(f_1))$. We get $\Lambda_{\hbar,2} = \mathrm{id} + \sum_{i=2}^{\infty} T_i'\hbar^{2i}$. Repeating the procedure we obtain a presentation $\Lambda_\hbar = A\exp(\frac{1}{\hbar^2}\operatorname{ad}(f))\exp(\operatorname{ad}(f_1))\exp(\hbar^2\operatorname{ad}(f_2))\ldots$. Using the Campbell-Hausdorff formula, we get a presentation of $\Lambda_\hbar$ in the form required in the statement of the proposition. $\square$

By taking the $\underline{G}$-invariants, we get the following commutative diagram (the embeddings $\mathcal{W}_\hbar\hookrightarrow\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]], \mathcal{U}_\hbar\hookrightarrow\mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]]$ come from the quantum Hamiltonian reduction).

(6.6)



Step 3. Recall from [Lo2] that for a flat $\mathbb{K}[\hbar]$-algebra $\mathcal{A}_\hbar$ equipped with a $\mathbb{K}^\times$-action one defines the set $\mathfrak{Id}_\hbar(\mathcal{A}_\hbar)$ of two-sided $\hbar$-saturated $\mathbb{K}^\times$-stable ideals.

Now let $\mathcal{A}$ be an algebra equipped with an increasing exhaustive separated filtration $\mathrm{F}_i\,\mathcal{A}$. Then to $\mathcal{A}$ we can assign the Rees algebra $R_\hbar(\mathcal{A}) = \bigoplus_{i\in\mathbb{Z}}(\mathrm{F}_i\,\mathcal{A})\hbar^i \subset \mathcal{A}[\hbar^{-1},\hbar]$. The group $\mathbb{K}^\times$ naturally acts on $R_\hbar(\mathcal{A})$ by algebra automorphisms. Also to any two-sided ideal $\mathcal{I}\subset\mathcal{A}$ we can assign the two-sided ideal $R_\hbar(\mathcal{I}) = \bigoplus_{i\in\mathbb{Z}}(\mathrm{F}_i\,\mathcal{A}\cap\mathcal{I})\hbar^i$. As we have seen in [Lo2, Subsection 3.2], the sets $\mathfrak{Id}(\mathcal{A})$ and $\mathfrak{Id}_\hbar(R_\hbar(\mathcal{A}))$ are identified via the maps $\mathcal{I}\mapsto R_\hbar(\mathcal{I}), \mathcal{I}_\hbar\mapsto\mathcal{I}_\hbar/(\hbar-1)\mathcal{I}_\hbar$. In particular, we have identifications $\mathfrak{Id}(\mathcal{W})\xrightarrow{\sim}\mathfrak{Id}_\hbar(\mathcal{W}_\hbar), \mathfrak{Id}(\mathcal{U})\xrightarrow{\sim}\mathfrak{Id}_\hbar(\mathcal{U}_\hbar), \mathfrak{Id}(\underline{\mathcal{W}})\xrightarrow{\sim}\mathfrak{Id}(\underline{\mathcal{W}}_\hbar)$.

Also we have the following result whose proof is straightforward.

Lemma 6.4.5. *Let $\mathcal{A},\mathcal{B}$ be filtered algebras and $\Phi:\mathcal{A}\to\mathcal{B}$ be a homomorphism preserving the filtrations. Let $\Phi_\hbar : R_\hbar(\mathcal{A})\to R_\hbar(\mathcal{B})$ be the induced homomorphism of the Rees algebras. Then $\Phi_\hbar^{-1}(R_\hbar(\mathcal{I})) = R_\hbar(\Phi^{-1}(\mathcal{I}))$ for any two-sided ideal $\mathcal{I}\subset\mathcal{B}$.*

In [Lo2, Subsection 3.4], we established a natural identification $\mathfrak{Id}_\hbar(\mathcal{W}_\hbar)\xrightarrow{\sim}\mathfrak{Id}_\hbar(\mathcal{W}^\wedge_\hbar)$. We have a similar identification for $\underline{\mathcal{W}}$. Also we have a natural identification $\mathfrak{Id}_\hbar(\mathcal{A}_\hbar)\xrightarrow{\sim}\mathfrak{Id}_\hbar(\mathbf{A}^\wedge_{V,\hbar}(\mathcal{A}_\hbar))$ provided $\mathcal{A}_\hbar$ is complete in the $\hbar$-adic topology. More precisely, the maps

$\mathcal{I}_\hbar \mapsto \mathbf{A}^\wedge_{V,\hbar}\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathcal{I}_\hbar, \mathcal{J}_\hbar \mapsto \mathcal{J}_\hbar \cap \mathcal{A}_\hbar$ are mutually inverse bijections between $\mathfrak{Id}_\hbar(\mathcal{A}_\hbar)$ and $\mathfrak{Id}_\hbar(\mathbf{A}^\wedge_{V,\hbar}(\mathcal{A}_\hbar))$.

Lemma 6.4.6. *There is an identification* $\mathfrak{Id}_\hbar(\underline{\mathcal{U}}^\wedge_\hbar) \xrightarrow{\sim} \mathfrak{Id}_\hbar(\mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}})$.

Proof. Recall the identification $\mathbb{K}[\widetilde{Y}][\hbar]^{\underline{G}} \xrightarrow{\sim} \underline{\mathcal{U}}_\hbar \otimes_{\mathbb{K}[\hbar]} \mathcal{D}_\hbar(N^-)$. Taking the completions at the point $(0,1,0) \in \underline{\mathfrak{g}}^* \times N^- \times \mathfrak{n}^{-*} = \underline{\mathfrak{g}}^* \times T^*N^-$, we get an isomorphism $\mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}} \xrightarrow{\sim} \underline{\mathcal{U}}^\wedge_\hbar\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathcal{D}_\hbar(N^-)^\wedge$, where the last factor is the completion of $\mathcal{D}_\hbar(N^-)$ at $(1,0)$. This isomorphism is $\mathbb{K}^\times$-equivariant. So we get a natural map

$$\mathfrak{Id}_\hbar(\mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}) \to \mathfrak{Id}_\hbar(\underline{\mathcal{U}}^\wedge_\hbar) \tag{6.7}$$

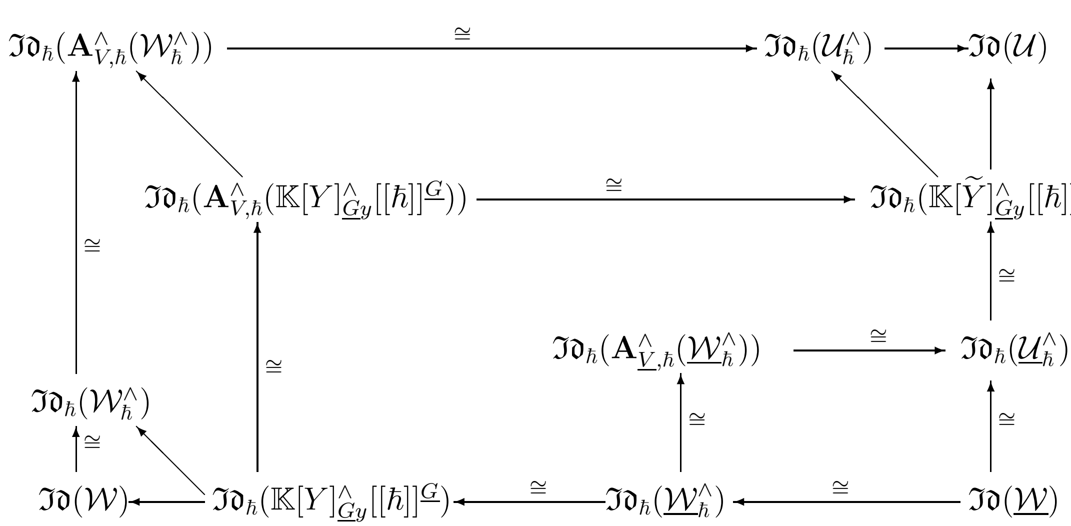
given by taking the intersection with $\underline{\mathcal{U}}^\wedge_\hbar$.

The algebra $\mathcal{D}_\hbar(N^-)^\wedge$ is isomorphic to $\mathbf{A}^\wedge_{\mathfrak{n}\oplus\mathfrak{n}^*,\hbar}$. Hence the map $\underline{\mathcal{J}}'_\hbar \mapsto \mathcal{D}_\hbar(N^-)^\wedge\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\underline{\mathcal{J}}'_\hbar$ is inverse to (6.7). □

Note that, thanks to Lemma 6.4.4, $\Lambda_\hbar$ acts trivially on $\mathfrak{Id}_\hbar\left(\mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}})\right)$.

Summarizing, we have the following commutative diagram.

(6.8)



Here almost all arrows are either natural identifications or are obtained from isomorphisms of algebras. The maps

$$\mathfrak{Id}_\hbar(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}) \to \mathfrak{Id}_\hbar(\mathcal{W}^\wedge_\hbar), \quad \mathfrak{Id}_\hbar(\mathbf{A}^\wedge_{V,\hbar}(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}})) \to \mathfrak{Id}_\hbar(\mathbf{A}^\wedge_{V,\hbar}(\mathcal{W}^\wedge_\hbar)),$$

$$\mathfrak{Id}_\hbar(\mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}) \to \mathfrak{Id}_\hbar(\mathcal{U}^\wedge_\hbar), \quad \mathfrak{Id}_\hbar(\mathcal{U}^\wedge_\hbar) \to \mathfrak{Id}(\mathcal{U})(= \mathfrak{Id}_\hbar(\mathcal{U}_\hbar))$$

are obtained by pull-backs with respect to the corresponding inclusions. Finally, the maps

$$\mathfrak{Id}_\hbar(\mathbb{K}[Y]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}) \to \mathfrak{Id}(\mathcal{W}), \quad \mathfrak{Id}_\hbar(\mathbb{K}[\widetilde{Y}]^\wedge_{\underline{G}y}[[\hbar]]^{\underline{G}}) \to \mathfrak{Id}(\mathcal{U})$$

are completely determined by the condition that the diagram is commutative.

Step 4. Let $\underline{\mathcal{I}}_\hbar$ be the ideal in $\underline{\mathcal{W}}_\hbar$ corresponding to $\underline{\mathcal{I}}$. Let $\underline{\mathcal{I}}^\wedge_\hbar$ denote the closure of $\underline{\mathcal{I}}_\hbar$ in $\underline{\mathcal{W}}^\wedge_\hbar$. One has the equality

$$\underline{\mathcal{I}}' \cap \underline{\mathcal{W}}^\heartsuit = (\underline{\mathcal{I}}^\wedge_\hbar)_{\mathbb{K}^\times-l.f.}/(\hbar-1)(\underline{\mathcal{I}}^\wedge_\hbar)_{\mathbb{K}^\times-l.f.},$$

where $\underline{\mathcal{I}}'$ is the closure of $\underline{\mathcal{I}}$ in $\underline{\mathcal{W}}'$. Indeed, any ideal in $\underline{\mathcal{W}}^{\heartsuit}$ is generated by its intersection with $\underline{\mathcal{W}}$, compare with the proof of [Lo4, Lemma 5.3], and both sides of the previous equality intersect $\underline{\mathcal{W}}$ in $\underline{\mathcal{I}}$.

Since the embedding $\mathcal{W} \hookrightarrow \underline{\mathcal{W}}'$ factors through $\mathcal{W} \hookrightarrow \underline{\mathcal{W}}^{\heartsuit}$, we see that $\mathcal{I}$ is the image of $\underline{\mathcal{I}}$ under the maps of the bottommost row of the commutative diagram (6.8). From the construction, $\mathcal{I}^{\dagger}$ is the image of $\mathcal{I}$ under the maps of the leftmost column and the topmost row of the diagram. So it remains to check that

(*) $\mathcal{J}$ is the image of $\underline{\mathcal{I}}$ under the maps of the rightmost column.

We have the following commutative diagram, where all vertical arrows are natural embeddings.

$$\begin{array}{ccccccc}
\underline{\mathcal{U}}_{\hbar}^{\wedge} & \longrightarrow & \mathbf{A}^{\wedge}_{\mathfrak{n}\oplus\mathfrak{n}^*,\hbar}(\underline{\mathcal{U}}_{\hbar}^{\wedge}) & \xrightarrow{\cong} & \mathbb{K}[\widetilde{Y}]^{\wedge}_{\underline{G}y}[[\hbar]]^{\underline{G}} & \longleftarrow & \mathcal{U}_{\hbar}^{\wedge} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\underline{\mathcal{U}}_{\hbar} & \longrightarrow & \mathcal{D}_{\hbar}(N^-)\otimes_{\mathbb{K}[\hbar]}\underline{\mathcal{U}}_{\hbar} & \xrightarrow{\cong} & \mathbb{K}[\widetilde{Y}][\hbar]^{\underline{G}} & \longleftarrow & \mathcal{U}_{\hbar}
\end{array} \tag{6.9}$$

The diagram (6.9) gives rise to the following commutative diagram of maps between the sets of ideals.

$$\begin{array}{ccccccc}
\mathfrak{Id}_{\hbar}(\underline{\mathcal{U}}_{\hbar}^{\wedge}) & \xrightarrow{\cong} & \mathfrak{Id}_{\hbar}(\mathbf{A}^{\wedge}_{\mathfrak{n}\oplus\mathfrak{n}^*,\hbar}(\underline{\mathcal{U}}_{\hbar}^{\wedge})) & \xrightarrow{\cong} & \mathfrak{Id}_{\hbar}(\mathbb{K}[\widetilde{Y}]^{\wedge}_{\underline{G}y}[[\hbar]]) & \longrightarrow & \mathfrak{Id}_{\hbar}(\mathcal{U}_{\hbar}^{\wedge}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{Id}_{\hbar}(\underline{\mathcal{U}}_{\hbar}) & \xrightarrow{\cong} & \mathfrak{Id}_{\hbar}(\underline{\mathcal{U}}_{\hbar}\otimes_{\mathbb{K}[\hbar]}\mathcal{D}_{\hbar}(N^-)) & \xrightarrow{\cong} & \mathfrak{Id}_{\hbar}(\mathbb{K}[\widetilde{Y}][\hbar]) & \longrightarrow & \mathfrak{Id}_{\hbar}(\mathcal{U}_{\hbar}) \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong \\
\mathfrak{Id}(\underline{\mathcal{U}}) & \xrightarrow{\cong} & \mathfrak{Id}(\underline{\mathcal{U}})\otimes\mathcal{D}(N^-) & & \longrightarrow & & \mathfrak{Id}(\mathcal{U})
\end{array} \tag{6.10}$$

The map in the rightmost column of diagram (6.8) is the composition of the identification $\mathfrak{Id}(\underline{\mathcal{W}}) \xrightarrow{\sim} \mathfrak{Id}_{\hbar}(\underline{\mathcal{U}}_{\hbar}^{\wedge})$ and the map $\mathfrak{Id}_{\hbar}(\underline{\mathcal{U}}_{\hbar}^{\wedge}) \to \mathfrak{Id}_{\hbar}(\mathcal{U}_{\hbar})$ from diagram (6.10). Now the claim (*) follows from Lemma 6.4.3. $\square$

6.5. **Parabolic induction and representation schemes.** In this subsection we study the morphism of representation schemes induced by the parabolic induction functor.

Let us recall some generalities on representation schemes.

Let $\mathcal{A}$ be a finitely generated associative algebra with generators $x_1, \ldots, x_n$. Consider the ideal $\mathcal{J}$ of relations for $\mathcal{A}$ in the free algebra $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ so that $\mathcal{A} \cong \mathbb{K}\langle x_1, \ldots, x_n\rangle/\mathcal{J}$. Fix some positive integer d and consider the subscheme $X \subset \operatorname{Mat}_d(\mathbb{K})^n = \{(X_1, \ldots, X_n), X_i \in \operatorname{Mat}_d(\mathbb{K})\}$ defined by the equations $f(X_1, \ldots, X_n) = 0$ for $f \in \mathcal{J}$. By definition the *representation scheme* $\operatorname{Rep}(\mathcal{A}, d)$ is the categorical quotient $X/\!/\operatorname{GL}_d$. The points of $\operatorname{Rep}(\mathcal{A}, d)$ are in bijection with isomorphism classes of semisimple $\mathcal{A}$-modules of dimension d. A homomorphism $\mathcal{A}_1 \to \mathcal{A}_2$ induces a morphism $\operatorname{Rep}(\mathcal{A}_2, d) \to \operatorname{Rep}(\mathcal{A}_1, d)$.

For elements $x_1,\ldots,x_{2n}$ in an associative algebra we put

$$s_{2n}(x_1,\ldots,x_{2n}) := \sum_{\sigma\in\mathfrak{S}_{2n}} \operatorname{sgn}(\sigma)x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(2n)}.$$

An algebra is said to satisfy the identity s_{2n} if $s_{2n}(x_1,\ldots,x_{2n})=0$ for all elements $x_1,\ldots,x_{2n}$ of this algebra. According to the Amitsur-Levitzki Theorem (see e.g. [McCR, Theorem 13.3.3]), the algebra of $\operatorname{Mat}_d(\mathbb{K})$ satisfies s_{2n} provided $d\leqslant n$.

For an algebra $\mathcal{A}$ let $\mathcal{A}^{(n)}$ denote the quotient of $\mathcal{A}$ by the two-sided ideal generated by all elements $s_{2n}(a_1,\ldots,a_{2n}), a_i\in\mathcal{A}$. Clearly, the schemes $\operatorname{Rep}(\mathcal{A},d)$ and $\operatorname{Rep}(\mathcal{A}^{(d)},d)$ are canonically isomorphic.

Proposition 6.5.1. (1) *If $\underline{e}$ is rigid, then the algebra $U([\underline{\mathfrak{g}},\underline{\mathfrak{g}}],\underline{e})^{(d)}$ is finite dimensional for any d.*
(2) *We have an isomorphism $\underline{\mathcal{W}}^{(d)}\xrightarrow{\sim} U(\mathfrak{z}(\underline{\mathfrak{g}}))\otimes U([\underline{\mathfrak{g}},\underline{\mathfrak{g}}],\underline{e})^{(d)}$.*
(3) *The inclusion $\underline{\mathcal{W}}\hookrightarrow\underline{\mathcal{W}}'$ induces an isomorphism $\underline{\mathcal{W}}^{(d)}\to\underline{\mathcal{W}}'^{(d)}$.*

Proof. We derive assertion (1) from Corollary 7.2.3 proved below in the Appendix. To do this we need to verify that the Poisson variety $\underline{S}_0:=\underline{S}\cap[\underline{\mathfrak{g}},\underline{\mathfrak{g}}]=\operatorname{Spec}(\operatorname{gr}U([\underline{\mathfrak{g}},\underline{\mathfrak{g}}],\underline{e}))$ has only one 0-dimensional symplectic leaf. Symplectic leaves in $\underline{S}_0$ are exactly the irreducible components of the intersections of $\underline{S}_0$ with coadjoint orbits of $\underline{G}$, see, for example, [GG, 3.1]. So we need to check that the only orbit in $[\underline{\mathfrak{g}},\underline{\mathfrak{g}}]$ of dimension $\dim\underline{\mathbb{O}}$ that intersects $\underline{S}_0$ is $\underline{\mathbb{O}}$ itself.

Assume the converse. Thanks to the contraction property of the Kazhdan action, an orbit $\underline{\mathbb{O}}'$ intersects $\underline{S}'$ if and only if $\underline{\mathbb{O}}\subset\overline{\mathbb{K}^\times\underline{\mathbb{O}}'}$. Recall that by a sheet in $[\underline{\mathfrak{g}},\underline{\mathfrak{g}}]^*$ one means an irreducible component of the locally closed subvariety $\{\alpha\in[\underline{\mathfrak{g}},\underline{\mathfrak{g}}]^*|\dim\underline{G}\alpha=k\}$ for some fixed k. A sheet in a semisimple algebra containing a rigid nilpotent orbit consists only of this orbit, see, for example, [McG, §5.5]. So if $\underline{\mathbb{O}}\subset\overline{\mathbb{K}^\times\underline{\mathbb{O}}'}$ and $\underline{\mathbb{O}}$ is rigid, then $\dim\underline{\mathbb{O}}'>\dim\underline{\mathbb{O}}$.

It follows that the only zero-dimensional symplectic leaf in $\underline{S}_0$ is $\underline{e}$. So Corollary 7.2.3 does apply in the present situation.

Assertion (2) follows from the decomposition $\underline{\mathcal{W}}=U(\mathfrak{z}(\underline{\mathfrak{g}}))\otimes U([\underline{\mathfrak{g}},\underline{\mathfrak{g}}],\underline{e})$.

Let us proceed to assertion (3). Let $\mathcal{I}$, resp. $\mathcal{I}'$, denote the ideal in $\underline{\mathcal{W}}$ (resp., $\underline{\mathcal{W}}'$) generated by $s_{2d}(x_1,\ldots,x_{2d})$ with $x_i\in\underline{\mathcal{W}}$, (resp., $x_i\in\underline{\mathcal{W}}'$). Clearly, $\mathcal{I}'$ is the closure of $\mathcal{I}$ in $\underline{\mathcal{W}}'$. We need to check that $\mathcal{I}'\cap\underline{\mathcal{W}}=\mathcal{I}$ and $\underline{\mathcal{W}}+\mathcal{I}'=\underline{\mathcal{W}}'$. Let $\underline{\mathcal{W}}=\bigoplus_{\alpha\in\mathbb{Z}}\underline{\mathcal{W}}_\alpha$ denote the eigenspace decomposition with respect to $\operatorname{ad}h_0$. We claim that $\mathcal{W}_\alpha\subset\mathcal{I}$ for all α less than some number α_0 depending on d.

Let Y denote the union of all sheets containing the orbit $\underline{\mathbb{O}}$. By Katsylo's results, [Kat], the group Q° acts trivially on $Y\cap S$. By Theorem 7.2.1, the maximal spectrum $\operatorname{Specm}\operatorname{gr}\mathcal{W}^{(d)}$ is contained in $Y\cap S$ so the $\mathbb{K}^\times$-action on $\operatorname{Specm}(\operatorname{gr}\mathcal{W}^{(d)})$ (via γ_0) is trivial. It follows that $\operatorname{ad}h_0$ has finitely many eigenvalues on $\operatorname{gr}\underline{\mathcal{W}}^{(d)}$ and hence on $\underline{\mathcal{W}}^{(d)}$.

Therefore $\mathcal{I}'=\mathcal{I}+\prod_{\alpha\leqslant\alpha_0}\underline{\mathcal{W}}_\alpha$. So $\mathcal{I}=\mathcal{I}'\cap\underline{\mathcal{W}}$ and $\mathcal{I}'+\underline{\mathcal{W}}=\underline{\mathcal{W}}'$. □

So we have a homomorphism $\mathcal{W}^{(d)}\to\underline{\mathcal{W}}^{(d)}$. It gives rise to a morphism $\operatorname{Rep}(\underline{\mathcal{W}},d)\to\operatorname{Rep}(\mathcal{W},d)$. Below we assume that the element $\underline{e}$ is rigid.

Theorem 6.5.2. *The morphism $\operatorname{Rep}(\underline{\mathcal{W}},d)\to\operatorname{Rep}(\mathcal{W},d)$ is finite.*

Proof. Recall that the centers $\underline{\mathcal{Z}},\mathcal{Z}$ of $\underline{\mathcal{U}},\mathcal{U}$ are identified with the centers of $\underline{\mathcal{W}},\mathcal{W}$. By Corollary 6.3.3, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{W}^{(d)} \\ \big\downarrow & & \big\downarrow \\ \underline{\mathcal{Z}} & \longrightarrow & \underline{\mathcal{W}}^{(d)} \end{array}$$

By Proposition 6.5.1, $\underline{\mathcal{W}}^{(d)} = U(\mathfrak{z}(\underline{\mathfrak{g}})) \otimes \mathcal{A}$, where $\mathcal{A}$ is a finite dimensional algebra. Let us check that the morphism $\operatorname{Rep}(\underline{\mathcal{W}}, d) = \operatorname{Rep}(\underline{\mathcal{W}}^{(d)}, d) \to \operatorname{Rep}(\underline{\mathcal{Z}}, d)$ is finite. For this we will describe the varieties in interest. Clearly, a morphism of schemes is finite if the induced morphism of the underlying varieties is so.

As a variety, $\operatorname{Rep}(\underline{\mathcal{Z}}, d)$ is just $\operatorname{Spec}(\underline{\mathcal{Z}})^d/S_d$. Indeed, this variety parameterizes semisimple $\underline{\mathcal{Z}}$-modules of dimension d. Such a module is determined up to an isomorphism by an unordered d-tuple of characters of $\underline{\mathcal{Z}}$.

Proceed to $\operatorname{Rep}(\underline{\mathcal{W}}^{(d)}, d)$. As we have seen above, $\underline{\mathcal{W}}^{(d)} = U(\mathfrak{z}(\underline{\mathfrak{g}})) \otimes \mathcal{A}$, where $\mathcal{A}$ is a finite dimensional algebra over $\mathcal{Z}([\underline{\mathfrak{g}}, \underline{\mathfrak{g}}])$. An irreducible $\underline{\mathcal{W}}^{(d)}$-module has the form $\mathbb{K}_\chi \otimes V$, where V is an irreducible $\mathcal{A}$-module and $\mathbb{K}_\chi$ is the one-dimensional $U(\mathfrak{z}(\underline{\mathfrak{g}}))$-module corresponding to a character χ. So $\operatorname{Rep}(\underline{\mathcal{W}}^{(d)})$ is a disjoint union of irreducible components $\operatorname{Rep}(\underline{\mathcal{W}}^{(d)})_v$, where v is an unordered collection of irreducible $\mathcal{A}$-modules with sum of dimensions equal d. Let $|v|$ denote the total number of representations in v. The component is naturally identified with the quotient of $(\mathfrak{z}(\underline{\mathfrak{g}})^*)^{|v|}$ by an action of an appropriate product of symmetric groups. The natural morphism $\operatorname{Rep}(U(\mathfrak{z}(\underline{\mathfrak{g}}))\otimes\mathcal{A}, d) \to \operatorname{Rep}(\underline{\mathcal{Z}}, d) = \operatorname{Rep}(U(\mathfrak{z}(\underline{\mathfrak{g}}))\otimes\mathcal{Z}([\underline{\mathfrak{g}}, \underline{\mathfrak{g}}]), d)$ can be read from the characters appearing in the restrictions of irreducible $\mathcal{A}$-modules to $\mathcal{Z}([\underline{\mathfrak{g}}, \underline{\mathfrak{g}}])$. It is easy to see that this morphism is finite.

From the description of $\operatorname{Rep}(\underline{\mathcal{Z}}, d)$ (the variety $\operatorname{Rep}(\mathcal{Z}, d)$ can be described similarly) it is clear that the morphism $\operatorname{Rep}(\underline{\mathcal{Z}}, d) \to \operatorname{Rep}(\mathcal{Z}, d)$ is also finite. So the composition $\operatorname{Rep}(\underline{\mathcal{W}}^{(d)}, d) \to \operatorname{Rep}(\mathcal{Z}, d)$ is finite. □

6.6. **Adjoint functors.** The goal of this subsection is to construct the left and right adjoint functors of the parabolic induction functor ρ.

Let M be a finite dimensional $\mathcal{W}$-module. Pick an ideal $\mathcal{J} \subset \mathcal{Z}$ of finite codimension annihilating M. So M is a module over $\mathcal{W}_{\mathcal{J}} := \mathcal{W}/\mathcal{W}\mathcal{J}$.

Proposition 6.6.1. *There is a minimal two-sided ideal $\mathcal{I}_0$ of finite codimension in $\mathcal{W}_{\mathcal{J}}$.*

Proof. It is enough to check that the $\mathcal{W}$-bimodule $\mathcal{W}_{\mathcal{J}}$ has finite length. This will follow for an arbitrary $\mathcal{J}$ if we prove it for a *maximal* one. The claim for maximal $\mathcal{J}$ is [Gi, Theorem 4.2.2,(i)]. □

In particular, the sequence $\ker(\mathcal{W}_{\mathcal{J}} \to \mathcal{W}_{\mathcal{J}}^{(d)})$ stabilizes for any given $\mathcal{J}$. Similarly, for $\underline{\mathcal{W}}_{\mathcal{J}} := \underline{\mathcal{W}}/\underline{\mathcal{W}}\mathcal{J}$ (where we consider $\mathcal{J}$ as a subspace in $\underline{\mathcal{Z}}$ via the embedding $\mathcal{Z} \hookrightarrow \underline{\mathcal{Z}}$) the sequence $\underline{\mathcal{W}}_{\mathcal{J}}^{(d)}$ stabilizes. So there is $N \in \mathbb{N}$ such that $\mathcal{W}_{\mathcal{J}}^{(d)} = \mathcal{W}_{\mathcal{J}}^{(N)}$ and $\underline{\mathcal{W}}_{\mathcal{J}}^{(d)} = \underline{\mathcal{W}}_{\mathcal{J}}^{(N)}$ for all $d > N$.

Let $\mathcal{W}_{\mathcal{J}}\text{-Mod}^{fin}, \underline{\mathcal{W}}_{\mathcal{J}}\text{-Mod}^{fin}$ denote the categories of finite dimensional modules for the corresponding algebras. Let $\rho_{\mathcal{J}} : \underline{\mathcal{W}}_{\mathcal{J}}\text{-Mod}^{fin} \to \mathcal{W}_{\mathcal{J}}\text{-Mod}^{fin}$ be the corresponding pullback functor.

Set $\kappa^l_{\mathcal{J}}(\bullet) := \underline{\mathcal{W}}_{\mathcal{J}}^{(N)} \otimes_{\mathcal{W}_{\mathcal{J}}^{(N)}} \bullet, \kappa^r_{\mathcal{J}}(\bullet) := \operatorname{Hom}_{\mathcal{W}_{\mathcal{J}}^{(N)}}(\underline{\mathcal{W}}_{\mathcal{J}}^{(N)}, \bullet)$. These are the left and right adjoint functors of $\rho_{\mathcal{J}}$.

Now let $\mathcal{J}_1 \subset \mathcal{J}_2$ be two ideals of $\mathcal{Z}$ of finite codimension. Then we have the natural embedding $\iota_{12} : \mathcal{W}_{\mathcal{J}_2}\text{-Mod}^{fin} \to \mathcal{W}_{\mathcal{J}_1}\text{-Mod}^{fin}$ and the similar embedding $\underline{\iota}_{12}$ for $\underline{\mathcal{W}}$.

It is clear that $\kappa^l_{\mathcal{J}_1} \circ \iota_{12} = \underline{\iota}_{12} \circ \kappa^l_{\mathcal{J}_2}$ and $\kappa^r_{\mathcal{J}_1} \circ \iota_{12} = \underline{\iota}_{12} \circ \kappa^r_{\mathcal{J}_2}$. Since the category $\mathcal{W}\text{-Mod}^{fin}$ is the direct limit of its full subcategories $\mathcal{W}_{\mathcal{J}}\text{-Mod}^{fin}$ (and similarly for $\underline{\mathcal{W}}$) we get the well-defined direct limit functors $\kappa^l, \kappa^r : \mathcal{W}\text{-Mod}^{fin} \to \underline{\mathcal{W}}\text{-Mod}^{fin}$ that are left and right adjoint to ρ.

7. Appendices

7.1. **Fedosov quantization vs twisted differential operators.** Let X_0 be a smooth affine variety. Consider the cotangent bundle $X := T^*X_0$. Then X has a canonical symplectic form ω. Our first goal in this section is to relate the Fedosov quantization corresponding to the zero curvature form to a certain algebra of twisted differential operators on X_0.

For computations we will need a precise description of the Poisson structure on $\mathbb{K}[X]$. Recall that the symplectic form ω on X equals $-d\lambda$, where λ is the canonical 1-form on X defined as follows. A typical point of X is $x = (x_0, \alpha)$, where $x_0 \in X_0, \alpha \in T^*_x X_0$. Equip X with the $\mathbb{K}^\times$-action by setting $t.(x_0, \alpha) = (x_0, t^{-2}\alpha)$. Let π denote the natural projection $X \twoheadrightarrow X_0$. Then for $v \in T_x X$ we set $\langle \lambda_x, v\rangle := \langle \alpha, d_x\pi(v)\rangle$.

Identify $\mathbb{K}[X_0]$ with a subalgebra in $\mathbb{K}[X]$ via π^* and also embed $\operatorname{Vect}(X_0)$ into $\mathbb{K}[X]$ as the space of functions of degree 2 with respect to the $\mathbb{K}^\times$-action. Then the Poisson bracket on $\mathbb{K}[X]$ is determined by the following equalities:

$$\begin{aligned} &\{f_1, f_2\} = 0,\\ &\{f, v\} = \partial_v f = -L_v f,\\ &\{v_1, v_2\} = [v_1, v_2],\\ &f, f_1, f_2 \in \mathbb{K}[X_0], v, v_1, v_2 \in \operatorname{Vect}(X), \end{aligned} \tag{7.1}$$

where L_v denotes the Lie derivative w.r.t v. With this sign convention the Cartan magic formula has the form

$$L_v\lambda = -d\iota_v\lambda - \iota_v d\lambda, v \in \operatorname{Vect}(X_0), \lambda \in \Omega^\bullet(X_0).$$

Consider the algebra $\mathcal{D}(X_0)$ of linear differential operators on X_0 and equip it with the filtration $\mathcal{D}^{\leqslant i}(X_0)$, where $\mathcal{D}^{\leqslant i}(X_0)$ consists of differential operators of order $\leqslant \frac{i}{2}$. Form the Rees algebra $D_\hbar(X_0) := \bigoplus_{i=0}^\infty \mathcal{D}^{\leqslant i}(X_0)\hbar^i$. This algebra has a natural $\mathbb{K}^\times$-action. Note also that $\mathcal{D}_\hbar(X_0)$ can be considered as the algebra of global section of the *sheaf* of homogeneous differential operators on X_0.

Consider the bundle $\Omega^{top}(= \Omega^{top}_{X_0})$ of top differential forms on X_0, and let $\widetilde{X}_0$ denote the total space of this bundle. This is a smooth algebraic variety acted on by the torus $\mathbb{K}^\times$. The algebra $\mathbb{K}[\widetilde{X}_0]^{\mathbb{K}^\times}$ is naturally identified with $\mathbb{K}[X_0]$, while the space of functions of degree 1 is nothing else but $\Gamma(X_0, \Omega^{top})$. The $\mathbb{K}^\times$-action gives rise to the Euler vector field $\mathbf{eu}$ on $\widetilde{X}_0$. Consider the algebra $\mathcal{D}^{\frac{1}{2}\Omega^{top}}(X_0)$ of twisted differential operators on $\frac{1}{2}\Omega^{top}$, i.e. the algebra $\mathcal{D}(\widetilde{X}_0)^{\mathbb{K}^\times}/(\mathbf{eu} - \frac{1}{2})$. This algebra has a filtration $\mathrm{F}_i\,\mathcal{D}^{\frac{1}{2}\Omega^{top}}(X_0)$ similar to the one above and we can form the Rees algebra

$$\mathcal{D}_\hbar^{\frac{1}{2}\Omega^{top}}(X_0) := \bigoplus_{i=0}^\infty \mathrm{F}_i\,\mathcal{D}^{\frac{1}{2}\Omega^{top}}(X_0)\hbar^i = \mathcal{D}_\hbar(\widetilde{X}_0)^{\mathbb{K}^\times}/(\mathbf{eu} - \frac{1}{2}\hbar^2).$$

Let ϱ denote the projection $\mathcal{D}(\widetilde{X}_0)^{\mathbb{K}^\times} \twoheadrightarrow \mathcal{D}^{\frac{1}{2}\Omega^{top}}(X_0)$. We have the natural embedding $\iota : \mathbb{K}[X_0] \hookrightarrow \mathcal{D}(\widetilde{X}_0)$. For $v \in \operatorname{Vect}(X_0)$ define a $\mathbb{K}^\times$-invariant vector field $\iota(v)$ on $\widetilde{X}_0$ by

$$L_{\iota(v)} f = L_v f, L_{\iota(v)}\sigma = L_v\sigma, f \in \mathbb{K}[X_0], \sigma \in \Gamma(X_0, \Omega^{top}).$$

For brevity, put $\theta = \varrho \circ \iota$.

Proposition 7.1.1. *Equip $\mathbb{K}[X][\hbar]$ with a homogeneous Fedosov star-product corresponding to the zero curvature form. Then there is a unique homomorphism $\mathbb{K}[X][\hbar] \to \mathcal{D}_\hbar^{\frac{1}{2}\Omega^{top}}(X_0)$ mapping $f \in \mathbb{K}[X_0]$ to $\theta(f)$ and $v \in \operatorname{Vect}(X_0)$ to $\hbar^2\theta(v)$. This homomorphism is an isomorphism.*

Proof. Let $f * g = \sum_{i=0}^\infty D_i(f,g)\hbar^{2i}$ be the star-product on $\mathbb{K}[X][\hbar]$. Then it is well known, see, for example, [BW, Lemma 3.3], that $D_i(f,g) = (-1)^i D_i(g,f)$ for all $f, g \in \mathbb{K}[X]$. In particular, $D_1(f,g) = \frac{1}{2}\{f,g\}$ and D_2 is symmetric. So we have

$$\begin{aligned} &f_1 * f_2 = f_1 f_2.\\ &f * v = fv - \frac{1}{2}L_v f\hbar^2.\\ &v * f = fv + \frac{1}{2}L_v f\hbar^2.\\ &v_1 * v_2 - v_2 * v_1 = [v_1, v_2]\hbar^2,\\ &f, f_1, f_2 \in \mathbb{K}[X_0], v, v_1, v_2 \in \operatorname{Vect}(X_0). \end{aligned} \tag{7.2}$$

Consider the $\mathbb{K}[\hbar]$-algebra $\widehat{\mathcal{D}}_\hbar$ generated by $\mathbb{K}[X_0]$ and $\operatorname{Vect}(X_0)$ subject to the relations (7.2). Equip $\widehat{\mathcal{D}}_\hbar$ with the filtration "by the order of a differential operator": $\mathrm{F}_k \widehat{\mathcal{D}}_\hbar = \mathbb{K}[X_0][\hbar] \operatorname{Vect}(X_0)^k$. We have a natural epimorphism $\widehat{\mathcal{D}}_\hbar \to \mathbb{K}[X][\hbar]$. Passing to the associated graded algebras we get a homogeneous homomorphism $\operatorname{gr}\widehat{\mathcal{D}}_\hbar \to \mathbb{K}[X][\hbar]$ (where the last space is considered as an algebra with respect to the commutative product). This homomorphism has an inverse, a natural epimorphism $\mathbb{K}[X][\hbar] \twoheadrightarrow \operatorname{gr}\widehat{\mathcal{D}}_\hbar$, because the algebra $\mathbb{K}[X][\hbar]$ is naturally identified with $S_{\mathbb{K}[X_0][\hbar]}(\operatorname{Vect}(X_0) \otimes \mathbb{K}[\hbar])$. It follows that the natural epimorphism $\widehat{\mathcal{D}}_\hbar \to \mathbb{K}[X][\hbar]$ (here $\mathbb{K}[X][\hbar]$ is a quantum algebra) is an isomorphism.

We are going to check that in $\mathcal{D}^{\frac{1}{2}\Omega^{top}}(X_0)$ the relations analogous to (7.2) hold. Clearly, $\iota(f_1)\iota(f_2) = \iota(f_1 f_2)$. Let us compute now $\iota(f)\iota(v)$. We have

$$\iota(f) \circ \iota(v) g = f L_v g, g \in \mathbb{K}[X_0], \tag{7.3}$$

$$\iota(f) \circ \iota(v)\sigma = f L_v \sigma = L_{fv}\sigma + df \wedge \iota_v \sigma = L_{vf}\sigma - (L_v f)\sigma. \tag{7.4}$$

So we see that $\iota(f)\iota(v) = \iota(fv) - \iota(L_v f)\mathbf{eu}$. Therefore in $\mathcal{D}^{\frac{1}{2}\Omega^{top}}(X_0)$ we have

$$\theta(f)\theta(v) = \theta(fv) - \frac{1}{2}\theta(L_v f).$$

Similarly, we have

$$\theta(v)\theta(f) = \theta(fv) + \frac{1}{2}\theta(L_v f).$$

Finally, from the definition of θ we have $\theta([v_1, v_2]) = [\theta(v_1), \theta(v_2)]$.

Since $\mathbb{K}[X][\hbar]$ is identified with $\widehat{\mathcal{D}}_\hbar$ as above, we have a unique homomorphism $\mathbb{K}[X][\hbar] \to \mathcal{D}_\hbar^{\frac{1}{2}\Omega^{top}}(X_0)$ sending $f \in \mathbb{K}[X_0], v \in \operatorname{Vect}(X_0)$ to $\theta(f), \hbar^2\theta(v)$. Define a filtration on $\mathcal{D}_\hbar^{\frac{1}{2}\Omega^{top}}(X_0)$

by

$$\mathrm{G}_i\,\mathcal{D}_\hbar^{\frac{1}{2}\Omega^{top}}(X_0)=\theta(\mathbb{K}[X_0])[\hbar](\hbar^2\theta(\mathrm{Vect}(X_0)))^i$$

(i.e., again by the order of a differential operator). Then the associated graded algebra is again naturally identified with the commutative algebra $\mathbb{K}[X][\hbar]$. Moreover, under this identification, the associated graded of the homomorphism under consideration is the identity. So we see that the homomorphism is actually an isomorphism. □

Suppose that we have an action of G on X_0. Then the map $\xi\mapsto \widehat{H}_\xi:=\xi_{X_0}\hbar^2\in\mathcal{D}^{\leqslant 2}(X_0)\hbar^2$, where ξ_{X_0} is the velocity vector field associated to ξ, is a quantum comoment map.

The G-action on X_0 lifts naturally to $\widetilde{X}_0$. The quantum moment map for the corresponding G-action on $\mathcal{D}_\hbar(\widetilde{X}_0)$ is $\xi\mapsto\iota(\xi_{X_0})\hbar^2$. The G-action descends to $\mathcal{D}_\hbar^{\frac{1}{2}\Omega^{top}}(X_0)$ with a quantum comoment map $\xi\mapsto\theta(\xi_{X_0})\hbar^2$.

In particular, if Ω^{top} is G-equivariantly trivial, then there is a $G\times\mathbb{K}^\times$-equivariant Hamiltonian isomorphism $\mathbb{K}[X][\hbar]\to\mathcal{D}_\hbar(X_0)$.

7.2. **A general result about filtered algebras.** Let $\mathcal{A}$ be an associative algebra with unit equipped with an increasing exhaustive filtration $\mathrm{F}_i\mathcal{A}, i\geqslant 0$. Let $A:=\sum_i A_i$, where $A_i:=\mathrm{F}_i\mathcal{A}/\mathrm{F}_{i-1}\mathcal{A}$, be the corresponding associated graded algebra. We assume that A is finitely generated over $\mathbb{K}$. Suppose that there is $d>0$ with $[\mathrm{F}_i\mathcal{A},\mathrm{F}_j\mathcal{A}]\subset\mathrm{F}_{i+j-d}\mathcal{A}$ for all i,j. Then A has a canonical Poisson bracket induced from $\mathcal{A}$ with $\{A_i,A_j\}\subset A_{i+j-d}$. Consider the Poisson ideal $I:=A\{A,A\}$. Then I is the minimal ideal of A such that the algebra A/I is Poisson commutative.

Fix a positive integer n. Recall the quotient $\mathcal{A}^{(n)}$ of $\mathcal{A}$ defined in Subsection 6.5. Let $\mathcal{I}_n$ be the kernel of the natural epimorphism $\mathcal{A}\twoheadrightarrow\mathcal{A}^{(n)}$.

Theorem 7.2.1. *In the above notation,* $\sqrt{\operatorname{gr}\mathcal{I}_n}\supset I$.

Proof. The filtration on $\mathcal{A}$ induces a filtration $\mathrm{F}_i\mathcal{A}^{(n)}$ on $\mathcal{A}^{(n)}$. Set $B:=\operatorname{gr}\mathcal{A}^{(n)}$. We need to show that all symplectic leaves of the Poisson subscheme $\operatorname{Spec}B\subset\operatorname{Spec}A$ are 0-dimensional. Assume the converse, pick a point $x\in\operatorname{Spec}B$ whose symplectic leaf has positive dimension. Without loss of generality, we may assume that x lies in the smooth locus of the reduced scheme associated with B.

The algebra $\mathcal{A}^{(n)}$ satisfies the identities s_{2n}. Consider the Rees algebra $R_\hbar(\mathcal{A}^{(n)})=\bigoplus_{i\geqslant 0}\hbar^i\,\mathrm{F}_i\mathcal{A}^{(n)}$. Then $R_\hbar(\mathcal{A}^{(n)})/(\hbar-a)\cong\mathcal{A}^{(n)}$ for $a\in\mathbb{K}, a\neq 0$, and $R_\hbar(\mathcal{A}^{(n)})/(\hbar)\cong B$. In particular, the algebras $R_\hbar(\mathcal{A}^{(n)})/(\hbar-a)$ satisfy s_{2n} for all $a\in\mathbb{K}$. So $R_\hbar(\mathcal{A}^{(n)})$ also satisfies s_{2n}. Let us note that $R_\hbar(\mathcal{A}^{(n)})$ is flat over $\mathbb{K}[\hbar]$.

Let $\mathfrak{m}_x$ be the maximal ideal of x in B and let $\widetilde{\mathfrak{m}}_x$ be the inverse image of $\mathfrak{m}_x$ in $R_\hbar(\mathcal{A}^{(n)})$. Consider the completion $R_\hbar^\wedge(\mathcal{A}^{(n)})$ of $R_\hbar(\mathcal{A}^{(n)})$ w.r.t $\widetilde{\mathfrak{m}}_x$, i.e., $R_\hbar^\wedge(\mathcal{A}^{(n)}):=\varprojlim R_\hbar(\mathcal{A}^{(n)})/\widetilde{\mathfrak{m}}_x^n$.

Lemma 7.2.2. $R_\hbar^\wedge(\mathcal{A}^{(n)})$ *is* $\mathbb{K}[[\hbar]]$*-flat.*

Proof. Consider the $\hbar$-adic completions $\widetilde{\mathfrak{m}}'_x, R'_\hbar(\mathcal{A}^{(n)})$ of $\widetilde{\mathfrak{m}}_x, R_\hbar(\mathcal{A}^{(n)})$. The assumptions of [Lo3, Lemma 2.4.2] hold for $\widetilde{\mathfrak{m}}_x'^2\subset R'_\hbar(\mathcal{A}^{(n)})$. So the *blow-up* algebra $\mathrm{Bl}_{\widetilde{\mathfrak{m}}'_x}(R'_\hbar(\mathcal{A}^{(n)})):=\bigoplus_{i\geqslant 0}\widetilde{\mathfrak{m}}_x'^{2i}$ is Noetherian. Now we can repeat the argument used in the proof of [Lo3, Proposition 2.4.1]. □

It follows from the construction that $R_\hbar^\wedge(\mathcal{A}^{(n)})$ satisfies s_{2n}. Also note that $R_\hbar^\wedge(\mathcal{A}^{(n)})/(\hbar)=B_x^\wedge:=\varprojlim B/\mathfrak{m}_x^k$. Let us check that there are elements $\widetilde{a},\widetilde{b}\in R_\hbar^\wedge(\mathcal{A}^{(n)})$ with $[\widetilde{a},\widetilde{b}]=\hbar^d$.

Recall that the symplectic leaf of $\mathrm{Spec}(B)$ passing through x has positive dimension. Therefore, [Kal, Proposition 3.3], implies that $B^\wedge_x$ can be decomposed into the completed tensor product of the algebra $\mathbb{K}[[a,b]]$ with $\{a,b\}=1$ and of some other Poisson algebra B'. Lift a,b to some elements $\widetilde{a},\widetilde{b}'$ of $R^\wedge_\hbar(\mathcal{A}^{(n)})$. Then $[\widetilde{a},\widetilde{b}']\in \hbar^d+\hbar^{d+1}R^\wedge_\hbar(\mathcal{A}^{(n)})$. The map $\{a,\bullet\}: B^\wedge_x\rightarrow B^\wedge_x$ is surjective. This observation easily implies that we can find an element $y\in \hbar R^\wedge_\hbar(\mathcal{A}^{(n)})$ such that $\widetilde{a}$ and $\widetilde{b}=\widetilde{b}'+\hbar y$ will satisfy $[\widetilde{a},\widetilde{b}]=\hbar^d$.

Now consider the Weyl algebra $\mathbf{A}_\hbar$ with generators u,v and the relation $[u,v]=\hbar^d$ (recall that previously we had $d=2$). We have an obvious homomorphism $\mathbf{A}_\hbar\rightarrow R^\wedge_\hbar(\mathcal{A}^{(n)})$. This homomorphism is injective because $R^\wedge_\hbar(\mathcal{A}^{(n)})$ is $\mathbb{K}[[\hbar]]$-flat. So $\mathbf{A}_\hbar$ satisfies s_{2n}. Hence the Weyl algebra $\mathbf{A}=\mathbf{A}_\hbar/(\hbar-1)$ also satisfies s_{2n}. This is impossible, for example, because the center of $\mathbf{A}$ is trivial, (see, for instance, [McCR, Proposition 13.6.11]). □

Corollary 7.2.3. *Suppose that the scheme* $\mathrm{Spec}(A)$ *has only one 0-dimensional symplectic leaf. Then* $\mathcal{I}_n$ *has finite codimension in* $\mathcal{A}$ *for any* n*. In particular,* $\mathcal{A}$ *has finitely many irreducible representations of any given dimension.*

Proof. The ideal I is just the maximal ideal of a point in A. So $\mathrm{gr}\,\mathcal{I}_n$ is of finite codimension in A. It follows that $\mathcal{I}_n$ is of finite codimension in $\mathcal{A}$. □

References

[BGK] J. Brundan, S. Goodwin, A. Kleshchev. *Highest weight theory for finite W-algebras.* IMRN 2008, no. 15, Art. ID rnn051.

[BeKa] R. Bezrukavnikov, D. Kaledin. *Fedosov quantization in algebraic context.* Moscow Math. J. 4(2004), 559-592.

[BrKl] J. Brundan, A. Kleshchev. *Shifted Yangians and finite W-algebras.* Adv. Math. 200(2006), 136-195.

[Bry] R. Brylinski. *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I.* Prog. Math. 213 (2003), Birkhäuser, Boston, 49-67.

[BV] D. Barbash, D. Vogan. *Unipotent representations.* Ann. Math. 121(1985), 41-110.

[BW] M. Bordemann, S. Waldmann, *A Fedosov star product of the Wick type for Kähler manifolds.* Lett. Math. Phys. 41 (1997), no. 3, 243-253.

[C] R.W. Carter. *Finite groups of Lie type. Conjugacy classes and complex characters.* John Wiley & Sons, 1985.

[CM] D. Collingwood, W. McGovern. *Nilpotent elements in semisimple Lie algebras.* Van Nostrand Reinhold, NY, 1993.

[F] B. Fedosov. *Deformation quantization and index theory,* in Mathematical Topics 9, Akademie Verlag, 1996.

[GG] W.L. Gan, V. Ginzburg. *Quantization of Slodowy slices.* IMRN, 5(2002), 243-255.

[Gi] V. Ginzburg. *Harish-Chandra bimodules for quantized Slodowy slices.* Repres. Theory 13(2009), 236-271.

[GR] S. Gutt, J. Rawnsley. *Natural star products on symplectic manifolds and quantum moment maps.* Preprint (2003), arXiv:math.SG/0304498.

[GRU] S. Goodwin, G. Röhrle, G. Ubly. *On 1-dimensional representations of finite W-algebras associated to simple Lie algebras of exceptional type.* LMS Journal of Computation and Mathematics 13(2010), 357-369.

[GS] V. Guillemin, Sh. Sternberg. *Symplectic techniques in physics.* Cambridge University Press, 1984.

[H] J.E. Humphreys. *Modular representations of simple Lie algebras,* Bull. Amer. Math. Soc. (N.S.) 35(1998), 105-122.

[Ja] J.C. Jantzen. *Einhüllende Algebren halbeinfacher Lie-Algebren.* Ergebnisse der Math., Vol. 3, Springer, New York, Tokio etc., 1983.

[Jo1] A. Joseph. *Gelfand-Kirillov dimension for the annihilators of simple quotients of Verma modules.* J. London Math. Soc. (2), 18(1978), N1, p. 50-60.

[Jo2] A. Joseph. *Goldie rank in the enveloping algebra of a semisimple Lie algebra, II.* J. Algebra 65(1980), 284-306.

[Kac1] V. Kac. *Classification of supersymmetries*. Proceedings of ICM 2002.

[Kac2] V. Kac, Review of [Pr1], MR1345285 (96g:17007).

[KRW] V. Kas, S.-S. Roan, M. Wakimoto. *Quantum reduction for affine superalgebras*. Comm. Math. Phys.241(2003), 307-342.

[Kal] D. Kaledin. *Symplectic singularities from the Poisson point of view*. J. Reine Angew. Math. 600(2006), 135-156.

[Kat] P.I. Katsylo, *Sections of sheets in a reductive algebraic Lie algebra*, Izv. Acad. Nauk SSSR Ser. Mat. 46 (1982) 477486 (in Russian); English transl. in: Math. USSR-Izv. 20 (1983) 449458.

[Kn] F. Knop. *Weyl groups of Hamiltonian manifolds*. Preprint, arXiv:dg-ga/9712010.

[Lo1] I.V. Losev. *Symplectic slices for reductive groups*. Mat. Sbornik 197(2006), N2, 75-86 (in Russian). English translation in: Sbornik Math. 197(2006), N2, 213-224.

[Lo2] I.V. Losev. *Quantized symplectic actions and W-algebras*. J. Amer. Math. Soc. 23(2010), 35-59.

[Lo3] I. Losev. *Finite dimensional representations of W-algebras*. arXiv:0807.1023. To appear in Duke Math. J.

[Lo4] I. Losev. *On the structure of the category $\mathcal{O}$ for W-algebras*. Séminars et Congrès 25(2010), 351-368.

[LS] G. Lusztig, N. Spaltenstein. *Induced unipotent classes*. J. London Math. Soc. (2), 19(1979), 41-52.

[Ly] T.E. Lynch. *Generalized Whittaker vectors and representation theory*. Thesis, M.I.T., 1979.

[McG] W. McGovern. *The adjoint representation and the adjoint action*. Encyclopaedia of mathematical sciences, 131. Invariant theory and algebraic transformation groups, II, Springer Verlag, Berlin, 2002.

[McCR] J.C. McConnel, J.C. Robson. *Noncommutative noetherian rings*. With the cooperation of L. W. Small. 2nd ed, Graduate texts in Math. 30, AMS 2001.

[Mo1] C. Moeglin. *Modèles de Whittaker et idéaux primitifs complètement premiers dans les algèbres enveloppantes.* C.R. Acad. Sci. Paris, Sér. I 303(1986), No. 17, 845848.

[Mo2] C. Moeglin. *Modèles de Whittaker et idéaux primitifs complètement premiers dans les algèbres enveloppantes.* Math. Scand. 63 (1988), 535.

[OV] A.L. Onishchik, E.B. Vinberg. *Seminar on Lie groups and algebraic groups*. Moscow, Nauka 1988 (in Russian). English translation: Berlin, Springer, 1990.

[Pr1] A. Premet. *Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture*. Invent. Math. 121, 79-121 (1995).

[Pr2] A. Premet. *Special transverse slices and their enveloping algebras*. Adv. Math. 170(2002), 1-55.

[Pr3] A. Premet. *Enveloping algebras of Slodowy slices and the Joseph ideal*. J. Eur. Math. Soc, 9(2007), N3, 487-543.

[Pr4] A. Premet. *Primitive ideals, non-restricted representations and finite W-algebras*. Moscow Math. J. 7(2007), 743-762.

[Pr5] A. Premet. *Commutative quotients of finite W-algebras*. Preprint (2008), Adv. Math 225(2010), 269-306.

[S] S. Skryabin. An appendix to [Pr2].