AN ENERGY GAP FOR YANG-MILLS CONNECTIONS

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ABSTRACT. Consider a Yang-Mills connection over a Riemann manifold $M=M^n,\, n\geq 3$, where M may be compact or complete. Then its energy must be bounded from below by some positive constant, if M satisfies certain conditions, unless the connection is flat.

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1. Introduction

We consider the problem: When is a Yang-Mills connection non-flat? Of course, the trivial answer $F_{\mu\lambda} \not\equiv 0$ is unsatisfactory. Bourguignon and Lawson proved in [3, Theorem C], among other results, that any Yang-Mills connection over S^n , $n \geq 3$, the field strength of which satisfies the pointwise estimate

(1.1)
$$F^2 = -\operatorname{tr}(F_{\mu\lambda}F^{\mu\lambda}) < \binom{n}{2}$$

is flat.

We want to prove that under certain assumptions on the base space M, which is supposed to be a Riemannian manifold of dimension $n \geq 3$, the energy of a Yang-Mills connection has to satisfy

$$\left(\int_{M} |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} \ge \kappa_0 > 0,$$

where κ_0 depends only on the Sobolev constants of M, n and the dimension of the Lie group \mathcal{G} , unless the connection is flat.

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Here,

$$(1.3) |F| = \sqrt{F^2},$$

and we also call the left-hand side of (1.2) energy though this label is only correct when n=4. However, this norm is also the crucial norm, which has to be (locally) small, used to prove regularity of a connection, cf. [4, Theorem 1.3].

The exponent $\frac{n}{2}$ naturally pops up when Sobolev inequalities are applied to solutions of differential equations satisfied by the field strength or the energy density of a connection in the adjoint bundle.

We distinguish two cases: M compact and M complete and non-compact. When M is compact, we require

$$(1.4) \bar{R}_{\alpha\beta}\Lambda^{\alpha}_{\lambda}\Lambda^{\beta\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}\Lambda^{\alpha\beta}\Lambda^{\mu\lambda} \ge c_0\Lambda_{\alpha\beta}\Lambda^{\alpha\beta}$$

for all skew-symmetric $\Lambda_{\alpha\beta} \in T^{0,2}(M)$, where $0 < c_0$, while for non-compact M the weaker assumption

(1.5)
$$\bar{R}_{\alpha\beta}\Lambda_{\lambda}^{\alpha}\Lambda^{\beta\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}\Lambda^{\alpha\beta}\Lambda^{\mu\lambda} \ge 0$$

and in addition

(1.6)
$$\left(\int_{M} u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \le c_1 \int_{M} |Du|^2 \qquad \forall u \in H^{1,2}(M)$$

should be satisfied.

1.1. **Remark.** (i) If M is a space of constant curvature

(1.7)
$$\bar{R}_{\alpha\beta\mu\lambda} = K_M(\bar{g}_{\alpha\mu}\bar{g}_{\beta\lambda} - \bar{g}_{\alpha\lambda}\bar{g}_{\beta\mu}),$$

then

(1.8)
$$\bar{R}_{\alpha\beta}\Lambda_{\lambda}^{\alpha}\Lambda^{\beta\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}\Lambda^{\alpha\beta}\Lambda^{\mu\lambda} = (n-2)K_{M}\Lambda_{\alpha\beta}\Lambda^{\alpha\beta}.$$

In case n=2 the curvature term therefore vanishes, and this result is also valid for an arbitrary two-dimensional Riemannian manifold, since the curvature tensor then has the same structure as in (1.7) though K_M is not necessarily constant.

- (ii) If $M = \mathbb{R}^n$, $n \geq 3$, the conditions (1.5) and (1.6) are always valid.
- 1.2. **Theorem.** Let $M = M^n$, $n \ge 3$, be a compact Riemannian for which the condition (1.4) with $c_0 > 0$ holds. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or satisfies (1.2) for some constant $\kappa_0 > 0$ depending on the Sobolev constants of M, n, c_0 , and the dimension of the Lie group.
- 1.3. **Theorem.** Let $M = M^n$, $n \ge 3$, be complete, non-compact and assume that the conditions (1.5) and (1.6) hold. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or the estimate (1.2) is valid. The constant $\kappa_0 > 0$ in (1.2) depends on the constant c_1 in (1.6), n, and the dimension of the Lie group.

2. The compact case

Let $(P, M, \mathcal{G}, \mathcal{G})$ be a principal fiber bundle where $M = M^n$, $n \geq 3$ is a compact Riemannian manifold with metric $\bar{g}_{\alpha\beta}$ and \mathcal{G} a compact, semi-simple Lie group with Lie algebra \mathfrak{g} . Let $f_c = (f_c^a)$ be a basis of ad \mathfrak{g} and

$$(2.1) A_{\mu} = f_c A_{\mu}^c$$

a Yang-Mills connection in the adjoint bundle $(E, M, \mathfrak{g}, \mathrm{Ad}(\mathcal{G}))$.

The curvature tensor of the connection is given by

$$R^a_{b\mu\lambda} = f^a_{cb} F^c_{\mu\lambda},$$

where

$$(2.3) F_{\mu\lambda} = f_c F_{\mu\lambda}^c$$

is the field strength of the connection, and

(2.4)
$$F^2 \equiv \gamma_{ab} F^a_{\mu\lambda} F^{b\mu\lambda} = R_{ab\mu\lambda} R^{ab\mu\lambda}$$

the energy density of the connection—at least up to a factor $\frac{1}{4}$.

Here, γ_{ab} is the Cartan-Killing metric acting on elements of the fiber \mathfrak{g} , and Latin indices are raised or lowered with respect to the inverse γ^{ab} or γ_{ab} , and Greek indices with respect to the metric of M.

2.1. **Definition.** The adjoint bundle E is vector bundle; let E^* be the dual bundle, then we denote by

(2.5)
$$T^{r,s}(E) = \Gamma(\underbrace{E \otimes \cdots \otimes E}_{r} \otimes \underbrace{E^* \otimes \cdots \otimes E^*}_{s})$$

the sections of the corresponding tensor bundle.

Thus, we have

(2.6)
$$F_{\mu\lambda}^{a} \in T^{1,0}(E) \otimes T^{0,2}(M).$$

Since A_{μ} is a Yang-Mills connection it solves the Yang-Mills equation

$$(2.7) F^{a\alpha}_{\lambda;\alpha} = 0,$$

where we use Einstein's summation convention, a semi-colon indicates covariant differentiation, and where we stipulate that a covariant derivative is always a *full* tensor, i.e.,

$$F^a_{\mu\lambda;\alpha} = F^a_{\mu\lambda,\alpha} + f^a_{bc} A^b_{\alpha} F^c_{\mu\lambda} - \bar{\Gamma}^{\gamma}_{\alpha\mu} F^a_{\gamma\lambda} - \bar{\Gamma}^{\gamma}_{\alpha\lambda} F^a_{\mu\gamma},$$

where $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ are the Christoffel symbols of the Riemannian connection; a comma indicates partial differentiation.

Before we formulate the crucial lemma let us note that $\bar{R}_{\alpha\beta\gamma\delta}$ resp. $\bar{R}_{\alpha\beta}$ symbolize the Riemann curvature tensor resp. the Ricci tensor of $\bar{g}_{\alpha\beta}$.

2.2. **Lemma.** Let A_{μ} be a Yang-Mills connection, then its energy density F^2 solves the equation

$$(2.9) \qquad -\frac{1}{4}\Delta F^2 + \frac{1}{2}F_{a\mu\lambda;\alpha}F^{a\mu\lambda}{}^{\alpha}{}^{\alpha} + \bar{R}_{\beta\mu}F^{a\beta}{}_{\lambda}F_a{}^{\mu\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}F_a{}^{\alpha\beta}F^{a\mu\lambda} = -f^a_{cb}F^c_{\alpha\mu}F^{b\alpha}{}_{\lambda}F_a{}^{\mu\lambda}.$$

Proof. Differentiating (2.7) covariantly with respect to x^{μ} and using the Ricci identities we obtain

(2.10)
$$0 = -F^{a\alpha}_{\lambda;\alpha\mu}$$

$$= -F^{a\alpha}_{\lambda;\mu\alpha} + R^{a}_{b\alpha\mu}F^{b\alpha}_{\lambda} + \bar{R}^{\alpha}_{\beta\alpha\mu}F^{a\beta}_{\lambda} + \bar{R}^{\beta}_{\lambda\mu\alpha}F^{a\alpha}_{\beta}.$$

On the other hand, differentiating the second Bianchi identities

$$(2.11) 0 = F_{\alpha\lambda;\mu}^a + F_{\mu\alpha;\lambda}^a + F_{\lambda\mu;\alpha}^a$$

we infer

$$(2.12) 0 = F^{a\alpha}_{\lambda;\mu\alpha} + F^{a\alpha}_{\mu;\lambda\alpha} + \Delta F^{a}_{\lambda\mu},$$

and we deduce further

$$(2.13) -\Delta F_{\mu\lambda}^a F_a^{\ \mu\lambda} = -2F_{\lambda:\mu\alpha}^{a\alpha} F_a^{\ \mu\lambda}.$$

In view of (2.10) we then conclude

$$(2.14) 0 = -\frac{1}{2} \Delta F^a_{\mu\lambda} F_a^{\ \mu\lambda} + R^a_{\ b\alpha\mu} F^{b\alpha}_{\ \lambda} F_a^{\ \mu\lambda} + \bar{R}_{\beta\mu} F^{a\beta}_{\ \lambda} F_a^{\ \mu\lambda} + \bar{R}^{\beta}_{\beta\mu} F^{a\beta}_{\ \lambda} F_a^{\ \mu\lambda},$$

which is equivalent to

$$(2.15) 0 = -\frac{1}{2} \Delta F_{\mu\lambda}^a F_a^{\ \mu\lambda} + f_{cb}^a F_{\alpha\mu}^c F^{b\alpha}_{\ \lambda} F_a^{\ \mu\lambda} + \bar{R}_{\beta\mu} F^{a\beta}_{\ \lambda} F_a^{\ \mu\lambda} - \bar{R}_{\alpha\mu\beta\lambda} F^{a\alpha\beta} F_a^{\ \mu\lambda},$$

in view of (2.2).

Finally, using the first Bianchi identities,

$$\bar{R}_{\alpha\beta\mu\lambda} + \bar{R}_{\alpha\mu\lambda\beta} + \bar{R}_{\alpha\lambda\beta\mu} = 0,$$

we deduce

$$(2.17) \qquad \bar{R}_{\alpha\beta\mu\lambda}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} + \bar{R}_{\alpha\mu\lambda\beta}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} + \bar{R}_{\alpha\lambda\beta\mu}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} = 0,$$

and hence

(2.18)
$$\bar{R}_{\alpha\beta\mu\lambda}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} = 2\bar{R}_{\alpha\mu\beta\lambda}F^{a\alpha\beta}F_{a}^{\ \mu\lambda},$$

from which the equation (2.9) immediately follows.

Proof of Theorem 1.2 on page 2. Define

$$(2.19) u = F^2,$$

then

$$(2.20) \bar{R}_{\beta\mu}F^{a\beta}_{\ \lambda}F_a^{\ \mu\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}F_a^{\ \alpha\beta}F^{a\mu\lambda} \ge c_0 u,$$

where $c_0 > 0$, in view of the assumption (1.4) on page 2.

Multiplying (2.9) with u and integrating by part we obtain

(2.21)
$$\frac{3}{8} \int_{M} |Du|^{2} + c_{0} \int_{M} u^{2} \le c \int_{M} \sqrt{u} u^{2},$$

where we used the simple estimate

$$(2.22) |Du|^2 \le 4F_{a\mu\lambda;\alpha}F^{a\mu\lambda}{}_{;}^{\alpha}u^2$$

and where c depends on n and the dimension of \mathfrak{g} ; note that

$$(2.23) f_c \in SO(\mathfrak{g}, \gamma_{ab}).$$

The integral on the right-hand side of (2.21) is estimated by

(2.24)
$$\int_{M} \sqrt{u} u^{2} \leq \left(\int_{M} u^{\frac{n}{4}} \right)^{\frac{2}{n}} \left(\int_{M} u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}},$$

where

$$\left(\int_{M} u^{\frac{n}{4}} \right)^{\frac{2}{n}} = \left(\int_{M} |F|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

Applying then the Sobolev inequality

(2.26)
$$\left(\int_{M} u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \le c_1 \int_{M} |Du|^2 + c_2 \int_{M} u^2,$$

cf. [1], we obtain

$$(2.27) \qquad \left(\int_{M} u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le c_3 \left(\int_{M} |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_{M} u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}},$$

where c_3 depends on c_1, c_2, c_0 and c. Hence, we deduce $u \equiv 0$ or

(2.28)
$$c_3^{-1} \le \left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

Setting

(2.29)
$$\kappa_0 = c_3^{-1}$$

finishes the proof.

3. The non-compact case

We now suppose that ${\cal M}={\cal M}^n$ is a complete, non-compact Riemannian manifold. Then there holds

(3.1)
$$H^{1,2}(M) = H_0^{1,2}(M),$$

i.e., the test functions $C_c^\infty(M)$ are dense in the Sobolev space $H^{1,2}(M)$, see [1, Lemme 4] or [2, Theorem 2.6].

Since we do not a priori

$$(3.2) F^2 \in H^{1,2}(M),$$

but only

(3.3)
$$F^2 \in H^{1,2}_{loc}(M),$$

the preceding proof has to be modified.

Let $\eta = \eta(t)$ be defined through

(3.4)
$$\eta(t) = \begin{cases} 1, & t \le 1, \\ (2-t)^q, & 1 \le t \le 2, \\ 0, & t \ge 2, \end{cases}$$

where

$$(3.5) q = \max(1, \frac{8}{n}).$$

Fix a point $x_0 \in M$ and let r be the Riemannian distance function with center in x_0

$$(3.6) r(x) = d(x_0, x).$$

Then r is Lipschitz such that

$$(3.7) |Dr| = 1$$

almost everywhere.

For k > 1 define

(3.8)
$$\eta_k(x) = \eta(k^{-1}r).$$

The functions

(3.9)
$$u^{p-1}\eta_{k}^{p}$$

where

$$(3.10) p = \frac{n}{4},$$

then have compact support, and multiplying (2.9) on page 4 with $u^{p-1}\eta_k^p$ yields

$$(3.11) \qquad (\frac{p}{4} + \frac{1}{8} - \epsilon) \int_{m} |Du|^{2} u^{p-2} \eta_{k}^{p} \le c \left(\int_{M} |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_{M} (u \eta_{k})^{\frac{n}{n-2}p} \right)^{\frac{n-2}{n}} + c_{\epsilon} \int_{M} |D\eta_{k}|^{2} \eta_{k}^{p-2} u^{p},$$

where $0 < \epsilon$ is supposed to be small.

Furthermore, there holds

(3.12)
$$\int_{M} |D(u\eta_{k})^{\frac{p}{2}}|^{2} = \frac{p^{2}}{4} \int_{M} |Du\eta_{k} + uD\eta_{k}|^{2} (u\eta_{k})^{p-2}$$

$$\leq (1+\epsilon) \frac{p^{2}}{4} \int_{M} |Du|^{2} u^{p-2} \eta_{k}^{p} + c_{\epsilon} \frac{p^{2}}{4} \int_{M} |D\eta_{k}|^{2} \eta_{k}^{p-2} u^{p}.$$

Now, choosing ϵ so small that

$$(3.13) (1+\epsilon)^{\frac{p^2}{4}} \le p(^{\frac{p}{4}} + \frac{1}{8} - \epsilon)$$

and setting

$$(3.14) \varphi = (u\eta_k)^{\frac{p}{2}}$$

we obtain

$$(3.15) \quad \int_{M} |D\varphi|^{2} \le pc \left(\int_{M} |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_{M} \varphi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + c_{\epsilon} \int_{M} |D\eta_{k}|^{2} \eta_{k}^{p-2} u^{p},$$

where c_{ϵ} is a new constant.

We furthermore observe that

$$(3.16) |D\eta_k|^2 \eta_k^{p-2} \le q^2 k^{-2} (2 - k^{-1} r)^{qp-2},$$

subject to

$$(3.17) 1 \le k^{-1}r \le 2.$$

In view of (3.5) and (3.10)

$$(3.18) qp - 2 \ge 0$$

and hence

$$(3.19) |D\eta_k|^2 \eta_k^{p-2} \le q^2 k^{-2}.$$

Applying now the Sobolev inequality (1.6) on page 2 to φ and choosing

we conclude $|F| \equiv 0$, if

$$\left(\int_{M} |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \kappa_0.$$

Indeed, if the preceding inequality is valid, then we deduce from (3.15)

$$(3.22) \qquad \left(1 - \kappa_0^{-1} \left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right) \left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le c_{\epsilon} q^2 k^{-2} \int_M |F|^{\frac{n}{2}}.$$

In the limit $k \to \infty$ we obtain

$$\left(\int_{M} |u|^{\frac{pn}{n-2}}\right)^{\frac{n-2}{n}} \le 0.$$

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