### AN ENERGY GAP FOR YANG-MILLS CONNECTIONS

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Abstract. Consider a Yang-Mills connection over a Riemann manifold  $M = M^n$ ,  $n \geq 3$ , where M may be compact or complete. Then its energy must be bounded from below by some positive constant, if M satisfies certain conditions, unless the connection is flat.

#### CONTENTS



#### 1. INTRODUCTION

<span id="page-0-0"></span>We consider the problem: When is a Yang-Mills connection non-flat? Of course, the trivial answer  $F_{\mu\lambda} \neq 0$  is unsatisfactory. Bourguignon and Lawson proved in [\[3,](#page-6-1) Theorem C], among other results, that any Yang-Mills connection over  $S<sup>n</sup>$ ,  $n \geq 3$ , the field strength of which satisfies the pointwise estimate

(1.1) 
$$
F^2 = -\operatorname{tr}(F_{\mu\lambda}F^{\mu\lambda}) < \binom{n}{2}
$$

is flat.

We want to prove that under certain assumptions on the base space  $M$ , which is supposed to be a Riemannian manifold of dimension  $n \geq 3$ , the energy of a Yang-Mills connection has to satisfy

<span id="page-0-1"></span>(1.2) 
$$
\left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} \ge \kappa_0 > 0,
$$

where  $\kappa_0$  depends only on the Sobolev constants of M, n and the dimension of the Lie group  $G$ , unless the connection is flat.

Date: July 18, 2018.

<sup>2000</sup> Mathematics Subject Classification. 35J60, 53C21, 53C44, 53C50, 58J05.

Key words and phrases. energy gap, Yang-Mills connections.

This work has been supported by the DFG.

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Here,

$$
|F| = \sqrt{F^2},
$$

and we also call the left-hand side of [\(1.2\)](#page-0-1) energy though this label is only correct when  $n = 4$ . However, this norm is also the crucial norm, which has to be (locally) small, used to prove regularity of a connection, cf. [\[4,](#page-6-2) Theorem 1.3].

The exponent  $\frac{n}{2}$  naturally pops up when Sobolev inequalities are applied to solutions of differential equations satisfied by the field strength or the energy density of a connection in the adjoint bundle.

We distinguish two cases: M compact and M complete and non-compact. When  $M$  is compact, we require

<span id="page-1-3"></span>(1.4) 
$$
\bar{R}_{\alpha\beta} \Lambda^{\alpha}_{\lambda} \Lambda^{\beta\lambda} - \frac{1}{2} \bar{R}_{\alpha\beta\mu\lambda} \Lambda^{\alpha\beta} \Lambda^{\mu\lambda} \geq c_0 \Lambda_{\alpha\beta} \Lambda^{\alpha\beta}
$$

for all skew-symmetric  $\Lambda_{\alpha\beta} \in T^{0,2}(M)$ , where  $0 < c_0$ , while for non-compact M the weaker assumption

<span id="page-1-1"></span>(1.5) 
$$
\bar{R}_{\alpha\beta} \Lambda^{\alpha}_{\lambda} \Lambda^{\beta\lambda} - \frac{1}{2} \bar{R}_{\alpha\beta\mu\lambda} \Lambda^{\alpha\beta} \Lambda^{\mu\lambda} \ge 0
$$

and in addition

<span id="page-1-2"></span>(1.6) 
$$
\left(\int_M u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le c_1 \int_M |Du|^2 \quad \forall u \in H^{1,2}(M)
$$

should be satisfied.

<span id="page-1-0"></span>1.1. **Remark.** (i) If  $M$  is a space of constant curvature

(1.7) 
$$
\bar{R}_{\alpha\beta\mu\lambda} = K_M(\bar{g}_{\alpha\mu}\bar{g}_{\beta\lambda} - \bar{g}_{\alpha\lambda}\bar{g}_{\beta\mu}),
$$

then

(1.8) 
$$
\bar{R}_{\alpha\beta}A_{\lambda}^{\alpha}A^{\beta\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}A^{\alpha\beta}A^{\mu\lambda} = (n-2)K_M A_{\alpha\beta}A^{\alpha\beta}
$$

In case  $n = 2$  the curvature term therefore vanishes, and this result is also valid for an arbitrary two-dimensional Riemannian manifold, since the curvature tensor then has the same structure as in  $(1.7)$  though  $K_M$  is not necessarily constant.

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(ii) If  $M = \mathbb{R}^n$ ,  $n \geq 3$ , the conditions [\(1.5\)](#page-1-1) and [\(1.6\)](#page-1-2) are always valid.

<span id="page-1-4"></span>1.2. **Theorem.** Let  $M = M^n$ ,  $n \geq 3$ , be a compact Riemannian for which the condition [\(1.4\)](#page-1-3) with  $c_0 > 0$  holds. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or satisfies [\(1.2\)](#page-0-1) for some constant  $\kappa_0 > 0$  depending on the Sobolev constants of M, n, c<sub>0</sub>, and the dimension of the Lie group.

1.3. Theorem. Let  $M = M^n$ ,  $n \geq 3$ , be complete, non-compact and assume that the conditions  $(1.5)$  and $(1.6)$  hold. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or the estimate [\(1.2\)](#page-0-1) is valid. The constant  $\kappa_0 > 0$  in (1.2) depends on the constant  $c_1$  in [\(1.6\)](#page-1-2), n, and the dimension of the Lie group.

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### <span id="page-2-2"></span>2. The compact case

<span id="page-2-0"></span>Let  $(P, M, \mathcal{G}, \mathcal{G})$  be a principal fiber bundle where  $M = M^n$ ,  $n \geq 3$  is a compact Riemannian manifold with metric  $\bar{g}_{\alpha\beta}$  and G a compact, semi-simple Lie group with Lie algebra  $\mathfrak{g}$ . Let  $f_c = (f_{cb}^a)$  be a basis of ad  $\mathfrak{g}$  and

$$
(2.1)\t\t\t A_{\mu} = f_c A_{\mu}^c
$$

a Yang-Mills connection in the adjoint bundle  $(E, M, \mathfrak{g}, \mathrm{Ad}(\mathcal{G}))$ . The curvature tensor of the connection is given by

(2.2) 
$$
R^a{}_{b\mu\lambda} = f^a_{cb} F^c_{\mu\lambda},
$$

where

$$
(2.3) \t\t F_{\mu\lambda} = f_c F_{\mu\lambda}^c
$$

is the field strength of the connection, and

(2.4) 
$$
F^2 \equiv \gamma_{ab} F^a_{\mu\lambda} F^{b\mu\lambda} = R_{ab\mu\lambda} R^{ab\mu\lambda}
$$

the energy density of the connection—at least up to a factor  $\frac{1}{4}$ .

Here,  $\gamma_{ab}$  is the Cartan-Killing metric acting on elements of the fiber g, and Latin indices are raised or lowered with respect to the inverse  $\gamma^{ab}$  or  $\gamma_{ab}$ , and Greek indices with respect to the metric of M.

2.1. **Definition.** The adjoint bundle E is vector bundle; let  $E^*$  be the dual bundle, then we denote by

(2.5) 
$$
T^{r,s}(E) = \Gamma(\underbrace{E \otimes \cdots \otimes E}_{r} \otimes \underbrace{E^* \otimes \cdots \otimes E^*}_{s})
$$

the sections of the corresponding tensor bundle.

Thus, we have

(2.6) 
$$
F^a_{\mu\lambda} \in T^{1,0}(E) \otimes T^{0,2}(M).
$$

<span id="page-2-1"></span>Since  $A_\mu$  is a Yang-Mills connection it solves the Yang-Mills equation

$$
F^{a\alpha}_{\quad \ \lambda;\alpha} = 0,
$$

where we use Einstein's summation convention, a semi-colon indicates covariant differentiation, and where we stipulate that a covariant derivative is always a full tensor, i.e.,

(2.8) 
$$
F^a_{\mu\lambda;\alpha} = F^a_{\mu\lambda,\alpha} + f^a_{bc} A^b_{\alpha} F^c_{\mu\lambda} - \bar{\Gamma}^{\gamma}_{\alpha\mu} F^a_{\gamma\lambda} - \bar{\Gamma}^{\gamma}_{\alpha\lambda} F^a_{\mu\gamma},
$$

where  $\bar{\Gamma}_{\alpha\beta}^{\gamma}$  are the Christoffel symbols of the Riemannian connection; a comma indicates partial differentiation.

Before we formulate the crucial lemma let us note that  $\bar{R}_{\alpha\beta\gamma\delta}$  resp.  $\bar{R}_{\alpha\beta}$ symbolize the Riemann curvature tensor resp. the Ricci tensor of  $\bar{g}_{\alpha\beta}$ .

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2.2. Lemma. Let  $A_\mu$  be a Yang-Mills connection, then its energy density  $F^2$  solves the equation

<span id="page-3-1"></span>(2.9) 
$$
- \frac{1}{4} \Delta F^2 + \frac{1}{2} F_{a\mu\lambda;\alpha} F^{a\mu\lambda}{}_{;}^{\alpha} + \bar{R}_{\beta\mu} F^{a\beta}{}_{\lambda} F_a{}^{\mu\lambda} - \frac{1}{2} \bar{R}_{\alpha\beta\mu\lambda} F_a{}^{\alpha\beta} F^{a\mu\lambda} = -f^a_{cb} F^c_{\alpha\mu} F^{b\alpha}{}_{\lambda} F_a{}^{\mu\lambda}.
$$

*Proof.* Differentiating [\(2.7\)](#page-2-1) covariantly with respect to  $x^{\mu}$  and using the Ricci identities we obtain

(2.10) 
$$
0 = -F^{a\alpha}_{\lambda;\alpha\mu} = -F^{a\alpha}_{\lambda;\mu\alpha} + R^{a}_{\ b\alpha\mu}F^{b\alpha}_{\ \lambda} + \bar{R}^{\alpha}_{\ \beta\alpha\mu}F^{a\beta}_{\ \lambda} + \bar{R}^{\beta}_{\ \lambda\mu\alpha}F^{a\alpha}_{\ \beta}.
$$

<span id="page-3-0"></span>On the other hand, differentiating the second Bianchi identities

(2.11) 
$$
0 = F^a_{\alpha\lambda;\mu} + F^a_{\mu\alpha;\lambda} + F^a_{\lambda\mu;\alpha}
$$

we infer  
(2.12) 
$$
0 = F^{a\alpha}_{\lambda;\mu\alpha} + F^{a}_{\mu}{}^{\alpha}_{;\lambda\alpha} + \Delta F^{a}_{\lambda\mu},
$$

and we deduce further

(2.13) 
$$
- \Delta F^a_{\mu\lambda} F_a^{\ \mu\lambda} = -2F^{a\alpha}_{\ \ \lambda;\mu\alpha} F_a^{\ \mu\lambda}.
$$

In view of [\(2.10\)](#page-3-0) we then conclude

(2.14) 
$$
0 = -\frac{1}{2} \Delta F^a_{\mu\lambda} F_a^{\mu\lambda} + R^a_{\phantom{a}b\alpha\mu} F^{b\alpha}_{\phantom{b}\lambda} F_a^{\mu\lambda} + \bar{R}_{\beta\mu} F^{a\beta}_{\phantom{a}\lambda} F_a^{\mu\lambda} + \bar{R}^{\beta}_{\phantom{\beta}\lambda\mu\alpha} F^{a\alpha}_{\phantom{a}\beta} F_a^{\mu\lambda},
$$

which is equivalent to

(2.15) 
$$
0 = -\frac{1}{2}\Delta F^a_{\mu\lambda}F_a^{\ \mu\lambda} + f^a_{cb}F^c_{\alpha\mu}F^{b\alpha}_{\ \lambda}F_a^{\ \mu\lambda} + \bar{R}_{\beta\mu}F^{a\beta}_{\ \lambda}F_a^{\ \mu\lambda} - \bar{R}_{\alpha\mu\beta\lambda}F^{a\alpha\beta}F_a^{\ \mu\lambda},
$$

in view of [\(2.2\)](#page-2-2).

Finally, using the first Bianchi identities,

(2.16) 
$$
\bar{R}_{\alpha\beta\mu\lambda} + \bar{R}_{\alpha\mu\lambda\beta} + \bar{R}_{\alpha\lambda\beta\mu} = 0,
$$

we deduce

(2.17) 
$$
\bar{R}_{\alpha\beta\mu\lambda}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} + \bar{R}_{\alpha\mu\lambda\beta}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} + \bar{R}_{\alpha\lambda\beta\mu}F^{a\alpha\beta}F_{a}^{\ \mu\lambda} = 0,
$$

and hence

(2.18) 
$$
\bar{R}_{\alpha\beta\mu\lambda}F^{a\alpha\beta}F_{a}^{\ \mu\lambda}=2\bar{R}_{\alpha\mu\beta\lambda}F^{a\alpha\beta}F_{a}^{\ \mu\lambda},
$$

from which the equation [\(2.9\)](#page-3-1) immediately follows.  $\square$ 

Proof of Theorem [1.2](#page-1-4) on page [2](#page-1-4). Define

 $(2.19)$ 2 ,

then

(2.20) 
$$
\bar{R}_{\beta\mu}F^{a\beta}_{\quad \lambda}F_{a}^{\ \mu\lambda}-\frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}F_{a}^{\ \alpha\beta}F^{a\mu\lambda}\geq c_{0}u,
$$

where  $c_0 > 0$ , in view of the assumption [\(1.4\)](#page-1-3) on page [2.](#page-1-3)

<span id="page-4-1"></span>Multiplying  $(2.9)$  with u and integrating by part we obtain

(2.21) 
$$
\frac{3}{8} \int_M |Du|^2 + c_0 \int_M u^2 \leq c \int_M \sqrt{u} u^2,
$$

where we used the simple estimate

(2.22) 
$$
|Du|^2 \le 4F_{a\mu\lambda;\alpha}F^{a\mu\lambda}{}^{\alpha}_{\quad;\alpha}u^2
$$

and where  $c$  depends on  $n$  and the dimension of  $\mathfrak{g}$ ; note that

(2.23) 
$$
f_c \in \mathrm{SO}(\mathfrak{g}, \gamma_{ab}).
$$

The integral on the right-hand side of [\(2.21\)](#page-4-1) is estimated by

(2.24) 
$$
\int_{M} \sqrt{u}u^{2} \leq \left(\int_{M} u^{\frac{n}{4}}\right)^{\frac{2}{n}} \left(\int_{M} u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}},
$$

where

(2.25) 
$$
\left(\int_M u^{\frac{n}{4}}\right)^{\frac{2}{n}} = \left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}}.
$$

Applying then the Sobolev inequality

(2.26) 
$$
\left(\int_M u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le c_1 \int_M |Du|^2 + c_2 \int_M u^2,
$$

cf. [\[1\]](#page-6-3), we obtain

$$
(2.27) \qquad \left(\int_M u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le c_3 \left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}},
$$

where  $c_3$  depends on  $c_1, c_2, c_0$  and c. Hence, we deduce  $u \equiv 0$  or

(2.28) 
$$
c_3^{-1} \le \left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}}.
$$

Setting

$$
\kappa_0 = c_3^{-1}
$$

<span id="page-4-0"></span>finishes the proof.  $\Box$ 

### 3. The non-compact case

We now suppose that  $M = M^n$  is a complete, non-compact Riemannian manifold. Then there holds

(3.1) 
$$
H^{1,2}(M) = H_0^{1,2}(M),
$$

i.e., the test functions  $C_c^{\infty}(M)$  are dense in the Sobolev space  $H^{1,2}(M)$ , see [\[1,](#page-6-3) Lemme 4] or [\[2,](#page-6-4) Theorem 2.6].

Since we do not a priori

(3.2) 
$$
F^2 \in H^{1,2}(M),
$$

but only

(3.3) 
$$
F^2 \in H^{1,2}_{loc}(M),
$$

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the preceding proof has to be modified.

Let  $\eta = \eta(t)$  be defined through

(3.4) 
$$
\eta(t) = \begin{cases} 1, & t \le 1, \\ (2-t)^q, & 1 \le t \le 2, \\ 0, & t \ge 2, \end{cases}
$$

where

<span id="page-5-0"></span>
$$
(3.5) \t\t q = \max(1, \frac{8}{n}).
$$

Fix a point  $x_0 \in M$  and let r be the Riemannian distance function with center in  $x_0$ 

$$
(3.6) \t\t\t r(x) = d(x_0, x).
$$

Then  $r$  is Lipschitz such that

$$
(3.7) \t|Dr| = 1
$$

almost everywhere.

For  $k \geq 1$  define

$$
(3.8) \qquad \qquad \eta_k(x) = \eta(k^{-1}r).
$$

The functions

$$
(3.9) \t\t u^{p-1} \eta_k^p,
$$

where

<span id="page-5-1"></span>
$$
(3.10) \t\t\t p = \frac{n}{4},
$$

then have compact support, and multiplying [\(2.9\)](#page-3-1) on page [4](#page-3-1) with  $u^{p-1}\eta_k^p$ yields

$$
(3.11) \qquad \frac{(\frac{p}{4} + \frac{1}{8} - \epsilon) \int_m |Du|^2 u^{p-2} \eta_k^p \le c \Big( \int_M |F|^{\frac{n}{2}} \Big)^{\frac{2}{n}} \Big( \int_M (u \eta_k)^{\frac{n}{n-2}p} \Big)^{\frac{n-2}{n}}}{+ c_{\epsilon} \int_M |D\eta_k|^2 \eta_k^{p-2} u^p,
$$

where  $0<\epsilon$  is supposed to be small.

Furthermore, there holds

$$
(3.12) \quad \int_M |D(u\eta_k)^{\frac{p}{2}}|^2 = \frac{p^2}{4} \int_M |Du\eta_k + uD\eta_k|^2 (u\eta_k)^{p-2}
$$
  

$$
\leq (1+\epsilon)\frac{p^2}{4} \int_M |Du|^2 u^{p-2} \eta_k^p + c_\epsilon \frac{p^2}{4} \int_M |D\eta_k|^2 \eta_k^{p-2} u^p
$$

.

Now, choosing  $\epsilon$  so small that

(3.13) 
$$
(1+\epsilon)\frac{p^2}{4} \le p(\frac{p}{4} + \frac{1}{8} - \epsilon)
$$

and setting

$$
\varphi = (u\eta_k)^{\frac{p}{2}}
$$

we obtain

<span id="page-6-5"></span>
$$
(3.15)\quad \int_M |D\varphi|^2 \le pc \Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_M \varphi^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} + c_\epsilon \int_M |D\eta_k|^2 \eta_k^{p-2} u^p,
$$

where  $c_{\epsilon}$  is a new constant. We furthermore observe that

(3.16) 
$$
|D\eta_k|^2 \eta_k^{p-2} \le q^2 k^{-2} (2 - k^{-1}r)^{qp-2},
$$
subject to  
(3.17) 
$$
1 \le k^{-1} r \le 2.
$$

In view of [\(3.5\)](#page-5-0) and [\(3.10\)](#page-5-1)

(3.18) qp − 2 ≥ 0

and hence

(3.19) 
$$
|D\eta_k|^2 \eta_k^{p-2} \le q^2 k^{-2}.
$$

Applying now the Sobolev inequality [\(1.6\)](#page-1-2) on page [2](#page-1-2) to  $\varphi$  and choosing (3.20)  $\kappa_0 = (c_1cp)^{-1}$ 

we conclude  $|F| \equiv 0$ , if

$$
(3.21)\qquad \qquad \left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \kappa_0.
$$

Indeed, if the preceding inequality is valid, then we deduce from [\(3.15\)](#page-6-5)

$$
(3.22)\qquad \left(1 - \kappa_0^{-1} \Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\right) \Big(\int_M |\varphi|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le c_{\epsilon} q^2 k^{-2} \int_M |F|^{\frac{n}{2}}.
$$

In the limit  $k \to \infty$  we obtain

$$
(3.23)\qquad \qquad \left(\int_M |u|^{\frac{pn}{n-2}}\right)^{\frac{n-2}{n}} \le 0.
$$

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