

# A pseudolocality theorem for Ricci flow

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## Abstract

In this paper we will give a simple proof of a modification of a result on pseudolocality for the Ricci flow by P. Lu [16] without using the pseudolocality theorem 10.1 of Perelman [18]. We also obtain an extension of a result of Hamilton [8] on the compactness of a sequence of complete pointed Riemannian manifolds  $\{(M_k, g_k(t), x_k)\}_{k=1}^{\infty}$  evolving under Ricci flow with uniform bounded sectional curvatures on  $[0, T]$  and uniform positive lower bound on the injectivity radii at  $x_k$  with respect to the metric  $g_k(0)$ .

Key words: pseudolocality, complete Riemannian manifold, Ricci flow, locally bounded Riemannian curvature

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A time dependent metric  $g_{ij}(t)$  on an  $n$ -dimensional manifold  $M$  is said to evolve by the Ricci flow on  $(0, T)$  if it satisfies

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \quad (1)$$

on  $(0, T)$  where  $R_{ij}(x, t)$  is the Ricci tensor with respect to the metric  $g_{ij}(t)$ . In 1982 R.S. Hamilton [6] used Ricci flow to prove that any compact 3-dimensional Riemannian manifold with strictly positive Ricci curvature also admits a metric of constant positive curvature. Recently there are many research on Ricci flow by A. Chau, L.F. Tam and C. Yu [1], P. Daskalopoulos, R.S. Hamilton and N. Sesum [4], [6], [8], S.Y. Hsu [9], [10], [11], B. Kleiner and J. Lott [12], J. Morgan and G. Tang [15], L. Ni [17], G. Perelman [18], [19], S. Kuang and Q.S. Zhang [13], [22] etc. Interested readers

can read the survey article [7] by R.S. Hamilton and the book Hamilton's Ricci flow [3] by B. Chow, P. Lu, and L. Ni for more results on Ricci flow.

In a recent paper [16] P. Lu proved the following pseudolocality theorem for Ricci flow.

**Theorem 1.** *For any  $n \in \mathbb{Z}^+$  and  $\delta > 0$  there exists a constant  $\varepsilon_0 > 0$  with the following property. For any  $r_0 > 0$  and  $0 < \varepsilon < \varepsilon_0$  suppose  $(M, g(t))$  is an  $n$ -dimensional complete solution to the Ricci flow on  $[0, (\varepsilon r_0)^2]$  with bounded sectional curvature, and assume that there exists  $x_0 \in M$  such that*

$$|Rm|(x, 0) \leq r_0^{-2} \quad \forall x \in B_{g(0)}(x_0, r_0) \quad (2)$$

and

$$\text{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq \delta r_0^n. \quad (3)$$

Then

$$|Rm|(x, t) \leq (\varepsilon r_0)^{-2} \quad \forall x \in B_{g(t)}(x_0, \varepsilon r_0), 0 \leq t \leq (\varepsilon r_0)^2. \quad (4)$$

As observed by P. Lu [16] Theorem 1 is implied by the following theorem.

**Theorem 2.** *For any  $n \in \mathbb{Z}^+$  and  $\delta > 0$  there exists a constant  $\varepsilon_0 > 0$  with the following property. For any  $r_0 > 0$  and  $0 < \varepsilon < \varepsilon_0$  suppose  $(M, g(t))$  is an  $n$ -dimensional complete solution to the Ricci flow on  $[0, (\varepsilon r_0)^2]$  with bounded sectional curvature, and assume that there exists  $x_0 \in M$  such that (2) and (3) hold. Then*

$$|Rm|(x, t) \leq (\varepsilon r_0)^{-2} \quad \forall x \in B_{g(t)}(x_0, e^{n-1} \varepsilon r_0), 0 \leq t \leq (\varepsilon r_0)^2. \quad (5)$$

The proof of Theorem 2 in [16] uses the pseudolocality theorem Theorem 10.1 of Perelman [18]. However a careful examination of the proof of Theorem 10.1 of Perelman [18] shows that the proof of Theorem 10.1 of [18] is not correct. The reason is as follows. In the proof of Theorem 10.1 of [18] Perelman constructed a sequence of pointed Ricci flow  $(M_k, g_k(t), (x_{0,k}, 0))$ ,  $0 \leq t \leq \varepsilon_k$ , with  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  and a sequence  $\delta_k \rightarrow 0$  as  $k \rightarrow \infty$  that satisfies

$$|Rm_{g_k}|(x, t) \leq \alpha t^{-1} + 2\varepsilon_k^{-2} \quad \forall d_{g_k(t)}(x, x_{0,k}) \leq \varepsilon_k, 0 \leq t \leq \varepsilon_k^2 \quad (6)$$

for some constant  $\alpha > 0$  and a sequence  $(x_k, t_k)$  with  $0 < t_k \leq \varepsilon_k^2$  and  $d_{g_k(t_k)}(x_{0,k}, x_k) < \varepsilon_k$  such that

$$|Rm_{g_k}(x_k, t_k)| > \alpha t_k^{-1} + \varepsilon_k^{-2}.$$

Perelman [18] also constructed a sequence  $(\bar{x}_k, \bar{t}_k)$  with

$$0 < \bar{t}_k \leq \varepsilon_k^2 \quad \text{and} \quad d_{g_k(\bar{t}_k)}(x_{0,k}, \bar{x}_k) < (2A_k + 1)\varepsilon_k$$

where  $A_k = 1/(100n\varepsilon_k)$  such that

$$|Rm_{g_k}(x, t)| \leq 4Q_k \quad \forall d_{g_k(\bar{t}_k)}(x, \bar{x}_k) < \frac{1}{10}Q_k^{-\frac{1}{2}}, \bar{t}_k - \frac{1}{2}\alpha Q_k^{-1} \leq t \leq \bar{t}_k$$

where  $Q_k = |Rm_{g_k(t)}(\bar{x}_k, \bar{t}_k)|$ . On the third paragraph on P.26 of [18] Perelman claimed that the sequence of metrics  $\hat{g}_k(t) = \frac{1}{2\bar{t}_k}g(2\bar{t}_k t)$  converges to some solution of Ricci flow  $\hat{g}_\infty(t)$  on  $0 \leq t \leq 1/2$  as  $k \rightarrow \infty$ . Perelman then concluded that there is a contradiction to the logarithmic Sobolev inequality on  $\mathbb{R}^n$  by passing to the limit a rescaled version of the equation on P.26 of [18] for  $t = 0$  as  $k \rightarrow \infty$ . However by (6),

$$|Rm_{\hat{g}_k}|(x, t) \leq 2\bar{t}_k |Rm_{g_k}|(x, 2\bar{t}_k t) \leq 2\bar{t}_k (\alpha(2\bar{t}_k t)^{-1} + 2\varepsilon_k^{-2}) = \alpha t^{-1} + 4$$

for any  $d_{\hat{g}_k(t)}(x, x_0) \leq \varepsilon_k / \sqrt{2\bar{t}_k}$  and  $0 \leq t \leq 1/2$ . Hence  $|Rm_{\hat{g}_k}|(x, t)$  are not uniformly bounded near  $t = 0$ . Thus one cannot apply Hamilton's compactness theorem [8] to conclude that the sequence  $\hat{g}_k(t)$  converges to some solution of Ricci flow  $\hat{g}_\infty(t)$  on  $0 \leq t \leq 1/2$  as  $k \rightarrow \infty$ . It is also not known why one can pass to the limit for the inequality on P.26 of [18] as  $k \rightarrow \infty$ .

Hence the proof of Theorem 10.1 of [18] is not correct and the validity of Theorem 10.1 of [18] is not known. On the other hand in the more detailed explanation of the proof of Theorem 10.1 of [18] on P.179 of [2] it is hard to check that the function  $\psi_i^2(x) = (2\pi)^{-n/2} e^{-\tilde{f}_i(x, 0)}$  defined there belong to  $W^{1,2}$  which is the required condition for the validity of the logarithmic Sobolev inequality (Theorem 22.16 of [2]) for manifolds satisfying the isoperimetric inequality.

In this paper we prove that under a mild additional hypothesis Theorem 2 holds without using Theorem 10.1 of [18]. More specifically we will prove that the following result holds.

**Theorem 3.** *For any  $n \in \mathbb{Z}^+$ ,  $C_0 > 0$  and  $\delta > 0$  there exists a constant  $\varepsilon_0 > 0$  with the following property. For any  $r_0 > 0$  and  $0 < \varepsilon < \varepsilon_0$  suppose  $(M, g(t))$  is an  $n$ -dimensional complete solution to the Ricci flow on  $[0, (\varepsilon r_0)^2]$  with bounded sectional curvature, and assume that there exists  $x_0 \in M$  such that (2), (3) and*

$$|Rm|(x, t) \leq \frac{C_0}{t} \quad \forall x \in B_{g(0)}(x_0, r_0), 0 < t \leq (\varepsilon r_0)^2 \quad (7)$$

*hold. Then (4) holds.*

Similar to [16] Theorem 3 is implied by the following theorem.

**Theorem 4.** *For any  $n \in \mathbb{Z}^+$ ,  $C_0 > 0$  and  $\delta > 0$  there exists a constant  $\varepsilon_0 > 0$  with the following property. For any  $r_0 > 0$  and  $0 < \varepsilon < \varepsilon_0$  suppose  $(M, g(t))$  is an  $n$ -dimensional complete solution to the Ricci flow on  $[0, (\varepsilon r_0)^2]$  with bounded sectional curvature, and assume that there exists  $x_0 \in M$  such that (2), (3) and (7) hold. Then (5) holds.*

*Proof of Theorem 4:* By rescaling the metric by  $1/r_0^2$  we may assume without loss of generality that  $r_0 = 1$ . Suppose the theorem is not true. Then there exist  $n \in \mathbb{Z}^+$ ,  $C_0 > 0$ ,  $\delta > 0$ , a sequence of positive numbers  $0 < \varepsilon_k < e^{1-n}/5$  with  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ ,

and a sequence of  $n$ -dimensional complete manifolds  $(M_k, g_k(t))$ ,  $0 \leq t \leq \varepsilon_k^2$ , with  $g_k$  satisfying the Ricci flow on  $[0, \varepsilon_k^2]$  with bounded sectional curvature, a sequence  $x_{0,k} \in M_k$  and a sequence  $(x_k, t_k) \in B_{g_k(0)}(x_{0,k}, e^{n-1}\varepsilon_k) \times (0, \varepsilon_k^2]$  such that

$$|Rm_{g_k(0)}|(x, 0) \leq 1 \quad \forall x \in B_{g_k(0)}(x_{0,k}, 1), \quad (8)$$

$$|Rm_{g_k(t)}|(x, t) \leq \frac{C_0}{t} \quad \forall x \in B_{g_k(0)}(x_{0,k}, 1), 0 < t \leq \varepsilon_k^2, \quad (9)$$

$$\text{Vol}_{g_k(0)}(B_{g_k(0)}(x_{0,k}, 1)) \geq \delta \quad (10)$$

and

$$|Rm_{g_k}|(x_k, t_k) > \varepsilon_k^{-2} \quad (11)$$

holds for all  $k \in \mathbb{Z}^+$ .

Then by [16] we have the following result.

**Claim 1** (Claim A of [16]): For any  $k \in \mathbb{Z}^+$ , there exists  $(\bar{x}_k, \bar{t}_k) \in B_{g_k(0)}(x_{0,k}, (2A_k + e^{n-1})\varepsilon_k) \times (0, \varepsilon_k^2]$  with  $Q_k = |Rm|(\bar{x}_k, \bar{t}_k) > \varepsilon_k^{-2}$  such that

$$|Rm|(x, t) \leq 4Q_k \quad \forall (x, t) \in B_{g_k(0)}(\bar{x}_k, A_k Q_k^{-\frac{1}{2}}) \times (0, \bar{t}_k] \quad (12)$$

where  $A_k = 1/(100n\varepsilon_k)$ .

Let  $\hat{g}_k(t) = Q_k g_k(t/Q_k)$  and  $\hat{t}_k = \bar{t}_k Q_k$ . By passing to a subsequence if necessary we may assume without loss of generality that  $d_{g_k(0)}(\bar{x}_k, x_{0,k}) < 1/10$  and  $A_k \geq 2$  for all  $k \in \mathbb{Z}^+$  and

$$T = \lim_{k \rightarrow \infty} \hat{t}_k \in [0, C_0]$$

exists. Then

$$\begin{cases} |Rm_{\hat{g}_k}|(\bar{x}_k, \hat{t}_k) = 1 \\ |Rm_{\hat{g}_k}|(x, t) \leq 4 \quad \forall (x, t) \in B_{\hat{g}_k(0)}(\bar{x}_k, A_k) \times (0, \hat{t}_k] \\ |Rm_{\hat{g}_k}|(x, 0) \leq Q_k^{-1} \quad \forall x \in B_{\hat{g}_k(0)}(\bar{x}_k, Q_k^{1/2}/2) \end{cases} \quad (13)$$

hold for all  $k \in \mathbb{Z}^+$ . By (8), (10), and the Bishop volume comparison theorem there exists a constant  $\delta_1 > 0$  such that

$$\text{Vol}_{g_k(0)}(B_{g_k(0)}(x_{0,k}, 1/10)) \geq \delta_1 \quad \forall k \in \mathbb{Z}^+.$$

Since  $B_{g_k(0)}(x_{0,k}, 1/10) \subset B_{g_k(0)}(\bar{x}_k, 1/4)$ ,

$$\text{Vol}_{g_k(0)}(B_{g_k(0)}(\bar{x}_k, 1/4)) \geq \delta_1 \quad \forall k \in \mathbb{Z}^+. \quad (14)$$

Since  $B_{g_k(0)}(\bar{x}_k, 1/4) \subset B_{g_k(0)}(x_{0,k}, 1)$ , by (8), (14) and Lemma 1 of [16] (cf. [21] and Theorem 4.10 of [8]) there exist constants  $\delta_0 > 0$  and  $0 < r_0 < 1/4$  such that

$$\text{Area}_{g_k(0)}(\partial\Omega)^n \geq (1 - \delta_0)\text{Vol}_{g_k(0)}(\Omega)^{n-1}$$

holds for any regular domain  $\Omega \subset B_{g_k(0)}(\bar{x}_k, r_0)$  and  $k \in \mathbb{Z}^+$ . Hence

$$\text{Area}_{\hat{g}_k(0)}(\partial\Omega)^n \geq (1 - \delta_0)\text{Vol}_{\hat{g}_k(0)}(\Omega)^{n-1} \quad (15)$$

holds for any regular domain  $\Omega \subset B_{g_k(0)}(\bar{x}_k, Q_k^{1/2}r_0)$  and  $k \in \mathbb{Z}^+$ . Since  $Q_k \rightarrow \infty$  as  $k \rightarrow \infty$ , without loss of generality we may assume that  $Q_k^{1/2}r_0 > 1$  for all  $k \in \mathbb{Z}^+$ . Then by (15) there exists a positive constant  $\delta_2 > 0$  such that

$$\text{Vol}_{\hat{g}_k(0)}(B_{\hat{g}_k(0)}(\bar{x}_k, 1)) \geq \delta_2 \quad \forall k \in \mathbb{Z}^+. \quad (16)$$

We now divide the proof into two cases.

**Case 1:**  $T = 0$

This case can be shown to be impossible by the same argument as the proof of case 3 on P.8–9 of [16] using Theorem 8.3 of [18] and a modification of the argument of Perelman [18]. For the sake of completeness we will give a simple different proof here. For any  $k \in \mathbb{Z}^+$  let  $\delta_{\bar{x}_k}$  be the delta mass at  $\bar{x}_k$  and  $\eta_k$  be the solution of

$$\begin{cases} \eta_{k,t} + \Delta_{\hat{g}_k(t)}\eta_k + C_1\eta_k = 0 & \text{in } B_{\hat{g}_k(0)}(\bar{x}_k, 1) \times (0, \hat{t}_k] \\ \eta_k(x, \hat{t}_k) = \delta_{\bar{x}_k} & \text{in } B_{\hat{g}_k(0)}(\bar{x}_k, 1) \\ \eta_k(x, t) = 0 & \text{on } \partial B_{\hat{g}_k(0)}(\bar{x}_k, 1) \times (0, \hat{t}_k] \end{cases} \quad (17)$$

where  $C_1 = 64 + 4n(n-1)$ . Then by the maximum principle  $\eta_k \geq 0$  in  $B_{\hat{g}_k(0)}(\bar{x}_k, 1) \times (0, \hat{t}_k]$  and  $\partial\eta_k/\partial\nu \geq 0$  on  $\partial B_{\hat{g}_k(0)}(\bar{x}_k, 1) \times (0, \hat{t}_k]$  where  $\partial/\partial\nu$  is the derivative with respect to the unit inward normal  $\nu$  on  $\partial B_{\hat{g}_k(0)}(\bar{x}_k, 1) \times (0, \hat{t}_k]$ . We extend  $\eta_k$  by letting  $\eta_k = 0$  on  $(\bar{B}_{\hat{g}_k(0)}(\bar{x}_k, 1) \times [\hat{t}_k, \infty)) \setminus \{(\bar{x}_k, \hat{t}_k)\}$  and we extend  $\hat{g}_k(t)$  by letting  $\hat{g}_k(t) = \hat{g}_k(\hat{t}_k)$  for all  $t \geq \hat{t}_k$ . Then  $\eta_k$  satisfies

$$\eta_{k,t} + \Delta_{\hat{g}_k(t)}\eta_k + C_1\eta_k = 0 \quad \text{in } (\bar{B}_{\hat{g}_k(0)}(\bar{x}_k, 1) \times (0, \infty)) \setminus \{(\bar{x}_k, \hat{t}_k)\}.$$

Now by (1), (13), and (17),

$$\begin{aligned} \frac{\partial}{\partial t} \left( \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \eta_k d\hat{V}_k(t) \right) &= \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} (\eta_{k,t} - R_{\hat{g}_k}\eta_k) d\hat{V}_k(t) \\ &= \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} (-\Delta_{\hat{g}_k(t)}\eta_k - (C_1 + R_{\hat{g}_k})\eta_k) d\hat{V}_k(t) \\ &\geq -C_4 \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \eta_k d\hat{V}_k(t) + \int_{\partial B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \frac{\partial\eta_k}{\partial\nu} d\sigma_k(t) \\ &\geq -C_4 \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \eta_k d\hat{V}_k(t) \end{aligned} \quad (18)$$

for any  $0 \leq t < \hat{t}_k$  where  $C_4 = 64 + 8n(n-1)$ . Integrating (18) over  $(0, \hat{t}_k)$ ,

$$\int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \eta_k(x, t) d\hat{V}_k(t) \leq e^{C_4\hat{t}_k} \quad \forall 0 \leq t < \hat{t}_k. \quad (19)$$

Hence by (13), (19) and the parabolic Schauder estimates [14] (cf.[1],[13]) there exists a constant  $C_2 > 0$  such that

$$|\partial\eta_k/\partial\nu| \leq C_2 \quad \text{on } \partial B_{\hat{g}_k(0)}(\bar{x}_k, 1) \times [0, \hat{t}_k] \quad \forall k \in \mathbb{Z}^+. \quad (20)$$

Since the curvature  $Rm_{\hat{g}_k}$  satisfies (cf. [16], [7]),

$$(|Rm_{\hat{g}_k}|^2)_t \leq \Delta_{\hat{g}_k(t)} |Rm_{\hat{g}_k}|^2 - 2|\nabla_{\hat{g}_k} Rm_{\hat{g}_k}|^2 + 16|Rm_{\hat{g}_k}|^3,$$

by (1), (13), (17), and (20) we have

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} |Rm_{\hat{g}_k}|^2 \eta_k d\hat{V}_k(t) \right) \\ &= \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} [ (|Rm_{\hat{g}_k}|^2)_t \eta_k + |Rm_{\hat{g}_k}|^2 \eta_{k,t} - R_{\hat{g}_k} |Rm_{\hat{g}_k}|^2 \eta_k ] d\hat{V}_k(t) \\ &\leq \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} [ (\Delta_{\hat{g}_k(t)} |Rm_{\hat{g}_k}|^2 + 64|Rm_{\hat{g}_k}|^2) \eta_k + |Rm_{\hat{g}_k}|^2 \eta_{k,t} - R_{\hat{g}_k} |Rm_{\hat{g}_k}|^2 \eta_k ] d\hat{V}_k(t) \\ &\leq \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} |Rm_{\hat{g}_k}|^2 (\eta_{k,t} + \Delta_{\hat{g}_k(t)} \eta_k + C_1 \eta_k) d\hat{V}_k(t) + 16 \int_{\partial B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \frac{\partial \eta_k}{\partial \nu} d\sigma_k(t) \\ &\leq 16C_2 |\partial B_{\hat{g}_k(0)}(\bar{x}_k, 1)| \end{aligned} \quad (21)$$

for any  $0 \leq t < \hat{t}_k$  where  $R_{\hat{g}_k}$ ,  $d\hat{V}_k(t)$ ,  $d\sigma_k(t)$  are the scalar curvature, volume element, and surface element with respect to the metric  $\hat{g}_k(t)$ .

By (13), (16), and Cheeger-Gromov's compactness theorem ([5], [20]) the sequence of pointed manifold  $(M_k, \hat{g}_k(0), \bar{x}_k)$  has a subsequence which we may assume without loss of generality to be the sequence itself that converges to some pointed manifold  $(M_0, \hat{g}_0, \bar{x}_0)$  as  $k \rightarrow \infty$  (cf. [20],[8]). Then there exists a constant  $C_3 > 0$  such that

$$|\partial B_{\hat{g}_k(0)}(\bar{x}_k, 1)| \leq \frac{C_3}{16C_2} \quad \forall k \in \mathbb{Z}^+. \quad (22)$$

Hence by (13), (17), (21) and (22),

$$\begin{aligned} 1 &= \lim_{t \nearrow \hat{t}_k} \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} |Rm_{\hat{g}_k}(x, t)|^2 \eta_k(x, t) d\hat{V}_k(t) \\ &\leq \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} |Rm_{\hat{g}_k}(x, 0)|^2 \eta_k(x, 0) d\hat{V}_k(0) + C_3 \hat{t}_k \\ &\leq Q_k^{-2} \int_{B_{\hat{g}_k(0)}(\bar{x}_k, 1)} \eta_k(x, 0) d\hat{V}_k(0) + C_3 \hat{t}_k \end{aligned} \quad (23)$$

By (19) and (23),

$$1 \leq Q_k^{-2} e^{C_4 \hat{t}_k} + C_3 \hat{t}_k \quad \forall k \in \mathbb{Z}^+. \quad (24)$$

Letting  $k \rightarrow \infty$  in (24) we get  $1 \leq 0$  and contradiction arises. Hence  $T > 0$ .

**Case 2:**  $T \in (0, C_0]$

By passing to a subsequence if necessary we may assume without loss of generality that  $T/2 < \hat{t}_k < 3T/2$  and  $A_k > e^{9nT}$  for all  $k \in \mathbb{Z}^+$ . By (1) and (13),

$$e^{-6nT} \hat{g}_k(0) \leq \hat{g}_k(t) \leq e^{6nT} \hat{g}_k(0) \quad \text{in } B_{\hat{g}_k(0)}(\bar{x}_k, A_k) \times (0, \hat{t}_k] \quad \forall k \in \mathbb{Z}^+. \quad (25)$$

Then

$$B_{\hat{g}_k(0)}(\bar{x}_k, 1) \subset B_{\hat{g}_k(t)}(\bar{x}_k, e^{3nT}) \subset B_{\hat{g}_k(0)}(\bar{x}_k, e^{9nT}) \quad \forall 0 \leq t \leq \hat{t}_k, k \in \mathbb{Z}^+. \quad (26)$$

Hence by (16) and (26),

$$\text{Vol}_{\hat{g}_k(0)}(B_{\hat{g}_k(T/2)}(\bar{x}_k, e^{3nT})) \geq \delta_2 \quad \forall k \in \mathbb{Z}^+. \quad (27)$$

Now by (1) and (13),

$$\left| \frac{\partial}{\partial t} (\log(d\hat{V}_k(t))) \right| \leq 4n(n-1) \quad \text{in } B_{\hat{g}_k(0)}(\bar{x}_k, A_k) \times (0, \hat{t}_k] \quad \forall k \in \mathbb{Z}^+. \quad (28)$$

Hence by (26) and (28),

$$\text{Vol}_{\hat{g}_k(T/2)}(B_{\hat{g}_k(T/2)}(\bar{x}_k, e^{3nT})) \geq e^{-2n(n-1)T} \text{Vol}_{\hat{g}_k(0)}(B_{\hat{g}_k(T/2)}(\bar{x}_k, e^{3nT})) \quad \forall k \in \mathbb{Z}^+. \quad (29)$$

By (27) and (29),

$$\text{Vol}_{\hat{g}_k(T/2)}(B_{\hat{g}_k(T/2)}(\bar{x}_k, e^{3nT})) \geq e^{-2n(n-1)T} \delta_2 \quad \forall k \in \mathbb{Z}^+. \quad (30)$$

Let  $M_k = B_{\hat{g}_k(0)}(\bar{x}_k, A_k)$ . By (13) and (30) the injectivity radii of  $(M_k, \hat{g}_k(T/2))$  at  $\bar{x}_k$  are uniformly bounded below by some positive constant for all  $k \in \mathbb{Z}^+$ . Hence by (13) and the Hamilton compactness theorem [8] there exists a subsequence of  $(M_k, \hat{g}_k(t), (\bar{x}_k, T/2))$  which we may assume without loss of generality to be the sequence itself that converges to some pointed complete manifold  $(M_\infty, g_\infty(t), (x_\infty, T/2))$ ,  $0 < t \leq T$ , as  $k \rightarrow \infty$ .  $g_\infty$  satisfies the Ricci flow equation (1) with

$$|Rm_\infty|(x, t) \leq 4 \quad \forall x \in M_\infty, 0 < t \leq T \quad (31)$$

where  $Rm_\infty(x, t)$  is the Riemannian curvature of  $M_\infty$  with respect to the metric  $g_\infty(t)$ . Let

$$h(t) = \sup_{x \in M_\infty} |Rm_\infty|(x, t).$$

Then  $0 \leq h(t) \leq 4$  on  $(0, T]$ . Since  $|Rm_{\hat{g}_k(\hat{t}_k)}(\bar{x}_k, \hat{t}_k)| = 1$  for all  $k \in \mathbb{Z}^+$ ,

$$|Rm_\infty(x_\infty, T)| = 1. \quad (32)$$

Hence  $h(T) \geq 1$ . By continuity there exists a constant  $\delta_1 > 0$  such that  $h(t) > 0$  on  $[T - \delta_1, T]$ . Let  $T_1 \geq 0$  be the minimal time such that  $h(t) > 0$  for any  $t \in (T_1, T]$ . Then  $T_1 < T - \delta_1$ . Suppose  $T_1 > 0$ . Then

$$Rm_\infty(x, T_1) \equiv 0 \quad \text{on } M_\infty. \quad (33)$$

By (1) and (33),

$$Rm_\infty(x, t) \equiv 0 \quad \text{on } M_\infty \times (0, T]. \quad (34)$$

By (32) and (34) contradiction arises. Hence  $T_1 = 0$ . Let  $\delta_2 = \inf_{0 < t \leq T} h(t)$ .

We now divide the proof of case 2 into two subcases:

**Case (a):**  $\delta_2 = 0$ .

Then there exists a sequence  $\{s_i\}_{i=1}^\infty$  with  $0 < s_i < T$  for all  $i \in \mathbb{Z}^+$  and  $s_i \rightarrow 0$  as  $i \rightarrow \infty$  such that  $h(s_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Since the curvature  $Rm_\infty$  satisfies (cf. [16], [7]),

$$(|Rm_\infty|^2)_t \leq \Delta_\infty |Rm_\infty|^2 - 2|\nabla_\infty Rm_\infty|^2 + 16|Rm_\infty|^3 \quad \text{in } M_\infty \times (0, T],$$

by (31),

$$\begin{aligned} & (|Rm_\infty|^2)_t \leq \Delta_\infty |Rm_\infty|^2 + 64|Rm_\infty|^2 \quad \text{in } M_\infty \times (0, T] \\ \Rightarrow & (e^{-64t}|Rm_\infty|^2)_t \leq \Delta_\infty (e^{-64t}|Rm_\infty|^2) \quad \text{in } M_\infty \times (0, T]. \end{aligned} \quad (35)$$

By (35) and the maximum principle [10],

$$e^{-64t}|Rm_\infty(x, t)|^2 \leq e^{-64s_i} \sup_{x \in M} |Rm_\infty(x, s_i)|^2 \leq h(s_i)^2 \quad \text{in } M_\infty \times (s_i, T] \quad \forall i \in \mathbb{Z}^+. \quad (36)$$

Letting  $i \rightarrow \infty$  in (36) we get (34). This contradicts (32). Hence case (a) cannot occur.

**Case (b):**  $\delta_2 > 0$ .

Let  $C_3$  and  $C_4$  be as in case 1 and let

$$T_2 = \min(T/2, \delta_2/(4C_3)). \quad (37)$$

Since  $\delta_2 > 0$ , there exists  $y_\infty \in M$  such that

$$|Rm_\infty|(y_\infty, T_2) \geq \delta_2/2. \quad (38)$$

Then there exists a sequence  $y_k \in M_k = B_{\hat{g}_k(0)}(\bar{x}_k, A_k)$  such that

$$d_{\hat{g}_k(T_2)}(y_k, y_\infty) \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (39)$$

and

$$|Rm_k|(y_k, T_2) \rightarrow |Rm_\infty|(y_\infty, T_2) \quad \text{as } k \rightarrow \infty. \quad (40)$$



Let  $r_0 = d_{g_\infty(T_2)}(x_\infty, y_\infty) + 1$ . Since  $d_{\hat{g}_k(T_2)}(\bar{x}_k, y_k) \rightarrow d_{g_\infty(T_2)}(x_\infty, y_\infty)$  and  $A_k \rightarrow \infty$ ,  $Q_k \rightarrow \infty$ , as  $k \rightarrow \infty$ , by passing to a subsequence if necessary we may assume without loss of generality that

$$d_{\hat{g}_k(T_2)}(\bar{x}_k, y_k) < r_0 \quad \text{and} \quad \min(A_k, Q_k^{1/2}/2) > 1 + e^{3nT} r_0 \quad \forall k \in \mathbb{Z}^+. \quad (41)$$

Then by (25) and (41),

$$B_{\hat{g}_k(t)}(\bar{x}_k, r_0) \subset B_{\hat{g}_k(0)}(\bar{x}_k, e^{3nT} r_0) \quad \forall 0 \leq t \leq \hat{t}_k, k \in \mathbb{Z}^+. \quad (42)$$

By (41) and (42),

$$B_{\hat{g}_k(0)}(y_k, 1) \subset B_{\hat{g}_k(0)}(\bar{x}_k, e^{3nT} r_0 + 1) \subset B_{\hat{g}_k(0)}(\bar{x}_k, \min(A_k, Q_k^{1/2}/2)) \quad (43)$$

holds for any  $k \in \mathbb{Z}^+$ . By (13), (43), and an argument similar to the proof of Case 1 but with  $(y_k, T/2)$  replacing  $(\bar{x}_k, \hat{t}_k)$  in the proof there we get

$$|Rm_{\hat{g}_k(T_2)}|(y_k, T_2)| \leq Q_k^{-2} e^{C_4 T_2} + C_3 T_2 \quad \forall k \in \mathbb{Z}^+. \quad (44)$$

Letting  $k \rightarrow \infty$  in (44), by (37), (38) and (40), we get

$$\delta_2/2 \leq C_3 T_2 \leq \delta_2/4$$

and contradiction arises. Hence case (b) is false. Thus no such sequence of manifolds  $(M_k, g_k, x_{0,k})$  exists and the theorem follows.  $\square$

By the result of [8] and an argument similar to the proof of case 2 we have the following extension of the compactness result of Hamilton [8].

**Theorem 5.** *Let  $\{(M_k, g_k(t), x_k)\}_{k=1}^\infty$  be a sequence of complete pointed Riemannian manifolds evolving under Ricci flow (1) with sectional curvatures uniform bounded above by some constant  $B > 0$  on  $[0, T]$  and uniform positive lower bound on the injectivity radii at  $x_k$  with respect to the metric  $g_k(0)$ . Then there exists a subsequence which converges to some complete pointed Riemannian manifold  $(M, g(t), x_0)$  on  $(0, T]$  that evolves by the Ricci flow on  $(0, T]$  with sectional curvatures uniform bounded above by  $B$ .*

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