The analysis of the stochastic stability for an economic game

A. L. Ciurdariu[a], M. Neamţu[b], A. Sandru[c], D. Opriş[d]

[a] Department of Mathematics, Politehnica University of Timişoara

P-ţa. Victoriei, nr 2, 300004, Timişoara, Romania, e-mail: cloredana43@yahoo.com,

[b]Department of Economic Informatics, Mathematics and Statistics,

Faculty of Economics, West University of Timişoara,

str. Pestalozzi, nr. 16A, 300115, Timişoara, Romania, e-mail:mihaela.neamtu@fse.uvt.ro

[c] Faculty of Exact Sciences, Aurel Vlaicu University of Arad,

str. Elena Dragoi nr. 2, 310330, Arad, e-mail: sandruandrea@gmail.com,

[d] Department of Applied Mathematics, Faculty of Mathematics,

West University of Timişoara, Bd. V. Parvan, nr. 4, 300223, Timişoara, Romania,

E-mail: opris@math.uvt.ro

Abstract

In this paper we investigate a stochastic model for an economic game. To describe this model we have used a Wiener process, as the noise has a stabilization effect. The dynamics are studied in terms of stochastic stability in the stationary state, by constructing the Lyapunov exponent, depending on the parameters that describe the model. Also, the Lyapunov function is determined in order to analyze the mean square stability. The numerical simulation that we did justifies the theoretical results.

1

1   Introduction.

Stochastic modeling plays an important role in many branches of science. In many practical situations, perturbations are expressed in terms of white noise, modeled by brownian motion. The behavior of a deterministic dynamical system which is disturbed by noise may be modeled by a stochastic differential equation (SDE), [8]. The stochastic stability has been introduced by Bertram and Sarachik and is characterized by the negativeness of Lyapunov exponents. In general, it is not possible to determine this exponents explicitly. Many numerical approaches have been proposed, which generally used the simulation of the stochastic trajectories. In the present paper, we study a stochastic dynamical system that is used in economy, in describing a Counot duopoly game.

In 1838, Cournot introduced the first formal theory of oligopoly, which treated the case of naive expectations, where each player assumes the last values taken by the competitors without estimation of their future reactions [5]. Recently, a lot of articles have shown that the Cournot model may lead to a cyclic or chaotic behavior [3], [4], [9], [11], [12], [13]. Also, in [14], Rosser reviews the development of the theory of complex oligopoly dynamics.

In the present paper we have studied a stochastic Cournot economic game. In Section 2 we present the Lyapunov exponent and stability in stochastic 2d dynamical structures. Section 3 describes the Lyapunov function method for the stochastic stability analysis. Section 4 studies the Lyapunov exponent for an economic game with stochastic dynamics. The Lyapunov function method for the stochastic game is given in Section 5. Some numerical simulations are done in Section 6. Finally, Section 7 draws some conclusions.

2   The Lyapunov exponent and stability in stochastic 2d dynamical structures.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space [8]. It is assumed that the $\sigma-$algebra $\mathcal{F}$ is a filtration that is, $\mathcal{F}$ is generated by a family of $\sigma-$algebra $\mathcal{F}_t (t \geq 0)$ such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad \forall s \leq t, s, t \in I,$$

where $I = [0, T]$, $T \in (0, \infty)$.

Let $\{x(t) = (x_1(t), x_2(t))\}_{t \geq 0}$ be a stochastic process. The system of Itô equations:

$$dx_i(t, \omega) = f_i(t, x(t, \omega))dt + g_i(x(t, \omega))dw(t, \omega), i = 1, 2, \qquad (1)$$

with the initial condition $x(0) = x_0$ is written as:

$$x_i(t, \omega) = x_{i0}(\omega) + \int_0^t f_i(x(s, \omega))ds + \int_0^t g_i(x(s, \omega))dw(s, \omega), i = 1, 2, \quad (2)$$

for almost all $\omega \in \Omega$ and for each $t > 0$, where $f_i(x)$ is a drift function, $g_i(x)$ is a diffusion function, $\int_0^t f_i(x(s))ds$, $i = 1, 2$ is a Riemann integral and $\int_0^t g_i(x(s))dw(s)$ is an Itô integral. It is assumed that f_i and g_i, $i = 1, 2$ satisfy the conditions of existence of solution for this SDE with initial condition $x(0) = a_0 \in \mathbb{R}^n$.

Let $x_0 = (x_{10}, x_{20}) \in \mathbb{R}^2$ be a solution of the system:

$$f_i(x_0) = 0, i = 1, 2. \qquad (3)$$

The functions g_i, $i = 1, 2$ are chosen so that:

$$g_i(x_0) = 0, i = 1, 2.$$

In what follows, we consider:

$$g_i(x) = \sum_{j=1}^{2} b_{ij}(x_j - x_{0j}), i = 1, 2,$$

where $b_{ij} \in \mathbb{R}, i, j = 1, 2$.

The linearized system of (2) in x_0, is given by:

$$X(t) = \int_0^t AX(s)ds + \int_0^t BX(s)dw(s),$$

where

$$X(t) = \begin{pmatrix} u_1(t, \omega) \\ u_2(t, \omega) \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$a_{ij} = \frac{\partial f_i}{\partial x_j}|_{x_0}, b_{ij} = \frac{\partial g_i}{\partial x_j}|_{x_0}.$$

3

The Oseledec multiplicative ergodic theorem [10] asserts the existence of 2 non-random Lyapunov exponents $\lambda_2 \leq \lambda_1 = \lambda$. The top Lyapunov exponent is given by:

$$\lambda = \lim_{t\to\infty} \sup \log \sqrt{u_1(t)^2 + u_2(t)^2}.$$

Applying the change to polar coordinates:

$$x(t) = r(t)cos\theta(t), y(t) = r(t)sin\theta(t)$$

by writing the Itô formula for

$$h_1(u_1, u_2) = \frac{1}{2}\log(u_1^2 + u_2^2) = \log(r), h_2(u_1, u_2) = arctg(\frac{u_2}{u_1}) = \theta.$$

we get:

Proposition 1 *[8]. The formulas*

$$\log\left(\frac{r(t)}{r(0)}\right) = \int_0^t q_1(\theta(s)) + \frac{1}{2}(q_4(\theta(s))^2 - q_2(\theta(s))^2)ds + \int_0^t q_2(\theta(s))dw(s), \quad (4)$$

$$\theta(t) = \theta(0) + \int_0^t q_3(\theta(s)) - q_2(\theta(s)q_4(\theta(s))ds + \int_0^t q_4(\theta(s))dw(s), \quad (5)$$

hold, where

$$\begin{aligned}
q_1(\theta) &= a_{11}cos^2(\theta) + (a_{12} + a_{21})cos\theta \sin\theta + a_{22}sin^2\theta, \\
q_2(\theta) &= b_{11}cos^2(\theta) + (b_{12} + b_{21})cos\theta \sin\theta + b_{22}sin^2\theta, \\
q_3(\theta) &= a_{21}cos^2(\theta) + (a_{22} - a_{11})cos\theta \sin\theta - a_{12}sin^2\theta, \\
q_4(\theta) &= b_{21}cos^2(\theta) + (b_{22} - b_{11})cos\theta \sin\theta - b_{12}sin^2\theta.
\end{aligned} \quad (6)$$

As the expectation of the Itô stochastic integral is null

$$E\int_0^t q_2(\theta(s))dw(s) = 0,$$

the Lyapunov exponent is given by:

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\log\left(\frac{r(t)}{r(0)}\right) = \lim_{t\to\infty} \frac{1}{t}E\int_0^t (q_1(\theta(s)) + \frac{1}{2}(q_4(\theta(s)))^2 - q_2(\theta(s)))ds.$$

4

Applying the Oseledec theorem, if $r(t)$ is ergodic, we get:

$$\lambda = \int_0^t (q_1(\theta) + \frac{1}{2}(q_3(\theta)^2 - q_2(\theta)))p(\theta)d\theta,$$

where $p(\theta)$ is the density of probability of the process θ.

An approximation of this density is calculated by solving the Fokker-Planck equation.

The Fokker-Planck (FPE) equation associated with equation (5) for $p = p(t, \theta)$ is

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial \theta}((q_3(\theta) - q_2(\theta)q_4(\theta))p) - \frac{1}{2}\frac{\partial^2}{\partial \theta^2}(q_4(\theta)^2 p) = 0. \qquad (7)$$

From (7), it results that the solution $p(\theta)$ of the FPE is a solution of the following first order equation:

$$(-q_3(\theta) + q_1(\theta)q_4(\theta) + q_2(\theta)g_5(\theta))p(\theta) + \frac{1}{2}q_4(\theta)^2 p'(\theta) = p_0, \qquad (8)$$

where $p'(\theta) = \dfrac{dp}{d\theta}$ and

$$q_5(\theta) = -(b_{12} + b_{21})sin2\theta - (b_{22} - b_{11})cos2\theta.$$

Proposition 2 *[8]. If $q_4(\theta) \neq 0$, the solution of the equation (8) is given by:*

$$p(\theta) = \frac{k}{D(\theta)q_4(\theta)^2}\left(1 + \eta \int_0^\theta D(u)du\right)$$

where k is determined by the normality condition

$$\int_0^{2\pi} p(\theta)d\theta = 1$$

and

$$\eta = \frac{D(2\pi) - 1}{\int_0^{2\pi} D(u)du}.$$

The function D is given by:

$$D(\theta) = \exp(-2\int_0^\theta \frac{q_3(u) - q_2(u)q_4(u) - q_4(u)q_5(u)}{q_4(u)^2}du)$$

A numerical solution of the phase distribution could be given by a simple backward difference scheme.

We consider $N \in \mathbb{R}_+$, $h = \dfrac{\pi}{N}$ and

$$q_1(i) = a_{11}\cos^2(ih) + (a_{12} + a_{21})\cos(ih)\sin(ih) + a_{22}\sin^2(ih),$$
$$q_2(i) = b_{11}\cos^2(ih) + (b_{12} + b_{21})\cos(ih)\sin(ih) + b_{22}\sin^2(ih),$$
$$q_3(i) = a_{21}\cos^2(ih) + (a_{22} - a_{11})\cos(ih)\sin(ih) - a_{12}sin^2(ih),$$
$$q_4(i) = b_{21}\cos^2(ih) + (b_{22} - b_{11})\cos(ih)\sin(ih) - b_{12}\sin^2(ih),$$
$$q_5(i) = -(b_{12} + b_{21})\sin(2ih) - (b_{22} - b_{11})cos(2ih), i = 0, ..., N$$

The function $p(i), i = 0, ..., N$ is given by the following relations:

$$p(i) = \left(p(0) + \frac{q_4(i)^2 p(i-1)}{2h}\right)F(i)$$

where

$$F(i) = \frac{2h}{2h(-q_3(i) + q_2(i)q_4(i) + q_4(i)q_5(i)) + q_4(i)^2}.$$

The Lyapunov exponent is $\lambda = \lambda(N)$, where

$$\lambda(N) = \sum_{i=0}^{N}\left(q_1(i) + \frac{1}{2}(q_4(i)^2 - q_2(i)^2)\right)p(i)h.$$

From Proposition 2 we obtain:

Proposition 3 *If the matrix B is given by:*

$$b_{11} = \alpha, b_{12} = -\beta, b_{21} = \beta, b_{22} = \alpha$$

then

$$p(\theta) = \frac{k}{\beta^2}\exp\left\{\frac{1}{\beta^2}\left((a_{21} - a_{12} - \alpha\beta)\theta + \frac{1}{2}(a_{11} - a_{22})\cos 2\theta + \frac{1}{2}(a_{21} - a_{12})\sin 2\theta\right)\right\}$$

$$k = \frac{\beta^2}{\int_0^{2\pi}\exp\{\frac{1}{\beta^2}((a_{21}-a_{12}-\alpha\beta)\theta+\frac{1}{2}(a_{11}-a_{22})\cos 2\theta+\frac{1}{2}(a_{21}-a_{12})\sin 2\theta)d\theta}$$

$$\lambda = \frac{1}{2}(a_{11} + a_{22} + \beta^2 - \alpha^2) + \frac{1}{2}(a_{11} - a_{22})c_2 + \frac{1}{2}(a_{21} + a_{12})s_2,$$

where

$$c_2 = \int_0^{2\pi}cos(2\theta)p(\theta)d\theta, \quad s_2 = \int_0^{2\pi}sin(2\theta)p(\theta)d\theta.$$

3   The Lyapunov function method for the stochastic stability analysis.

We consider the system of stochastic equations, SDE, given by:

$$dx_i(t) = f_i(x(t))dt + g_i(x(t))dB^i(t), \quad i = 1, 2, \tag{9}$$

where $x(t) = x(t, \omega)$ and $B^1(t)$, $B^2(t)$ are Wiener processes.

Let $V : D = (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with respect to the first variable and a C^2 class function with respect to the other variables.

Let:

$$LV(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{2} f_i(x)\frac{\partial V(t, x)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2} g_i(x)g_j(x)\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j} \tag{10}$$

be a differential operator.

We suppose that $x_e = 0$ is the stationary state of (9), that means:

$$f_i(0) = g_{i\alpha}(0) = 0, \quad i = 1, 2, \alpha = 1, 2.$$

The theorem which gives us conditions for the stability of the trivial solution $x_e = 0$ in the terms of the Lyapunov function is:

Theorem 4 *[15] Under the above conditions, if there is a function $V : D \to \mathbb{R}$ and two continuous functions $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ and $k > 0$ so that for $||x|| < k$ the relation:*

$$u(||x||) < V(t, x) < v(||x||)$$

holds, then:

(i) if $LV(t, x) \leq 0$, $x \in (0, k)$, then solution of (9) $x_e = 0$ is stable in probability;

(ii) if there is a continuous function $c : \mathbb{R}_+ \to \mathbb{R}_+$ so that $LV(t, x) \leq -c(||x||)$ then solution $x_e = 0$ of (9) is asymptotically stable.

Let $V : D = (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with respect to the first variable and a C^2 class function with respect to the other variables.

The theorem that gives us the exponential p-stability condition of the trivial solution (11) is:

Theorem 5 *[15] If function V satisfies the following inequalities:*

$$k_1||u||^p \leq V(t,x) \leq k_2||u||^p$$
$$LV(t,u) \leq -k_3||u||^p, \quad k_i > 0, p > 0, i = 1,2,$$

then the trivial solution of (11) is exponential p-stable for $t \geq 0$.

For the concrete problems the following theorem is used:

Theorem 6 *[6] If function V satisfies the following inequality:*
(i) $LV(u) \leq 0$, then the trivial solution is stable in probability;
(ii) $LV(u) \leq -c(||u||)$, where $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function, then the trivial solution is asymptotically stable;
(iii) $LV(u) \leq -q^T Qq$, where Q is a symmetric matrix positively defined, then the trivial solution is mean square stable.

In general, the functions f_i, $g_{i\alpha}$, $i = 1,2$, $\alpha = 1,2$ are nonlinear functions and the above theorem is difficult to use. Therefore, the linearization method of system (9), in the neighborhood of the equilibrium point is used.

The linearized stochastic differential system SDEL of (9) is given by:

$$du_1(t) = (a_{11}u_1(t) + a_{12}u_2(t))dt + (b_{11}u_1(t) + b_{12}u_2(t))dB^1(t)$$
$$du_2(t) = (a_{21}u_1(t) + a_{22}u_2(t))dt + (b_{21}u_1(t) + b_{22}u_2(t))dB^2(t). \tag{11}$$

For (11) expression LV is given by:

$$LV = (a_{11}u_1 + a_{12}u_2)\frac{\partial V}{\partial u_1} + (a_{21}u_1 + a_{22}u_2)\frac{\partial V}{\partial u_2} +$$
$$\frac{1}{2}[(b_{11}u_1 + b_{12}u_2)^2\frac{\partial^2 V}{\partial u_1^2} + (b_{21}u_1 + b_{22}u_2)^2\frac{\partial^2 V}{\partial u_2^2}] \tag{12}$$

4 The Lyapunov exponent for an economic game with stochastic dynamics.

Two firms enter the market with a homogenous consumption product. The elements which describe the model are: the quantities which enter the market from the two firms $x_i \geq 0$, $i = \overline{1,2}$; the inverse demand function

$p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (p is a derivable function with $p'(x) < 0$, $\lim_{x \to a_1} p(x) = 0$, $\lim_{x \to 0} p(x) = b_1$, $(a_1 \in \overline{\mathbb{R}}, b_1 \in \overline{\mathbb{R}})$; the cost functions $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (C_i are derivable functions with $C_i'(x_i) > 0$, $C_i'' \geq 0$, $i = \overline{1,2}$).

In our study we consider $p(x) = \dfrac{1}{x}$, $x > 0$ and $C_i(x_i) = c_i x_i + d_i$, $i = 1, 2$.

The mathematical model of the stochastic dynamic economic game is described by the stochastic system of equations:

$$x_1(t) = x_1(0) + k_1 \int_0^t \left(\frac{x_2(s)}{(x_1(s) + x_2(s))^2} - c_1 \right) ds + \int_0^t (b_{11}x_1(s) + b_{12}x_2(s) + \gamma_1) dw(s)$$

$$x_2(t) = x_2(0) + k_2 \int_0^t \left(\frac{x_1(s)}{(x_1(s) + x_2(s))^2} - c_2 \right) ds + \int_0^t (b_{21}x_1(s) + b_{22}x_2(s) + \gamma_2) dw(s)$$

$$(13)$$

where $b_{ij} \in \mathbb{R}$, $i, j = 1, 2$, $k_1 > 0, k_2 > 0$, $x_i(t) = x_i(t, \omega)$, $i = 1, 2$.

$$\gamma_1 = -\frac{b_{11}c_2 + b_{12}c_1}{(c_1 + c_2)^2}, \gamma_2 = -\frac{b_{21}c_2 + b_{22}c_1}{(c_1 + c_2)^2}.$$

For $b_{ij} = 0$, $i, j = 1, 2$ model (13) is reduced to the classical model of the economic game [3], [9].

The system of stochastic equations (13), has the form (2) from section 2, where:

$$f_1(x_1, x_2) = \frac{x_2}{(x_1 + x_2)^2} - c_1, g_1(x_1, x_2) = b_{11}x_1 + b_{12}x_2 + \gamma_1,$$

$$f_2(x_1, x_2) = \frac{x_1}{(x_1 + x_2)^2} - c_2, g_2(x_1, x_2) = b_{21}x_1 + b_{22}x_2 + \gamma_2.$$

Applying the results from section 2, we have:

Proposition 7 *(i) The stationary state of (SDE) (13) is given by:*

$$x_{10} = \frac{c_2}{(c_1 + c_2)^2}, x_{20} = \frac{c_1}{(c_1 + c_2)^2};$$

(ii) The elements of the matrix A, which characterize linearized equation (13) in (x_{10}, x_{20}) are:

$$a_{11} = -2k_1 c_1 (c_1 + c_2), a_{12} = -k_1(c_1^2 - c_2^2)$$

$$a_{21} = k_2(c_1^2 - c_2^2), a_{22} = -2k_2 c_2(c_1 + c_2);$$

9

(iii) The roots of the characteristic equation:

$$\mu^2 - (a_{11} + a_{22})\mu + a_{11}a_{22} - a_{12}a_{21} = 0 \qquad (14)$$

have the real part:

$$Re(\mu_{1,2}) = -(k_1c_1 + k_2c_2)(c_1 + c_2);$$

(iv) If $b_{11} = \alpha$, $b_{12} = -\beta$, $b_{21} = \beta$, $b_{22} = \alpha$, $\beta \neq 0$, then the Lyapunov coefficient of (SDE) (3) is:

$$\lambda = -(k_1c_1 + k_2c_2)(c_1 + c_2) + \frac{1}{2}(\beta^2 - \alpha^2) - (k_1c_1 - k_2c_2)(c_1 + c_2)D_2 +$$
$$+ \frac{1}{2}(k_2 - k_1)(c_1^2 - c_2^2)E_2 \qquad (15)$$

where

$$D_2 = \int_0^{2\pi} cos(2\theta)p(\theta)d\theta, \; E_2 = \int_0^{2\pi} sin(2\theta)p(\theta)d\theta$$

and

$$p(\theta) = kg(\theta), k = \frac{1}{\int_0^{2\pi} g(\theta)d\theta},$$

$$g(\theta) = \frac{1}{\beta^2} \exp\{\frac{1}{\beta^2}((k_1 + k_2)(c_1^2 - c_2^2) + \alpha\beta)\theta - (k_1c_1 - k_2c_2)(c_1 + c_2)\cos(2\theta) +$$

$$+ \frac{1}{2}(k_1 + k_2)(c_1^2 - c_2^2)\sin(2\theta)\}.$$

5 Numerical Simulations.

We have done the numerical simulations using a program in Maple 12. For $c_1 = 0.2$, $c_2 = 2$, $k_1 = 0.2$, $k_2 = 0.4$, $\beta = 2$, in figure 1 is displayed $(\alpha, \lambda(\alpha))$, where $\lambda(\alpha)$ is given by (15). For $\alpha \in (-\infty, -1.2) \cup (1.1, \infty)$, the Lyapunov exponent is negative, then (SDE) has an asymptotically stable stationary state. For $\alpha \in (-1.2, 1.1)$, the Lyapunov exponent is positive and (SDE) has an asymptotically unstable stationary state.

10

Fig 1. $(\alpha, \lambda(\alpha))$
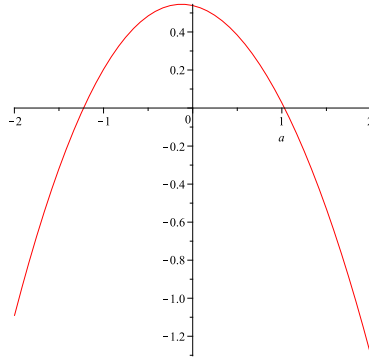
If β is a real parameter and $\alpha = 2$, in figure 2 we have: $(\beta, \lambda(\beta))$.

Fig 2. $(\beta, \lambda(\beta))$
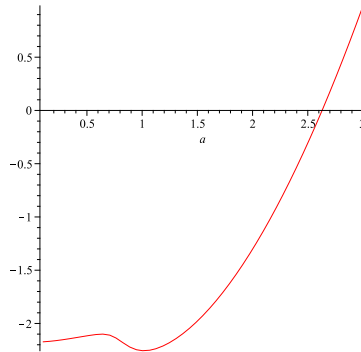
For $\beta \in (-\infty, -2.6) \cup (2.6, \infty)$ the Lyapunov exponent is positive and (SDE) has an asymptotically unstable stationary state. For $\beta(-2.6, 2.6)$ the Lyapunov exponent is negative and (SDE)equation has an asymptotically stable stationary state.

11

The Euler second order scheme for (SDE) (13) is given by:

$$x_1(n+1) = x_1(n) + h\left(\frac{x_2(n)}{(x_1(n)+x_2(n))^2} - c_1\right) + (b_{11}x_1(n) + b_{12}x_2(n) + \gamma_1)$$

$$\cdot G(n) + b_{11}(b_{11}x_1(n) + b_{12}x_2(n) + \gamma_1)\frac{G(n)^2 - h}{2} + \left(-\frac{2x_1(n)x_2(n)}{(x_1(n)+x_2(n))^3}\right.$$

$$\cdot\left(\frac{x_2(n)}{(x_1(n)+x_2(n))^2} - c_1\right) + (b_{11}x_1(n) + b_{12}x_2(n) + \gamma_1)\frac{x_1(n)x_2(n)}{(x_1(n)+x_2(n))^3}\left.\right)\frac{h^2}{2} +$$

$$(b_{11} - \frac{2x_2(n)}{(x_1(n)+x_2(n))^3})(b_{11}x_1(n) + b_{12}x_2(n) + \gamma_1)\frac{hG(n)}{2},$$

$$x_2(n+1) = x_2(n) + h\left(\frac{x_1(n)}{(x_1(n)+x_2(n))^2} - c_2\right) + (b_{21}x_1(n) + b_{22}x_2(n) + \gamma_2)$$

$$\cdot G(n) + b_{22}(b_{21}x_1(n) + b_{22}x_2(n) + \gamma_2)\frac{G(n)^2 - h}{2} + \left(-\frac{2x_1(n)x_2(n)}{(x_1(n)+x_2(n))^3}\right.$$

$$\cdot\left(\frac{x_1(n)}{(x_1(n)+x_2(n))^2} - c_2\right) + (b_{21}x_1(n) + b_{22}x_2(n) + \gamma_2)\frac{x_1(n)x_2(n)}{(x_1(n)+x_2(n))^3}\left.\right)\frac{h^2}{2} +$$

$$(b_{21} - \frac{2x_1(n)}{(x_1(n)+x_2(n))^3})(b_{21}x_1(n) + b_{22}x_2(n) + \gamma_2)\frac{hG(n)}{2},$$

where $G(n) = w((n+1)h) - w(nh)$, $n = 1, 2, ...$, and $x_i(n) = x_i(nh, \omega)$, $i = 1, 2$.

In figures 3 and 4 the orbits: $(n, x_1(n, \omega))$ for (SDE) and $(n, x_1(n))$ for (ODE) are displayed:
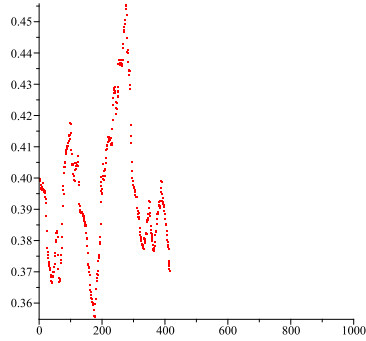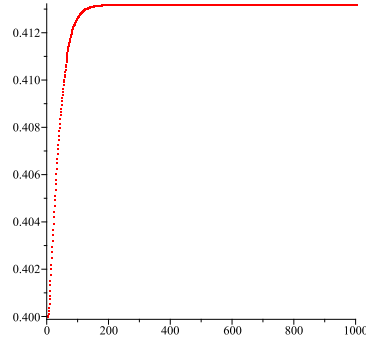
Fig 3. $(n, x_1(n, \omega))$ Fig 4. $(n, x_1(n))$

12

In figures 5 and 6 the orbits: $(n, x_2(n, \omega))$ for (SDE) and $(n, x_2(n))$ for (ODE) are displayed:
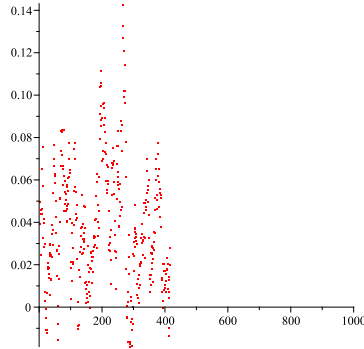
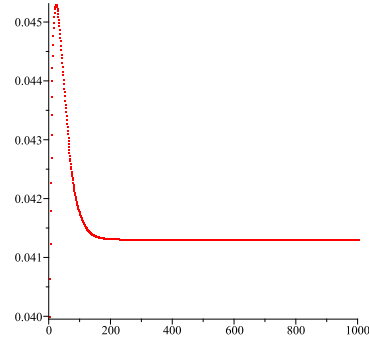Fig 5. $(n, x_2(n, \omega))$

Fig 6. $(n, x_2(n))$

In figures 7 and 8 the orbits: $(x_1(n, \omega), x_2(n, \omega))$ for (SDE) and $(x_1(n), x_2(n))$ for (ODE) are displayed:

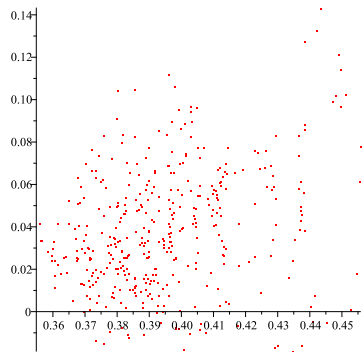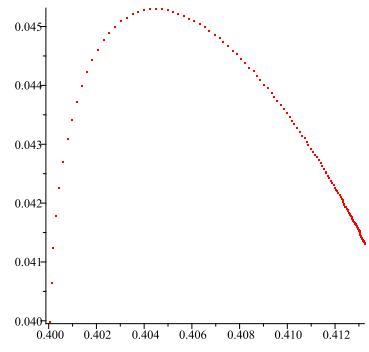Fig 7. $(x_1(n, \omega), x_2(n, \omega))$

Fig 8. $(x_1(n), x_2(n))$

6   The Lyapunov function method for the stochastic economic game.

Theorem 6 is used for the analysis of the stability with the help of the Lyapunov function.

13

Let $V : D = \{[0, \infty) \times \mathbb{R}^2\} \to \mathbb{R}$ be the function given by:

$$V(u) = \frac{1}{2}(\omega_1 u_1^2 + \omega_2 u_2^2),$$

where $\omega_i > 0, i = 1, 2$.

Using formula (12) for the linearized system of (13):

$$
\begin{aligned}
du_1(t) &= (a_{11}u_1(t) + a_{12}u_2(t))dt + (b_{11}u_1(t) + b_{12}u_2(t))dB(t) \\
du_2(t) &= (a_{21}u_1(t) + a_{22}u_2(t))dt + (b_{21}u_1(t) + b_{22}u_2(t))dB(t),
\end{aligned}
\tag{16}
$$

we obtain:

$$
\begin{aligned}
LV(u(t)) =& (a_{11}u_1 + a_{12}u_2)\omega_1 u_1 + (a_{21}u_1 + a_{22}u_2)\omega_2 u_2 + \\
& \frac{1}{2}[(b_{11}u_1 + b_{12}u_2)^2\omega_1 + (b_{21}u_1 + b_{22}u_2)^2\omega_2] = \\
=& (a_{11}\omega_1 + \frac{1}{2}b_{11}^2\omega_1 + \frac{1}{2}b_{21}^2\omega_2)u_1^2 + (a_{22}\omega_2 + \frac{1}{2}b_{12}^2\omega_1 + \frac{1}{2}b_{22}^2\omega_2)u_2^2 + \\
& + (a_{12}\omega_1 + a_{21}\omega_2 + b_{11}b_{12}\omega_1 + b_{21}b_{22}\omega_2)u_1 u_2.
\end{aligned}
\tag{17}
$$

If

$$
\begin{aligned}
A_1 &= -\frac{a_{21} + b_{21}b_{22}}{a_{12} + b_{11}b_{12}}, \quad a_{12} + b_{11}b_{12} \neq 0, \omega_1 = -A_1\omega_2, \\
q_1 &= (a_{11} + ds\frac{1}{2}b_{11})A_1 - \frac{1}{2}b_{21}^2, q_2 = -a_{22} - \frac{1}{2}b_{22}^2 + \frac{1}{2}b_{12}^2 A_1,
\end{aligned}
\tag{18}
$$

then from (17) and (18) we get:

$$LV(u) = -q_1\omega_2 u_1^2 - q_2\omega_2 u_2^2.$$

Form the above relations and Theorem 6 we obtain:

Proposition 8 *If $b_{ij}, i, j = 1, 2$ satisfy the relations:*

$$a_{12} + b_{11}b_{12} \neq 0, A_1 < 0, q_1 > 0, q_2 > 0,$$

then the trivial solution of (16) is mean square stable.

7 Conclusions.

In the present paper we investigate an economic game with stochastic dynamics. We focus on a particular game and determine the Lyapunov exponent for the stochastic system of equations that describes the mathematical model and the Lyapunov function for the analysis of the mean square stability. The calculation of the top Lyapunov exponent allows us to decide whether a stochastic system is stable or not. Using a program in Maple 12, we display the Lyapunov exponent and the system orbits. Conditions for the solution of the stochastic game to be asymptotically mean square stable are established.

References

[1] Afanasev, V.N, Kolmanowskii, V.B, Nosov, N.V., *Mathematical Theory of Control Systems Design*, Kluwer,Dordrecht, 1996.

[2] Arnold, L., *Random dynamical systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[3] Bundău O., Neamţu M., Opriş D., *Rent seeking games with tax evasion*, Pannonian Applied Mathematical Meetings, B.A.M-CX, 2007; 2295-2315: 121-128; arXiv:math.DS. 0706.0664v1, 6 june, 2007.

[4] Bischi G.I., Naimzada A., *Global analysis of a dynamic duopoly game with bounded rationality.* In Advances in Dynamic Games and Applications, Vol. 5., Birkhaur, Boston, 1999.

[5] Cournot A., *Researches into the principles of the theory of wealth.* Engl. trans., Chapter VII, Irwin Paper Back Classics in Economics, 1963.

[6] Hasminskii, R.Z., *Stochastic Stability of Differential Equations*, Sijthoff and Noordho, 1981.

[7] Hu. B.Y., Mahommed, S.E., Yan, F., *Discrete-time approximation of stochastic delay equations*, The Annals of Probability, vol. 32, Nr 1A (2004), 265-314.

[8] Kloeden, P.E., Platen, E., *Numerical solution of stochastic differential equations*, Springer-Verlag, 1992.

[9] Mircea G., Neamţu M., Opriş D.: *Dynamical systems from economy, mechanics, biology described by differential equations with time delay (in Romanian)*, Mirton Publishing house, (2003).

[10] Oseledec, V.I., *A multiplicative Ergodic theorem, Lyapunov characteristic numbers for dynamical systems,* Trans. Moscow Math. Soc. 1968, no.19, 197-231.

[11] Puu T., *Chaos in duopoly pricing.* Chaos, Solitons and Fractals 1991; 1: 573-581.

[12] Puu T., *The Chaotic duopolists revisited.* J. of Economic Behavior and Organization Vol. 1998; 37: 385-394.

[13] Puu T., *Attractors, bifurcations and chaos: Nonlinear phenomena in economics*, Springer-Verlag, 2000.

[14] Rosser J.B., *The development of complex oligopoly dynamics theory.* In Text Book Oligopoly Dynamics: Models and Tools Springer -Verleg, 2002.

[15] Schurz, H., *Moment contractivity and stability exponents of nonlinear stochastic dynamical systems*, IMA Print Series ♯1656, 1999.