

A Note on the Topology of a Generic Subspace of Riem

Henrique de A. Gomes*

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Abstract

For $\mathcal{M} := \text{Riem}(M)$ the space of Riemannian metrics over a compact 3-manifold without boundary M , we study topological properties of the dense open subspace \mathcal{M}' of metrics which possess no Killing vectors. Given the stratification of \mathcal{M} , we work under the condition that, in a sense defined in the text, the connected components of each stratum do not accumulate. Given this condition we find that one of the most fundamental results regarding the topology of \mathcal{M} , namely that it has trivial homotopy groups, would still be true for \mathcal{M}' . This would make the topology of \mathcal{M}' completely understood. Coupled with the fact that for \mathcal{M}' , $\text{Diff}(M) \hookrightarrow \mathcal{M}' \xrightarrow{\pi} \mathcal{M}'/\text{Diff}(M)$ is a principal fiber bundle, which makes $\mathcal{M}'/\text{Diff}(M)$ a proper manifold (as opposed to $\mathcal{M}/\text{Diff}(M)$), we would have that $\pi_n(\mathcal{M}'/\mathcal{D}(M)) = \pi_{n-1}(\mathcal{D}(M))$, which reflects the topology of M . These results would render the space of metrics with no symmetries subject to the above condition, \mathcal{M}' , as an ideal setting for geometrodynamical analysis.

1 Introduction

Geometrodynamics, as championed by Wheeler, is the study of gravitation through a primary focus on *space and changes therein* rather than on space-time itself. It is in essence merely a dynamical view of GR, technically taking form as its constrained Hamiltonian formulation.

To make such an analysis tractable (and for its numerous nice causality properties) space-time is assumed to be globally hyperbolic and hence to be homeomorphic to a topological product $M \times \mathbb{R}$, with M being a space-like hypersurface. As such the (unconstrained) configuration space is given by

$$\mathcal{M} := \text{Riem}(M) = \text{the space of all 3-Riemannian metrics over } M$$

In the main body of this paper this will be the space we will work with; the space of all possible descriptions of spatial configurations of the vacuum Universe.

Defining

$$\mathcal{D}(M) := \text{Diff}(M) \text{ the space of all 3-diffeomorphisms of } M$$

*University of Nottingham, School of Mathematical Sciences, gomes.ha@gmail.com, pmxhg3@nottingham.ac.uk

we identify in \mathcal{M} the 3-metrics¹ $h = f^*g$ and g , where $f \in \mathcal{D}(M)$ acts by pull-back. The resulting orbit space

$$\mathcal{S} := \mathcal{M}/\mathcal{D}(M) \quad \text{is called superspace, the space of geometries of } M$$

This purports to be the physical space, and projection leaves physical quantities invariant under spatial diffeomorphisms.

However a problem quickly arises; the space of spatial geometries, \mathcal{S} , is not properly a manifold [1], which makes the use of many structures inconvenient. However, we will very briefly describe earlier work showing that \mathcal{S} is a stratified manifold; where the strata correspond to manifolds of metric indexed by the (conjugacy classes of) their symmetry groups.

Thus the following is properly a manifold [2]: $\mathcal{M}'/\mathcal{D}(M) = \mathcal{S}'$, where \mathcal{M}' is a generic subspace (and a proper infinite-dimensional manifold) of the space of metrics consisting only of the metrics which possess no non-trivial symmetry group:

$$\mathcal{M}' := \{g \in \mathcal{M} \mid I_g(M) = \text{Id}\} \quad \text{space of metrics without non-trivial isometries.}$$

Then, it is furthermore possible to construct a principal fiber bundle structure

$$\mathcal{D}(M) \hookrightarrow \mathcal{M}' \xrightarrow{\pi} \mathcal{M}'/\mathcal{D}(M) =: \mathcal{S}'$$

which makes many tools from gauge theory available for the study of the dynamics of \mathcal{S}' .

The subspace \mathcal{M}' of \mathcal{M} has not been greatly explored in the literature (see [3] for an exception), in spite of these interesting properties. One of the reasons might be because \mathcal{M} has a very simple topology, namely it is contractible, a property which might not have extended to any non-trivial subspace. Another alternative has been to use the resolution of the singularities of stratification through the use of the augmented space $\mathcal{M} \times F(M)$ [4] to study the topology of \mathcal{M} , and its relation to the topology of M [5].

Here we prove that if the condition stated in the abstract², which we call condition \mathcal{C} in the text, holds true, then this contractibility property of \mathcal{M} , which is extremely valuable in all kinds of analysis on it, extends to \mathcal{M}' .

More specifically, using the genericity of \mathcal{M}' , the fact that \mathcal{M} is contractible, the slice theorem for \mathcal{M} [2], and using an infinite-dimensional Banach version of Thom's transversality theorem, we are able to prove that under condition \mathcal{C} n -spheres in \mathcal{M}' are contractible. Furthermore, since for \mathcal{M}' , unlike \mathcal{M} , we have a proper principal fibration, we get the interesting result that topological properties of \mathcal{S}' depend only on topological properties of $\mathcal{D}(M)$.

In section 2 we give the basic setting and some well-known theorems in the study of \mathcal{M} which will be relevant to our main theorem. In section 3, after briefly introducing the main tool of our theorem (which is not very present in the study of \mathcal{M}), Thom's transversality theorem, we proceed to the proof.

¹We reserve the g 's to the three metric in this paper since it will be the only type that appears throughout the text.

²Namely, whenever the number of connected components of each strata is infinite, no infinite sequence of points belonging to different connected components of the strata have an accumulation point in the same strata.

2 Basic Setting

In this section we introduce the necessary background necessary for our main result.

First of all, some notation. As mentioned in the introduction, M will be a 3-dimensional connected, compact manifold without boundary. We list some definitions along with some basic properties of the spaces we have to deal with in order to investigate the gauge structure of Riem:

- M is a compact oriented manifold without boundary.
- $L_S^2(TM) := TM^* \otimes_S TM^*$ and $L_S^{2+}(TM)$ its positive-definite subspace.
- $\Gamma^r(L_S^2(TM))$, $0 < r < \infty$, is a Banach space, separable in the C^r -weak (Whitney) topology (uniform convergence up to r -derivatives).³ We define the Banach manifold $\mathcal{M}^r = \Gamma^r(L_S^{2+}(TM))$ in the same way.
- $\bigcap_{r=0}^{\infty} \Gamma^r(L_S^2(TM)) =: \Gamma^\infty(L_S^2(TM)) := S_2(M)$ is a Frechét space (Metrizible Complete Locally Convex Topological Vector space), constructed from the inverse limit of separable Banach manifolds (ILB). $\mathcal{M} := \text{Riem}(M) = \bigcap_{r=0}^{\infty} \mathcal{M}^r$ is an open cone in $S_2(M)$ in the sense that for $\lambda_i > 0$ and $g_i \in \mathcal{M}$, $\sum_i g_i \in \mathcal{M}$.

2.1 Orbits

Let us also review the following general facts, which characterize the action of what will play the role of a Lie algebra and Lie group [2]:

- The set $\mathcal{D}(M) := \text{Diff}(M)$ of smooth diffeomorphisms of M is a regular Lie group, and it acts on \mathcal{M} on the right as a group of transformations by pulling back metrics:

$$\begin{aligned} \mu : \mathcal{M} \times \mathcal{D}(M) &\rightarrow \mathcal{M} \\ (g, f) &\mapsto f^*g \end{aligned}$$

an action which is smooth with respect to the C^∞ -structures of \mathcal{M} and $\mathcal{D}(M)$ ⁴. It is clear that two metrics are isometric if and only if they lie in the same orbit,

$$g_1 \sim g_2 \Leftrightarrow g_1, g_2 \in \mathcal{O}_g := \mu_g(\mathcal{D}(M))$$

- The derivative of the orbit map $\mu_g : \mathcal{D}(M) \rightarrow \mathcal{M}$ at the identity

$$\begin{aligned} T_{\text{Id}}\mu_g : \Gamma(TM) &\rightarrow T_g\mathcal{M} \\ X &\mapsto L_Xg \end{aligned} \tag{1}$$

where X is the infinitesimal generator of a given curve of diffeomorphisms of M . The spaces V_g , tangent to the orbits will be called vertical and are defined as:

$$V_g := T_g(\mathcal{O}_g) = \{L_Xg \mid X \in \Gamma(TM)\}$$

³It can be given the structure of a Hilbert space, with derivatives up to order r defined almost everywhere and each partial derivative being square integrable.

⁴The natural action is on the right since of course $(f_1 f_2)^*g = f_2^* f_1^* g$.

Since M is compact, every $X \in \Gamma(TM)$ is complete and $\Gamma(TM)$ forms a regular Lie algebra under the usual commutator of vector fields, $[X_1, X_2] \in \Gamma(TM)$.

- For an isometry group $I_g M \in \mathcal{D}(M)$, the Lie algebra $T_{\text{Id}} I_g M$ is a finite dimensional (and hence splitting) subspace of $\Gamma(TM)$, and so

$$T_{\text{Id}} \mu_g : \Gamma(TM)/T_{\text{Id}} I_g M \rightarrow V_g \quad \text{is an isomorphism}$$

Furthermore, not only is $\mu_g : \mathcal{D}(M)/I_g M \rightarrow \mathcal{M}$ an injective immersion onto \mathcal{O}_g , but more importantly, *it is also a homeomorphism* [2]. This is extremely important as it does not allow different strata to wind around infinitely close to each other, a fact which is needed for the slice theorem and which we will use for our result on the topology of \mathcal{M}' .

So, for example for $g \in \mathcal{M}'$ the map $T_{\text{Id}} \mu_g : \Gamma(TM) \rightarrow V_g$ is an isomorphism over the image.

We state one of the most fundamental results regarding the geometry of \mathcal{M} , which is needed to show \mathcal{M}' is open dense:

Theorem 1 (Slice for \mathcal{S} , [2]) *For each $g \in \mathcal{M}$ there exists a contractible submanifold S of \mathcal{M} containing g such that*

1. $f \in I_g M \Rightarrow f^* S = S$
2. $f \notin I_g M \Rightarrow f^* S \cap S = \emptyset$
3. *There exists a local cross section $\tau : Q \subset \mathcal{D}(M)/I_g(M) \rightarrow \mathcal{D}(M)$ where Q is an open neighborhood of the identity, such that the following map is a diffeomorphism*

$$F : Q \times S \rightarrow \mathcal{U}_g \tag{2}$$

$$([f], s) \mapsto \tau([f])^* s \tag{3}$$

where \mathcal{U}_g is an open neighborhood of $g \in \mathcal{M}$.

A straightforward corollary of this theorem is the following:

Corollary 1 *Given $g \in \mathcal{M}$, and an arbitrary neighborhood V of $\text{Id} \in \mathcal{D}(M)$, there exists a neighborhood \mathcal{N}_V of g such that for any $h \in \mathcal{N}_V$, there exists $f \in V$ such that*

$$f^{-1}(I_h M)f \subseteq I_g M$$

The proof follows easily by noticing that we can take $\mathcal{N}_V \subseteq \mathcal{U}_g$ and $\tau(Q) \subseteq V$ in the theorem above. Then by the third item in the theorem there exists an element $[f] \in Q$ such that $h = F([f], s)$, hence $h' = (\tau([f]^{-1}))^* h \in S$. Then by the second item, any element of $\mathcal{D}(M)$ that fixes h must be in $I_g M$, thus $I_{h'} M = f^{-1}(I_h M)f \subseteq I_g M$. This proves that there always exists a neighborhood of a metric g where all elements have less symmetry than g . Clearly this implies that \mathcal{M}' is open. To prove that it is dense, one must merely infinitesimally perturb the metric to dissolve any symmetries.

2.2 Stratification Theorem

Fischer, in his Stratification Theorem [1], has shown that \mathcal{S} is not actually a manifold, since geometries which possess an isometry beyond the identity don't have neighborhoods homeomorphic to neighborhoods of less symmetric geometries. The metrics that do allow isometries impede the quotient space $\mathcal{M}/\mathcal{D}(M)$ to have a manifold structure.

To be more specific, let $I_g(M) \subset \mathcal{D}(M)$ be the isometry group of g . For a representation $\rho_G : G \rightarrow \mathcal{D}(M)$, the conjugacy class of this representation denoted by $[\rho_G]$ is made up of the representations $\rho'_G : G \rightarrow \mathcal{D}(M)$ such that $\rho'_G = f \circ \rho_G \circ f^{-1}$ for some $f \in \mathcal{D}(M)$. In this way one defines inequivalent actions of G as given by non-conjugate representations of G . We denote the equivalence class of rho_G of conjugate representations by $[\rho_G]$.

Let $\mathcal{M}_{\rho_G} := \{g \in \mathcal{M} \mid I_g(M) = \rho_G(G)\}$ be the space of metrics with isometry $I_g(M)$. We define $\mathcal{M}_{[\rho_G]} := \{g \in \mathcal{M} \mid I_g(M) = f \circ \rho_G(G) \circ f^{-1} \mid f \in \mathcal{D}(M)\}$, is the subspace of metrics which have symmetry type $[\rho_G]$. The space of metrics that have G as a symmetry group is $\cup_{\beta} \mathcal{M}_{[\rho_G^\beta]}$ where $\rho_G^\beta : G \rightarrow \mathcal{D}(M)$ are inequivalent representations. The quotient space for each conjugacy class $[\rho_G]$, $\mathcal{M}_{[\rho_G]}/\mathcal{D}(M)$ does form a submanifold, called a stratum. Unfortunately, it is modeled over a different topological vector space for each symmetry group, namely,

$$\frac{S_2(M)}{(\Gamma^\infty(TM)/T_{\text{Id}}I_gM)} \quad (4)$$

Even if there exists only one equivalence class of actions of G , and thus $\mathcal{M}_G = \mathcal{M}_{[\rho_G]}$, we might still have that the number of components of $\mathcal{M}_{[\rho_G]}$ is infinite. For the moment, let us call the connected components of $\mathcal{M}_{[\rho_G]}$ by $C_i(\mathcal{M}_{[\rho_G]})$. *We will need the assumption that if the number of connected components is infinite, then no infinite sequence of points $c_i \in C_i(\mathcal{M}_{[\rho_G]})$ belonging to different connected components have an accumulation point in $\mathcal{M}_{[\rho_G]}$.*

Definition 1 *If for every stratum $\mathcal{M}_{[\rho_G]}$ the number of connected components $C_i(\mathcal{M}_{[\rho_G]})$ is finite, or if infinite, then no infinite sequence of points $c_i \in C_i(\mathcal{M}_{[\rho_G]})$ belonging to different connected components have an accumulation point in $\mathcal{M}_{[\rho_G]}$, we will say \mathcal{M} obeys condition \mathcal{C} .*

Using the above corollary of the slice theorem, 1 stating the hierarchical relation between the symmetry of metrics and their neighborhoods, Fischer showed that:

$$\mathcal{M}_{[\rho_G]} \cap \overline{\mathcal{M}_{[\rho'_G]}} \neq \emptyset \Leftrightarrow [\rho_G] \leq [\rho'_G] \text{ (or } [\rho'_G] \leq [\rho_G]) \Leftrightarrow \mathcal{M}_{[\rho'_G]} \subset \overline{\mathcal{M}_{[\rho_G]}} \text{ (resp. } \mathcal{M}_{[\rho_G]} \subset \overline{\mathcal{M}_{[\rho'_G]}}) \quad (5)$$

which is what gives it its stratification status.

One can resolve the singularities in this construction by considering the space $\mathcal{M} \times F(M)$ where $F(M)$ is the bundle of frames over M . One then has the PFB $\mathcal{D}(M) \hookrightarrow \mathcal{M} \times F(M) \rightarrow \mathcal{S}_{F(M)} = \{(g, b) \mid (g, b) \simeq (f^*g, f^*b) \in \mathcal{M} \times F(M)\}$ [4]. The main thing here is that even though the action of the diffeomorphisms is not free in either component, it is free on the product, since an isometry that fixes a frame has to be the identity [6]. Our initial aim for developing this work was to study the connection associated to $\mathcal{D}(M)$ and \mathcal{M} , which is why we disregard this auxiliary $\mathbf{SO}(3)$ construction.

For any M , the metrics that do not admit isometries beyond the identity, i.e. in \mathcal{M}' , form a generic, (open and dense), subset of \mathcal{M} [2]. The same is true of the respective projected subset, \mathcal{S}' , consisting of unsymmetrical geometries of \mathcal{S} . Clearly \mathcal{M}' contains all of its orbits. To emphasize, we work with \mathcal{M}' instead of \mathcal{M} since $\mathcal{M}'/\mathcal{D}(M)$ is properly a manifold, and \mathcal{M}' properly a principal fiber bundle above \mathcal{S}' , hence amenable to gauge theory.

3 Transversal Submanifolds and Contractibility of \mathcal{M}'

Now we develop the related topology of \mathcal{M}' , proving that it is in fact simply-connected. In fact, it will prove trivial to extend the simple-connectedness result to prove that all homotopy groups are trivial.

3.1 Transversal Banach Manifolds

We need first state a weaker version of Thom's transversality theorem [7], valid in finite dimensions. Two finite-dimensional submanifolds P, Q of a given smooth finite-dimensional manifold N will be said to intersect transversally, $P \pitchfork Q$, if at every point of intersection their separate tangent spaces at that point together generate the tangent space of the ambient manifold at that point: $T_x P \oplus T_x Q = T_x N$. Manifolds that do not intersect are vacuously transverse.

Theorem 2 (Thom) *Let M be a compact finite-dimensional manifold, N be any smooth manifold and $f_0 : M \rightarrow N$. If Z is an embedded submanifold of N then:*

- **Stability:** *If f_0 is transversal to Z , then for any homotopy f_t , there exists $\epsilon > 0$ such that for $t < \epsilon$, f_t is still transversal.*
- **Genericity:** *There exist a homotopy f_t of f_0 and an $\epsilon > 0$ arbitrarily small such that f_ϵ is transversal to Z .*

Usually, transversality theory in finite dimensions gives generic and stable properties of transversality between any two smooth maps from arbitrary manifold domains into a common target manifold (neither one is necessarily an embedding). Even though the whole of transversality theory does not convert to the infinite-dimensional Banach manifold setting⁵, we still have an effective version of Theorem 2 in [10](Theorem 1E), which will be sufficient for our purposes:

Theorem 3 (Transversality, [10]) *Let X and Y be smooth manifolds modeled on a separable Hilbert space E and a Banach space F , respectively. Let $B \subset Y$ be a closed direct submanifold (i.e. such that for every $y \in B$, $T_y B$ is a direct summand of $T_y Y$). Then for*

⁵Unless, that is, we have two extra conditions: i) one of the domains, say the manifold Q , is finite-dimensional, $f : Q \rightarrow \mathcal{N}$, and ii) the second map, between arbitrary Banach manifolds, $\mathcal{F} : \mathcal{P} \rightarrow \mathcal{N}$, is a smooth Fredholm map (which means its tangent map has everywhere a finite-dimensional kernel and cokernel) [8]. For more details on which properties of transversality theory can be transplanted to Banach manifolds see [9], Ch. 4 (Proposition 4.3.10). The proof of the following theorem 3 itself relies on the fact that Fredholm C^r maps are dense on the space of continuous maps between suitable Banach manifolds.

any smooth map $\phi : X \rightarrow Y$ and any continuous function $\epsilon : X \rightarrow \mathbb{R}(> 0)$ there is a smooth ϵ -approximation $\Psi : X \rightarrow Y$ of ϕ which is transversal over B . Furthermore if $K \subset X$ is a closed set on which ϕ is transversal over B , we can also require that $\Psi|_K = \phi|_K$.

Clearly the theorem gives genericity of transversality, and stability can be recovered as in the original version of the theorem (see sections 4A and 4B of [10]).

3.2 Contractibility of \mathcal{M}'

Now we can state the main theorem regarding the topology of \mathcal{M}' :

Theorem 4 (Homotopy groups of \mathcal{M}') *Under the condition \mathcal{C} , we have that any smooth loop in \mathcal{M}' is contractible. I.e. the fundamental group of \mathcal{M}' is trivial.*

First of all, since the above theorem only works for target Banach manifolds, and we have in our hands ILB manifolds, we do the usual adaptation of working with the extensions of $C^\infty(E)$ to $\mathcal{B}^r(E)$, which is the completion of the space $C^\infty(E)$ under the C^r -Whitney topology, and is thus a Banach space.

In the following then, $\mathcal{M}_{[\rho_\alpha]}^r$ will denote the completion of the manifold $\mathcal{M}_{[\rho_\alpha]}$ under the C^r -Whitney topology, and is thus a Banach manifold. Here, to simplify notation, α parametrizes all the equivalence classes of all the groups, and is countable [1]. In other words for $[\rho_\beta]$, we have the bijection $(\gamma, \beta) \equiv \alpha \in \mathbb{N}$.

Lemma 1 *Under the condition \mathcal{C} , any closed compact manifold of \mathcal{M}^r has a finite number of intersections with $\{\overline{\mathcal{M}_{[\rho_\alpha]}^r}, \alpha \in \mathbb{N}\}$.*

Proof of Lemma 1. First of all a comment is in order about the extendability of the Slice and Stratification theorems. The slice theorem [2] itself is proven first using such completions, i.e. for \mathcal{M}^r (see Theo. 7.1.), but in that case, completion with respect to an inner product induced by the r -th jet bundle, thus making it a Hilbert manifold. We do not need this extra structure for the present purpose. The Stratification theorem relies basically on the hierarchical structure present in the slice theorem (item iii), and is hence also easily extendable to the Banach completion.

Let $\Lambda \in \mathbb{N}$ label a particular set of $[\rho_\alpha]$, where $\mathcal{M}_{[\rho_\alpha]}^r$ are the maximal submanifolds (in the partial ordering of conjugacy classes of Lie groups mentioned above) that intersect a given compact submanifold K of \mathcal{M}^r . We are only interested in the number of intersections between each maximal submanifold $\overline{\mathcal{M}_{[\rho_\alpha]}^r}$ and K , intersections which, by (5) are disjoint for different $[\rho_\alpha]$ and non-accumulating for the same $[\rho_\alpha]$ by assumption. Under this assumption, there will be no difference in proof if we assume there is indeed only one connected component of each $\mathcal{M}_{[\rho_\alpha]}^r$.

By implicitly using the Axiom of Choice, we form a sequence of unique representatives from each intersection, $\{g_\alpha \mid g_\alpha \in K \cap \overline{\mathcal{M}_{[\rho_\alpha]}^r}, \alpha \in I \subset \Lambda\}$. Since K is compact, either the sequence is finite or it has an accumulation point in K . Suppose that there is an accumulation point g_∞ .

Then we have that there necessarily exist different $g_N \in \overline{\mathcal{M}_{[\rho_N]}^r}$ and $g_{N'} \in \overline{\mathcal{M}_{[\rho_{N'}]}^r}$ which lie in the same open set domain $U_{g_N}^r$ of a slice at g_N . But by corollary 1 we then have that

$I_{g_{\mathbb{N}}}(M)$ and $I_{g_{\mathbb{N}}}(M)$ are conjugate, which would imply an inclusion relation between $[\rho_{\mathbb{N}}]$ and $[\rho_{\mathbb{N}}]$. This is absurd, since we specifically chose the maximal representatives in each tree. If there are more than one connected component of the strata, as by assumption there is no accumulation point, the number of intersections of each strata with K will also be finite.

Thus, under the conditions \mathcal{C} , *any compact manifold of \mathcal{M}^r has a finite number of intersections with $\{\overline{\mathcal{M}_{[\rho_\alpha]}^r}, \alpha \in \mathbb{N}\}$.* \square

Proof of Theorem 4. Now take $\alpha : S^1 \rightarrow \mathcal{M}^{r'}$ to be an embedding. As $\mathcal{M}^{r'}$ is open α exists. We know that \mathcal{M}^r is a smooth manifold, it is in fact a contractible cone inside a topological locally convex vector space. Thus there exists an embedding of the disc D^2 given by $f_0 : D^2 \rightarrow \mathcal{M}^r$ such that $f_0(\partial D^2) = \alpha(S^1)$. We shall abbreviate notation and sometimes call f_0 the surface when in fact we mean the image of f_0 .

Let us start by considering the submanifold $\overline{\mathcal{M}_{(G_{\alpha_1})}^r}$, as in the above lemma. We take a homotopy of the embedding f_0 , call it $(f_0)_t$. By genericity of transversality⁶, there exists an arbitrarily small ϵ_1 such that we have an embedding $(f_0)_{\epsilon_1}$, which is homotopic to f_0 where $(f_0)_{\epsilon_1}$ is everywhere transversal to the closure of $\mathcal{M}_{(G_{\alpha_1})}^r$. We rename $(f_0)_{\epsilon_1} = f_{\epsilon_1}$. Furthermore, by theorem 3, we can choose it to satisfy $f_{\epsilon_1}|_{\partial D^2} = f_0|_{\partial D^2}$ which maintains the boundary of this embedding as α .

Thus we have a surface in \mathcal{M}^r whose boundary is α and which is everywhere transversal to $\overline{\mathcal{M}_{[\rho_{\alpha_1}]^r}$.

Now by the stability property of transversality, for each homotopy of f_{ϵ_1} , there exists a ‘stability distance’ $\epsilon > 0$ such that for $t < \epsilon$ it will remain transversal to $\overline{\mathcal{M}_{[\rho_{\alpha_1}]^r}$. Then we repeat the first process for $\overline{\mathcal{M}_{[\rho_{\alpha_2}]^r}$; for some homotopy $(f_{\epsilon_1})_t$, where by genericity we now find an $\epsilon_2 < \epsilon_1'$, such that ϵ_1' is within the stability distance for transversality between f_{ϵ_1} and $\overline{\mathcal{M}_{[\rho_{\alpha_1}]^r}$. Hence $(f_{\epsilon_1})_{\epsilon_2}$ is transversal to $\overline{\mathcal{M}_{[\rho_{\alpha_2}]^r}$ and we have kept it transversal to $\overline{\mathcal{M}_{[\rho_{\alpha_1}]^r}$ as well. Finite iteration will get us a surface, which we call f_λ , whose boundary is α and which is everywhere transversal to $\{\overline{\mathcal{M}_{[\rho_\alpha]}^r}, \alpha \in \mathbb{N}\}$.

Now we are left to prove that transversality between $\overline{\mathcal{M}_{[\rho_\alpha]}^r}$ and the constructed two-dimensional submanifold f_λ implies that their intersection is vacuous. Recall that the $\mathcal{M}_{[\rho_\alpha]}^r$ are splitting submanifolds (at each point $y \in \mathcal{M}_{[\rho_\alpha]}^r$ the entire tangent space $T_y \mathcal{M}^r$ is a direct summand of its tangent space $T_y \mathcal{M}_{[\rho_\alpha]}^r$), hence we can form the projection $\pi : T_y \mathcal{M}^r \rightarrow T_y \mathcal{M}^r / T_y \mathcal{M}_{[\rho_\alpha]}^r$.

Now, suppose that such a y is in the transversal surface, i.e. $y = f_\lambda(x)$. From theorem 3, we have that the composition

$$\pi \circ T_x f_\lambda : T_x D^2 \rightarrow T_y \mathcal{M}^r / T_y \mathcal{M}_{[\rho_\alpha]}^r$$

is surjective. But $T_y \mathcal{M}^r \simeq S_2(M)^r$ and $T_y \mathcal{M}_{[\rho_\alpha]}^r$ has infinite dimensionality and co-dimensionality.⁷ As D^2 is only two dimensional, the map cannot be surjective, which means $f_\lambda^{-1}(\mathcal{M}_{[\rho_\alpha]}^r) = \emptyset$.

⁶Note that even though one of the embeddings is finite-dimensional, and the kernel of the inclusion map of the $\mathcal{M}_{[\rho_\alpha]}^r$ is finite-dimensional, it is not necessarily Fredholm, since the cokernel may have infinite-dimension. This is why we resort to the weaker theorem in [10]. See the ‘Remark’ in [10], pg 5.

⁷ As a matter of fact, one can see that $f^* \mathcal{M}_{\rho_G} = \mathcal{M}_{\rho'_G}$ for $\rho'_G = f \circ \rho_G \circ f^{-1}$. In fact, we have that

$$T_y \mathcal{M}_{\rho_\alpha}^r \simeq \Gamma^r \left(\frac{TM}{T\rho(\mathfrak{g})} \otimes_S \frac{TM}{T\rho(\mathfrak{g})} \right)$$

Since $\mathcal{M}_{[\rho_\alpha]} \subset \mathcal{M}_{[\rho_\alpha]}^r$ this implies the theorem. \square

Furthermore, we can repeat the argument for D^k and assert that, under condition \mathcal{C} , *all homotopy groups of \mathcal{M}' are trivial.*

4 Conclusions

We have studied the topology of an open dense subset of riemannian metrics on a compact manifold without boundary, namely, the subset of metrics with trivial isometry group. For this study, we used intensively both the Ebin-Palais Slice theorem [2], which states the existence of a slice for the action of the diffeomorphism group, and Fischer's stratification theorem [4], which states that \mathcal{M} is partitioned into submanifolds of metrics with different symmetries.

If the different connected components of the strata are non-accumulating in the sense of definition 1, we proved that \mathcal{M}' is contractible. The strategy of the proof was to find discs B^{n+1} with fixed border S^n contained in \mathcal{M} , which were transversal to all such submanifolds of metrics possessing symmetry. Then finding that transversality meant vacuous intersection is a trivial step. To be able to apply the transversality theorem we had at hand [10], we had to use these theorems in the weaker, Banach completion domain.

It is known that differential topology of infinite-dimensional separable Hilbert manifolds is completely classified, in the sense that if H_1 and H_2 are infinite-dimensional separable Hilbert manifolds and have isomorphic homotopy groups, they are in fact diffeomorphic. Hence in our case, since all our homotopy groups are trivial we can in fact state that \mathcal{M}' is itself contractible.

Furthermore, since we now also would have properly a principal fiber bundle $\text{Diff}(M) \hookrightarrow \mathcal{M}' \xrightarrow{\pi} \mathcal{M}'/\text{Diff}(M)$, following the long exact homotopy sequence we would have that $\pi_n(\mathcal{M}'/\mathcal{D}(M)) = \pi_{n-1}(\mathcal{D}(M))$. I.e. if \mathcal{M} obeys condition \mathcal{C} , $\mathcal{M}'/\mathcal{D}(M)$ is a classifying space for $\mathcal{D}(M)$. The dependence of related invariants on the topology of M are studied in [11]. For some classes of M , $\mathcal{D}(M)$ is contractible as well, hence $\mathcal{M}' \simeq \mathcal{S}' \times \mathcal{D}(M)$. If this is the case it can have no Gribov ambiguities, and coordinate choices such as the harmonic ones are smoothly well-defined over \mathcal{M}' .

The study of the dependence of condition \mathcal{C} on the topology of M is left for future work. We also leave for future work the ascertainment of the question if condition \mathcal{C} is satisfied for all compact M without boundary, but only for certain symmetry groups G_α . In this case, their complement $\mathcal{M}_{[\rho_G]}$, would have to be open dense subset of M' for the work done here to remain valid.

As a last remark, it is interesting in fact to compare our result to [5]. There, Giulini studies the case of an open manifold, let us call it Σ and its one point compactification $\bar{\Sigma} = \Sigma \cup \{\infty\}$. By resolving the singularities of stratification using the extended space $\text{Riem}(\bar{\Sigma}) \times$

where $u, v \in TM$ are equivalent if $u = v - T_{\text{Id}}\rho(X)$ for $X \in \mathfrak{g}$. So $T_y\mathcal{M}_{[\rho_\alpha]}^r$ is modelled on a subspace of

$$\mu_{\mathfrak{g}}((\mathcal{D}(M)/I_{\mathfrak{g}}M))_* \left(\Gamma^r \left(\frac{TM}{T\rho(\mathfrak{g})} \otimes_S \frac{TM}{T\rho(\mathfrak{g})} \right) \right)$$

where one has to quotient out the action of the diffeomorphism group by the isotropy subgroup.

$F(M)$, as we mentioned in passing in section 2, one finds a principal fiber bundle $\text{Diff}_F(\bar{\Sigma}) \hookrightarrow \text{Riem}(\bar{\Sigma}) \times F(M) \xrightarrow{\pi} \text{Riem}(\bar{\Sigma})/\text{Diff}_F(\bar{\Sigma})$. By using the same techniques as we did one arrives at $\pi_n(\text{Riem}(\bar{\Sigma})/\text{Diff}_F(\bar{\Sigma})) = \pi_{n-1}(\text{Diff}_F(\bar{\Sigma}))$, where $\text{Diff}_F(\bar{\Sigma})$ are diffeomorphisms f such that $f(\infty) = f(\infty)$ and that $f_*(\infty) = \text{Id}$. So in that case, there is no restriction to a subset of Riem , but there is a restriction on the group of diffeomorphisms, i.e. $\text{Riem}(\bar{\Sigma})/\text{Diff}_F(\bar{\Sigma})$ is a classifying space for the group $\text{Diff}_F(\bar{\Sigma})$. Note that dependence of these homotopy groups on the topology of Σ differs from the dependence of $\pi_n(\mathcal{D}(M))$ on the topology of M .

Furthermore, we mention in passing that by studying the homotopy of $\mathcal{M}'/\mathcal{D}(M)$ one in principle could distinguish between homotopy equivalent manifolds M_1, M_2 which are not homeomorphic [12]. See also this last reference for numerous possible applications of these concepts to quantum gravity and for a literature review.

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