Hypersurfaces with constant sectional curvature of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

Fernando Manfio & Ruy Tojeiro

Abstract

We classify the hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ with constant sectional curvature and dimension $n \geq 3$.

1 Introduction

The submanifold geometry of the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ has been extensively studied in the last years. Here $\mathbb{S}^n$ and $\mathbb{H}^n$ denote the sphere and hyperbolic space of dimension n, respectively. Emphasis has been given on minimal and constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, starting with the work in [1] and [15], among others. See [11] for an updated list of references on this topic.

Surfaces of constant *Gaussian* curvature of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ were investigated in [2] and [3], with special attention to their global properties (see also [12] for a local study in $\mathbb{H}^2 \times \mathbb{R}$). In particular, nonexistence of *complete* surfaces of constant Gaussian curvature c in $\mathbb{S}^2 \times \mathbb{R}$ (respectively, $\mathbb{H}^2 \times \mathbb{R}$) was established for $c < -1$ and $0 < c < 1$ (respectively, $c < -1$). It was also shown that a complete surface of constant Gaussian curvature $c > 1$ in $\mathbb{S}^2 \times \mathbb{R}$ (respectively, $c > 0$ in $\mathbb{H}^2 \times \mathbb{R}$) must be a rotation surface. Moreover, the profile curves of such surfaces have been explicitly determined.

Our aim in this paper is to classify all hypersurfaces with constant sectional curvature and dimension $n \geq 3$ of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. It turns out that for $n \geq 4$ a hypersurface of constant sectional curvature c in $\mathbb{S}^n \times \mathbb{R}$ (respectively, $\mathbb{H}^n \times \mathbb{R}$) only exists, even locally, if $c \geq 1$ (respectively, $c \geq -1$), and for any such values of c it must be an open subset of a complete rotation hypersurface. In the case $n = 3$, exactly one class of nonrotational hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ with constant sectional curvature arises. Each hypersurface in this class in $\mathbb{S}^3 \times \mathbb{R}$ (respectively, $\mathbb{H}^3 \times \mathbb{R}$) has constant sectional curvature $c \in (0, 1)$ (respectively, $c \in (-1, 0)$), and is constructed in an explicit way by means of a family of parallel flat surfaces in $\mathbb{S}^3$ (respectively, $\mathbb{H}^3$). An interesting property of such a hypersurface is that its unit normal vector field makes a constant angle with the unit vector field spanning the factor $\mathbb{R}$. All surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

with this property were classified in [8] and [9], where they were called *constant angle surfaces*. Here we give a simple proof of a generalization of this result to constant angle hypersurfaces of arbitrary dimension of both $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

2 Preliminaries

Let $\mathbb{Q}_\epsilon^n$ denote either the sphere $\mathbb{S}^n$ or hyperbolic space $\mathbb{H}^n$, according as $\epsilon = 1$ or $\epsilon = -1$, respectively. In order to study hypersurfaces $f\colon M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$, our approach is to regard f as an isometric immersion into $\mathbb{E}^{n+2}$, where $\mathbb{E}^{n+2}$ denotes either Euclidean space or Lorentzian space of dimension $(n+2)$, according as $\epsilon = 1$ or $\epsilon = -1$, respectively. More precisely, let $(x_1, \ldots, x_{n+2})$ be the standard coordinates on $\mathbb{E}^{n+2}$ with respect to which the flat metric is written as

$$ds^2 = \epsilon\, dx_1^2 + dx_2^2 + \ldots + dx_{n+2}^2.$$

Regard $\mathbb{E}^{n+1}$ as

$$\mathbb{E}^{n+1} = \{(x_1, \ldots, x_{n+2}) \in \mathbb{E}^{n+2} : x_{n+2} = 0\}$$

and

$$\mathbb{Q}_\epsilon^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n+1} : \epsilon\, x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = \epsilon\} \;\; (\text{with } x_1 > 0 \text{ if } \epsilon = -1).$$

Then we consider the inclusion

$$i\colon \mathbb{Q}_\epsilon^n \times \mathbb{R} \to \mathbb{E}^{n+1} \times \mathbb{R} = \mathbb{E}^{n+2}$$

and study the composition $i \circ f$, which we also denote by f.

Given a hypersurface $f : M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$, let N denote a unit normal vector field to f and let $\frac{\partial}{\partial t}$ be a unit vector field tangent to the second factor. Then, a vector field T and a smooth function ν on M^n are defined by

$$\frac{\partial}{\partial t} = f_* T + \nu N.$$

Notice that T is the gradient of the height function $h = \langle f, \frac{\partial}{\partial t} \rangle$.

Two trivial classes of hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ arise if either ν or T vanishes identically:

Proposition 1 *Let $f : M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ be a hypersurface.*

(i) If T vanishes identically, then $f(M^n)$ is an open subset of a slice $\mathbb{Q}_\epsilon^n \times \{t\}$.

(ii) If ν vanishes identically, then $f(M^n)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a hypersurface of $\mathbb{Q}_\epsilon^n$.

Let ∇ and R be the Levi-Civita connection and the curvature tensor of M^n, respectively, and let A be the shape operator of f with respect to N. Then the Gauss and Codazzi equations are

$$R(X,Y)Z = (AX \wedge AY)Z + \epsilon((X \wedge Y)Z - \langle Y,T\rangle(X \wedge T)Z + \langle X,T\rangle(Y \wedge T)Z), \tag{2}$$

and

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = \epsilon\, \nu(X \wedge Y)T, \tag{3}$$

respectively, where $X, Y, Z \in TM$. Moreover, the fact that $\frac{\partial}{\partial t}$ is parallel in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ yields for all $X \in TM$ that

$$\nabla_X T = \nu AX, \tag{4}$$

and

$$X(\nu) = -\langle AX, T\rangle. \tag{5}$$

3 A basic lemma

Our main goal in this section is to prove the following lemma.

Lemma 2 *Let $f : M_c^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ be a hypersurface of dimension $n \geq 3$ and constant sectional curvature $c \neq 0$. Assume that $T \neq 0$ at $x \in M_c^n$. Then T is a principal direction at x.*

Lemma 2 will follow by putting together Lemma 3 and Proposition 4 below:

Lemma 3 *Let $f : M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ be a hypersurface. Suppose that $T \neq 0$ at $x \in M^n$. Then f has flat normal bundle at x as an isometric immersion into $\mathbb{E}^{n+2}$ if and only if T is a principal direction at x.*

Proposition 4 *Any isometric immersion $g : M_c^n \to \mathbb{E}^{n+2}$ of a Riemannian manifold with dimension $n \geq 3$ and constant sectional curvature $c \neq 0$ has flat normal bundle.*

Lemma 3 was first proved in [7] for $n = 2$ and $\epsilon = 1$. A proof of the general case can be found in [16]. For the proof of Proposition 4 we make use of standard facts from [13] on the theory of flat bilinear forms. Recall that a symmetric bilinear form $\beta\colon V \times V \to W$, where V and W are finite-dimensional vector spaces, is said to be *flat* with respect to an inner product $\langle\ ,\ \rangle\colon W \times W \to \mathbb{R}$ if

$$\langle \beta(X,Y), \beta(Z,T)\rangle - \langle \beta(X,T), \beta(Z,Y)\rangle = 0$$

for all $X, Y, Z, T \in V$. Clearly, the standard example of a flat bilinear form is the second fundamental form of an isometric immersion between space forms with the same constant sectional curvature.

Denote by $N(\beta) \subset V$ the *nullity subspace* of β, given by

$$N(\beta) = \{X \in V \colon \beta(X, Y) = 0 : Y \in V\},$$

and by $S(\beta) \subset W$ its *image subspace*

$$S(\beta) = \text{span}\{\beta(X, Y) : X, Y \in V\}.$$

The next result is a basic fact on flat bilinear forms (cf. Corollary 1 and Corollary 2 in [13]) :

Theorem 5 [13] *Let $\beta \colon V \times V \to W$ be a flat bilinear form with respect to an inner product $\langle\ ,\ \rangle$ on W. Assume that $\langle\ ,\ \rangle$ is either positive-definite or Lorentzian and, in the latter case, suppose that $S(\beta)$ is a nongenerate subspace of W, i.e., $S(\beta) \cap S(\beta)^{\perp} = \{0\}$. Then*

$$\dim N(\beta) \geq \dim V - \dim S(\beta).$$

Another fact we will need in order to handle the case $n = 3$ in Proposition 4 is the following consequence of Theorem 2 in [13]:

Theorem 6 [13] *Let $\beta \colon V \times V \to W$ be a flat bilinear form with respect to an inner product $\langle\ ,\ \rangle$ on W. Assume that $\dim V = \dim W$, that $N(\beta) = \{0\}$ and that $\langle\ ,\ \rangle$ is either positive-definite or Lorentzian. Moreover, in the latter case suppose that there exists a vector $e \in W$ such that $\langle \beta(\ ,\ ), e\rangle$ is positive definite. Then there exists a diagonalyzing basis $\{e_1, \ldots, e_n\}$ for β, i.e., $\beta(e_i, e_j) = 0$ for $1 \leq i \neq j \leq n$.*

Proof of Proposition 4: First recall that $\mathbb{R}^{n+2}$ admits an umbilical inclusion i into both hyperbolic space $\mathbb{H}_c^{n+3}$ and the Lorentzian sphere $\mathbb{S}_c^{n+2,1}$ of constant sectional curvature c, according as $c < 0$ or $c > 0$, respectively, i.e., its second fundamental form α is

$$\alpha(X, Y) = \sqrt{|c|}\langle X, Y\rangle\eta,$$

where η is one of the two normal vectors such that $\langle \eta, \eta\rangle = -\text{sgn(c)}$, where $\text{sgn(c)} = c/|c|$. Similarly, Lorentzian space $\mathbb{L}^{n+2}$ admits umbilical inclusions into $\mathbb{H}_c^{n+2,1}$ or $\mathbb{S}_c^{n+1,2}$, according as $c < 0$ or $c > 0$, respectively.

Then, the second fundamental form $\alpha_\phi = g^*\alpha + i_*\alpha_g$ of $\phi = i \circ g$ at every $x \in M_c^n$ is a flat bilinear form with respect to the inner product $\langle\ ,\ \rangle$ on its three-dimensional normal space. The inner product $\langle\ ,\ \rangle$ is positive-definite if $c < 0$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$, Lorentzian if either $c > 0$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$ or if $c < 0$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$, and has index two if $c > 0$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$. In the latter case, α_ϕ is also flat with respect to the Lorentzian inner product $-\langle\ ,\ \rangle$. Moreover, since

$$\langle \alpha_\phi(,), i_*\eta \rangle = \langle \alpha(,), \eta \rangle = -\mathrm{sgn(c)}\sqrt{|c|}\langle\ ,\ \rangle,$$

it follows that $N(\alpha_\phi) = \{0\}$. Let us consider the two possible cases:

(i) $S(\alpha_\phi)$ *is nondegenerate :* in this case Theorem 5 gives

$$\dim S(\alpha_\phi) \geq n - \dim N(\alpha_\phi) = n.$$

Since $\dim S(\alpha_\phi) \leq 3$, this implies that $n = 3 = \dim S(\alpha_\phi)$. Since $\langle \alpha_\phi(,), -\mathrm{sgn(c)} i_*\eta \rangle$ is positive definite, it follows from Theorem 6 that there exists a basis $\{e_1, \ldots, e_n\}$ of $T_x M^n_c$ such that $\alpha_\phi(e_i, e_j) = 0$ for $i \neq j$. In particular, we have

$$0 = \langle \alpha_\phi(e_i, e_j), i_*\eta \rangle = -\mathrm{sgn(c)}\sqrt{|c|}\langle e_i, e_j \rangle \ \text{ for } \ i \neq j,$$

that is, $\{e_1, \ldots, e_n\}$ is an orthogonal basis. Since $\{e_1, \ldots, e_n\}$ also diagonalizes α_g, we conclude that g has flat normal bundle.

(ii) $S(\alpha_\phi)$ *is degenerate :* in this case, there exists a nonzero vector $\rho \in S(\alpha_\phi) \cap S(\alpha_\phi)^\perp$. Writing $\rho = \eta + i_*\zeta$, with ζ a unit normal vector to g, we obtain from $0 = \langle \alpha_\phi(X, Y), \rho \rangle$ for all $X, Y \in T_x M^n_c$ that

$$\langle \alpha_g(X, Y), \zeta \rangle = \mathrm{sgn(c)}\sqrt{|c|}\langle X, Y \rangle,$$

for all $X, Y \in T_x M^n_c$, i.e., g has an umbilical normal direction. Since g has codimension two, the Ricci equation implies that its normal bundle is flat. ∎

The flat case $c = 0$ can also be handled by means of Theorem 5:

Lemma 7 *Let* $f : M^n_0 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ *be a flat hypersurface of dimension* $n \geq 3$. *Assume that* $T \neq 0$ *at* $x \in M^n_0$.

(i) *If* $\epsilon = 1$, *then* ν *vanishes at* x.

(ii) *If* $\epsilon = -1$, *then either* ν *vanishes at* x *or* $A_N = A_\xi$ *for one of the two possible choices of a unit normal vector* N *to* f *at* x.

In any case, T *is a principal direction of* f *at* x.

Proof: Regard f as an isometric immersion into $\mathbb{E}^{n+2}$. Then, its second fundamental form α is a flat bilinear map by the Gauss equation. Let ξ denote the outward pointing unit normal vector field to $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Then it is easily seen that the shape operator of f with respect to ξ is given by

$$A_\xi T = -\nu^2 T \ \text{ and } \ A_\xi X = -X \ \text{ for } X \in \{T\}^\perp. \tag{6}$$

Assume that $\nu \neq 0$ at $x \in M_0^n$. Then A_ξ, and hence α, has trivial kernel by (6). If $\epsilon = 1$, it follows from Theorem 5 that

$$2 \geq \dim S(\alpha) \geq n,$$

a contradiction that proves (i). If $\epsilon = -1$, Theorem 5 in the Lorentzian case implies that $S(\alpha)$ is a degenerate subspace of the two-dimensional normal space of f in $\mathbb{E}^{n+2}$ at x. Hence $S(\alpha)$ is spanned by the light-like vector $i_*N + \xi$ for one of the two unit normal vectors N to f in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ at x. But the fact that $i_*N + \xi \in S(\alpha)^\perp$ just means that $A_N = A_\xi$.

For the last assertion, notice that a point where ν vanishes is a local minimum for ν, hence $A_N T = 0$ at x by (5). ∎

4 Rotation hypersurfaces

Rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ have been defined and their principal curvatures computed in [6], as an extension of the work in [4] on rotation hypersurfaces of space forms.

With notations as in Section 2, let P^3 be a three-dimensional subspace of $\mathbb{E}^{n+2}$ containing the $\frac{\partial}{\partial x_1}$ and the $\frac{\partial}{\partial x_{n+2}}$ directions. Then $(\mathbb{Q}_\epsilon^n \times \mathbb{R}) \cap P^3 = \mathbb{Q}_\epsilon^1 \times \mathbb{R}$. Denote by $\mathcal{I}$ the group of isometries of $\mathbb{E}^{n+2}$ that fix pointwise a two-dimensional subspace $P^2 \subset P^3$ also containing the $\frac{\partial}{\partial x_{n+2}}$-direction. Consider a curve α in $\mathbb{Q}_\epsilon^1 \times \mathbb{R} \subset P^3$ that lies in one of the two half-spaces of P^3 determined by P^2.

Definition 8 A *rotation hypersurface in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ with profile curve α and axis P^2* is the orbit of α under the action of $\mathcal{I}$.

We will always assume that P^3 is spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$. In the case $\epsilon = 1$, we also assume that P^2 is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$, and that the curve α is parametrized by arc length as

$$\alpha(s) = (\sin(k(s)), 0, \ldots, 0, \cos(k(s)), h(s)),$$

where s runs over an interval I where $\cos(k(s)) \geq 0$, so that $\alpha(I)$ is contained in a closed half-space determined by P^2. Here $k, h \colon I \to \mathbb{R}$ are smooth functions satisfying

$$k'(s)^2 + h'(s)^2 = 1 \ \text{ for all } s \in I. \tag{7}$$

In this case, the rotation hypersurface in $\mathbb{S}^n \times \mathbb{R}$ with profile curve α and axis P^2 can be parametrized by

$$f(s,t) = (\sin(k(s)), \cos(k(s))\varphi_1(t), \ldots, \cos(k(s))\varphi_n(t), h(s)), \tag{8}$$

where $t=(t_1,\ldots,t_{n-1})$ and $\varphi=(\varphi_1,\ldots,\varphi_n)$ parametrizes $\mathbb{S}^{n-1}\subset\mathbb{R}^n$. The metric induced by f is

$$d\sigma^2=ds^2+\cos^2(k(s))dt^2, \tag{9}$$

where dt^2 is the standard metric of $\mathbb{S}^{n-1}$.

For $\epsilon=-1$, one has three distinct possibilities, according as P^2 is Lorentzian, Riemannian or degenerate, respectively. We call f, accordingly, a rotation hypersurface of *spherical*, *hyperbolic* or *parabolic* type, because the orbits of $\mathcal{I}$ are spheres, hyperbolic spaces or horospheres, respectively. In the first case, we can assume that P^2 is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$ and that the curve α is parametrized by

$$\alpha(s)=(\cosh(k(s)),0,\ldots,0,\sinh(k(s)),h(s)). \tag{10}$$

Then f can be parametrized by

$$f(s,t)=(\cosh(k(s)),\sinh(k(s))\varphi_1(t),\ldots,\sinh(k(s))\varphi_n(t),h(s)). \tag{11}$$

The induced metric is

$$d\sigma^2=ds^2+\sinh^2(k(s))dt^2, \tag{12}$$

where dt^2 is the standard metric of $\mathbb{S}^{n-1}$.

In the second case, assuming that P^2 is spanned by $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$, the curve α can also be parametrized as in (10), and a parametrization of f is

$$f(s,t)=(\cosh(k(s))\varphi_1(t),\ldots,\cosh(k(s))\varphi_n(t),\sinh(k(s)),h(s)), \tag{13}$$

where $t=(t_1,\ldots,t_{n-1})$ and $\varphi=(\varphi_1,\ldots,\varphi_n)$ parametrizes $\mathbb{H}^{n-1}\subset\mathbb{L}^n$. The induced metric is

$$d\sigma^2=ds^2+\cosh^2(k(s))dt^2, \tag{14}$$

where dt^2 is the standard metric of $\mathbb{H}^{n-1}$.

Finally, when P^2 is degenerate, we choose a pseudo-orthonormal basis

$$e_1=\frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_{n+1}}\right),\quad e_{n+1}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_{n+1}}\right),\quad e_j=\frac{\partial}{\partial x_j},$$

for $j\in\{2,\ldots,n,n+2\}$, and assume that P^2 is spanned by e_{n+1} and e_{n+2}. Notice that $\langle e_1,e_1\rangle=0=\langle e_{n+1},e_{n+1}\rangle$ and $\langle e_1,e_{n+1}\rangle=1$. Then, we can parametrize α by

$$\alpha(s)=\left(k(s),0,\ldots,0,-\frac{1}{2k(s)},h(s)\right),$$

with

$$k(s)>0\ \text{ and }\ (\ln k)'^2(s)+h'(s)^2=1, \tag{15}$$

and a parametrization of f is

$$f(s,t_2,\ldots,t_n)=\left(k(s),k(s)t_2,\ldots,k(s)t_n,-\frac{1}{2k(s)}-\frac{k(s)}{2}\sum_{i=2}^{n}t_i^2,h(s)\right), \tag{16}$$

whose induced metric is

$$d\sigma^2=ds^2+k^2(s)dt^2, \tag{17}$$

where dt^2 is the standard metric of $\mathbb{R}^{n-1}$.

Remark 9 Our definition of a rotation hypersurface in $\mathbb{Q}_\epsilon^n\times\mathbb{R}$ was taken from [6], and it naturally extends the one given in [4] for space forms. For $\epsilon=-1$, it differs from that used in [2], where only rotation surfaces of spherical type were considered.

We are now in a position to classify rotation hypersurfaces of $\mathbb{Q}_\epsilon^n\times\mathbb{R}$ with constant sectional curvature c and dimension $n\geq 3$. We state separately the cases $\epsilon=1$ and $\epsilon=-1$:

Theorem 10. *Let $f\colon M_c^n\to\mathbb{S}^n\times\mathbb{R}$ be a rotation hypersurface with constant sectional curvature c and dimension $n\geq 3$. Then $c\geq 1$. Moreover,*

(i) if $c=1$ then $f(M_c^n)$ is an open subset of a slice $\mathbb{S}^n\times\{t\}$.

(ii) if $c>1$ then $f(M_c^n)$ is an open subset of a complete hypersurface that can be parametrized by (8), with

$$k(s)=\arccos\left(\frac{1}{\sqrt{c}}\sin(\sqrt{c}\,s)\right) \tag{18}$$

and

$$h(s)=-\sqrt{\frac{c-1}{c}}\ln\left(\frac{\cos(\sqrt{c}\,s)+\sqrt{c-\sin^2(\sqrt{c}\,s)}}{1+\sqrt{c}}\right),\quad s\in[0,\pi/\sqrt{c}]. \tag{19}$$

Theorem 11. *Let $f\colon M_c^n\to\mathbb{H}^n\times\mathbb{R}$ be a rotation hypersurface with constant sectional curvature c and dimension $n\geq 3$. Then $c\geq -1$. Moreover,*

(i) if $c=-1$ then $f(M^n)$ is an open subset of a slice $\mathbb{H}^n\times\{t\}$.

(ii) if $c\in(-1,0)$ then one of the following possibilities holds:

(a) $f(M^n)$ is an open subset of a complete hypersurface of spherical type that can be parametrized by (11), with

$$k(s)=\operatorname{arcsinh}\left(\frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}\,s)\right) \tag{20}$$

and

$$h(s) = \sqrt{\frac{c+1}{-c}} \ln \left(\frac{\cosh(\sqrt{-c}\, s) + \sqrt{-c + \sinh^2(\sqrt{-c}\, s)}}{1 + \sqrt{-c}} \right). \tag{21}$$

(b) *$f(M^n)$ is an open subset of a complete hypersurface of hyperbolical type that can be parametrized by (13), with*

$$k(s) = \operatorname{arccosh} \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c}\, s) \tag{22}$$

and

$$h(s) = \sqrt{\frac{c+1}{-c}} \ln \left(\sinh(\sqrt{-c}\, s) + \sqrt{c + \cosh^2(\sqrt{-c}\, s)} \right). \tag{23}$$

(c) *$f(M^n)$ is an open subset of a complete hypersurface of parabolical type that can be parametrized by (16), with*

$$k(s) = \exp \sqrt{-c}\, s \tag{24}$$

and

$$h(s) = \sqrt{1+c}\, s. \tag{25}$$

(iii) *if $c = 0$, then one of the following possibilities holds:*

(a) *$f(M^n)$ is an open subset of a complete hypersurface of spherical type that can be parametrized by (11), with*

$$k(s) = \operatorname{arcsinh}(s) \tag{26}$$

and

$$h(s) = -1 + \sqrt{1+s^2}. \tag{27}$$

(b) *$f(M^n)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a horosphere of $\mathbb{H}^n$.*

(iv) *if $c > 0$, then $f(M^n)$ is an open subset of a complete hypersurface of spherical type that can be parametrized by (11), with*

$$k(s) = \operatorname{arcsinh} \left(\frac{1}{\sqrt{c}} \sin(\sqrt{c}\, s) \right) \tag{28}$$

and

$$h(s) = -\sqrt{\frac{c+1}{c}} \arctan \left(\frac{\cos(\sqrt{c}\, s)}{\sqrt{c + \sin^2(\sqrt{c}\, s)}} \right). \tag{29}$$

Remark 12 The hypersurfaces in Theorems 10 and 11 also occur in dimension $n = 2$. In particular, those in parts $(ii) - b)$ and $(ii) - c)$ of Theorem 11 provide examples of complete surfaces of constant Gaussian curvature $c \in (-1, 0)$ in $\mathbb{H}^2 \times \mathbb{R}$ that do not appear in [2].

For the proof of Theorems 10 and 11 we make use of the following fact:

Proposition 13. *Assume that the warped product $I \times_\rho \mathbb{Q}^n_\delta$, $n \geq 2$, $\delta \in \{-1, 0, 1\}$, has constant sectional curvature c.*

(i) If $c > 0$, then $\delta = 1$ and $\rho(s) = \frac{1}{\sqrt{c}} \sin(\sqrt{c}\, s + \theta_0)$, $\theta_0 \in \mathbb{R}$.

(ii) If $c = 0$, then one of the following possibilities holds:

(a) $\delta = 1$ and $\rho(s) = \pm s + s_0$, $s_0 \in \mathbb{R}$.

(b) $\delta = 0$ and $\rho(s) = A \in \mathbb{R}$.

(iii) If $c < 0$, then one of the following possibilities holds:

(a) $\delta = -1$ and $\rho(s) = \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c}\, s + \theta_0)$, $\theta_0 \in \mathbb{R}$.

(b) $\delta = 0$ and $\rho(s) = \exp(\pm\sqrt{-c}\, s + s_0)$, $s_0 \in \mathbb{R}$.

(c) $\delta = 1$ and $\rho(s) = \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}\, s + \theta_0)$, $\theta_0 \in \mathbb{R}$.

Proof: In a warped product $I \times_\rho \mathbb{Q}^n_\delta$, $n \geq 2$, the sectional curvature along a plane tangent to $\mathbb{Q}^n_\delta$ is $(\delta - (\rho')^2)/\rho^2$, whereas the sectional curvature along a plane spanned by unit vectors $\partial/\partial s$ and X tangent to I and $\mathbb{Q}^n_\delta$, respectively, is $-\rho''/\rho$. Therefore, $I \times_\rho \mathbb{Q}^n_\delta$ has constant sectional curvature c if and only if

$$(\rho')^2 + c\rho^2 = \delta. \tag{30}$$

Notice that $-\rho''/\rho = c$, or equivalently,

$$\rho'' + c\rho = 0, \tag{31}$$

follows by differentiating (30). If $c > 0$, we obtain from (30) that $\delta = 1$. Moreover, by (31) we have that

$$\rho(s) = A \cos\sqrt{c}\, s + B \sin\sqrt{c}\, s$$

for some $A, B \in \mathbb{R}$, which gives $(\rho')^2 + c\rho^2 = c(A^2 + B^2)$. From (30) we get $c(A^2 + B^2) = 1$, hence we may write

$$A = \frac{1}{\sqrt{c}} \sin\theta_0 \quad \text{and} \quad B = \frac{1}{\sqrt{c}} \cos\theta_0$$

for some $\theta_0 \in \mathbb{R}$. It follows that

$$\rho(s) = \frac{1}{\sqrt{c}} \sin(\sqrt{c}\, s + \theta_0).$$

The remaining cases are similar. ∎

Proof of Theorems 10 and 11: First we determine the possible values of c for a rotation hypersurface $f\colon M_c^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ with constant sectional curvature c and dimension $n \geq 3$. If T vanishes on an open subset, then $c = \epsilon$ by Proposition 1. Otherwise, we can assume that T is nowhere vanishing. Then f has exactly two distinct principal curvatures λ and $\mu \neq 0$, the first one being simple with T as principal direction (cf. [6]). Let $\{T, X_1, \ldots, X_{n-1}\}$ be an orthogonal basis of eigenvectors of A at x, with

$$AT = \lambda T \quad \text{and} \quad AX_i = \mu X_i, \;\; 1 \leq i \leq n-1.$$

From the Gauss equation (2) of f for $X = X_i$ and $Y = Z = X_j$, $i \neq j$, we get

$$c - \epsilon = \mu^2,$$

and hence $c > \epsilon$. This proves the first assertions in Theorems 10 and 11.

Now assume that $\epsilon = 1$. Then f can be parametrized by (8), with $k(s)$ and $h(s)$ satisfying (7), and the metric induced by f is given by (9). Since $c \geq 1$, by Proposition 13 we must have

$$\cos(k(s)) = \frac{1}{\sqrt{c}} \sin(\sqrt{c}\, s + \theta_0)$$

for some $\theta_0 \in \mathbb{R}$. Replacing s by $s - \theta_0/\sqrt{c}$, we can assume that $\theta_0 = 0$. If $c = 1$, then f just parametrizes an open subset of a slice $\mathbb{S}^n \times \{t\}$. If $c > 1$, we obtain that $k(s)$ and $h(s)$ are given by (18) and (19), respectively. The corresponding profile curve is exactly that of the complete surface of constant sectional curvature c in $\mathbb{S}^2 \times \mathbb{R}$ determined in [2], and their argument also applies to show the completeness of f in any dimension $n \geq 3$.

From now on we deal with the case $\epsilon = -1$. Assume first that f is of spherical type. Then f can be parametrized by (11), with $k(s)$ and $h(s)$ satisfying (7), and the metric induced by f is given by (12). By Proposition 13, the warping function $\sinh(k(s))$ must be equal to

$$\frac{1}{\sqrt{c}} \sin(\sqrt{c}\, s + \theta_0), \quad \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}\, s + \theta_0), \;\; \theta_0 \in \mathbb{R}, \quad \text{or} \quad \pm s + s_0, \;\; s_0 \in \mathbb{R},$$

according as $c > 0$, $c < 0$ or $c = 0$, respectively. After suitably replacing the parameter s, we can assume that $\theta_0 = 0$ in the first two cases, and that $\sinh(k(s)) = s$ in the last one. Each possibility gives rise to the expressions (20), (28) and (26) for $k(s)$, and (21), (29) and (27) for $h(s)$, respectively. The corresponding profile curves are exactly those of the complete rotation surfaces with constant sectional curvature of spherical type determined in [2], and the completeness of the corresponding hypersurfaces can be seen in the same way as in [2].

Now suppose that f is of hyperbolical type. Then, it can be parametrized by (13), with $k(s)$ and $h(s)$ satisfying (7), and the induced metric is (14). Since $c \geq -1$, by Proposition 13 we must have $c \in [-1, 0)$ and

$$\cosh(k(s)) = \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c}\, s + \theta_0), \quad \theta_0 \in \mathbb{R}.$$

As before, we can assume that $\theta_0 = 0$. If $c = -1$, then $f(M^n)$ is an open subset of a slice $\mathbb{H}^n \times \{t\}$. Otherwise, k and h are given by (22) and (23), respectively.

Finally, suppose that f is of parabolical type. Then, it can be parametrized by (16), with $k(s)$ and $h(s)$ satisfying (15), and the induced metric is (17). By Proposition 13, we must have $c \leq 0$ and

$$k(s) = A \in \mathbb{R} \quad \text{or} \quad k(s) = \exp(\pm\sqrt{-c}\, s + s_0), \quad s_0 \in \mathbb{R},$$

according as $c = 0$ or $c < 0$, respectively. In the first case, f just parametrizes an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a horosphere of $\mathbb{H}^n$. In the second case, we can assume that $k(s) = \exp \sqrt{-c}\, s$ and then h is given by (25). Completeness of the hypersurfaces in this and the preceding case is straightforward. ∎

5 Constant angle hypersurfaces

Let $g\colon M^{n-1} \to \mathbb{Q}^n_\epsilon$ be a hypersurface and let $g_s\colon M^{n-1} \to \mathbb{Q}^n_\epsilon$ be the family of parallel hypersurfaces to g, that is,

$$g_s(x) = C_\epsilon(s) g(x) + S_\epsilon(s) N(x), \tag{32}$$

where N is a unit normal vector field to g,

$$S_\epsilon(s) = \begin{cases} \cos s, & \text{if } \epsilon = 1 \\ \cosh s, & \text{if } \epsilon = -1 \end{cases} \qquad \text{and} \qquad S_\epsilon(s) = \begin{cases} \sin s, & \text{if } \epsilon = 1 \\ \sinh s, & \text{if } \epsilon = -1. \end{cases}$$

For $\epsilon = 1$, write the principal curvatures of g as

$$\lambda_i = \cot\theta_i, \quad 0 < \theta_i < \pi, \quad 1 \leq i \leq m,$$

where the θ_i form an increasing sequence. For X in the eigenspace of the shape operator A_N of g corresponding to the principal curvature λ_i, $1 \leq i \leq m$, we have

$$g_{s*}X = g_*(\cos s\, X - \sin s\, A_N X) = (\cos s - \sin s \cot\theta_i) X = \frac{\sin(\theta_i - s)}{\sin\theta_i} X,$$

Thus, g_s is an immersion at x if and only if $s \neq \theta_i(x) (\text{mod } \pi)$ for any $1 \leq i \leq m$.

For $\epsilon = -1$, write the principal curvatures of g with absolute value greater than 1 as

$$\lambda_i = \coth\theta_i, \quad \theta_i \neq 0, \quad 1 \leq i \leq m.$$

As in the preceding case, for X in the eigenspace of the shape operator A_N corresponding to the principal curvature λ_i, $1 \leq i \leq m$, we have

$$g_{s*}X = \frac{\sinh(\theta_i - s)}{\sinh\theta_i}X,$$

Thus, g_s is an immersion at x if and only if $s \neq \theta_i(x)$ for any $1 \leq i \leq m$.

In the case $\epsilon = 1$, set

$$U := \{(x, s) \in M^{n-1} \times \mathbb{R} \,:\, s \in (\theta_m(x) - \pi, \theta_1(x))\}. \tag{33}$$

For $\epsilon = -1$, let θ_+ (respectively, θ_-) be the least (respectively, greater) of the θ_i that is greater than 1 (respectively, less than -1), and set

$$U := \{(x, s) \in M^{n-1} \times \mathbb{R} \,:\, s \in (\theta_-(x), \theta_+(x))\}. \tag{34}$$

In both cases, if $V \subset M^{n-1}$ is an open subset and I is an open interval containing 0 such that $V \times I \subset U$, then g_s is an immersion on V for every $s \in I$, with

$$N_s(x) = -\epsilon S_\epsilon(s)g(x) + C_\epsilon(s)N(x) \tag{35}$$

as a unit normal vector at x.

Now define

$$f\colon M^n := V \times I \to \mathbb{Q}^n_\epsilon \times \mathbb{R} \subset \mathbb{E}^{n+2}$$

by

$$f(x, s) = g_s(x) + Bs\frac{\partial}{\partial t}, \quad B > 0. \tag{36}$$

Then

$$f_*X = g_{s*}X, \quad \text{for any } X \in TM^{n-1},$$

and

$$f_*\frac{\partial}{\partial s} = N_s + B\frac{\partial}{\partial t},$$

where

$$N_s(x) = -\epsilon S_\epsilon(s)g(x) + C_\epsilon(s)N(x). \tag{37}$$

Since g_s is an immersion on V for every $s \in I$, it follows that f is an immersion on M^n with

$$\eta(x, s) = -\frac{B}{a}N_s(x) + \frac{1}{a}\frac{\partial}{\partial t}, \quad a = \sqrt{1 + B^2} \tag{38}$$

as a unit normal vector field. Thus, f has the property that

$$\langle \eta, \frac{\partial}{\partial t} \rangle = \frac{1}{a}$$

is constant on M^n. Following [8], f was called in [16] a *constant angle hypersurface*. Constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been classified in [8] and [9], respectively. The next result was obtained in [16] as a consequence of a more general theorem. For the sake of completeness we provide here a simple and direct proof.

Theorem 14. *Any constant angle hypersurface $f\colon M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is either an open subset of a slice $\mathbb{Q}^n_\epsilon \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, an open subset of a product $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a hypersurface of $\mathbb{Q}^n_\epsilon$, or it is locally given by the preceding construction.*

Proof: Let η be a unit normal vector field to f. By assumption, $\nu = \langle \eta, \partial/\partial t \rangle$ is a constant on M^n, which we can assume to belong to $[0,1]$. Since $\|T\|^2 + \nu^2 = 1$, the vector field T has also constant length. By Proposition 1, the cases $\nu = 1$ and $\nu = 0$ correspond to the first two possibilities in the statement, respectively. From now on, we assume that $\nu \in (0,1)$, hence T is a vector field whose length is also a constant in $(0,1)$. Since T is a gradient vector field, its integral curves are (not unit-speed) geodesics in M^n. The fact that T is a gradient also implies that the orthogonal distribution $\{T\}^\perp$ is integrable. Thus, there exists locally a diffeomorphism $\psi\colon M^{n-1} \times I \to M^n$, where I is an open interval containing 0, such that $\psi(x,\cdot)\colon I \to M^n$ are integral curves of T and $\psi(\cdot,s)\colon M^{n-1} \to M^n$ are integral manifolds of $\{T\}^\perp$. Set $F = f \circ \psi$, with f being regarded as an isometric immersion into $\mathbb{E}^{n+2}$. Then

$$X\langle F, \frac{\partial}{\partial t} \rangle = \langle f_*\psi_* X, \frac{\partial}{\partial t} \rangle = \langle \psi_* X, T \rangle = 0$$

for any $X \in TM^{n-1}$. Thus $\langle F(x,s), \frac{\partial}{\partial t} \rangle = \rho(s)$ for some smooth function ρ on I.

On the other hand, it follows from

$$0 = d\nu(X) = -\langle AX, T \rangle \;\text{ for all }\; X \in TM^n$$

that $AT = 0$, hence $F(x,\cdot)\colon I \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ are geodesics in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, where $F = f \circ \psi$. Therefore, the projections $\Pi_1 \circ F(x,\cdot)\colon I \to \mathbb{Q}^n_\epsilon$ and $\Pi_2 \circ F(x,\cdot)\colon I \to \mathbb{R}$ are geodesics of $\mathbb{Q}^n_\epsilon$ and $\mathbb{R}$, respectively.

That $\Pi_2 \circ F(x,\cdot)\colon I \to \mathbb{R}$ are geodesics in $\mathbb{R}$ just means that $\rho(s) = Bs$, for some constant $B > 0$, after possibly a translation in the parameter s and changing s by $-s$. Now define $g\colon M^{n-1} \to \mathbb{Q}^n_\epsilon$ by

$$g(x) = \Pi_1 \circ F(x,0).$$

Rescaling the parameter s so that the geodesics $\Pi_1 \circ F(x,\cdot)\colon I \to \mathbb{Q}^n_\epsilon$ have unit speed, the fact that they are normal to g at $g(x)$ for any $x \in M^{n-1}$ just says that

$$\Pi_1 \circ F(x,s) = g_s(x),$$

where g_s denotes the parallel hypersurface to g at a distance s. ∎

Remark 15 The proof of Theorem 14 also applies to hypersurfaces of $\mathbb{R}^{n+1}$ whose unit normal vector field makes a constant angle with a fixed direction $\partial/\partial t$. Namely, writing $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, with the second factor being spanned by $\partial/\partial t$, it shows that any such hypersurface is either an open subset of an affine subspace $\mathbb{R}^n \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, an open subset of a product $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a hypersurface of $\mathbb{R}^n$, or it is locally given by (36), where g_s is the family of parallel hypersurfaces to some hypersurface g in the first factor $\mathbb{R}^n$, namely, $g_s(x) = g(x) + sN(x)$ for a unit vector field N to g. A proof of this fact for surfaces in $\mathbb{R}^3$ was given in [14].

6 Nonrotational examples in dimension three

Here we use the construction of the previous section to produce a family of nonrotational hypersurfaces of $\mathbb{S}^3 \times R$ (respectively, $\mathbb{H}^3 \times \mathbb{R}$) with constant sectional curvature c for any $c \in (0, 1)$ (respectively, $c \in (-1, 0)$).

Given a hypersurface $g\colon M^{n-1} \to \mathbb{Q}^n_\epsilon$ and the family $g_s\colon M^{n-1} \to \mathbb{Q}^n_\epsilon$ of parallel hypersurfaces to g, an easy computation shows that, whenever $\cot_\epsilon(s) := C_\epsilon(s)/S_\epsilon(s)$ is not a principal curvature of g at any $x \in M^{n-1}$, the shape operator A_s of g_s with respect to the unit normal vector field N_s given by (37) is

$$A_s = (\cot_\epsilon s\, I - A)^{-1}(\cot_\epsilon s\, A + \epsilon I). \tag{39}$$

Let $g\colon M^2 \to \mathbb{Q}^3_\epsilon$ be a surface and let

$$f\colon M^3 := V \times I \subset M^2 \times \mathbb{R} \to \mathbb{Q}^3_\epsilon \times \mathbb{R} \subset \mathbb{E}^5$$

be defined as in the previous section in terms of g. The normal space of f, as a submanifold of $\mathbb{E}^5$, is spanned by the unit normal vector field η given by (38) and by the unit normal vector field $\xi(x, s) = g_s(x)$, which is normal to $\mathbb{Q}^3_\epsilon \times \mathbb{R}$ at $f(x, s)$. We have

$$a\tilde{\nabla}_X \eta = B g_{s*} A^s X = B f_* A^s X$$

and

$$a\tilde{\nabla}_{\frac{\partial}{\partial s}} \eta = \epsilon B g_s = \epsilon B \xi,$$

hence the principal curvatures of A^f_η are

$$-\frac{B}{a} k_1^s, \quad -\frac{B}{a} k_2^s \ \text{ and } \ 0,$$

where k_1^s and k_2^s are the principal curvatures of g_s, the principal curvature 0 corresponding to the principal direction $\partial/\partial s$. On the other hand,

$$\tilde{\nabla}_X \xi = g_{s*} X = f_* X$$

and

$$\tilde{\nabla}_{\frac{\partial}{\partial s}}\xi = N_s = \frac{1}{a^2} f_* \frac{\partial}{\partial s} - \frac{B}{a}\eta.$$

Thus, the principal curvatures of A^f_ξ are $-1/a^2$ and -1, the first being simple with $\partial/\partial s$ as principal direction, and the second having multiplicity two with TV as eigenbundle.

Now assume that $M^2 = M^2_0$ is flat. Then, the principal curvatures k_1 and k_2 of g satisfy $k_1 k_2 = -\epsilon$ everywhere. By (39), the principal curvatures of g_s with respect to N_s are

$$k^s_i = \frac{\cot_\epsilon s k_i + \epsilon}{\cot_\epsilon s - k_i}, \;\; 1 \leq i \leq 2,$$

hence $k^s_1 k^s_2 = -\epsilon$, that is, g_s is also a flat surface. It follows that the sectional curvature of M^3 along TV is

$$(-\frac{B}{a}k^s_1)(-\frac{B}{a}k^s_2) + \epsilon = \frac{\epsilon}{a^2},$$

which is also the sectional curvature of M^3 along any plane spanned by $\partial/\partial s$ and a vector $X \in TV$.

Remark 16 It is easily seen that if the hypersurface f just constructed is regarded as a submanifold of $\mathbb{R}^5$ for $\epsilon = 1$, then it does not have any umbilical normal direction at any point. Hence it provides a new example of a constant curvature submanifold of $\mathbb{R}^5$ with codimension two that is free of weak-umbilic points in the sense of [13].

Example 17 As an explicit example, consider the Clifford torus

$$g\colon M^2_0 := \mathbb{S}^1(\cos\theta_0) \times \mathbb{S}^1(\sin\theta_0) \to \mathbb{S}^3$$

parametrized by

$$g(t_1, t_2) = (\cos\theta_0 \cos t_1, \cos\theta_0 \sin t_1, \sin\theta_0 \cos t_2, \sin\theta_0 \sin t_2),$$

which has

$$N(t_1, t_2) = (-\sin\theta_0 \cos t_1, -\sin\theta_0 \sin t_1, \cos\theta_0 \cos t_2, \cos\theta_0 \sin t_2)$$

as a unit normal vector field in $\mathbb{S}^3$. Then,

$$f\colon M^2_0 \times \mathbb{R} \to \mathbb{S}^3$$

given by (36) can be reparametrized by

$$f(t_1, t_2, s) = (\cos s \cos t_1, \cos s \sin t_1, \sin s \cos t_2, \sin s \sin t_2, Bs),$$

after replacing $s + \theta_0$ by s and a translation in the $\partial/\partial t$-direction. This hypersurface appears in [5] as an example of a weak-umbilic free doubly-rotation surface with constant sectional curvature having the helix $s \mapsto (\cos s, \sin s, Bs)$ as profile, in the sense of [10].

A similar example can be constructed in $\mathbb{H}^3 \times \mathbb{R}$, starting with the flat surface

$$g\colon M_0^2 := \mathbb{H}^1(\cosh\theta_0) \times \mathbb{S}^1(\sinh\theta_0) \to \mathbb{H}^3$$

parametrized by

$$g(t_1, t_2) = (\cosh\theta_0 \cos t_1, \cosh\theta_0 \sin t_1, \sinh\theta_0 \cos t_2, \sinh\theta_0 \sin t_2).$$

In this case, the corresponding constant curvature hypersurface of $\mathbb{H}^3 \times \mathbb{R}$ is

$$f(t_1, t_2, s) = (\cosh s \cos t_1, \cosh s \sin t_1, \sinh s \cos t_2, \sinh s \sin t_2, Bs),$$

These examples can be characterized as the only constant curvature hypersurfaces of $\mathbb{Q}_\epsilon^3 \times \mathbb{R}$ with 0 as principal curvature in the T-direction and whose two remaining principal curvatures are constant along $\{T\}^\perp$.

7 The main result

In this section we prove our main result, namely, we provide a complete classification of all hypersurfaces with constant sectional curvature of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, $n \geq 3$. We state separately the cases $\epsilon = 1$ and $\epsilon = -1$. For $\epsilon = 1$ we have:

Theorem 18. *Let $f\colon M_c^n \to \mathbb{S}^n \times \mathbb{R}$, $n \geq 3$, be an isometric immersion of a Riemannian manifold of constant sectional curvature c. Then $c \geq 0$. Moreover,*

- *(i) if $c = 0$ then $n = 3$ and $f(M_0^3)$ is an open subset of a Riemannian product $M_0^2 \times \mathbb{R}$, where M_0^2 is a flat surface of $\mathbb{S}^3$.*
- *(ii) if $c \in (0, 1)$ then $n = 3$ and f is locally given by the construction described in Section 6.*
- *(iii) if $c = 1$ then $f(M_1^n)$ is an open subset of a slice $\mathbb{S}^n \times \{t\}$.*
- *(iv) if $c > 1$ then $f(M_c^n)$ is an open subset of a rotation hypersurface given by Theorem 10-(ii).*

The classification of constant curvature hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ with dimension $n \geq 3$ reads as follows:

Theorem 19. *Let $f\colon M_c^n \to \mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, be an isometric immersion of a Riemannian manifold of constant sectional curvature c. Then $c \geq -1$. Moreover,*

- *(i) if $c = -1$ then $f(M_{-1}^n)$ is an open subset of a slice $\mathbb{H}^n \times \{t\}$.*

(*ii*) *if* $c \in (-1, 0)$ *then either* $n = 3$ *and* f *is locally given by the construction described in Section* 6*, or* $f(M_0^n)$ *is an open subset of one of the rotation hypersurfaces given by Theorem 11-(ii).*

(*iii*) *if* $c = 0$ *then one of the following possibilities holds:*

(*a*) $n = 3$ *and* $f(M_0^3)$ *is an open subset of a Riemannian product* $M_0^2 \times \mathbb{R}$*, where* M_0^2 *is a flat surface of* $\mathbb{H}^3$*.*

(*b*) $f(M_0^n)$ *is an open subset of a Riemannian product* $M_0^{n-1} \times \mathbb{R}$*, where* M_0^{n-1} *is a horosphere of* $\mathbb{H}^n$*.*

(*c*) $f(M_0^n)$ *is an open subset of the spherical rotation hypersurface given by Theorem 11-(iii)-(a).*

(*iv*) *if* $c > 0$ *then* $f(M_c^n)$ *is an open subset of the spherical rotation hypersurface given by Theorem 11-(iv).*

Proof of Theorems 18 and 19: Assume that the vector field T does not vanish at $x \in M^n$. Then T is a principal direction of f by Lemma 2 and Lemma 7. Let $\{T, X_1, \ldots, X_{n-1}\}$ be an orthogonal basis of eigenvectors of A_N at x, with

$$A_N T = \lambda T \quad \text{and} \quad A_N X_i = \lambda_i X_i, \;\; 1 \leq i \leq n-1.$$

From the Gauss equation (2) of f for $X = X_i$ and $Y = Z = X_j$, $i \neq j$, we get

$$c - \epsilon = \lambda_i \lambda_j, \quad i \neq j. \tag{40}$$

On the other hand, for $X = T$ and $Y = Z = X_i$ the Gauss equation yields

$$c - \epsilon = \lambda \lambda_i - \epsilon ||T||^2. \tag{41}$$

Assume first that $c = \epsilon$. By (40), we can assume that $\lambda_i = 0$ for all $2 \leq i \leq n-1$. Then, applying (41) for $i \geq 2$ yields a contradiction with $T \neq 0$. We conclude that for $c = \epsilon$ the vector field T vanishes identically, and this gives part (*iii*) of Theorem 18 and part (*i*) of Theorem 19.

Now suppose that $c \neq \epsilon$. Then T can not vanish on any open subset. Thus, we can assume without loss of generality that it is nowhere vanishing. If $n \geq 4$, we obtain from (40) that all $\lambda_i' s$ coincide for $2 \leq i \leq n-1$. Denote all of them by μ. Then, the Gauss equations now read

$$c - \epsilon = \mu^2 \tag{42}$$

and

$$c - \epsilon = \lambda \mu - \epsilon \|T\|^2, \tag{43}$$

which can also be written as

$$c = \lambda \mu + \epsilon \nu^2. \tag{44}$$

In particular, it follows from (42) that $c > \epsilon$.

Now, since $T \neq 0$, it follows from (42) and (43) that $\lambda \neq \mu$. Moreover, since T is a principal direction, we obtain from (5) that ν is constant along the leaves of $\{T\}^\perp$, and hence the same holds for λ by (44) (since μ has multiplicity greater than one, one can show using the Codazzi equation that it is constant along its eigenbundle; cf. the proof of Theorem 1 in [6]). Then, one can use the following result to conclude that f is a rotation hypersurface. It slightly generalizes Theorem 1 in [6], but actually follows from its proof.

Proposition 20 *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a hypersurface with $n \geq 3$ and $T \neq 0$. Assume that f has exactly two principal curvatures λ and μ everywhere, the first one being simple with T as a principal direction. If λ is constant along the leaves of the eigenbundle $\{T\}^\perp$ of μ, then $f(M^n)$ is an open subset of a rotation hypersurface.*

Thus, the proofs of Theorems 18 and 19 for $n \geq 4$ are completed by Theorems 10 and 11. This also applies to the case $n = 3$ when we have $\lambda_2 = \lambda_3$ everywhere. By (40) and (41), this is not the case only if $\lambda = 0$. In this situation, equation (44) reduces to

$$\epsilon \nu^2 = c. \tag{45}$$

If $c = 0$, then ν vanishes identically, and thus $f(M^3_0)$ must be an open subset of a Riemannian product $M^2_0 \times \mathbb{R}$, where M^2_0 is a flat surface in either $\mathbb{S}^3$ or $\mathbb{H}^3$, according as $\epsilon = 1$ or $\epsilon = -1$, respectively. If $c \neq 0$, it follows from (45) that f is a constant angle hypersurface. Therefore, by Theorem 14 it is locally given by (36) for some surface $g\colon M^2 \to \mathbb{Q}^3_\epsilon$. Moreover, if we write $\nu = 1/a$, it was shown in Section 6 that the principal curvatures of f are

$$-\frac{B}{a}k_1^s - \frac{B}{a}k_2^s \text{ and } 0,$$

where k_1^s and k_2^s are the principal curvatures of g_s. By the Gauss equation (40), we have

$$c - \epsilon = (-\frac{B}{a}k_1^s)(-\frac{B}{a}k_2^s).$$

Replacing $c = \epsilon/a^2$ and using that $B^2 + 1 = a^2$, it follows that $k_1^s k_2^s = -\epsilon$, hence g is a flat surface. ∎

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