On Fall Colorings of Graphs

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Abstract

A fall k-coloring of a graph G is a proper k-coloring of G such that each vertex of G sees all k colors on its closed neighborhood. We denote Fall(G) the set of all positive integers k for which G has a fall k-coloring. In this paper, we study fall colorings of lexicographic product of graphs and categorical product of graphs and answer a question of [3] about fall colorings of categorical product of complete graphs. Then, we study fall colorings of union of graphs. Then, we prove that fall k-colorings of a graph can be reduced into proper k-colorings of graphs in a specified set. Then, we characterize fall colorings of Mycielskian of graphs. Finally, we prove that for each bipartite graph G, $\mathrm{Fall}(G^c) \subseteq \{ \chi(G^c) \}$ and it is polynomial time to decision whether or not $\mathrm{Fall}(G^c) = \{ \chi(G^c) \}$.
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1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless and without multiple edges). Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$ and $[k] := \{i|\ i \in \mathbb{N},\ 1 \leq i \leq k\}$. A k-coloring (proper k-coloring) of G is a function $f : V \to [k]$ such that for each $1 \leq i \leq k$, $f^{-1}(i)$ is an independent set. We say that G is k-colorable whenever G admits a k-coloring f, in this case, we denote $f^{-1}(i)$ by V_i and call each $1 \leq i \leq k$, a color (of f) and each V_i, a color class (of f). The minimum integer k for which G has a k-coloring, is called the chromatic number of G and is denoted by $\chi(G)$.

Let G be a graph, f be a k-coloring of G and v be a vertex of G. The vertex v is called colorful (or color-dominating or b-dominating) if each color $1 \leq i \leq k$ appears on the closed neighborhood of v ($f(N[v]) = [k]$). The k-coloring f is said to be a fall k-coloring (of G) if each vertex of G is colorful. There are graphs G for which G has no fall k-coloring for any positive integer k. For example, C_5 (a cycle with 5 vertices) and graphs with at least one edge and one isolated vertex, have not any fall k-colorings for any positive integer k. The notation Fall(G) stands for the set of all positive integers k for which G has a fall k-coloring. Whenever $\mathrm{Fall}(G) \neq \emptyset$, we call min(Fall($G$)) and max(Fall($G$)), fall chromatic number of G and fall achromatic number of G and denote them by $\chi_f(G)$ and $\psi_f(G)$, respectively. The terminology fall coloring was firstly introduced in 2000 in [3] and has received attention recently, see [1],[2],[3],[5].

2 Fall colorings of lexicographic product of graphs

Let G and H be graphs. The lexicographic product of G and H is defined the graph with vertex set $V(G) \times V(H)$ and edge set $\{ \ \{(x_1, y_1), (x_2, y_2)\} \mid x_1, x_2 \in V(G)$ and $y_1, y_2 \in V(H)$ and $[\ (\{x_1, x_2\} \in E(G))$ or $(x_1 = x_2, \{y_1, y_2\} \in E(H)) \ ] \ \}$. For each $x \in V(G)$, the induced subgraph of $G[H]$ on $\{x\} \times V(H)$ is denoted by H_x.

Note that $G[H]$ and $H[G]$ are not necessarily isomorphic. For example, let $G := K_2$ and H be the complement of G. $G[H]$ has 4 edges and $H[G]$ has 2 edges and therefore, they are not isomorphic. But lexicographic product of graphs is associative up to isomorphism (For arbitrary graphs G_1, G_2 and G_3, $(G_1[G_2])[G_3]$ and $G_1[G_2[G_3]]$ are isomorphic.).

Theorem 1. *Let G and H be graphs and $k \in \mathrm{Fall}(G[H])$ and f be a fall k-coloring of $G[H]$. Then, for each $x \in V(G)$, $S_x := f(V(H_x))$ forms a fall $|S_x|$-coloring of H_x.*

Proof. Let $x \in V(G)$ and (x, y) be an arbitrary vertex of H_x and its color be α. Then, for each $\beta \in S_x \setminus \{\alpha\}$, there exists a vertex (a, b) of $G[H]$ adjacent with (x, y) which is colored β. Obviously $a = x$, otherwise, since $\beta \in S_x$, there exists a vertex $(x, z) \in V(H_x)$ colored β. (x, y) is adjacent with (a, b) and $x \neq a$, so $\{x, a\} \in E(G)$ and therefore, (x, z) and (a, b) are adjacent in $G[H]$ and both of them are colored β, which is a contradiction. Therefore, $a = x$ and $(a, b) \in V(H_x)$. Hence, S_x forms a fall $|S_x|$-coloring of H_x. ■

Corollary 1. *Let G and H be graphs. Then,* $\mathrm{Fall}(G[H]) \neq \emptyset \Rightarrow \mathrm{Fall}(H) \neq \emptyset$, *or equivalently,* $\mathrm{Fall}(H) = \emptyset \Rightarrow \mathrm{Fall}(G[H]) = \emptyset$.

Corollary 2. *Let G and H be graphs such that* $\mathrm{Fall}(G[H]) \neq \emptyset$. *Then,* $\mathrm{Fall}(H) \neq \emptyset$ *and for each fall k-coloring f of $G[H]$ and each $x \in V(G)$, $\chi_f(H) \leq |f(V(H_x))| \leq \psi_f(H)$.*

There are pairs of graphs (G, H) for which $\mathrm{Fall}(G) = \emptyset$ but $\mathrm{Fall}(G[H]) \neq \emptyset$. For example, $\mathrm{Fall}(C_5) = \emptyset$ but $C_5[K_2]$ has a fall 5-coloring. First let's label the vertices of $C_5[K_2]$ lexicographically: $1 := (1, 1), \ 2 := (1, 2), \ 3 := (2, 1), \ldots, \ 10 := (5, 2)$. Here is a fall 5-coloring f of $C_5[K_2]$: $f(1) = 1, \ f(2) = 2, \ f(3) = 3, \ f(4) = 4, \ f(5) = 1, \ f(6) = 5, \ f(7) = 2, \ f(8) = 4, \ f(9) = 5, \ f(10) = 3$. Also, there are pairs of graphs (G, H) for which $\mathrm{Fall}(G) = \emptyset$ and $\mathrm{Fall}(H) \neq \emptyset$ and $\mathrm{Fall}(G[H]) = \emptyset$. For example, $\mathrm{Fall}(C_5) = \emptyset$ and $\mathrm{Fall}(K_1) \neq \emptyset$ and $\mathrm{Fall}(C_5[K_1]) = \mathrm{Fall}(C_5) = \emptyset$. The next theorem shows that if $\mathrm{Fall}(G) \neq \emptyset$ and $\mathrm{Fall}(H) \neq \emptyset$, then, $\mathrm{Fall}(G[H]) \neq \emptyset$.

Theorem 2. *Let G and H be graphs for which* $\mathrm{Fall}(G) \neq \emptyset$ *and* $\mathrm{Fall}(H) \neq \emptyset$. *Then,* $\{ \ \sum_{i=1}^{s} k_i \mid s \in \mathrm{Fall}(G), \ \forall 1 \leq i \leq s : \ k_i \in \mathrm{Fall}(H) \ \} \subseteq \mathrm{Fall}(G[H])$.

Proof. Let $s \in \mathrm{Fall}(G)$ and $g : V(G) \to [s]$ be a fall s-coloring of G and for each $1 \leq i \leq s$, $k_i \in \mathrm{Fall}(H)$ and h_i be a fall k_i-coloring of H. Let's color each vertex (x, y) of $G[H]$ by color $(g(x), h_{g(x)}(y))$. Indeed, let's consider the function $f : V(G[H]) \to S := \{ \ (g(x), h_{g(x)}(y)) \mid (x, y) \in V(G) \times V(H) \ \}$ which assigns to each (x,y) of $G[H]$, $(g(x), h_{g(x)}(y))$. For each adjacent vertices (x,y) and (a,b) in $G[H]$, $\{x,a\} \in E(G)$ or $(x = a$ and $\{y,b\} \in E(H))$. So, $g(x) \neq g(a)$ or $(g(x) = g(a)$ and $h_{g(x)}(y) \neq h_{g(a)}(b))$. Therefore, $(g(x), h_{g(x)}(y)) \neq (g(a), h_{g(a)}(b))$. This shows that f is a $(\sum_{i=1}^{s} k_i)$-coloring of $G[H]$ such that uses exactly $\sum_{i=1}^{s} k_i$ colors. Now let's show that f is a fall $(\sum_{i=1}^{s} k_i)$-coloring of $G[H]$. For each $(x,y) \in V(G[H])$ and each $(\alpha, \beta) \in S \setminus \{\ (g(x), h_{g(x)}(y))\ \}$, there is a vertex (u,v) of $G[H]$ colored (α,β), or equivalently, $(g(u), h_{g(u)}(v)) = (\alpha, \beta)$. Now, there are two cases:

Case I) The case that $g(x) = g(u)$. In this case, $h_{g(x)} = h_{g(u)}$ and $h_{g(x)}(y) \neq h_{g(u)}(v)$. Since $h_{g(x)}$ is a fall $k_{g(x)}$-coloring of H, there exists a vertex $z \in V(H)$ such that $\{z,y\} \in E(H)$ and $h_{g(x)}(z) = h_{g(u)}(v)$. The vertex (x,z) of $G[H]$ is adjacent with (x,y) and its color is $f((x,z)) = (g(x), h_{g(x)}(z)) = (g(u), h_{g(u)}(v)) = (\alpha, \beta)$.

Case II) The case that $g(x) \neq g(u)$. Since g is a fall s-coloring of G, there exists a vertex $z \in V(G)$ such that $\{x,z\} \in E(G)$ and $g(z) = g(u)$. So, $h_{g(u)}(v) = h_{g(z)}(v)$. The vertex (z,v) is adjacent with (x,y) in $G[H]$ and $f((z,v)) = (g(z), h_{g(z)}(v)) = (g(u), h_{g(u)}(v)) = (\alpha, \beta)$.

Hence, f is a fall $(\sum_{i=1}^{s} k_i)$-coloring of $G[H]$. Therefore, $\{\ \sum_{i=1}^{s} k_i \mid s \in \text{Fall}(G),\ \forall 1 \leq i \leq s$: $k_i \in \text{Fall}(H)\ \} \subseteq \text{Fall}(G[H])$. ■

Corollary 3. *Let G and H be graphs for which* $\text{Fall}(G) \neq \emptyset$ *and* $\text{Fall}(H) \neq \emptyset$. *Then,* $\chi_f(G[H]) \leq \chi_f(G)\chi_f(H) \leq \psi_f(G)\psi_f(H) \leq \psi_f(G[H])$.

$\chi_f(G[H])$ and $\chi_f(G)\chi_f(H)$ are not necessarily equal. For example, $\chi_f(C_9) = 3$ and $\chi_f(K_2) = 2$. Therefore, $\chi_f(C_9)\chi_f(K_2) = 6$, but $\chi_f(C_9[K_2]) \leq 5$, first let's label the vertices of $C_9[K_2]$ lexicographically: 1:=(1,1), 2:=(1,2), 3:=(2,1), ..., 18:=(9,2). Here is a fall 5-coloring f of $C_9[K_2]$: $f(1) = 1,\ f(2) = 4,\ f(3) = 2,\ f(4) = 3,\ f(5) = 5,\ f(6) = 1,\ f(7) = 4,\ f(8) = 2,\ f(9) = 3,\ f(10) = 1,\ f(11) = 5,\ f(12) = 2,\ f(13) = 4,\ f(14) = 3, f(15) = 1,\ f(16) = 2,\ f(17) = 5,\ f(18) = 3$. Also, $\psi_f(G)\psi_f(H)$ and $\psi_f(G[H])$ are not necessarily equal. For example, $\psi_f(C_8) = 2$ and $\psi_f(K_2) = 2$ and therefore, $\psi_f(C_8)\psi_f(K_2) = 4$. But $\psi_f(C_8[K_2]) \geq 5$. First let's label the vertices of $C_8[K_2]$ lexicographically: $1 := (1,1),\ 2 := (1,2),\ 3 := (2,1), \ldots,\ 16 := (8,2)$. Here is a fall 5-coloring f of $C_8[K_2]$: $f(1) = 1,\ f(2) = 2,\ f(3) = 3,\ f(4) = 4,\ f(5) = 5,\ f(6) = 1,\ f(7) = 2,\ f(8) = 3,\ f(9) = 4,\ f(10) = 1,\ f(11) = 5,\ f(12) = 2,\ f(13) = 3,\ f(14) = 1,\ f(15) = 5,\ f(16) = 4$.

Theorem 2 says that if G and H are graphs for which $\text{Fall}(G) \neq \emptyset$ and $\text{Fall}(H) \neq \emptyset$, Then, $\{\ \sum_{i=1}^{s} k_i \mid s \in \text{Fall}(G),\ \forall 1 \leq i \leq s:\ k_i \in \text{Fall}(H)\ \} \subseteq \text{Fall}(G[H])$. Since $5 \in \text{Fall}(C_9[K_2])$ and $5 \notin \{\ \sum_{i=1}^{s} k_i \mid\ s \in \text{Fall}(C_9),\ \forall\ 1 \leq i \leq s$: $k_i \in \text{Fall}(K_2)\ \}$, $\text{Fall}(G[H])$ and $\{\ \sum_{i=1}^{s} k_i \mid s \in \text{Fall}(G),\ \forall\ 1 \leq i \leq s:\ k_i \in \text{Fall}(H)\}$ are not necessarily equal in this theorem.

Theorem 3. *There are pairs of graphs (G,H) for which* $\text{Fall}(G) \neq \emptyset$ *and* $\text{Fall}(H) \neq \emptyset$ *and the following strictly inequality holds.*

$\chi_f(G[H]) < \chi_f(G)\chi_f(H) < \psi_f(G)\psi_f(H) < \psi_f(G[H])$.

Proof. Set $G := C_6 \bigvee C_8 \bigvee C_9$ (the join of C_6 and C_8 and C_9) and $H := K_2$. Since $(C_6 \bigvee C_8 \bigvee C_9)[K_2]$ and ($C_6[K_2]$)$\bigvee$($C_8[K_2]$)$\bigvee$($C_9[K_2]$) are isomorphic, $\chi_f((C_6 \bigvee C_8 \bigvee C_9)[K_2]) = \chi_f(C_6[K_2]) + \chi_f(C_8[K_2]) + \chi_f(C_9[K_2]) \leq 4 + 4 + 5 = 13$ and $\psi_f((C_6 \bigvee C_8 \bigvee C_9)[K_2]) = \psi_f(C_6[K_2]) + \psi_f(C_8[K_2]) + \psi_f(C_9[K_2]) \geq 6 + 5 + 6 =$

17 . Also, $\chi_f(C_6 \bigvee C_8 \bigvee C_9) = 7$ and $\psi_f(C_6 \bigvee C_8 \bigvee C_9) = 8$ and $\chi_f(K_2) = \psi_f(K_2) = 2$, as desired. ■

Theorem 4. *For each $\varepsilon > 0$, There exists a pair of graphs (S,T) for which $\min\{\psi_f(S[T]) - \psi_f(S)\psi_f(T), \psi_f(S)\psi_f(T) - \chi_f(S)\chi_f(T), \chi_f(S)\chi_f(T) - \chi_f(S[T])\} \geq \varepsilon$.*

Proof. With no loss of generality, we can assume that ε is a natural number. Set $G := C_6 \bigvee C_8 \bigvee C_9$ and $S := K_\varepsilon[G]$ and $T := K_2$. Since $S[T]$ and $K_\varepsilon[G[T]]$ are isomorphic and $\chi_f(K_\varepsilon[G[T]]) = \varepsilon\chi_f(G[T])$ and $\psi_f(K_\varepsilon[G[T]]) = \varepsilon\psi_f(G[T])$, the theorem implies. ■

One can easily observe that if G and H are graphs such that $\mathrm{Fall}(G[H]) \neq \emptyset$, then, $\chi_f(G[H]) \geq \omega(G)\chi_f(H)$. The next clear proposition introduces a sufficient condition for equality.

Proposition 1. *Let G and H be graphs such that $\mathrm{Fall}(G) \neq \emptyset$ and $\mathrm{Fall}(H) \neq \emptyset$ and $\chi_f(G) = \omega(G)$. Then, $\chi_f(G[H]) = \chi_f(G)\chi_f(H) = \omega(G)\chi_f(H)$.*

Corollary 4. *If G is a tree or a complete graph or C_{2k} (for some $k \in \mathbb{N} \setminus \{1\}$) and H is a graph such that $\mathrm{Fall}(H) \neq \emptyset$, then, $\chi_f(G[H]) = \chi_f(G)\chi_f(H) = \omega(G)\chi_f(H)$.*

Corollary 1 says that in every fall k-coloring of $G[H]$ and each $x \in V(G)$, the number of colors appear on $V(H_x)$ is at most $\psi_f(H)$. Hence, $\psi_f(G[H]) \leq (\delta(G) + 1)\psi_f(H)$. The following clear proposition introduces a condition for equality.

Proposition 2. *Let G and H be graphs for which $\mathrm{Fall}(G) \neq \emptyset$ and $\mathrm{Fall}(H) \neq \emptyset$ and $\psi_f(G) = \delta(G) + 1$. Then, $\psi_f(G[H]) = \psi_f(G)\psi_f(H) = (\delta(G) + 1)\psi_f(H)$.*

Corollary 5. *If G is a tree or a complete graph or C_{3k} (for some $k \in \mathbb{N}$) and H is a graph such that $\mathrm{Fall}(H) \neq \emptyset$, then, $\psi_f(G[H]) = \psi_f(G)\psi_f(H) = (\delta(G) + 1)\psi_f(H)$.*

3 Type-II graph homomorphisms and lexicographic product of graphs

Now we study a type of graph homomorphisms that is related to fall colorings of graphs.

Definition 1. Let G and H be graphs. A function $f : V(G) \to V(H)$ is called a type-II graph homomorphism from G to H if f satisfies the following two conditions.

1) $\{u, v\} \in E(G) \Rightarrow \{f(u), f(v)\} \in E(H)$.
2) $\{u_1, v_1\} \in E(H) \Rightarrow \forall v \in f^{-1}(v_1) : \exists u \in f^{-1}(u_1)$ s.t $\{u, v\} \in E(G)$. ♠

Type-II graph homomorphisms introduced by Laskar and Lyle in 2009 in [5]. They showed that for any graph G, $k \in \mathrm{Fall}(G)$ iff there exists a type-II graph homomorphism from G to K_k. Note that every type-II graph homomorphism from a graph G to a complete graph, is surjective. If f_1 is a type-II graph homomorphism from G to H and f_2 is a type-II graph homomorphism from H to I, then, $f_2 o f_1$ is a type-II graph homomorphism from G to I. Also, if there exists a type-II graph homomorphism from G to H and $k \in \mathrm{Fall}(H)$, then, $k \in \mathrm{Fall}(G)$. If there exists a type-II graph homomorphism from G_1 to G_2 and a type-II graph homomorphism from H_1 to H_2, then, there exists a type-II graph homomorphism from $G_1 \square H_1$ to $G_2 \square H_2$. We prove a similar theorem for lexicographic product of graphs.

Theorem 5. *Let G_1, G_2, H_1 and H_2 be graphs and f_1 be a type-II graph homomorphism from G_1 to G_2 and f_2 be a surjective type-II graph homomorphism from H_1 to H_2. Then, there exists a type-II graph homomorphism f_3 from $G_1[H_1]$ to $G_2[H_2]$.*

Proof. Let $f_3 : V(G_1[H_1]) \to V(G_2[H_2])$ be defined the function which assigns to each $(g,h) \in V(G_1[H_1])$, $f_3((g,h)) = (f_1(g), f_2(h))$. For each $\{\ (x_1,y_1),(x_2,y_2)\ \} \in E(G_1[H_1])$, $\{x_1,x_2\} \in E(G_1)$ or ($x_1 = x_2$ and $\{y_1,y_2\} \in E(H_1)$). Therefore, $\{\ f_1(x_1), f_1(x_2)\ \} \in E(G_2)$ or ($f_1(x_1) = f_1(x_2)$ and $\{\ f_2(y_1), f_2(y_2)\ \} \in E(H_2)$). Hence, $\{\ (f_1(x_1), f_2(y_1)), (f_1(x_2), f_2(y_2))\ \} \in E(G_2[H_2])$ and consequently, the property 1 holds. Now for each $\{\ (\alpha_1,\beta_1),(\alpha_2,\beta_2)\ \} \in E(G_2[H_2])$ and each $(u_1,v_1) \in f_3^{-1}((\alpha_1,\beta_1))$, there are two cases:

Case I) The case that $\{\alpha_1,\alpha_2\} \in E(G_2)$. Since f_1 is a type-II graph homomorphism and $u_1 \in f_1^{-1}(\alpha_1)$, there exists $u_2 \in f_1^{-1}(\alpha_2)$ such that $\{u_1,u_2\} \in E(G_1)$. Surjectivity of f_2 implies that there exists $v_2 \in f_2^{-1}(\beta_2)$. Therefore, $(u_2,v_2) \in f_3^{-1}((\alpha_2,\beta_2))$ and $\{\ (u_1,v_1),(u_2,v_2)\ \} \in E(G_1[H_1])$ and accordingly, the property 2 holds.

Case II) The case that $\alpha_1 = \alpha_2$ and $\{\beta_1,\beta_2\} \in E(H_2)$. In this case, $u_1 \in f_1^{-1}(\alpha_2)$ and since f_2 is a type-II graph homomorphism and $v_1 \in f_2^{-1}(\beta_1)$, there exists $v_2 \in f_2^{-1}(\beta_2)$ such that $\{v_1,v_2\} \in E(H_1)$. Hence, $(u_1,v_2) \in f_3^{-1}((\alpha_2,\beta_2))$ and $\{\ (u_1,v_1),(u_1,v_2)\ \} \in E(G_1[H_1])$ and therefore, the property 2 holds. Thus, f_3 is a type-II graph homomorphism. ∎

Corollary 6. *If G and H are graphs such that $r_1 \in \mathrm{Fall}(G)$ and $r_2 \in \mathrm{Fall}(H)$, then*

$\chi_f(G[H]) \leq \chi_f(G[K_{r_2}]) \leq \chi_f(K_{r_1}[K_{r_2}]) \leq \psi_f(K_{r_1}[K_{r_2}]) \leq \psi_f(G[K_{r_2}]) \leq \psi_f(G[H])$.

4 Fall colorings of categorical product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The graph $G_1 \times G_2 := (V_1 \times V_2, \{\ \{\ (x_1,y_1),(x_2,y_2)\ \} \mid \{x_1,x_2\} \in E(G_1) \text{ and } \{y_1,y_2\} \in E(G_2)\ \})$ is called the categorical product of G and H.

Categorical product of graphs is commutative and associative up to isomorphism (For each arbitrary graphs G_1, G_2 and G_3, $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic, also, $(G_1 \times G_2) \times G_3$ and $G_1 \times (G_2 \times G_3)$ are isomorphic.). For arbitrary graphs G and H, if $E(G) = \emptyset$ or $E(H) = \emptyset$, then, $E(G \times H) = \emptyset$ and therefore, $G \times H$ has only a fall 1-coloring and $\mathrm{Fall}(G \times H) = \{1\}$. Thus, hereafter, let's focus on nonempty edge set graphs, unless stated otherwise. Firstly, note that $\mathrm{Fall}(G :=$

$(\{a,b,c,d\}\ ,\ \{\ \{a,b\},\{b,c\},\{c,a\},\{d,a\}\ \}\ ) = \emptyset$ and $\mathrm{Fall}(G\times G)=\emptyset$. Secondly, note that $\mathrm{Fall}(C_5 := (\ \{0,1,2,3,4\}\ ,\ \{\ \{0,1\},\{1,2\},\{2,3\},\{3,4\},\{4,0\}\ \}\ )) = \emptyset$, but the function $f : V(C_5 \times C_5) \to [5]$ which assigns to each (i,j) of $V(C_5\times C_5)$, $f((i,j))$:=(the arithmetic residue of $i+2j$ modulo 5)+1 where the last $+$ is the natural summation in $\mathbb{Z}$, is a fall 5-coloring of $C_5\times C_5$, and therefore, $\mathrm{Fall}(C_5 \times C_5) \neq \emptyset$. The next theorem implies that if $\mathrm{Fall}(G)\neq\emptyset$ or $\mathrm{Fall}(H)\neq\emptyset$, then, $\mathrm{Fall}(G\times H)\neq\emptyset$.

Theorem 6. *For each $n\in\mathbb{N}$ and each arbitrary graphs $G_1,\ldots,G_n$,*
$\forall 1\leq i\leq n : \mathrm{Fall}(G_i)\subseteq \mathrm{Fall}(\times_{i=1}^{n} G_i)$.

Proof. Since categorical product of graphs is commutative and associative up to isomorphism, it suffices to prove that $\mathrm{Fall}(G_1)\subseteq\mathrm{Fall}(G_1\times G_2)$. If $\mathrm{Fall}(G_1)=\emptyset$, the theorem holds trivially. For each $k\in\mathrm{Fall}(G_1)$, there exists a fall k-coloring f of G_1. Now, the function $g : V(G_1\times G_2)\to[k]$ which assigns to each $(u,v)\in V(G_1\times G_2)$, $g((u,v)) = f(u)$ is a fall k-coloring of $G_1\times G_2$ and therefore, $k\in\mathrm{Fall}(G_1\times G_2)$. Hence, $\mathrm{Fall}(G_1)\subseteq\mathrm{Fall}(G_1\times G_2)$. ∎

Corollary 7. *For each $n\in\mathbb{N}$ and each arbitrary graphs $G_1,\ldots,G_n$ such that for each $i\in[n]$, $\mathrm{Fall}(G_i)\neq\emptyset$, the following inequalities hold.*
$\chi_f(\times_{i=1}^{n} G_i)\leq \min\{\ \chi_f(G_i)\mid i\in[n]\ \}\leq \max\{\ \psi_f(G_i)\mid i\in[n]\ \}\leq \psi_f(\times_{i=1}^{n} G_i)$.

Now again type-II graph homomorphisms:

Theorem 7. *Let G_1, G_2, H_1 and H_2 be graphs and f_1 be a type-II graph homomorphism from G_1 to G_2 and f_2 be a type-II graph homomorphism from H_1 to H_2. Then, there exists a type-II graph homomorphism f_3 from $G_1\times H_1$ to $G_2\times H_2$.*

Proof. Let $f_3 : V(G_1\times H_1)\to V(G_2\times H_2)$ be defined the function which assigns to each $(g,h)\in V(G_1\times H_1)$, $f_3((g,h)) = (f_1(g), f_2(h))$. For each $\{\ (x_1,y_1),(x_2,y_2)\ \}\in E(G_1\times H_1)$, $\{x_1,x_2\}\in E(G_1)$ and $\{y_1,y_2\}\in E(H_1)$. Therefore, $\{\ f_1(x_1),f_1(x_2)\ \}\in E(G_2)$ and $\{\ f_2(y_1),f_2(y_2)\ \}\in E(H_2)$. Hence, $\{\ f_3((x_1,y_1)),f_3((x_2,y_2))\ \}\in E(G_2\times H_2)$ and therefore, the property 1 of type-II graph homomorphisms holds. Now for each $\{\ (a,b),(c,d)\ \}\in E(G_2\times H_2)$ and each $(\alpha,\beta)\in f_3^{-1}(\ (c,d)\ )$, $\alpha\in f_1^{-1}(c)$ and $\beta\in f_2^{-1}(d)$. So, there exist $x\in f_1^{-1}(a)$ and $y\in f_2^{-1}(b)$ such that $\{x,\alpha\}\in E(G_1)$ and $\{y,\beta\}\in E(H_1)$, hence, $(x,y)\in f_3^{-1}(\ (a,b)\ )$ and $\{\ (x,y),(\alpha,\beta)\ \}\in E(G_1\times H_1)$. So, the property 2 of type-II graph homomorphisms holds, too. Consequently, f_3 is a type-II graph homomorphism.

∎

We know that if f is a type-II graph homomorphism from G to H and $k\in\mathrm{Fall}(H)$, then, $k\in\mathrm{Fall}(G)$. Also, for each graph G and each natural number k, $k\in\mathrm{Fall}(G)$ iff there exists a type-II graph homomorphism from G to K_k. Therefore, the previous theorem implies the following corollary.

Corollary 8. *Let $n\in\mathbb{N}$ and for each $i\in[n]$, G_i be a graph and $k_i\in\mathrm{Fall}(G_i)$. Then, there exists a type-II graph homomorphism from $\times_{i=1}^{n} G_i$ to $\times_{i=1}^{n} K_{k_i}$ and*

$\mathrm{Fall}(\times_{i=1}^{n} K_{k_i}) \subseteq \mathrm{Fall}(\times_{i=1}^{n} G_i)$. *Also,* $\chi_f(\times_{i=1}^{n} G_i) \leq \chi_f(\times_{i=1}^{n} K_{k_i}) \leq \psi_f(\times_{i=1}^{n} K_{k_i}) \leq \psi_f(\times_{i=1}^{n} G_i)$. *These inequalities can easily extend to more inequalities in general. For example, in the case* $n = 2$, $\chi_f(G_1 \times G_2) \leq \left\{ \begin{array}{l} \chi_f(K_{k_1} \times G_2) \\ \chi_f(G_1 \times K_{k_2}) \end{array} \right. \leq \chi_f(K_{k_1} \times K_{k_2}) \leq \psi_f(K_{k_1} \times K_{k_2}) \leq \left\{ \begin{array}{l} \psi_f(K_{k_1} \times G_2) \\ \psi_f(G_1 \times K_{k_2}) \end{array} \right. \leq \psi_f(G_1 \times G_2)$.

Dunbar, et al. in [3] showed that for each $m, n \in \mathbb{N}\setminus\{1\}$, $\mathrm{Fall}(K_m \times K_n) = \{m, n\}$. They also showed that if $n \in \mathbb{N} \setminus \{1\}$ and for each $i \in [n]$, $r_i \in \mathbb{N} \setminus \{1\}$, then, $\{r_1, ..., r_n\} \subseteq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$. They constructed a fall 6-coloring of $K_2 \times K_3 \times K_4$ and asked for conditions for a finite and with more than two elements set $S := \{r_1, ..., r_n\} \subseteq \mathbb{N} \setminus \{1\}$ for which $S \subsetneqq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$.

Theorem 8. *Let* $n \geq 3$, $S := \{r_1, ..., r_n\} \subseteq \mathbb{N} \setminus \{1\}$, $r_1 < r_2 < ... < r_n$ *and* S *contain at least one even integer. Then,* $S \subsetneqq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$*, besides,* $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ *contains an integer greater than* r_n.

Proof. There are five cases.

Case I) The case that $r_1 = 2$. In this case, let $t \in \{r_1, ..., r_n\} \setminus \{r_1, r_n\}$. Consider $K_2 \times K_t \times K_{r_n}$. Let σ be a disarrangement of $[t]$ (a permutation σ of $[t]$ such that for each $i \in [t]$, $\sigma(i) \neq i$). Obviously, $\{ \{(1, i, 1), (1, \sigma(i), 2), (2, i, 2), (2, \sigma(i), 1)\} \mid 1 \leq i \leq t \} \bigcup \{ \{(x, y, z) \mid (x, y, z) \in K_2 \times K_t \times K_{r_n},\ z = i \} \mid 3 \leq i \leq r_n \}$ is the set of color classes of a fall $(r_n + t - 2)$-coloring of $K_2 \times K_t \times K_{r_n}$. But $r_n + t - 2 > r_n$ and therefore, in this case, $\mathrm{Fall}(K_2 \times K_t \times K_{r_n})$ contains an integer greater than r_n.

Case II) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains at least two distinct even integers such that one of them is r_n and the other is r_s that $s \in \{1, ..., n-1\}$. Let $r_j \in \{r_1, ..., r_n\} \setminus \{r_s, r_n\}$. Consider $K_{r_s} \times K_{r_j} \times K_{r_n}$ and a disarrangement σ of $[r_j]$. For each $1 \leq t \leq r_j$, color the vertices $(1, t, 1), (1, \sigma(t), 2), (2, t, 2)$ and $(2, \sigma(t), 1)$ with the color t and color each other vertex (x, y, z) with the color $\lfloor \frac{x-1}{2} \rfloor (\frac{r_j r_n}{2}) + \lfloor \frac{z-1}{2} \rfloor r_j +$ the color of $(x - 2\lfloor \frac{x-1}{2} \rfloor, y, z - 2\lfloor \frac{z-1}{2} \rfloor)$. This is a fall $\frac{r_s r_j r_n}{4}$-coloring of $K_{r_s} \times K_{r_j} \times K_{r_n}$. Since $2 < r_1$, $\frac{r_s r_j r_n}{4} > \max\{r_s, r_j, r_n\}$. Hence, Theorem 6 implies that $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than r_n.

Case III) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains at least two distinct even integers such that none of them is r_n. Similar to the case II, $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than r_n.

Case IV) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains exactly one even integer and r_n is even. In this case, consider $K_{r_{n-2}-1} \times K_{r_{n-1}} \times K_{r_n}$ and a disarrangement σ of $[r_{n-1}]$. For each $1 \leq t \leq r_{n-1}$, color the vertices $(1, t, 1), (1, \sigma(t), 2), (2, t, 2)$ and $(2, \sigma(t), 1)$ with the color t and color each other vertex (x, y, z) of $K_{r_{n-2}-1} \times K_{r_{n-1}} \times K_{r_n}$ with the color $\lfloor \frac{x-1}{2} \rfloor (\frac{r_{n-1} r_n}{2}) + \lfloor \frac{z-1}{2} \rfloor r_{n-1} +$ the color of $(x - 2\lfloor \frac{x-1}{2} \rfloor, y, z - 2\lfloor \frac{z-1}{2} \rfloor)$. Also, color each vertex (r_{n-2}, y, z) of $K_{r_{n-2}} \times K_{r_{n-1}} \times K_{r_n}$ with the color $\frac{(r_{n-2}-1) r_{n-1} r_n}{4} + 1$. Therefore, a fall $(\frac{(r_{n-2}-1) r_{n-1} r_n}{4} + 1)$-coloring of $K_{r_{n-2}} \times K_{r_{n-1}} \times K_{r_n}$ and also of $\times_{i=1}^{n} K_{r_i}$ yields. But, $\frac{(r_{n-2}-1) r_{n-1} r_n}{4} + 1 > r_n$. Thus, $\mathrm{Fall}(K_2 \times K_t \times K_{r_n})$ and therefore $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than r_n.

Case V) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains exactly one even integer and r_n is odd. In this case, similar to the case IV, $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than r_n.

Accordingly, in all cases, $\{r_1, ..., r_n\} \subsetneqq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$. Besides, $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than r_n.

■

Even though Dunbar, et al. in [3] constructed a fall 6-coloring of $K_2 \times K_3 \times K_4$, this theorem also shows that in the corollary 7, the inequality $\max\{\ \psi_f(G_i) \mid i \in [n]\ \} \leq \psi_f(\times_{i=1}^{n} G_i)$ can be strict in many cases. But we conjecture that the inequality $\chi_f(\times_{i=1}^{n} G_i) \leq \min\{\ \chi_f(G_i) \mid i \in [n]\ \}$ is always an equality.

Conjecture 1. *For each $n \in \mathbb{N}$ and for each arbitrary graphs $G_1, \ldots, G_n$ such that for each $i \in [n]$, $\mathrm{Fall}(G_i) \neq \emptyset$, the following equality holds.*
$\chi_f(\times_{i=1}^{n} G_i) = \min\{\chi_f(G_i) |\ i \in [n]\}$.

5 Fall colorings of union of graphs

Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, G_i be a graph. The graph $(\ \bigcup_{i=1}^{n} (\{i\} \times V(G_i))\ ,\ \bigcup_{i=1}^{n}\ \{\ \{(i,x),(i,y)\} \mid \{x,y\} \in E(G_i)\ \}\ )$ is called the union graph of $G_1, ..., G_n$ and is denoted by $\biguplus_{i=1}^{n} G_i$.

The following obvious theorem describes fall colorings of union of graphs.

Theorem 9. *Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, G_i be a graph. Then, the following three statements hold.*
1) If $\mathrm{Fall}(\biguplus_{i=1}^{n} G_i) \neq \emptyset$, then, for each $1 \leq i \leq n$, $\mathrm{Fall}(G_i) \neq \emptyset$.
2) $\mathrm{Fall}(\biguplus_{i=1}^{n} G_i) = \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.
3) If $\mathrm{Fall}(\biguplus_{i=1}^{n} G_i) \neq \emptyset$, then, $\chi_f(\biguplus_{i=1}^{n} G_i) = \min \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$ and $\psi_f(\biguplus_{i=1}^{n} G_i) = \max \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.

Since any graph G is isomorphic to any union graph of all its connected components, the following corollary yields immediately.

Corollary 9. *Let G be a graph and G_i $(1 \leq i \leq n)$ be all its connected components. Then, the following three statements hold.*
1) If $\mathrm{Fall}(G) \neq \emptyset$, then, for each $1 \leq i \leq n$, $\mathrm{Fall}(G_i) \neq \emptyset$.
2) $\mathrm{Fall}(G) = \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.
3) If $\mathrm{Fall}(G) \neq \emptyset$, then, $\chi_f(G) = \min \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$ and $\psi_f(G) = \max \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.

6 Restriction of fall t-colorings of a graph into proper t-colorings of graphs in a specified set

Now we prove that fall k-colorings of a graph can be reduced into proper k-colorings of graphs in a specified set.

Let G be a graph and $1 \leq t \leq \delta(G) + 1$ be a fixed natural number. For each $v \in V(G)$, choose $t - 1$ arbitrary elements of $N_G(v)$ and join these $t - 1$ vertices to each other and name the new graph H. Let $\widehat{G_t}$ be the set of all graphs H constructed like this.

Theorem 10. *For each* $1 \leq t \leq \delta(G)+1$, $t \in \mathrm{Fall}(G)$ *iff* $t \in \{\chi(H) |\ H \in \widehat{G_t}\}$. *Specially,* $\mathrm{Fall}(G) = \bigcup_{i=1}^{\delta(G)+1} (\ \{\chi(H) |\ H \in \widehat{G_i}\} \bigcap \{i\}\ )$.

Proof. Let $1 \leq t \leq \delta(G)+1$. If $t \in \{\chi(H) |\ H \in \widehat{G_t}\}$, then, there exists a graph H in $\widehat{G_t}$ such that $\chi(H) = t$ and there exists a t-coloring f of H. This coloring f of H, is obviously a fall t-coloring of G and therefore, $t \in \mathrm{Fall}(G)$. Conversely, if $t \in \mathrm{Fall}(G)$, then, there exists a fall t-coloring g of G. For each $v \in V(G)$, there exist $t-1$ elements of $N_G(v)$ such that the set of their colors and the color of v is equal to $[t]$, join all of them to each other to construct a new graph T in $\widehat{G_t}$. The fall t-coloring g of G is obviously a t-coloring of T, also $\omega(T) \geq t$, thus, $\chi(T) = t$ and $t \in \{\chi(H) |\ H \in \widehat{G_t}\}$. The second part of the theorem follows immediately.

■

Restricting this theorem into r-regular graphs and $t = r+1$, yields a beautiful proposition of [4] but in different terminologies.

Proposition 3. *For each r-regular graph G,* $r+1 \in \mathrm{Fall}(G)$ *iff* $\chi(G^{(2)}) = r+1$, *where* $G^{(2)} = (\ V(G)\ ,\ \{\ \{x,y\} \mid x,y \in V(G),\ x \neq y,\ d_G(x,y) \leq 2\ \}\ )$.

7 Fall Colorings of Mycielskian of graphs

Let $G := (\ \{x_1,\ldots,x_n\}\ ,\ E(G)\ )$ be a graph. The Mycielskian of G (denoted by $M(G)$) is a graph with $2n+1$ vertices $x_1,\ldots,x_n,y_1,\ldots,y_n,z$ with edge set $E(G) \bigcup \{\ \{y_i,x_j\} \mid i,j \in [n],\ \{x_i,x_j\} \in E(G)\ \} \bigcup \{\ \{z,y_i\} \mid i \in [n]\ \}$.

For example, $M(K_2)$ is C_5. We know that $\mathrm{Fall}(M(K_2)) = \mathrm{Fall}(C_5) = \emptyset$. Now we prove that for each graph G, $\mathrm{Fall}(M(G)) = \emptyset$.

Theorem 11. *For each graph G,* $\mathrm{Fall}(M(G)) = \emptyset$.

Proof. If $E(G) = \emptyset$, then, $M(G)$ has at least one isolated vertex and also at least one edge. Therefore, $\mathrm{Fall}(M(G)) = \emptyset$. Now we prove the theorem for the case $E(G) \neq \emptyset$. If $E(G) \neq \emptyset$ and $\mathrm{Fall}(M(G)) \neq \emptyset$, then, there exists a fall k-coloring f of $M(G)$ for some $k \in \mathbb{N}$. Since $E(G) \neq \emptyset$, there exists an integer $i_0 \in [n]$ such that $f(x_{i_0}) \neq f(z)$ and since for each $j \in [n]$, $f(y_j) \neq f(z)$ and f is a fall k-coloring, there exists $i_1 \in [n]$ such that $x_{i_1} \in N_G(x_{i_0})$ and $f(x_{i_1}) = f(z)$. Since for each $i \in [n]$ with $f(x_i) \neq f(z)$, $N(y_i) \setminus \{z\} \subseteq N(x_i)$, so $f(x_i) \in \{f(y_i), f(z)\}$, on the other hand, $f(x_i) \neq f(z)$, hence, $f(x_i) = f(y_i)$. This immediately shows that each color of $[k]$ appears on the neighborhood of y_{i_1}, which is a contradiction. Hence, $\mathrm{Fall}(M(G)) = \emptyset$.

■

8 Fall colorings of complement of bipartite graphs

Complements of bipartite graphs are very interesting graphs, because in each proper k-coloring, the cardinality of each color class is at most 2. The following theorem characterizes all fall colorings of this type of graphs.

Theorem 12. *Let G be a bipartite graph. Then,* $\mathrm{Fall}(G^c) \subseteq \{ \chi(G^c) \}$*. Besides, it is polynomial to decide whether or not* $\mathrm{Fall}(G^c) = \{ \chi(G^c) \}$.

Proof. If $\mathrm{Fall}(G^c) \neq \emptyset$, then, $\exists k \in \mathrm{Fall}(G^c)$. Suppose that f is a fall k-coloring of G^c. Obviously, each color class of f is either of the form $\{x\}$ or of the form $\{y, z\}$ such that $y \in A$ and $z \in B$. A color class is of the form $\{x\}$ iff x is an isolated vertex of the graph G. Therefore, the set of color classes of f is the union of $\{ \{x\} \mid x \in V(G),\ deg_G(x) = 0 \}$ and the set of edges of a perfect matching of the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$, also, $k = |V(G)| - \frac{1}{2}|\{ x \mid x \in V(G),\ deg_G(x) > 0 \}|$. Therefore, $\mathrm{Fall}(G^c) \subseteq \{ |V(G)| - \frac{1}{2}|\{ x \mid x \in V(G),\ deg_G(x) > 0 \}| \}$. Besides, if $\mathrm{Fall}(G^c) \neq \emptyset$, then, the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$ has a perfect matching, in this case, obviously, $\chi(G^c) = |V(G)| - \frac{1}{2}|\{ x \mid x \in V(G),\ deg_G(x) > 0 \}|$, and consequently, $\mathrm{Fall}(G^c) = \{ \chi(G^c) \}$. Therefore, for each bipartite graph G, $\mathrm{Fall}(G^c) \subseteq \{ \chi(G^c) \}$. We know that if $\mathrm{Fall}(G^c) \neq \emptyset$, then, the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$ has a perfect matching. Conversely, if the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$ has a perfect matching, then, the union of $\{ \{x\} \mid x \in V(G),\ deg_G(x) = 0 \}$ and the edge set of each perfect matching of the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$ is the set of color classes of a fall ($|V(G)| - \frac{1}{2}|\{ x \mid x \in V(G),\ deg_G(x) > 0 \}|$)-coloring of G^c and therefore, $\mathrm{Fall}(G^c) \neq \emptyset$. Accordingly, $\mathrm{Fall}(G^c) = \{ \chi(G^c) \}$ iff $\mathrm{Fall}(G^c) \neq \emptyset$ iff the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$ has a perfect matching. Since the problem of deciding whether or not the induced subgraph of G on $\{ x \mid x \in V(G),\ deg_G(x) > 0 \}$ has a perfect matching, is a polynomial time problem, thus, it is polynomial time to decide whether or not $\mathrm{Fall}(G^c) = \{ \chi(G^c) \}$.
■

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