New proofs of some formulas of Guillera-Ser-Sondow

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We present logarithmic series for $u, \ln u$ and the Euler-Mascheroni constant γ. It was indicated by J. Sondow that Theorem 4 and all proofs are new. All proofs are elementary. We present some conjectures.

1. Introduction and main results.

Theorem 1. *For each real $u > 0$*

$$1 = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{ku+1}.$$

Remark 2. For each real $u > 0$

$$u = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln(ku+1) = \sum_{m=1}^{\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{m} \ln(ku+1).$$

From the proof of Theorem 1 it follows that for each $u_0 > 0$ the convergence in Theorem 1 is uniform for $u \in [0, u_0]$. Hence the first formula of Remark 2 follows by integrating the formula of Theorem 1. The second formula of Remark 2 is deduced from the first one below. The second formula of Remark 2 is [GS06, Theorem 5.3, GS08]. In [GS06] the proof of Theorem 5.3 is easy: Theorem 5.3 follows from Theorem 5.2 (which is easy) and Example 2.4 (which is easy and well-known).

Corollary 3. *For each real $u > 0$*

$$u = \lim_{m\to\infty} \prod_{k=1}^{m} (k+u)^{\binom{m}{k}(-1)^{k+1}} = \prod_{m=0}^{\infty} \left(\prod_{k=0}^{m} (k+u+1)^{\binom{m}{k}(-1)^{k}} \right).$$

Take $u = 1, 2, 3$ in the second formula of Corollary 3

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{8}{9} \cdot \frac{128}{135} ..., \quad 2 = \frac{3}{1} \cdot \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{125}{128} ..., \quad 3 = \frac{4}{1} \cdot \frac{4}{5} \cdot \frac{24}{25} \cdot \frac{864}{875} ...$$

Recall that $\gamma = \lim\limits_{m\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{m} - \ln m\right)$.

Theorem 4.

$$\gamma = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^k}{k} \ln(k!).$$

Using this formula we reprove the following formula

Corollary 5.

$$\gamma = \sum_{j=1}^{\infty} \sum_{i=1}^{j} \binom{j-1}{i-1} \frac{(-1)^i}{j} \ln i.$$

The proof of this formula in [Se26] used analytic continuation of Riemann zeta function. Ideas of proofs of this formula in [So03] are explained [So03, remark before Proof 1].

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2. Conjecture.

1. For each real positive numbers $z_1, z_2, \ldots, z_n$

$$z_1 z_2 \ldots z_n = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln(1 + z_1 \ln(1 + z_2 \ldots \ln(1 + z_n k) \ldots)).$$

For example for $n = 2$, we obtain

$$z_1 z_2 = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln(1 + z_1 \ln(1 + k z_2)).$$

2. For each real $z \geq 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n!)^2} = \lim_{m\to\infty} \sum_{n=1}^{m} \binom{m}{n} \frac{(-1)^{n+1}}{z(n+1)+1}.$$

3.

$$\ln\frac{\pi}{2} = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^k}{k \sum\limits_{n=1}^{m} \frac{2^{n-1}}{n}} \ln\frac{k!!}{(k-1)!!} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{2^n} \left(\frac{2}{k} - \frac{2}{j} - \frac{j}{k} + 2\right) \ln\frac{k!!}{(k-1)!!}.$$

Where $k!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (k-2) \cdot k$ for k odd and $k!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (k-2) \cdot k$, for k even. See also [So05].

4. For all different positive integers $a_1, a_2, \ldots, a_m$, we define

$$f(a_k) = \prod_{n=1, n\neq k}^{m} \frac{a_n}{a_n - a_k}.$$

(For $m = 1$ we have $f(a_1) = 1$.) Then

$$\gamma = \sum_{n=1}^{m} f(a_n) \frac{\ln(a_n!)}{a_n} + \sum_{n=1}^{\infty} \int_0^{1/n} \frac{dx}{\prod\limits_{k=1}^{m} 1 + (a_k x)^{-1}}.$$

5. Define $y_k := \sum\limits_{n=1}^{\infty} \int\limits_0^{1/n} \frac{dx}{1+(kx)^{-1}}$. Then the numbers y_i, where i runs through positive integers, are linearly independent over $\mathbb{Z}$.

6. It follows from 4 and 5 that e^γ is a irrational number.

3. Proofs.

All Lemmas are essentially known (Lemma 9 and Lemma 10 can be found in [Wi08]). But we present proofs for completeness.

In order to prove Theorem 1 and Theorem 4 we need

Lemma 7. *For each $z > 0$ and $m = 1, 2, 3...$*

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{k+z} = \frac{g_m(z)}{z} \quad \text{where} \quad g_m(z) := \frac{m!}{(z+1)(z+2)\dots(z+m)}.$$

Proof. Take the decomposition $\frac{g_m(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z+1} + \frac{A_2}{z+2} + ... + \frac{A_m}{z+m}$ into simplest fractions. We have $m! = \sum\limits_{k=0}^{m} A_k \prod\limits_{i=0, i\neq k}^{m} (z+i)$. Taking $z = -k$ we obtain

$$m! = A_k(-k)(-k+1)...2 \cdot 1 \cdot 2...(m-k-1)(m-k) = A_k k!(m-k)!(-1)^k.$$

Hence $A_k = \binom{m}{k}(-1)^k$. QED

Lemma 8. For each $z_0 > 0$ we have that $g_m(z)/z$ converges to 0 uniformly for $z \in [z_0; +\infty)$ as m tends to infinity.

Proof. We have

$$\frac{g_m(z)}{z} = \frac{1}{z\,(1+z)\left(1+\frac{z}{2}\right)\dots\left(1+\frac{z}{m}\right)} < \frac{1}{z \cdot z\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m}\right)} \underset{m\to\infty}{\to} 0. \quad \mathit{QED}$$

Proof of Theorem 1. It follows from Lemma 7 and Lemma 8 that

$$\lim_{m\to\infty} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{k+z} = 0.$$

Taking $z = \frac{1}{u}$ and changing the limit of summation we obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{ku+1} = 1. \quad \mathit{QED}$$

Proof of Theorem 4. Let us prove that

$$\sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln(k!) \overset{(1)}{=} \lim_{n\to\infty} \left(\sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \ln n - \sum_{j=1}^{n} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln\left(1+\frac{k}{j}\right) \right) \overset{(2)}{=}$$

$$\overset{(2)}{=} \lim_{n\to\infty} \left(\ln n - \sum_{j=1}^{n} \frac{1}{j} \right) + \sum_{j=1}^{\infty} \int\limits_{0}^{1/j} g_m(1/u)du \overset{(3)}{=} -\gamma + \sum_{j=1}^{\infty} \int\limits_{0}^{1/j} g_m(1/u)du \overset{(*)}{\to} -\gamma \quad \text{as} \quad m \to \infty.$$

The first equality follows because

$$k! = k! \lim_{n\to\infty} \frac{n^k}{(n+1)(n+2)\dots(n+k)} = \lim_{n\to\infty} \frac{n^k n! k!}{(n+k)!} =$$

$$= \lim_{n\to\infty} \frac{n^k n!}{(k+1)(k+2)\dots(k+n)} = \lim_{n\to\infty} \frac{n^k}{\left(1+\frac{k}{1}\right)\left(1+\frac{k}{2}\right)\dots\left(1+\frac{k}{n}\right)}.$$

Let us prove the second equality. Taking $z = \frac{1}{u}$ in Lemma 7 and changing the limit of summation we have

$$1 - \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{ku+1} = g_m(1/u).$$

Hence (improperly) integrating this formula with from 0 to $1/j$, we obtain

$$\sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln(1+\frac{k}{j}) = \frac{1}{j} - \int_0^{1/j} g_m(1/u)du.$$

This and $\sum\limits_{k=1}^{m} \binom{m}{k}(-1)^{k+1} = 1$ imply the second equality.

The third equality is clear.

Let us prove (*). We have

$$0 < g_m(1/u) < \frac{g_{m-1}(1/u)}{1+\frac{1}{m}} \quad \text{for} \quad 0 < u \le 1.$$

Hence

$$0 < g_m(1/u) \le \frac{g_1(1/u)}{(1+\frac{1}{2})(1+\frac{1}{3})...(1+\frac{1}{m})} = \frac{2g_1(1/u)}{m+1} \quad \text{for} \quad 0 < u \le 1.$$

For each m the series of left-hand side of the equality (*) converges by the sum of limits theorem. Hence for the sum S_m in the left-hand side of (*) we have

$$0 < S_m < \frac{2S_1}{m+1} \overset{m\to\infty}{\to} 0. \quad QED$$

In order to prove Remark 2, Corollary 3 and Corollary 5 we need

Lemma 9.

$$\frac{\binom{m}{k}}{k} = \sum_{n=k}^{m} \frac{\binom{n}{k}}{n}.$$

Proof. We have

$$\binom{m}{k} = \sum_{n=1}^{m-k+1} \binom{m-n}{k-1} = \sum_{n=k}^{m} \binom{n-1}{k-1} = k \sum_{n=k}^{m} \frac{\binom{n}{k}}{n}.$$

Here the first equality holds because $\binom{m-n}{k-1}$ equals to the number of k-subsets of $\{1, 2, \ldots, m\}$ whose minimal element is n, the second equality holds because the summands in those sums are equal.

Proof the second formula of Remark 2. By Lemma 9 for $X_{n,k} = \binom{n}{k}\frac{(-1)^{k+1}}{n} \ln(ku+1)$ we have

$$\lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln(ku+1) = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{n=k}^{m} X_{n,k} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} X_{n,k}. \quad QED$$

Proof the first formula of Corollary 3. Take logarithms of both sides. Take $z = \frac{1}{u}$ in the first formula of Remark 2 we obtain

$$\frac{1}{z} = \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln\left(\frac{k}{z}+1\right).$$

By Lemma 8 the limit in this formula is uniform for $z \in [z_0; +\infty], z_0 \geq 0$. Hence integrating this formula from 1 to u with respect to z, we get

$$\ln u \overset{(1)}{=} \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \left(k \ln(k+u) - k + u \ln\left(\frac{k}{u}+1\right)\right) \overset{(2)}{=}$$

$$\overset{(2)}{=} \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \ln(k+u) - \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} + u \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^{k+1}}{k} \ln\left(\frac{k}{u}+1\right) \overset{(3)}{=}$$

$$\overset{(3)}{=} \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \ln(k+u).$$

The first equality follows because

$$\int_1^u \frac{dz}{z} = \ln u, \quad \int_1^u \ln(\frac{k}{z}+1) dz = (k+u)\ln(k+u) - k + u \ln u = k \ln(k+u) - k + u \ln\left(\frac{k}{u}+1\right)$$

The second equality equality is clear. The third equality follows by the second formula of Remark 2 and

$$\sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} = 1. \quad QED.$$

Proof the second formula of Corollary 3. Take logarithms of both sides. By Lemma 9 for $a_k = \ln(k+u)$ we obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} a_k = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{n=k}^{m} \binom{n}{k} (-1)^{k+1} \frac{k}{n} a_k =$$

$$= \sum_{1\leq k \leq n < \infty} \binom{n-1}{k-1} (-1)^{k+1} a_k = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{k+1} a_k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} a_{k+1}. \quad QED$$

In order to prove Corollary 5, we need

Lemma 10.

$$\sum_{k=n}^{j} (-1)^{k+n} \binom{j}{k} = \binom{j-1}{n-1}.$$

Proof. The proof is by induction on n. For $n = 1$ this is known.
Let us prove the inductive step. By Pascal's rule we have:

$$\sum_{k=n+1}^{j} \binom{j}{k} (-1)^{k+n} = \sum_{k=n}^{j} \binom{j}{k} (-1)^{k+n} - \binom{j}{n} = \binom{j-1}{n-1} - \binom{j}{n} = \binom{j-1}{n}. \quad QED$$

Proof of Corollary 5. We have

$$\gamma \overset{(1)}{=} \lim_{m\to\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^k}{k} \ln(k!) \overset{(2)}{=} \lim_{m\to\infty} \sum_{k=1}^{m} (-1)^k \left(\sum_{j=k}^{m} \binom{j}{k} \frac{1}{j} \right) \left(\sum_{n=1}^{k} \ln n \right) \overset{(3)}{=}$$

$$\stackrel{(3)}{=} \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{j=k}^{m} \sum_{n=1}^{k} a_{kjn} \stackrel{(4)}{=} \lim_{m\to\infty} \sum_{1\le n\le k\le j\le m} a_{kjn} \stackrel{(5)}{=} \lim_{m\to\infty} \sum_{j=1}^{m} \sum_{n=1}^{j} \sum_{k=n}^{j} a_{kjn} \stackrel{(6)}{=}$$

$$\stackrel{(6)}{=} \sum_{j=1}^{\infty} \sum_{n=1}^{j} \frac{(-1)^n}{j} \binom{j-1}{n-1} \ln n.$$

Here $a_{kjn} = \binom{j}{k} \frac{(-1)^k}{j} \ln n$. The first equality follows by theorem 4. The second equality follows by lemma 9. The third, the fourth and the fifth equality is clear. The sixth equality follows by lemma 10. QED

4. Acknowledgements.

The author is grateful to J. Sondow and A.Skopenkov for useful comments on this paper and references. This paper was presented at Moscow Mathematical Conference of High-school Students in 2009.

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