

Schreier graphs of the Basilica group

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Abstract

With any self-similar action of a finitely generated group G of automorphisms of a regular rooted tree T can be naturally associated an infinite sequence of finite graphs $\{\Gamma_n\}_{n \geq 1}$, where Γ_n is the Schreier graph of the action of G on the n -th level of T . Moreover, the action of G on ∂T gives rise to orbital Schreier graphs Γ_ξ , $\xi \in \partial T$. Denoting by ξ_n the prefix of length n of the infinite ray ξ , the rooted graph (Γ_ξ, ξ) is then the limit of the sequence of finite rooted graphs $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$ in the sense of pointed Gromov-Hausdorff convergence. In this paper, we give a complete classification (up to isomorphism) of the limit graphs (Γ_ξ, ξ) associated with the Basilica group acting on the binary tree, in terms of the infinite binary sequence ξ .¹

Mathematics Subject Classification (2010) Primary 20E08; Secondary 20F69, 05C63, 37E25.

1 Introduction

Schreier graphs arise naturally from the action of a group on a set. In this paper, we consider groups acting by automorphisms on rooted trees. Let T be a regular rooted tree and $G < \text{Aut}(T)$ be a finitely generated group of automorphisms of T . By fixing a finite set S of generators of G , we get naturally a sequence $\{\Gamma_n\}_{n \geq 1}$ of finite left Schreier graphs of the action of G on T . The vertex set of Γ_n coincides with the set L_n of vertices of the n -th level of T , and two vertices $v, v' \in L_n$ are connected by an edge if there exists $s \in S$ such that $s \cdot v = v'$. If G is transitive on each level, then the graphs Γ_n are connected.

Similarly, the action of G on the boundary ∂T of the tree gives rise to an uncountable family of infinite orbital Schreier graphs $\{\Gamma_\xi\}_{\xi \in \partial T}$, with $V(\Gamma_\xi) = G \cdot \xi$. It turns out that these orbital Schreier graphs, viewed as rooted graphs (Γ_ξ, ξ) , are exactly the limits in the pointed Gromov-Hausdorff topology of finite Schreier graphs (Γ_n, ξ_n) rooted at ξ_n , the prefix of length n of ξ (see Subsection 2.2). Moreover, if we choose the root of Γ_n uniformly at random for all $n \geq 1$, then the uniform measure on the set $\{(\Gamma_\xi, \xi); \xi \in \partial T\}$ is the random weak limit (in the sense of [4]) of this sequence of random rooted graphs.

Particularly interesting examples of such sequences of Schreier graphs come from the class of self-similar, or automata, groups. In this paper, we examine in detail finite and infinite Schreier graphs of the Basilica group B acting on the binary tree in a self-similar fashion. The Basilica group is an example of a group generated by a finite automaton (see Fig. 2). It was introduced by R. Grigorchuk and A. Żuk in [9], where they show that it does not belong to the closure of the set of groups of subexponential growth under the operations of group extension and direct limit. L. Bartholdi and B. Virág further showed it to be amenable, making Basilica the first example of an amenable but not subexponentially amenable group [3]. This group has also been described by V. Nekrashevych as the iterated monodromy group of the complex polynomial $z^2 - 1$ (see [12]), and there exists therefore a natural way to associate to it a compact limit space homeomorphic to the well-known Basilica fractal (see Fig. 1). Finite Schreier graphs $\{\Gamma_n\}_{n \geq 1}$ form an approximating sequence of this limit space.

The aim of this work is to classify (up to isomorphism of unrooted and unlabeled graphs) all limits

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$\{\Gamma_\xi\}_{\xi \in \partial T}$ of the sequence $\{\Gamma_n\}_{n \geq 1}$ of finite Schreier graphs of the Basilica group. In other words, the question that we answer here is: given an infinite binary sequence ξ , describe the infinite Schreier graph Γ_ξ of the action of B on the orbit of ξ . The limit graph Γ_ξ is shown to have one, two or four ends, the case of one end being generic with respect to the uniform measure on ∂T (this latter fact can also be deduced from general arguments, see [6]). The main result of the paper is an explicit classification of infinite Schreier graphs of the Basilica group in terms of the boundary point ξ (Theorems 4.1, 4.6, 4.8, 4.11 and 5.4.) There exist one isomorphism class of 4-ended graphs, all belonging to the same orbit; uncountably many isomorphism classes of 2-ended graphs, each consisting of two orbits; and uncountably many isomorphism classes of 1-ended graphs, almost all of which contain uncountably many orbits. This latter aspect is particularly interesting, as it is not the case in other examples where we were able to perform a similar analysis. For example in the case of the Hanoi towers group $H^{(3)}$, whose finite Schreier graphs form an approximating sequence for the Sierpiński gasket, there are uncountably many isomorphism classes of graphs with one end, but each of them contains at most 6 different orbits. We intend to pursue this investigation in a future work.

It follows (see Section 5.2 below) that in the case of the Basilica group there exist uncountably many isomorphism classes of (unlabeled) limit graphs, each of measure 0. In other words, the random weak limit of the sequence of finite Schreier graphs is a continuous measure with uncountable support. A. Vershik recently raised the question about existence of continuous ergodic probability measures on the lattice $L(G)$ of subgroups of a given group G , invariant under the action of G on $L(G)$ by conjugation [15]. For many self-similar groups including the Basilica (more precisely, for all weakly branched groups), the pull-back of the random weak limit of labeled finite Schreier graphs to $L(G)$ gives such a measure, concentrated on stabilizers of points in the boundary of the tree.

It would be interesting to perform a similar analysis systematically on sequences of finite Schreier graphs associated with self-similar groups, in particular with iterated monodromy groups of quadratic polynomials, whose infinite Schreier graphs are closely related to the Julia set of the polynomial (see the recent work [1] for a detailed description of this relation.) In particular, the inflation recursive process of constructing finite Schreier graphs introduced in [12] (and used in [5] to study growth of infinite Schreier graphs) might also give a possible general approach to classification of infinite Schreier graphs of contracting groups of automorphisms of rooted trees, in the spirit of the present work. For example, though there are examples of groups (the first Grigorchuk group of intermediate growth, the adding machine...) with only one or two isomorphism classes of infinite (unlabeled) Schreier graphs, it seems that those are exceptional cases and that in general the support of the random weak limit of Γ_n 's is uncountable. We plan to address this question in a future work.

Sequences of Schreier graphs of self-similar groups have been mainly studied in the literature from the viewpoint of spectral computations (see e.g. [2],[8],[13]...) However, in contrast with other well-known examples of self-similar groups, the spectral measure of the Laplace operator on infinite Schreier graphs of the Basilica group is not known. Explicit description of these graphs that we obtain in this paper can be useful in this aspect.

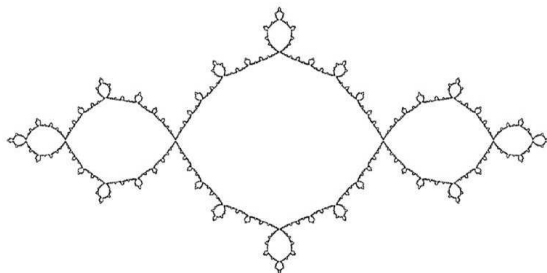


Fig. 1. the Julia set $J(z^2 - 1)$.

Our initial motivation comes from the work [11] that aims at constructing new examples of asymptotic behaviour for the Abelian Sandpile Model (ASM). The infinite Schreier graphs of the Basilica group provide uncountable families of examples, on one hand, of 2-ended graphs non quasi-isometric to \mathbb{Z} , on which the ASM is not critical and, on the other hand, of 1-ended graphs of quadratic growth, on which the ASM is critical.

2 Preliminaries

2.1 Groups acting on rooted trees

Let T be the regular rooted tree of degree q , i.e., the rooted tree in which each vertex has q children. Given a finite alphabet $X = \{0, 1, \dots, q-1\}$ of q elements, let us denote by X^n the set of words of length n in the alphabet X and put $X^* = \bigcup_{n \geq 0} X^n$, where the set X^0 consists of the empty word. Moreover, we denote by X^ω the set of infinite words in X . In this way, each vertex of the n -th level of the tree can be regarded as an element of X^n and the set X^ω can be identified with the set ∂T of infinite geodesic rays starting at the root of T . The set X^ω can be equipped with the direct product topology. The basis of open sets is the collection of all cylindrical sets $\{wX^\omega; w \in X^*\}$. The space X^ω is totally disconnected and homeomorphic to the Cantor set. The cylindrical sets generate a σ -algebra of Borel subsets of the space X^ω . We shall denote by λ the uniform measure on X^ω .

We denote by $Aut(T)$ the group of all automorphisms of T , i.e., the group of all bijections of the set of vertices of T preserving the incidence relation. Clearly, the root and hence the levels of the tree are preserved by any automorphism of T . A group $G \leq Aut(T)$ is said to be **spherically transitive** if it acts transitively on each level of the tree.

The **stabilizer** of a vertex $x \in T$ is the subgroup of G defined as $Stab_G(x) = \{g \in G : g(x) = x\}$; the **stabilizer of the n -th level** L_n of the tree is $Stab_G(L_n) = \bigcap_{x \in L_n} Stab_G(x)$; finally, the **stabilizer of a boundary point** $\xi \in X^\omega$ is $Stab_G(\xi) = \{g \in G : g(\xi) = \xi\}$. The following properties hold.

- For all $n \geq 1$, $Stab_G(L_n)$ is a normal subgroup of G of finite index.
- The subgroups $Stab_G(x)$, for $x \in L_n$, are all of index q^n . Moreover, if the action of G on T is spherically transitive, they are all conjugate.
- $\bigcap_{\xi \in \partial T} Stab_G(\xi)$ is trivial.
- Denote by ξ_n the prefix of ξ of length n . Then $Stab_G(\xi) = \bigcap_{n \in \mathbb{N}} Stab_G(\xi_n)$.
- $Stab_G(\xi)$ has infinite index in G and $Stab_G(\xi)/(Stab_G(\xi) \cap Stab_G(L_n)) \cong Stab_G(\xi_n)/Stab_G(L_n)$.

If $g \in Aut(T)$ and $v \in X^*$, define $g|_v \in Aut(T)$, called **the restriction of the action of g to the subtree rooted at v** , by $g(vw) = g(v)g|_v(w)$ for all $v, w \in X^*$. Every subtree of T rooted at a vertex is isomorphic to T . Therefore, every automorphism $g \in Aut(T)$ induces a permutation of the vertices of the first level of the tree and q restrictions, $g|_0, \dots, g|_{q-1}$, to the subtrees rooted at the vertices of the first level. It can be written as $g = \tau_g(g|_0, \dots, g|_{q-1})$, where $\tau_g \in S_q$ describes the action of g on L_1 . In fact, $Aut(T)$ is isomorphic to the wreath product $S_q \wr Aut(T)$ where S_q denotes the symmetric group on q letters, and thus $Aut(T) \cong \varprojlim_{i=1}^\infty S_q$.

Definition 2.1. [12] *A group G acting by automorphisms on a q -regular rooted tree T is **self-similar** if $g|_v \in G, \forall v \in X^*, \forall g \in G$.*

A self-similar group G can be embedded into the wreath product $S_q \wr G$. Consequently, an automorphism $g \in G$ can be represented as $g = \tau_g(g|_0, \dots, g|_{q-1})$, where $\tau_g \in S_q$ describes the action of g on L_1 , and $g|_i \in G$ is the restriction of the action of g on the subtree T_i rooted at the i -th vertex of the first level. So, if $x \in X$ and w is a finite word in X , we have $g(xw) = \tau_g(x)g|_x(w)$.

Definition 2.2. [12] A self-similar group G is **self-replicating** (or fractal) if it acts transitively on the first level of the tree and, for all $x \in X$, the map $g \mapsto g|_x$ from $\text{Stab}_G(x)$ to G is surjective.

2.2 Schreier graphs

Consider a finitely generated group G with a set S of generators such that $id \notin S$ and $S = S^{-1}$, and suppose that G acts on a set M . Then, one can consider a graph $\Gamma(G, S, M)$ with the set of vertices M , and two vertices m, m' joined by an edge (labeled by s) if there exists $s \in S$ such that $s(m) = m'$. Clearly, if the action of G on M is transitive, then $\Gamma(G, S, M)$ is the **Schreier graph** $\Gamma(G, S, \text{Stab}_G(m))$ of the group G with respect to the subgroup $\text{Stab}_G(m)$ for some (any) $m \in M$. If the action of G on M is not transitive, and $m \in M$, then we denote by $\Gamma(G, S, m)$ the Schreier graph of the action on the G -orbit of m , and we call such a graph an **orbital Schreier graph**.

Suppose now that G acts spherically transitively on a rooted tree T . Then, the n -th **Schreier graph** of G is by definition $\Gamma(G, S, X^n) = \Gamma(G, S, P_n)$ where P_n denotes the subgroup stabilizing some word $w \in X^n$. For each $n \geq 1$, let $\pi_{n+1} : \Gamma(G, S, X^{n+1}) \rightarrow \Gamma(G, S, X^n)$ be the map defined on the vertex set of $\Gamma(G, S, X^{n+1})$ by $\pi_{n+1}(x_1 \dots x_n x_{n+1}) = x_1 \dots x_n$. Since $P_{n+1} \leq P_n$, π_{n+1} induces a surjective morphism between $\Gamma(G, S, X^{n+1})$ and $\Gamma(G, S, X^n)$. This morphism is a graph covering of degree q .

We also consider the action of G on $\partial T \equiv X^\omega$ and the orbital Schreier graph $\Gamma(G, S, G \cdot \xi) = \Gamma(G, S, P_\xi)$ where P_ξ denotes the stabilizer of ξ for the action of G on X^ω . Recall that, given a ray ξ , we denote by ξ_n the prefix of ξ of length n , and that $P_\xi = \bigcap_n P_n$. It follows that the infinite Schreier graph $\Gamma_\xi = \Gamma(G, S, P_\xi)$ can be approximated (as a rooted graph) by finite Schreier graphs $\Gamma_n = \Gamma(G, S, P_n)$, as $n \rightarrow \infty$, in the compact space of rooted graphs of uniformly bounded degree endowed with **pointed Gromov-Hausdorff convergence** ([10], Chapter 3), provided, for example, by the following metric: given two rooted graphs (Γ_1, v_1) and (Γ_2, v_2) ,

$$\text{Dist}((\Gamma_1, v_1), (\Gamma_2, v_2)) := \inf \left\{ \frac{1}{r+1}; B_{\Gamma_1}(v_1, r) \text{ is isomorphic to } B_{\Gamma_2}(v_2, r) \right\}$$

where $B_\Gamma(v, r)$ is the ball of radius r in Γ centered in v .

2.3 Self-similar groups and automata

An **automaton** is a quadruple $\mathcal{A} = (\mathcal{S}, X, \mu, \nu)$, where \mathcal{S} is the set of states; X is an alphabet; $\mu : \mathcal{S} \times X \rightarrow \mathcal{S}$ is the transition map; and $\nu : \mathcal{S} \times X \rightarrow X$ is the output map. The automaton \mathcal{A} is **finite** if \mathcal{S} is finite and it is **invertible** if, for all $s \in \mathcal{S}$, the transformation $\nu(s, \cdot) : X \rightarrow X$ is a permutation of X . An automaton \mathcal{A} can be represented by its **Moore diagram**. This is a directed labeled graph whose vertices are identified with the states of \mathcal{A} . For every state $s \in \mathcal{S}$ and every letter $x \in X$, the diagram has an arrow from s to $\mu(s, x)$ labeled by $x|\nu(s, x)$. A natural action on the words over X is induced, so that the maps μ and ν can be extended to $\mathcal{S} \times X^*$:

$$\mu(s, xw) = \mu(\mu(s, x), w),$$

$$\nu(s, xw) = \nu(s, x)\nu(\mu(s, x), w), \tag{1}$$

where we set $\mu(s, \emptyset) = s$ and $\nu(s, \emptyset) = \emptyset$, for all $s \in \mathcal{S}, x \in X$ and $w \in X^*$. Moreover, (1) defines uniquely a map $\nu : \mathcal{S} \times X^\omega \rightarrow X^\omega$.

If we fix an initial state s in an automaton \mathcal{A} , then the transformation $\nu(s, \cdot)$ on the set $X^* \cup X^\omega$ is defined by (1); it is denoted by \mathcal{A}_s . The image of a word $x_1 x_2 \dots$ under \mathcal{A}_s can be easily found using the Moore diagram (see, for instance, Fig. 2 below). Consider the directed path starting at the state s with consecutive labels $x_1|y_1, x_2|y_2, \dots$; the image of the word $x_1 x_2 \dots$ under the transformation \mathcal{A}_s is

then $y_1y_2\dots$. More generally, given an invertible automaton $\mathcal{A} = (\mathcal{S}, X, \mu, \nu)$, one can consider the group generated by the transformations \mathcal{A}_s , for $s \in \mathcal{S}$; this group is called the **automaton group** generated by \mathcal{A} and is denoted by $G(\mathcal{A})$.

A basic theorem [12] states that the action of a group G on $X^* \cup X^\omega$ is self-similar if and only if G is generated by an invertible automaton.

Let \mathcal{A} be a finite automaton with the set of states \mathcal{S} and alphabet X and let us denote $\alpha(k, s)$, for $k \in \mathbb{N}$ and $s \in \mathcal{S}$, the number of words $w \in X^k$ such that $s|_w \neq id$. Sidki suggested to call \mathcal{A} **bounded**, if the sequence $\alpha(k, s)$ is bounded as a function of k for each state $s \in \mathcal{S}$. He showed in [14] that a finite invertible automaton is bounded if and only if any two non-trivial cycles in the Moore diagram of the automaton are disjoint and not connected by a directed path.

It can be shown moreover, that any group generated by a bounded automaton is **contracting** [7], which means, for a self-similar group, the existence of a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $k \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$, for all words v of length greater or equal to k .

2.4 The Basilica group

The Basilica group B was introduced by R. Grigorchuk and A. Żuk [9] as the group generated by the following three-state automaton.

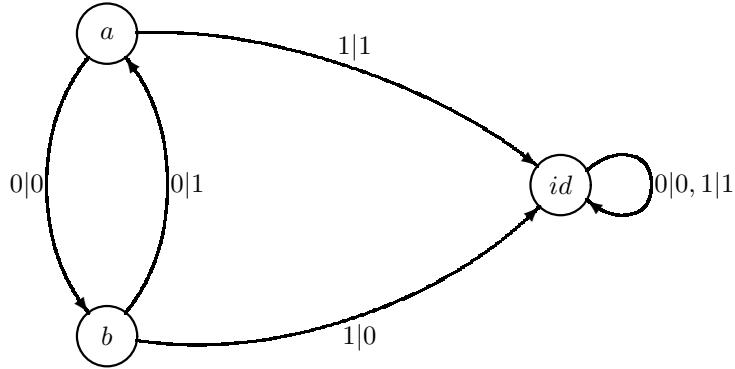


Fig. 2. The automaton generating the Basilica group.

It can be read from the automaton that the Basilica group is an automorphism group of the rooted binary tree generated by two automorphisms a and b of the following self-similar structure:

$$a = e(b, id) \quad b = \varepsilon(a, id),$$

where id denotes the trivial automorphism of the tree, while e and ε are respectively the identity and the nontrivial permutation in S_2 . One can verify directly that the stabilizer of the first level is the subgroup $Stab_B(L_1) = \langle a, a^b, b^2 \rangle$, with $a^b = b^{-1}ab = e(id, b^a)$ and $b^2 = e(a, a)$. This implies in particular, that B is self-replicating. Moreover, since the action of B on the first level of the tree is transitive, its action is also spherically transitive. It follows from the results cited in the previous subsection that the Basilica group is a contracting self-similar group generated by a bounded automaton.

We end this section by an observation about the action of the Basilica group on the boundary of the binary tree that we will need in Section 4 below.

Definition 2.3. Two right-infinite sequences $x_1x_2\dots, y_1y_2\dots \in X^\omega$ are called **cofinal** (denoted $x_1x_2\dots \sim y_1y_2\dots$) if they differ only in finitely many letters. *cofinality* is an equivalence relation. The respective equivalence classes are called the *cofinality classes* and they are denoted by $Cof(\cdot)$.

Proposition 2.4. *The union of the cofinality classes $Cof(0^\omega) \cup Cof((01)^\omega) \cup Cof((10)^\omega)$ constitutes one orbit of the action of the Basilica group on X^ω . Any other cofinality class is exactly one orbit.*

Proof. One directly verifies that $a(0^\omega) = (01)^\omega$, $a^{-1}(0^\omega) = 010^\omega$, $b(0^\omega) = (10)^\omega$ and $b^{-1}(0^\omega) = 10^\omega$. Moreover, it follows from the definition of the generators a and b , that 0^ω , $(01)^\omega$ and $(10)^\omega$ are the only infinite words which are mapped onto infinite words not cofinal to them by some of the generators (a, b for 0^ω , a^{-1} for $(01)^\omega$ and b^{-1} for $(10)^\omega$). Thus, the orbit $B \cdot 0^\omega$ is contained in the union $Cof(0^\omega) \cup Cof((01)^\omega) \cup Cof((10)^\omega)$. On the other hand, we show that any word $\eta \in Cof(0^\omega) \cup Cof((01)^\omega) \cup Cof((10)^\omega)$ belongs to the orbit of 0^ω by providing an automorphism in B mapping 0^ω to η . We only discuss the case of words of the type $\eta = w0^\omega$ (the other cases are analogous). Suppose that $|w| = k$. By transitivity, there exists $g \in B$ such that $g(0^k) = w$; set $g' := g|_{0^k}$. Since B is self-replicating, there exists $h \in Stab_B(w)$ such that $h|_w = (g')^{-1}$. This gives $hg(0^\omega) = w0^\omega$.

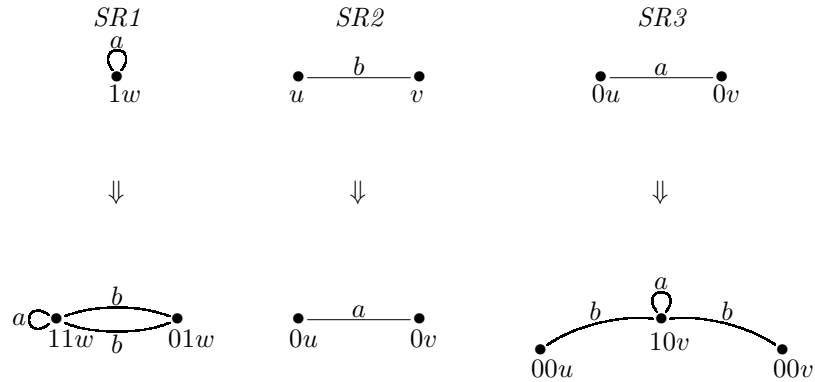
Since 0^ω , $(01)^\omega$ and $(10)^\omega$ are the only infinite words which are mapped onto infinite words not cofinal to them by some of the generators, the orbit of any $\xi \in X^\omega \setminus (Cof(0^\omega) \cup Cof((01)^\omega) \cup Cof((10)^\omega))$ is contained in one cofinality class. The other inclusion is proven by the same argument as used to show that any word $\eta \in Cof(0^\omega) \cup Cof((01)^\omega) \cup Cof((10)^\omega)$ belongs to the orbit of 0^ω . \square

3 Schreier graphs of the Basilica group

3.1 The structure of finite Schreier graphs

For each $n \geq 1$, let us denote by $\Gamma_n \equiv \Gamma(B, \{a, b\}, \{0, 1\}^n)$ the n -th Schreier graph of the action of the Basilica group. Recall that the edges of Γ_n are labeled by the generators a, b of the group B and that its vertices are encoded by words of length n in the alphabet $\{0, 1\}$. We begin this subsection by providing some convenient substitutional rules which allow to construct Γ_n 's recursively.

Proposition 3.1. *The Schreier graph Γ_{n+1} is obtained from Γ_n by applying to all subgraphs of Γ_n given by single edges the following substitutional rules **SR**:*



with

$$\Gamma_1 = \begin{array}{c} \text{a} \\ \circlearrowleft \\ \bullet \\ 0 \end{array} \begin{array}{c} \text{b} \\ \curvearrowright \\ \bullet \\ 1 \end{array} \begin{array}{c} \text{a} \\ \circlearrowright \\ \bullet \\ 1 \end{array}$$

Proof. By definition of a , we can distinguish two types of a -edges: a first type joining vertices $0w$ and $0b(w)$, and a second type which are loops at vertices $1w$, with $w \in \{0, 1\}^n$. Similarly, we distinguish the following types of b -edges:

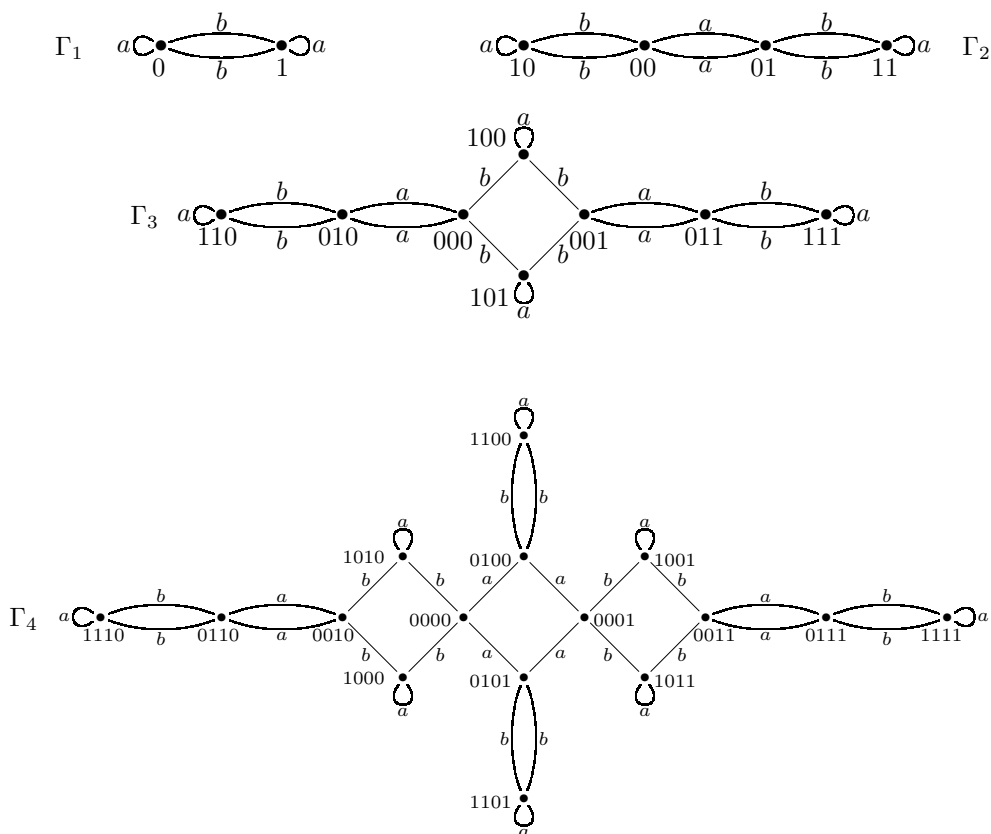
- $b(00w) = 10b(w)$ (note that $w \neq b(w)$);
- $b(01w) = 11w$;
- $b(10w) = 00w$;
- $b(11w) = 01w$.

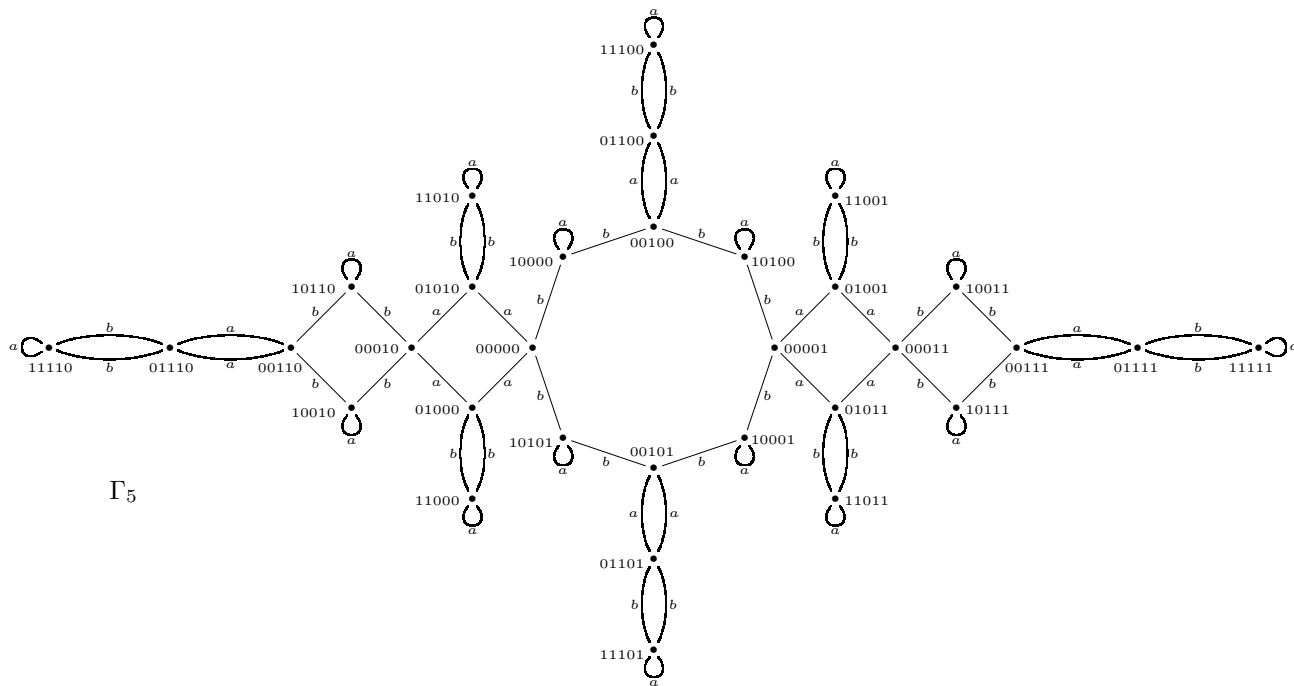
All b -edges can be partitioned in pairs. Pairs of b -edges of second and fourth type connecting vertices $11w$ and $01w$ in Γ_{n+1} arise from a -loops at the vertex $1w$ in Γ_n ; moreover, there must be an a -loop based at $11w$ by definition of a , hence SR1.

The b -edges of first and third type are paired in chains of length 2. Such a chain connecting $00w$ to $10b(w)$ to $00b(w)$ in Γ_{n+1} arises from the a -edge joining vertices $0w$ and $0b(w)$; moreover, there must be an a -loop based at $10b(w)$, hence SR3.

Finally, a -edges between vertices $0u$ and $0v$ in Γ_{n+1} are in bijection with b -edges joining u and v in Γ_n , hence SR2. □

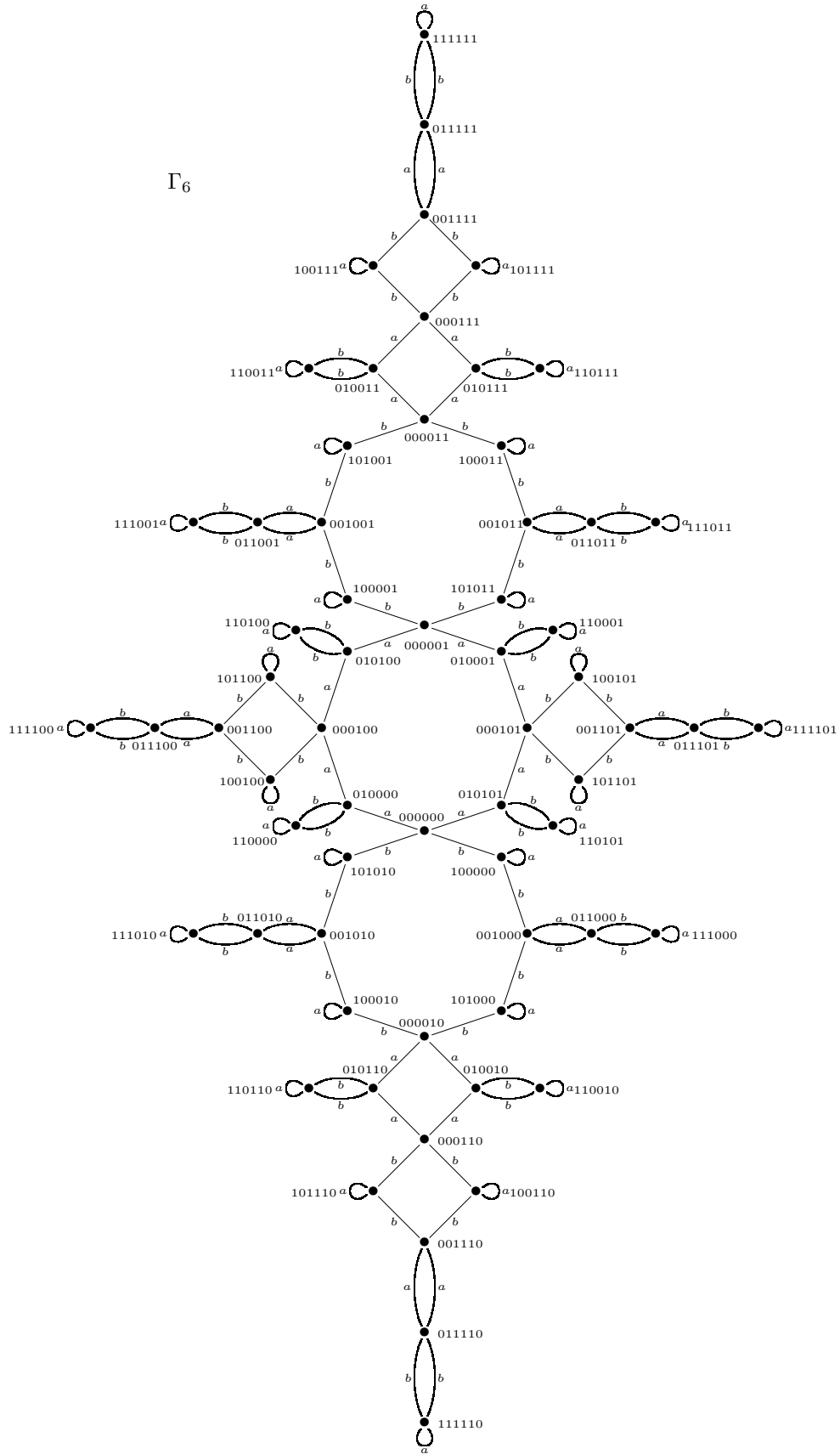
Below follow the pictures of the Schreier graphs Γ_n for some first values of $n \geq 1$.





Γ_5

Γ_6



Given $k \in \mathbb{N}$, recall that a graph G is k -connected if for every proper subset $Y \subset V(G)$ with $|Y| < k$, $G \setminus Y$ is connected. A connected graph G is **separable** if it can be disconnected by removing only one vertex. Such a vertex is called a **cut** vertex. The biggest 2-connected components of a separable graph are called **blocks**.

The following is an easy consequence of the substitutional rules (Proposition 3.1):

Proposition 3.2. *For every $n \geq 1$, Γ_n is a 4-regular separable graph whose blocks are cycles. Every vertex without a loop in Γ_n is a cut vertex. Moreover, removing any cut vertex disconnects Γ_n into exactly two components. Finally, the maximal length of a cycle in Γ_n is $2^{\lceil \frac{n}{2} \rceil}$.*

For the remainder of the paper, it will be convenient to consider Schreier graphs Γ_n , $n \geq 1$, embedded in the plane in such a way that each cycle is a regular polygon and the graph Γ_n has two symmetry axes, a horizontal one and a vertical one. The center of the central cycle of Γ_n coincides with the origin of the plane, which is the intersection of the axes of symmetry. By convention, the positive rotation by an angle α around the origin is performed in the counterclockwise direction.

From now on we will forget about the labels on the edges of the Schreier graphs and only consider unlabeled graphs. On one hand, our aim is to classify limits of Schreier graphs up to isomorphism of unlabeled graphs. On the other hand, in the case of the Basilica group the labeling is uniquely determined by the graph, so we are not losing any information by forgetting the labels.

Recall from Subsection 2.2 that the finite Schreier graphs $\{\Gamma_n\}_{n \geq 1}$ form a sequence of graph coverings and that we denote by $\pi_{n+1} : \Gamma_{n+1} \rightarrow \Gamma_n$ the covering projection given by $\pi_{n+1}(x_1 \dots x_n x_{n+1}) = x_1 \dots x_n$. In the remaining part of this subsection we shall describe in detail the structure of the graphs Γ_n and of the projections π_n .

Definition 3.3. *Let $v \in V(\Gamma_n) \setminus \{0^n\}$ be a cut vertex. Removing v splits Γ_n into two connected components, U_1 and U_2 , one of them (say U_1) containing the vertex 0^n . We call the **decoration** $\mathcal{D}(v)$ of v the subgraph of Γ_n induced by the vertex set $V(U_2) \cup \{v\}$. If $v \in V(\Gamma_n)$ has a loop, then $\mathcal{D}(v)$ is the subgraph induced by v . Finally, if $v = 0^n$, then $\mathcal{D}(0^n)$ is the subgraph induced by $V(U_i) \cup \{0^n\}$, where $0^{n-1}1 \notin U_i$.*

For any cut vertex v of Γ_n , we also consider the subgraph $\mathcal{D}(v)^c$ induced by $V(U_1) \cup \{v\}$. If v has a loop attached to it, then $\mathcal{D}(v)^c$ is just Γ_n with the corresponding loop erased.

*The decoration of a given vertex $v \in V(\Gamma_n)$ is called a **k-decoration** (or a **decoration of height k**) if it is isomorphic to the decoration of the vertex 0^k in the Schreier graph Γ_k for some $1 \leq k \leq n$.*

Proposition 3.4. 1. *Every decoration in Γ_n is a k -decoration for some $1 \leq k \leq n$.*

2. *Let $v \in V(\Gamma_n) \setminus \{0^n\}$. Then, the decoration $\mathcal{D}(v)$ of v is a k -decoration if and only if, while reading v from the left, we first encounter 1 in the k -th position.*

3. *Let $v \in V(\Gamma_{n+1}) \setminus \{0^{n+1}, 0^n1\}$ and let $\mathcal{D}(v)$ be its decoration. Then $\mathcal{D}(v)$ is mapped under π_{n+1} bijectively to the decoration $\mathcal{D}(\pi_{n+1}(v)) \in \Gamma_n$ of $\pi_{n+1}(v)$.*

Proof. Part 1. Observe that, given $v \in V(\Gamma_n)$ and its decoration $\mathcal{D}(v) \subset \Gamma_n$, applying SR to $\mathcal{D}(v)$ yields a graph isomorphic to $\mathcal{D}(0v) \subset \Gamma_{n+1}$. Indeed, if $v \neq 0^n$, then the subgraph of Γ_{n+1} resulting from applying the SR to $\mathcal{D}(v)$ does not contain 0^{n+1} . Since the vertex $0v$ is a cut vertex, the statement follows by definition of a decoration. On the other hand, if $v = 0^n$, we can repeat the previous argument replacing 0^n by $0^{n-1}1$. We now prove the assertion by induction on n . The decorations of the vertices of Γ_1 are both 1-decorations. Let $v \in V(\Gamma_{n+1})$. If $v = 1u$ for some u of length n , then it follows from the SR that v has a loop, hence the decoration of v is a 1-decoration. If $v = 0u$, then consider vertex u in Γ_n together with its decoration $\mathcal{D}(u)$. By induction hypothesis, $\mathcal{D}(u)$ is a k -decoration for some $1 \leq k \leq n$ that is, $\mathcal{D}(u)$ is isomorphic to $\mathcal{D}(0^k)$. Hence, applying the SR to $\mathcal{D}(u)$ yields a subgraph of Γ_{n+1} isomorphic to the subgraph of Γ_{k+1} obtained by applying the SR to $\mathcal{D}(0^k)$. On the other hand, applying the SR to $\mathcal{D}(u)$ yields $\mathcal{D}(0u) = \mathcal{D}(v)$. Hence, $\mathcal{D}(v)$ is isomorphic to $\mathcal{D}(0^{k+1})$.

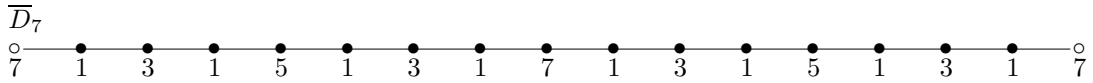
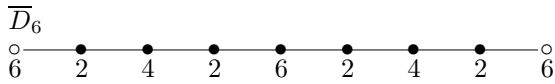
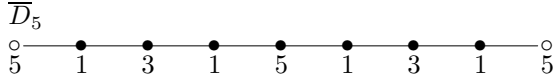
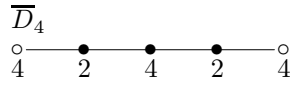
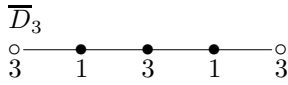
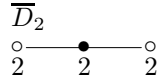
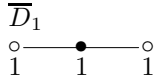
Part 2. Let us write $v = 0^l1u$ for some $l \geq 0$. Consider the decoration $\mathcal{D}(1u) \subset \Gamma_{n-l}$ of $1u$. By the SR, $\mathcal{D}(1u)$ is isomorphic to $\mathcal{D}(0) \subset \Gamma_1$. Starting from $\mathcal{D}(0)$ and applying the SR $k-1$ times yields $\mathcal{D}(0^k)$.

Similarly, starting from $\mathcal{D}(1u)$ and applying the SR l times yields $\mathcal{D}(v)$. Suppose now that $\mathcal{D}(v)$ is a k -decoration. This means that $\mathcal{D}(v)$ is isomorphic to $\mathcal{D}(0^k)$, thus by the observations we just made, l must be equal to $k - 1$. Conversely, if $l = k - 1$, then necessarily $\mathcal{D}(v)$ is isomorphic to $\mathcal{D}(0^k)$.

Part 3. Given $v \in V(\Gamma_{n+1}) \setminus \{0^{n+1}, 0^n 1\}$, $\pi_{n+1}(v) \in V(\Gamma_n)$ is obtained by erasing the last letter of v . On one hand, if $w \in V(\Gamma_{n+1})$ is a cut vertex, then $\pi_{n+1}(w) \in V(\Gamma_n)$ is a cut vertex too. On the other hand, for all $w \in \mathcal{D}(v)$, $\pi_{n+1}(w) \neq 0^n$. Thus, $\mathcal{D}(v)$ must be mapped (surjectively) on $\mathcal{D}(\pi_{n+1}(v))$. But we have proven in *Part 2* that $\mathcal{D}(v)$ is a k -decoration if and only if $\mathcal{D}(\pi_{n+1}(v))$ is a k -decoration. The statement follows. \square

Definition 3.5. For every $n \geq 1$, we call **central cycle** of Γ_n the unique cycle containing both vertices 0^n and $0^{n-1}1$. The decoration $\mathcal{D}(0^n)$ is the **left part** of Γ_n , the decoration $\mathcal{D}(0^{n-1}1)$ is the **right part** of Γ_n , the subgraph $\Gamma_n \setminus \{\mathcal{D}(0^n) \cup \mathcal{D}(0^{n-1}1)\}$ is the **central part** of Γ_n .

It is convenient to encode the graph Γ_n by a diagram, denoted \overline{D}_n , constructed as follows: consider a path containing $2^{\lceil \frac{n}{2} \rceil}$ edges. Its vertices are identified with vertices of the central cycle of Γ_n so that the left-half of \overline{D}_n encodes the upper-half of the central cycle of Γ_n , whereas the right-half of \overline{D}_n encodes the lower-half of the central cycle; and the vertices situated at the extremities of the path both encode the vertex 0^n . Denote by D_n the diagram without these two *boundary* vertices. Label every vertex by the height of the decoration attached to it. Here are some examples:



Proposition 3.6. The diagram \overline{D}_{n+1} encoding Γ_{n+1} is obtained from D_{n-1} by the following recursive rule (denoted by **RCR**):

$$\overline{D}_{n+1} : \quad \begin{array}{c} \circ \text{---} \text{---} \bullet \text{---} \text{---} \circ \\ n+1 \quad \quad \quad n+1 \quad \quad \quad n+1 \end{array}$$

$D_{n-1} \qquad \qquad \qquad D_{n-1}$

Proof. We know by Proposition 3.4 that when we project Γ_{n+1} on Γ_n under π_{n+1} , the decoration of a vertex $v \in V(\Gamma_{n+1}) \setminus \{0^{n+1}, 0^n 1\}$ is mapped onto a decoration of the same height in Γ_n . There are only two $(n+1)$ -decorations in Γ_{n+1} , namely those of the vertices 0^{n+1} and $0^n 1$, and they are both mapped under π_{n+1} to $\mathcal{D}(0^n)^c$ in such a way that $\mathcal{D}(0^{n+1})$ superposes onto $\mathcal{D}(0^n 1)$ after a rotation of -180° around the origin and then a translation (see Fig. 3).

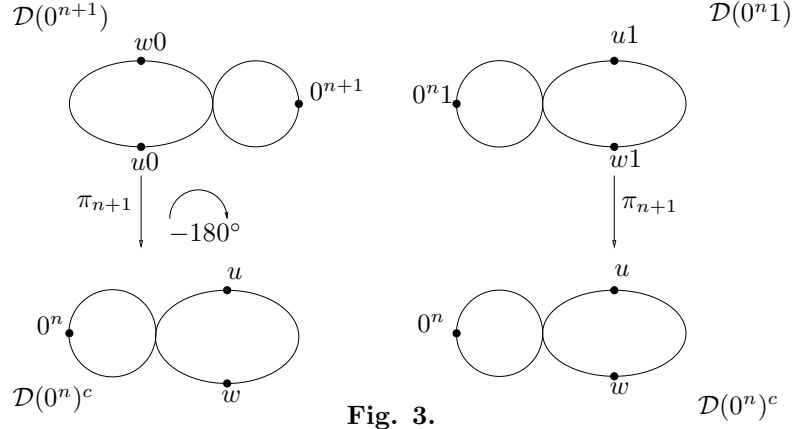


Fig. 3.

Thus, the cycle in each of these two $(n + 1)$ -decorations which contains respectively 0^{n+1} and 0^{n+1} is mapped bijectively to the central cycle of Γ_n . Moreover, pairs of opposite vertices of the central cycle of Γ_{n+1} are identified under the projection π_{n+1} . Hence, the central cycle of Γ_{n+1} is mapped to the twice shorter cycle in Γ_n containing the vertex 0^n but not the vertex 0^{n-1} in such a way that 0^{n+1} is identified with 0^{n+1} and the two halves of the central cycle are superposed (see Fig. 4).

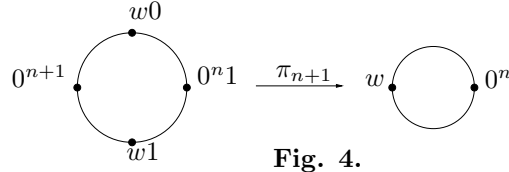


Fig. 4.

Since the decoration $\mathcal{D}(0^n)$ in Γ_n is bijectively mapped by π_n to $\mathcal{D}(0^{n-1})^c$ in Γ_{n-1} , the statement follows. \square

3.2 Converging sequences of Schreier graphs

Recall that for any $n \geq 1$, Γ_n is a separable graph whose blocks are cycles.

Definition 3.7. A *cycle-path* of length k , $k \in \mathbb{N} \cup \{\infty\}$, in a graph Γ is a sequence of k distinct cycles in Γ such that two consecutive cycles intersect at exactly one point.

Given $w, w' \in V(\Gamma_n)$, there is a unique cycle-path P_1, \dots, P_r in Γ_n of minimal length, such that $w \in P_1$ and $w' \in P_r$. We will say that P_1, \dots, P_r joins w to w' . Similarly, given $w \in V(\Gamma_n)$ and P a cycle of Γ_n , there is a unique cycle-path of minimal length $P_1, \dots, P_r = P$ such that $w \in P_1$. We will say then that P_1, \dots, P_r joins w to P .

Notation 3.8. Given an infinite word ξ and its prefix ξ_n of length n , write $\mathcal{CP}_{\xi_n} = \{P_i^n\}_{i=1}^t$ for the unique cycle-path joining ξ_n to the central cycle of Γ_n . Then we have $t = |\mathcal{CP}_{\xi_n}|$, the length of the cycle-path, and we denote by $\mathcal{LP}_{\xi_n} = \{k_i^n\}_{i=1}^t$ the sequence of lengths of the cycles in the cycle-path.

In order to identify the limit $\lim_{n \rightarrow \infty} (\Gamma_n, \xi_n)$ in the space of rooted graphs, it is important to keep track of the behaviour of the cycle-path \mathcal{CP}_{ξ_i} as i changes from n to $n + 1$. As we will see, there are three possibilities for the length of \mathcal{CP}_{ξ_i} : it can either increase/decrease by one, or stay stable. More precisely, we have

Proposition 3.9. Let $\xi \in \{0, 1\}^\omega$. For any $n \geq 1$, consider the cycle-path \mathcal{CP}_{ξ_n} (respectively $\mathcal{CP}_{\xi_{n+1}}$) in Γ_n (respectively in Γ_{n+1}). Then, the following situations may occur:

1. "Expansion", that is, $|\mathcal{CP}_{\xi_{n+1}}| = |\mathcal{CP}_{\xi_n}| + 1$; this occurs if and only if $\xi_n \in \mathcal{D}(0^n)^c$.
2. "Contraction", that is, $|\mathcal{CP}_{\xi_{n+1}}| = |\mathcal{CP}_{\xi_n}| - 1$; this occurs if and only if $\xi_n \in \mathcal{D}(0^n)$.
3. "Stability", that is, $|\mathcal{CP}_{\xi_{n+1}}| = |\mathcal{CP}_{\xi_n}|$; this occurs if and only if ξ_{n+1} is either 0^{n+1} or $0^n 1$.

Proposition 3.9 motivates the following definition:

Definition 3.10. For every word $\xi \in \{0, 1\}^\omega$, we define its index of stability as

$$z_\xi = \sup\{n \geq 1 : |\mathcal{CP}_{\xi_n}| = 1\}. \quad (2)$$

Moreover, define the **SEC-sequence** of ξ to be the infinite word in $\{S, E, C\}$ beginning with S^{z_ξ} and such that the $(n+1)$ -st letter (for $n \geq z_\xi$) of this word is either E or C depending on whether in the passage from Γ_n to Γ_{n+1} we observe an expansion or a contraction between the cycle-paths \mathcal{CP}_{ξ_n} and $\mathcal{CP}_{\xi_{n+1}}$.

Remark 3.11. We can consider the SEC-sequence of any finite prefix ξ_n of ξ by taking the restriction up to the n -th term of the SEC-sequence of ξ . The occurrence of a block of t expansions beginning at position n in the SEC-sequence of a word ξ encodes the fact that $|\mathcal{CP}_{\xi_{n+t-1}}| - |\mathcal{CP}_{\xi_{n-1}}| = t$ whereas the occurrence of a block of $m/2$ expansions-contractions EC beginning at position n means that $|\mathcal{CP}_{\xi_{n+m-1}}| - |\mathcal{CP}_{\xi_{n-1}}| = 0$. Moreover, observe (see the proof of Proposition 3.9 below) that the SEC-sequence associated with an infinite word ξ cannot contain two consecutive C 's.

We will also need to control the lengths of individual cycles in the cycle-path \mathcal{CP}_{ξ_i} as i changes from n to $n+1$. Namely, we have

Lemma 3.12. Let $\xi \in \{0, 1\}^\omega$. For any $n \geq 1$, consider the cycle-path \mathcal{CP}_{ξ_n} (respectively $\mathcal{CP}_{\xi_{n+1}}$) in Γ_n (respectively in Γ_{n+1}). Then, the sequence $\mathcal{LP}_{\xi_{n+1}}$ is obtained from \mathcal{LP}_{ξ_n} as:

1. if $\xi_n \in \mathcal{D}(0^n)^c$, then $k_i^{n+1} = k_i^n$ for every $i = 1, \dots, t$ and

$$\begin{cases} k_{t+1}^{n+1} = k_t^n & \text{if } n \text{ is odd,} \\ k_{t+1}^{n+1} = 2k_t^n & \text{if } n \text{ is even;} \end{cases}$$

2. if $\xi_n \in \mathcal{D}(0^n)$, then $k_i^{n+1} = k_i^n$ for every $i = 1, \dots, t-2$ and

$$\begin{cases} k_{t-1}^{n+1} = k_t^n & \text{if } n \text{ is odd,} \\ k_{t-1}^{n+1} = 2k_t^n & \text{if } n \text{ is even.} \end{cases}$$

Proof of Proposition 3.9 and Lemma 3.12. Let us look at how the graph Γ_{n+1} is obtained from Γ_n under π_{n+1}^{-1} . Recall that all Γ_n 's are embedded in the plane. Consider the decoration $\mathcal{D}(0^n)$ in Γ_n . Take two copies of $\mathcal{D}(0^n)$: the first one is rotated by -90° around the origin, whereas the second one is rotated by $+90^\circ$ around the origin. Now split the only vertex of degree 2 (which is 0^n) into two vertices v_1, v_1' for the first copy and v_2, v_2' for the second copy. Then glue v_1 with v_2 into a vertex v and v_1' with v_2' into a vertex v' (see Fig. 5).

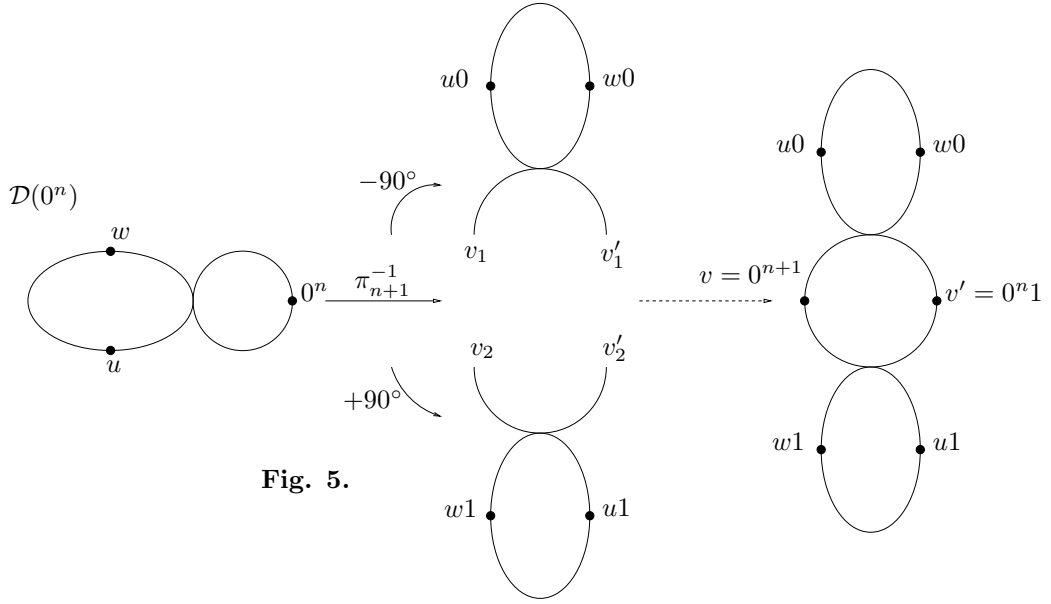


Fig. 5.

Finally, after a rotation by $+180^\circ$ around the origin, attach to v a copy of $\mathcal{D}(0^n)^c$ by gluing v with the only vertex of $\mathcal{D}(0^n)^c$ of degree 2 and attach to v' a copy of $\mathcal{D}(0^n)^c$ (see Fig. 6).

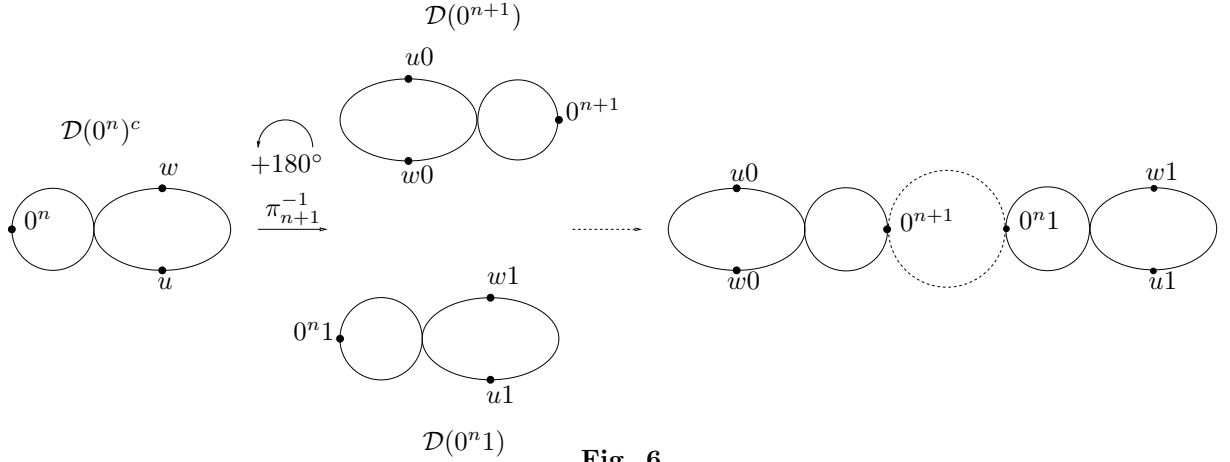


Fig. 6.

Accordingly, vertices of Γ_{n+1} are labeled as follows:

- add a 0 at the right of each vertex of the first copy of $\mathcal{D}(0^n)$ except for vertices v_1 and v'_1 ,
- add a 1 at the right of each vertex of the second copy of $\mathcal{D}(0^n)$ except for vertices v_2 and v'_2 ,
- put $v_1 \equiv v_2 \equiv v = 0^{n+1}$ and $v'_1 \equiv v'_2 \equiv v' = 0^n 1$,
- add a 0 at the right of each vertex of the left copy of $\mathcal{D}(0^n)^c$,
- add a 1 at the right of each vertex of the right copy of $\mathcal{D}(0^n)^c$.

Thus, if $\xi_n \in \mathcal{D}(0^n)^c$, then $|\mathcal{CP}_{\xi_{n+1}}| = |\mathcal{CP}_{\xi_n}| + 1$ (see Fig. 6). On the other hand, if $\xi_n \in \mathcal{D}(0^n) \setminus \{0^n\}$, then $\xi_{n+1} \in \mathcal{D}(0^{n+1})^c$ and $|\mathcal{CP}_{\xi_{n+1}}| = |\mathcal{CP}_{\xi_n}| - 1$ (see Fig. 5). Finally, if $\xi_{n+1} = 0^{n+1}$ or $\xi_{n+1} = 0^n 1$, then obviously, $|\mathcal{CP}_{\xi_{n+1}}| = |\mathcal{CP}_{\xi_n}| = 1$. Proposition 3.9 follows.

Lemma 3.12 follows from the above description of π_{n+1}^{-1} (see Fig. 5 and Fig. 6). \square

4 Infinite Schreier graphs

This section and the next one contain the main results of the paper. Our aim is to give a classification of the infinite Schreier graphs (Γ_ξ, ξ) , $\xi \in \partial T$, of the Basilica group acting on the boundary of the binary tree.

In this section, we show that a graph Γ_ξ has one, two or four ends (Theorem 4.1), and classify the graphs with four and two ends up to isomorphism (Theorems 4.6, 4.8, 4.11). It happens however that almost every infinite Schreier graph is one-ended (see Subsection 5.2 for the discussion), and the one-ended Schreier graphs are classified up to isomorphism in Subsection 5.1 (Theorem 5.4).

Given an infinite graph $\Gamma = (V, E)$, a **ray** is an infinite sequence of distinct vertices of Γ such that any two consecutive vertices of this sequence are adjacent in Γ . Consider an equivalence relation on the set of rays in Γ : two rays \mathcal{R} and \mathcal{R}' are equivalent if for any finite set $S \subset V$ both \mathcal{R} and \mathcal{R}' have a tail in the same component of $\Gamma \setminus S$. If two rays are equivalent, and only then, they can be linked by infinitely many disjoint paths. An **end** is an equivalence class of rays. Note that every infinite, locally finite graph must have at least one end.

Theorem 4.1. *Set $E_i = \{\xi \in \{0, 1\}^\omega \mid \text{the infinite Schreier graph } \Gamma_\xi \text{ has } i \text{ ends}\}$. Then*

1. $E_4 = \{w0^\omega, w(01)^\omega \mid w \in X^*\}$;
2. $E_1 = \{\alpha_1\beta_1\alpha_2\beta_2\dots, \alpha_i, \beta_j \in \{0, 1\} \mid \{\alpha_i\}_{i \geq 1} \text{ and } \{\beta_j\}_{j \geq 1} \text{ both contain infinitely many 1's}\}$;
3. $E_2 = \{0, 1\}^\omega \setminus \{E_1 \sqcup E_4\}$.

This theorem will be proven in the next subsection.

Observe that, by Proposition 2.4, E_4 consists of exactly one orbit, whereas E_1 and E_2 consist of infinitely many orbits. More precisely, our classification results proven below (Theorems 4.1, 4.6, 4.8, 5.4) imply the following:

- Corollary 4.2.**
1. *There exists only one class of isomorphism of 4-ended (unrooted) infinite Schreier graphs. It contains a single orbit.*
 2. *There exist uncountably many classes of isomorphism of 2-ended (unrooted) infinite Schreier graphs. Each of these classes contains exactly two orbits.*
 3. *There exist uncountably many classes of isomorphism of 1-ended (unrooted) infinite Schreier graphs. The isomorphism class of Γ_{1^ω} is a single orbit, and every other class contains uncountably many orbits.*

See also Propositions 5.11 and 5.12 for a measurable classification of infinite Schreier graphs $\{\Gamma_\xi \mid \xi \in \partial T\}$.

We end this subsection with an infinite analogue of Propositions 3.2 and 3.4 about the structure of the Basilica Schreier graphs:

- Proposition 4.3.**
1. *For any $\xi \in \{0, 1\}^\omega$, the infinite orbital Schreier graph Γ_ξ is separable and its blocks are either cycles or single edges.*
 2. *For any $\eta \in V(\Gamma_\xi)$ ($\eta \neq 0^\omega$), removing η splits Γ_ξ into several components among which one, denoted by U , is finite. Then, the subgraph induced by $V(U) \cup \{\eta\}$ is isomorphic to a k -decoration for some $k \geq 1$.*
 3. *The decoration of η is a k -decoration if and only if while reading η from the left, we first encounter 1 in the k -th position.*

Proof. Part 1. Since the sequence of finite rooted graphs (Γ_n, ξ_n) converges to (Γ_ξ, ξ) , we have that for all $r \geq 1$, there exists $N \geq 1$ such that for all $n \geq N$, $B_{\Gamma_n}(\xi_n, r) \simeq B_{\Gamma_\xi}(\xi, r)$. But the subgraph $B_{\Gamma_n}(\xi_n, r)$ of Γ_n is separable and its blocks are either cycles or single edges. Since $\lim_{r \rightarrow \infty} B_{\Gamma_\xi}(\xi, r) = (\Gamma_\xi, \xi)$, the statement follows.

Parts 2. and 3. Let $\eta \in V(\Gamma_\xi)$ ($\eta \neq 0^\omega$) and let η_k be the shortest prefix of η ending by 1. By Proposition 3.4, the decoration of η_n in Γ_n is a k -decoration for every $n \geq k + 1$. If $d \equiv d_{\Gamma_\xi}(\eta, \xi)$ denotes the usual graph distance in Γ_ξ between ξ and η , then there exists a radius $r_{(d,k)}$ and a $N(r_{(d,k)}) \geq 1$ such that for all $n \geq N(r_{(d,k)})$, the ball $B_{\Gamma_n}(\xi_n, r_{(d,k)})$ contains η_n together with its decoration and $B_{\Gamma_n}(\xi_n, r_{(d,k)}) \simeq B_{\Gamma_\xi}(\xi, r_{(d,k)})$. \square

4.1 Limit graph with four ends and Proof of Theorem 4.1

In this subsection, we determine for which infinite words $\xi \in \{0, 1\}^\omega$ the corresponding orbital infinite Schreier graph Γ_ξ has four ends, and we study the shape of these graphs.

Let $\xi \in \{0, 1\}^\omega$. By Proposition 3.2, for any $n > 1$, ξ_n belongs to exactly two cycles of Γ_n denoted by $C(\xi_n)$ and $C'(\xi_n)$; $C(\xi_n)$ is contained in $\mathcal{D}(\xi_n)$ whereas $C'(\xi_n)$ is contained in $\mathcal{D}(\xi_n)^c$. Finally, write $l_n(\xi) := |C(\xi_n)|$ and $l'_n(\xi) := |C'(\xi_n)|$. We have the following:

Lemma 4.4. *Given $\xi \in \{0, 1\}^\omega$, the sequence $\{l_n(\xi)\}_{n \geq 1}$ diverges as $n \rightarrow \infty$ if and only if $\xi = 0^\omega$.*

Proof. It follows from SR that $l_n(0^\omega) = 2^{\lceil \frac{n-1}{2} \rceil}$ and $l'_n(0^\omega) = 2^{\lceil \frac{n}{2} \rceil}$ for every $n \geq 1$. Conversely, consider $\eta \in \{0, 1\}^\omega$ such that $\eta \neq 0^\omega$. Then $\eta = 0^k 1 \eta'$ for some $k \geq 0$ and it follows from Proposition 3.4, Part 2., that a $(k + 1)$ -decoration is attached to η_n for every $n \geq k + 1$. Hence, $l_n(\eta)$ is constant for each $n \geq k + 1$. \square

The next lemma describes the decorations attached to $C(0^n)$ and $C'(0^n)$.

Lemma 4.5. *For $n \geq 2$, consider the graph Γ_n . Then, for $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$, to every vertex situated on $C'(0^n)$ at distance $2^k + l \cdot 2^{k+1}$ for some $l \geq 0$ from 0^n is attached a $(2k + 1)$ -decoration if n is odd and a $(2k + 2)$ -decoration if n is even. On the other hand, for $0 \leq k \leq \lfloor \frac{n-2}{2} \rfloor$, to every vertex situated on $C(0^n)$ at distance $2^k + l \cdot 2^{k+1}$ for some $l \geq 0$ from 0^n is attached a $(2k + 2)$ -decoration if n is odd and a $(2k + 1)$ -decoration if n is even.*

Proof. Observe that the cycle $C'(0^n)$ (which is the central cycle of Γ_n) and the decorations attached to the vertices of $C'(0^n)$ are encoded by the diagram D_n (see Subsection 3.1). Similarly, the cycle $C(0^n)$ and the decorations attached to the vertices of $C(0^n)$ are encoded by the diagram D_{n-1} . Induction on n and the RCR yield the statement. \square

Proof of Theorem 4.1, Part 1. Consider the vertex 0^n in Γ_n together with its neighbours. Consider the following four disjoint paths: each of them starts at a different neighbour of 0^n . The two first are included in $C(0^n)$ and they are of length $l_n(0^\omega)/2 - 1$ whereas the two others are included in $C'(0^n)$ and they are of length $l'_n(0^\omega)/2 - 1$. Since $l_n(0^\omega) = 2^{\lceil \frac{n-1}{2} \rceil}$ and $l'_n(0^\omega) = 2^{\lceil \frac{n}{2} \rceil}$, their respective lengths diverge as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} (\Gamma_n, 0^n) = (\Gamma_{0^\omega}, 0^\omega)$, the corresponding four rays are disjoint in $(\Gamma_{0^\omega}, 0^\omega)$ so that they are not equivalent. Hence, the graph $(\Gamma_{0^\omega}, 0^\omega)$ has at least four ends.

Consider now $\xi \in V(\Gamma_{0^\omega})$, $\xi \neq 0^\omega$. Then $\xi = 0^k 1 \xi'$ for some $k \geq 0$; by Proposition 3.4, a $(k + 1)$ -decoration is attached to ξ_n for every $n \geq k + 1$. Since every such decoration is finite, any ray must have infinitely many vertices in common with one of the four previous rays. We thus conclude that the graph $(\Gamma_{0^\omega}, 0^\omega)$ has exactly four ends.

Finally, Lemma 4.4 implies that the set of infinite words ξ such that the orbital Schreier graph Γ_ξ has four ends is exactly constituted by the orbit of 0^ω . Then the result follows from Proposition 2.4. \square

We describe now a 4-regular infinite separable graph $\Gamma_{(4)}$ whose blocks are either cycles or single edges. The number of ends of $\Gamma_{(4)}$ is 4. It will turn out that $\Gamma_{(4)}$ is isomorphic (as an unrooted graph) to Γ_ξ for each $\xi \in E_4$. $\Gamma_{(4)}$ is constructed as follows: consider two copies, \mathcal{R}_1 and \mathcal{R}_2 , of the double ray whose vertices are naturally identified with the integers. Let these two double rays intersect at vertex 0. For every $k \geq 0$, we define the subset of \mathbb{Z}

$$A_k := \{n \in \mathbb{Z} \mid n \equiv 2^k \pmod{2^{k+1}}\}.$$

For every $k \geq 0$, attach to each vertex of A_k in \mathcal{R}_1 (respectively in \mathcal{R}_2) a $(2k+1)$ -decoration (respectively a $(2k+2)$ -decoration) by its unique vertex of degree 2.

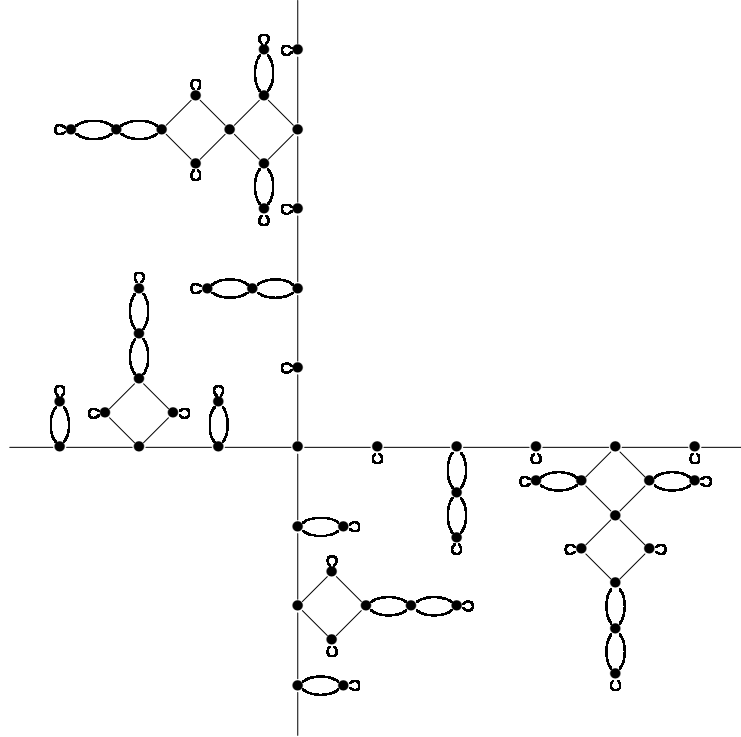


Fig. 7. A finite part of $\Gamma_{(4)}$.

Theorem 4.6. *If $\xi \in E_4$, then Γ_ξ and $\Gamma_{(4)}$ are isomorphic as unrooted graphs.*

Proof. We can suppose without loss of generality that $\xi = 0^\omega$ since we know that all words in E_4 belong to the same orbit. Recall that the length of $C(0^n)$ in Γ_n is $2^{\lceil \frac{n-1}{2} \rceil}$. Consider the ball $B_{\Gamma_n}(0^n, 2^{\lceil \frac{n-1}{2} \rceil - 1} - 1)$ centered on 0^n and of radius $2^{\lceil \frac{n-1}{2} \rceil - 1} - 1$ as well as the ball $B_{\Gamma_\xi}(0^\omega, 2^{\lceil \frac{n-1}{2} \rceil - 1} - 1)$ in Γ_ξ . By Lemma 4.5, these balls are isomorphic for every $n \geq 2$ and their radii tend to infinity as $n \rightarrow \infty$. \square

Proof of Theorem 4.1, Parts 2 and 3. Let $\xi \in E_2$. We can write $\xi = \alpha_1\beta_1\alpha_2\beta_2\dots$ where $\alpha_i, \beta_i \in \{0, 1\}$ and either $\{\alpha_i\}_{i \geq 1}$ or $\{\beta_i\}_{i \geq 1}$ (but not both) has only finitely many 1's. Suppose without loss of generality that $\{\alpha_i\}_{i \geq 1}$ contains finitely many 1's (the other case can be treated in a similar way). Let $l \geq 0$ be such that the prefix ξ_{2l-1} ends by the last non-zero value of the sequence $\{\alpha_i\}$, ξ_{-1} denoting by convention the empty prefix. Let $k = l + \inf\{i \geq 0 : \beta_{i+1} = 1\}$ so that β_k is the first 1 appearing in ξ after the prefix ξ_{2l-1} . The infinite word ξ can be rewritten as

$$\xi = \xi_{2k}0\beta_{k+1}0\beta_{k+2}\dots$$

The next observation follows from the proof of Proposition 3.9: since ξ_{2k} ends by a 1, $\xi_{2k} \in \mathcal{D}(0^{2k})^c$ and $\xi_{2k}0 \in \mathcal{D}(0^{2k+1})$. For each $r \geq 1$, on one hand, $\xi_{2k}0\beta_{k+1} \dots 0\beta_{k+r}0$ belongs to the left part of $\Gamma_{2k+2r+1}$ and there is an expansion in the passage from ξ_{2k+2r} to $\xi_{2k+2r+1}$, and on the other hand, $\xi_{2k}0\beta_{k+1} \dots 0\beta_{k+r}$ belongs to the central part of Γ_{2k+2r} and a contraction occurs in the passage from $\xi_{2k+2r-1}$ to ξ_{2k+2r} .

Let $\mathcal{CP}_{\xi_{2k}}$ be the cycle-path joining ξ_{2k} to the central cycle of Γ_{2k} and let $\mathcal{LP}_{\xi_{2k}} = \{k_i^{2k}\}_{i=1}^t$ be as defined in Notation 3.8. By the previous observation, $|\mathcal{CP}_{\xi_{2k+2r}}| = |\mathcal{CP}_{\xi_{2k}}| = t$ for each $r \geq 1$. Moreover, it follows from Lemma 3.12, that $k_i^{2k+2r} = k_i^{2k}$ for all $1 \leq i \leq t-1$ and each $r \geq 1$. Thus, we conclude that for N sufficiently large, the distance separating the vertex ξ_{2n} from the central cycle in Γ_{2n} is constant for every $n > N$. On the other hand, the length k_t^{2k+2r} of the central cycle of Γ_{2k+2r} tends to infinity as $r \rightarrow \infty$, so that, in the limit, the central cycle splits into two disjoint rays \mathcal{R} and \mathcal{R}' . Thus, Γ_ξ has at least two ends.

By Proposition 4.3, given any $\eta \in V(\Gamma_\xi)$, the height of the decoration $\mathcal{D}(\eta)$ is finite. Thus, any ray in Γ_ξ must be equivalent either to \mathcal{R} or to \mathcal{R}' . Thus, Γ_ξ has exactly two ends.

We show now that if $\xi \in E_1$, then Γ_ξ has one end. On one hand, the SEC-sequence associated with ξ cannot contain two consecutive C's (see Remark 3.11). On the other hand, if ξ_n belongs to the right part of Γ_n and n' is the first index greater than n such that $\xi_{n'}$ belongs again to the right part of $\Gamma_{n'}$, then $|\mathcal{CP}_{\xi_{n'}}| = |\mathcal{CP}_{\xi_n}| + 1$.

Lemma 4.7. *For any $\xi \in \{0, 1\}^\omega$, consider the subgraph of (Γ_ξ, ξ) , $(\mathcal{CP}_\xi, \xi) := \lim_{n \rightarrow \infty} (\mathcal{CP}_{\xi_n}, \xi_n)$. Then \mathcal{CP}_ξ is a cycle-path of infinite length, $\mathcal{CP}_\xi = \{P_i\}_{i \geq 1}$, if and only if $\xi \in E_1$. In this case, for any $n \geq 1$ such that ξ_n ends with a 1, the subgraph $\{P_i^n\}_{i=1}^{t-1}$ of $\mathcal{CP}_{\xi_n} = \{P_i^n\}_{i=1}^t$ is isomorphic to $\{P_i\}_{i=1}^{t-1} \subset \mathcal{CP}_\xi$.*

Proof of the lemma. Observe that, \mathcal{CP}_ξ is a cycle-path of infinite length if and only if there exist infinitely many indices n such that ξ_n is in the right part of Γ_n ; otherwise, we would observe from some point in the SEC-sequence associated with ξ an alternating sequence of expansions E and contractions C, so that $|\mathcal{CP}_{\xi_n}|$ would be bounded as $n \rightarrow \infty$.

It is easy to check that E_1 coincides with the set of words $\xi \in \{0, 1\}^\omega$ in which the number of subwords of type 10^k1 , with $k \geq 0$ even, is infinite. Hence, if $\xi \in E_1$, there are infinitely many indices n such that ξ_n belongs to the right part of the graph Γ_n . Conversely, suppose that $\xi \notin E_1$, so that the number of subwords in ξ of type 10^k1 , with $k \geq 0$ even, must be finite. Thus, the SEC-sequence associated with ξ contains from some point an alternating sequence of expansions E and contractions C. It follows that \mathcal{CP}_ξ is not a cycle-path of infinite length (it consists of a finite cycle-path joining ξ to a double-ray).

The second part of the lemma follows from the proof of Proposition 3.9. \square

End of Proof of Theorem 4.1. Let \mathcal{R} be a ray contained in \mathcal{CP}_ξ . Each cycle $P_i \in \mathcal{CP}_\xi$ is finite and, by Proposition 4.3, there is a finite decoration attached to each of its vertices. Hence, any other ray \mathcal{R}' in Γ_ξ must have infinitely many vertices in common with \mathcal{R} . We thus conclude that Γ_ξ has one end. \square

4.2 Limit graphs with two ends

Given $\xi \in E_2$, let w be the longest prefix of ξ ending by a subword of type 10^k1 , with k even (see *Proof of Theorem 4.1, Parts 2 and 3*). If $\{\alpha_i\} \equiv 0$, then we set w to be the empty word; if $\{\beta_i\} \equiv 0$, then w is the prefix of ξ of length 1. Thus, we can write $\xi = w0x_10x_2 \dots$, $x_i \in \{0, 1\}$. Introduce $\bar{\xi} = w0\bar{x}_10\bar{x}_2 \dots$, where $\bar{x}_i = 1 - x_i$ for every $i \geq 1$. Note that $\xi \in E_2$ if and only if $\bar{\xi} \in E_2$.

Theorem 4.8. *Let $\xi, \eta \in E_2$. Then Γ_ξ and Γ_η are isomorphic as unrooted graphs if and only if either $\xi \sim \eta$ or $\bar{\xi} \sim \eta$. In particular, each isomorphism class of 2-ended graphs consists of exactly two orbits.*

Proof. \Leftarrow : If $\xi \sim \eta$, then $\Gamma_\xi = \Gamma_\eta$ by Proposition 2.4. Suppose now that $\eta \sim \bar{\xi}$. This implies that there exist prefixes v, \hat{v}, \tilde{v} with $|v| = |\hat{v}| = |\tilde{v}| = n_0$ and where \tilde{v} ends by a 1, such that

$$\eta = v0z_10z_2\dots \quad \bar{\xi} = \hat{v}0z_10z_2\dots \quad \xi = \tilde{v}0\bar{z}_10\bar{z}_2\dots,$$

where $\bar{z}_i = 1 - z_i$ for each $i \geq 1$. It follows from the proof of Proposition 3.9 that ξ_{n_0+2} belongs to $\mathcal{D}(0^{n_0+2})^c \subset \Gamma_{n_0+2}$. If we encode Γ_{n_0+2} by the diagram \bar{D}_{n_0+2} (see Subsection 3.1), let u_{n_0+2} be the vertex of \bar{D}_{n_0+2} to which is attached the decoration containing ξ_{n_0+2} . Let $u'_{n_0+2} \in \bar{D}_{n_0+2}$ be the symmetric vertex with respect to the middle vertex of \bar{D}_{n_0+2} so that the decorations attached to u_{n_0+2} and u'_{n_0+2} have the same height (observe that the vertices of \bar{D}_n are labeled symmetrically with respect to the middle vertex labeled by n .) Finally, let $\tilde{\xi}_{n_0+2} \in \mathcal{D}(u'_{n_0+2})$ be symmetric to ξ_{n_0+2} as shown in Fig. 8.

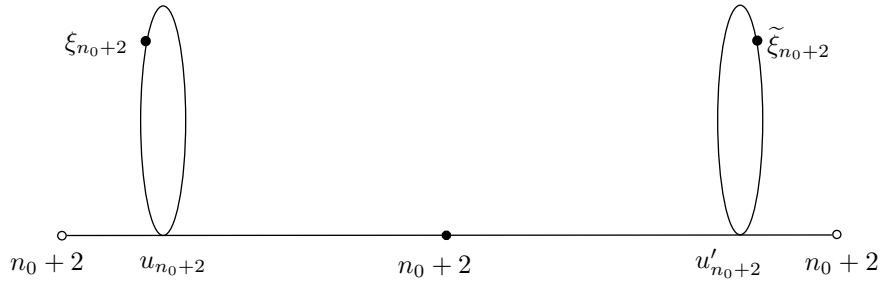


Fig. 8. \bar{D}_{n_0+2}

Consider $\varphi(\xi) = \varphi(\xi_{n_0+2}0\bar{z}_20\bar{z}_3\dots) := \tilde{\xi}_{n_0+2}0z_20z_3\dots$. Since $\varphi(\xi) \sim \eta$, then by Proposition 2.4, $\varphi(\xi) \in V(\Gamma_\eta)$.

For each $i \geq 1$, let $R_i = d(\xi_{i+2}, 0^{i+2})$ in Γ_{i+2} . Then, since $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$ (respectively $\{(\Gamma_n, \eta_n)\}_{n \geq 1}$) converges to (Γ_ξ, ξ) (respectively (Γ_η, η)), there exists $N(R_i) > 1$ such that for all $n > N(R_i)$, we have

$$B_{\Gamma_\xi}(\xi, R_i) \simeq B_{\Gamma_n}(\xi_n, R_i) \simeq B_{\Gamma_n}(\varphi(\xi)_n, R_i) \simeq B_{\Gamma_\eta}(\varphi(\xi), R_i),$$

where the second isomorphism follows from the previous observations and from the fact that if $\xi_n \in \mathcal{D}(0^n)^c$ and $\xi_{n+2} \in \mathcal{D}(0^{n+2})^c \setminus \mathcal{D}(0^{n+1}1)$, then $\xi_{n+2} = \xi_n 0x$ where $x \in \{0, 1\}$, so that x determines whether ξ_{n+2} belongs to a decoration situated in the left-half or in the right-half of D_{n+2} . Since $\xi \in E_2$, R_i tends to infinity as $i \rightarrow \infty$ by Proposition 2.4. We conclude that Γ_ξ and Γ_η are isomorphic.

\Rightarrow : Let $\eta, \xi \in E_2$ and suppose that $\xi \approx \eta$ and $\bar{\xi} \approx \eta$. For the sake of contradiction, suppose that there exists an isomorphism $\varphi : \Gamma_\xi \rightarrow \Gamma_\eta$ of unrooted graphs. We will show that there exists a radius R such that the balls $B_{\Gamma_\xi}(\xi, R)$ and $B_{\Gamma_\eta}(\varphi(\xi), R)$ are not isomorphic which will yield a contradiction.

By Proposition 2.4, $\varphi(\xi) \sim \eta$. There exist finite words u, \hat{u} such that

$$\xi = u0x_10x_2\dots \quad \varphi(\xi) = \hat{u}0y_10y_2\dots$$

and $|u| = |\hat{u}|$. If not, then ξ and $\varphi(\xi)$ belong to infinite graphs whose decorations have different parity, which is a contradiction. Indeed, depending on whether it is the sequence $\{\alpha_i\}$ or $\{\beta_j\}$ that takes only finitely many 1's, the corresponding infinite Schreier graph has decorations of only even or only odd height.

Since $\eta \approx \xi$, $\eta \approx \bar{\xi}$ but $\eta \sim \varphi(\xi)$, there exist infinitely many indices i such that $x_i \neq y_i$ and $x_{i+1} = y_{i+1}$. We can thus suppose without loss of generality that u and \hat{u} both end with a 1. Moreover, arguments as in Proof of Proposition 3.9 imply that there exists an infinite subsequence $\{n_k\}_{k \geq 0} \subset \mathbb{N}$ such that, for any $k \geq 0$, ξ_{n_k} and $\varphi(\xi)_{n_k}$ belong to decorations attached to vertices in different halves of D_{n_k} , while ξ_{n_k+2} and $\varphi(\xi)_{n_k+2}$ belong to decorations attached to distinct vertices v_{n_k+2} and v'_{n_k+2} in the same half of D_{n_k+2} . Thus $d(v_{n_k+2}, w_{n_k+2}) \neq d(v'_{n_k+2}, w_{n_k+2})$ where w_{n_k+2} denotes the vertex of \bar{D}_{n_k+2} encoding $0^{n_k+1}1$ (see Fig. 9).

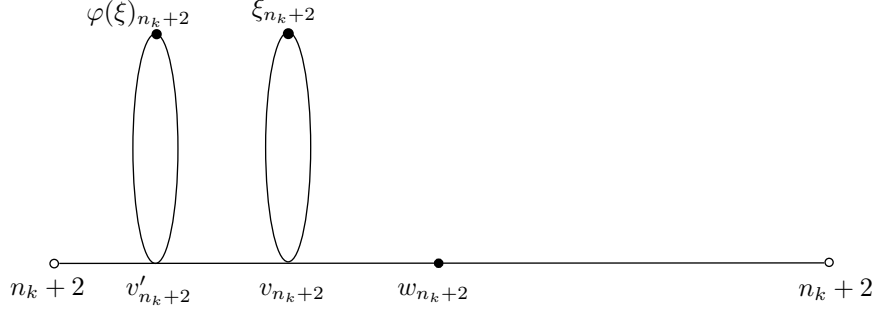


Fig. 9. \bar{D}_{n_k+2}

Suppose without loss of generality, that $d_{\Gamma_{n_k+2}}(v_{n_k+2}, w_{n_k+2}) < d_{\Gamma_{n_k+2}}(v'_{n_k+2}, w_{n_k+2})$. By RCR, for each $k \geq 0$, the minimal distance between v_{n_k+2} (respectively v'_{n_k+2}) and a vertex in \bar{D}_{n_k+2} labeled by $n_0 + 2$ is constant. On the other hand, since $\varphi(\xi) \in E_2$, the distance between v_{n_k+2} (respectively v'_{n_k+2}) and the boundary vertices of \bar{D}_{n_k+2} tends to infinity as $k \rightarrow \infty$.

Thus, for any sufficiently large k , there exists a radius $R(n_0)$ defined as the minimum over all radii r such that the ball $B_{\Gamma_{n_k+2}}(\xi_{n_k+2}, r)$ contains a $(n_0 + 2)$ -decoration attached to a vertex of D_{n_k+2} , and such that both $B_{\Gamma_{n_k+2}}(\xi_{n_k+2}, r)$ and $B_{\Gamma_{n_k+2}}(\varphi(\xi)_{n_k+2}, r)$ do not contain any boundary vertices of \bar{D}_{n_k+2} .

Hence, there exists $N(R(n_0)) > 1$ such that for all $n_k \geq N(R(n_0))$, we have

$$B_{\Gamma_\xi}(\xi, R(n_0)) \simeq B_{\Gamma_{n_k+2}}(\xi_{n_k+2}, R(n_0)) \not\simeq B_{\Gamma_{n_k+2}}(\varphi(\xi)_{n_k+2}, R(n_0)) \simeq B_{\Gamma_\eta}(\varphi(\xi), R(n_0)).$$

□

For $\xi \in E_2$, we will now construct explicitly a 4-regular, 2-ended graph $\Gamma(\xi)$ that we will show to be isomorphic (as unrooted graph) to Γ_ξ (see Theorem 4.11 below). Recall from Theorem 4.1 that $E_2 = \{\alpha_1\beta_1\alpha_2\beta_2\dots \mid \alpha_i, \beta_j \in \{0, 1\} \text{ such that either } \{\alpha_i\} \text{ or } \{\beta_j\} \text{ has finitely many 1's}\}$. Observe that in the case where the sequence $\{\alpha_i\}$ (respectively $\{\beta_j\}$) takes the value 1 a finite number of times, only decorations of even (respectively odd) height will appear in the limit graph. We can thus partition the set $E_2 = E_{2, \text{even}} \sqcup E_{2, \text{odd}}$. We will only consider the case of $\xi \in E_{2, \text{even}}$, (the other case can be treated similarly.) Suppose therefore that $\xi = \alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\dots \in E_{2, \text{even}}$, so that $\{\alpha_i\}_{i \geq 1}$ contains finitely many 1's and define the graph $\Gamma(\xi)$ associated with ξ .

Proposition 4.9. *Consider the following subsets of \mathbb{Z} :*

$$A'_0 := 2\mathbb{Z} \quad \text{and} \quad A'_k := \{n \in \mathbb{Z} : n \equiv 2^k - 1 - \sum_{i=1}^k 2^i \beta_{i+1} \pmod{2^{k+1}}\} \quad \text{for each } k \geq 1.$$

Construct $\Gamma(\xi)$ as a straight line with integer vertices with, for each $k \geq 0$, a $(2k+2)$ -decoration attached by its unique vertex of degree 2 to every vertex corresponding to an integer in A'_k . The constructed graph $\Gamma(\xi)$ is well defined.

The graph $\Gamma(\xi)$ for $\xi \in E_{2, \text{odd}}$ is defined similarly, replacing β by α in the definition of A'_k , and by attaching $(2k+1)$ -decorations instead of $(2k+2)$ -decorations. Proposition 4.9 follows from the following lemma.

Lemma 4.10. *Let $\{x_i\}_{i \geq 1}$ be a sequence with values in $\{0, 1\}$ supposing that it takes both values infinitely many times. For every $k \geq 1$, define*

$$L_k = \{n \in \mathbb{Z} : n \equiv 2^k - 1 - \sum_{i=1}^k 2^i x_i \pmod{2^{k+1}}\}.$$

Then $\coprod_{k \geq 1} L_k$ is a partition of $2\mathbb{Z} + 1$.

Proof. We first prove that $L_k \cap L_h = \emptyset$ if $k \neq h$. Suppose that $h > k$ and let $s, t \in \mathbb{Z}$ be such that

$$2^k - 1 - \sum_{i=1}^k 2^i x_i + 2^{k+1} t = 2^h - 1 - \sum_{i=1}^h 2^i x_i + 2^{h+1} s.$$

Hence, $2^k(1 + 2t) = 2^h(1 + 2s) - \sum_{i=k+1}^h 2^i x_i = 2^{k+1} m$ where $m \in \mathbb{Z}$ which is impossible.

We prove now that $2\mathbb{Z} + 1 \subset \prod_{k \geq 1} L_k$ by showing that for any $n \in \mathbb{Z}$ odd, there exist $k \geq 1$ and $t \in \mathbb{Z}$ such that

$$n = 2^k - 1 - \sum_{i=1}^k 2^i x_i + t \cdot 2^{k+1}. \quad (3)$$

Let $p_1 := \min\{i \geq 1 : x_i = 1\}$. If $n + 1 = 0$, then a solution of (3) is $k = p_1$ and $t = 0$. Suppose that $n + 1 \neq 0$ and write $n + 1 = 2^{u_1} m_1$, with $u_1 \geq 1$ and m_1 odd. If $p_1 \neq u_1$, a solution of (3) is $k = \min\{p_1, u_1\}$ and $t = \frac{m_1 - 1}{2}$ if $k = u_1$ or $t = 2^{u_1 - p_1 - 1} m_1$ if $k = p_1$. If $u_1 = p_1$, then necessarily, $k > p_1$ and (3) becomes

$$2^{p_1} \left(m_1 + 1 + \sum_{i=p_1+1}^k 2^{i-p_1} x_i \right) = 2^k(1 + 2t). \quad (4)$$

Let $p_2 := \min\{i \geq 1 : x_{p_1+i} = 1\}$. If $m_1 + 1 = 0$, then a solution of (4) is $k = p_1 + p_2$ and $t = 0$. Suppose that $m_1 + 1 \neq 0$ and write $m_1 + 1 = 2^{u_2} m_2$, with $u_2 \geq 1$ and m_2 odd. If $p_2 \neq u_2$, then setting $k = p_1 + \min\{p_2, u_2\}$, one can check that (4) has a solution. If $p_2 = u_2$, then necessarily $k > p_1 + p_2$ and (4) becomes

$$2^{p_1+p_2} \left(m_2 + 1 + \sum_{i=p_1+p_2+1}^k 2^{i-p_1-p_2} x_i \right) = 2^k(1 + 2t). \quad (5)$$

We iterate the above argument: at the $s + 1$ step, let $p_s := \min\{i \geq 1 : x_{\sum_{r=1}^{s-1} p_r + i} = 1\}$; if $m_s + 1 = 0$, set $k = \sum_{r=1}^s p_r$ and $t = 0$; if $m_s + 1 \neq 0$, write $m_s + 1 = 2^{u_{s+1}} m_{s+1}$, with $u_{s+1} \geq 1$ and m_{s+1} odd. If $p_s \neq u_s$, one can check as above that (3) has a solution. Clearly, the sequence $\{|m_s + 1|\}_{s \geq 1}$ is strictly decreasing. If $n < 0$, then there exists s_0 such that $m_{s_0} = -1$ and (3) has a solution. If $n > 0$, then there exists s_0 such that $m_s = 1$ for all $s \geq s_0$. In the latter case, $u_s = 1$ for all $s > s_0$. On the other hand, there are only finitely many indices s such that $p_s = u_s = 1$. Otherwise, there would exist i_0 such that $x_i = 1$ for every $i > i_0$ which would contradict the hypothesis on the sequence $\{x_i\}_{i \geq 1}$. \square

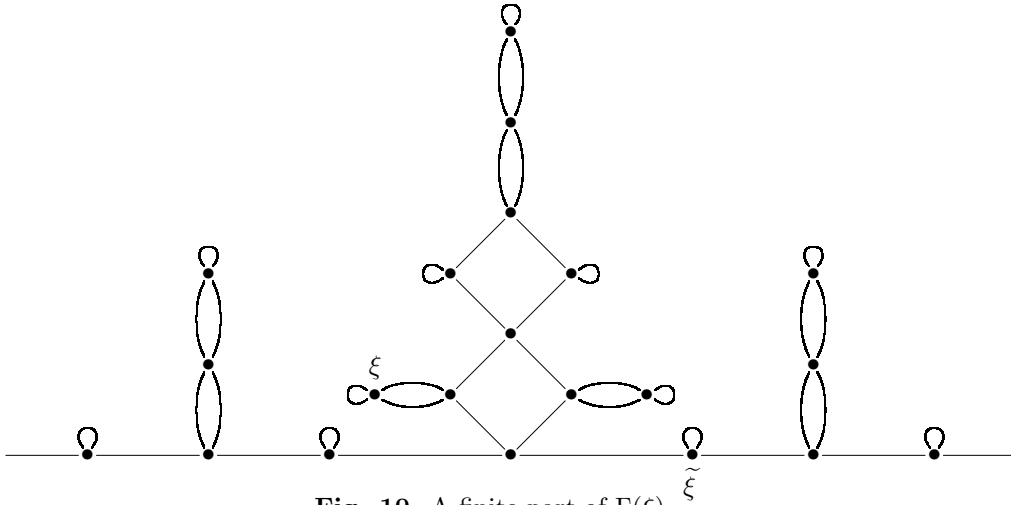


Fig. 10. A finite part of $\Gamma(\xi)$.

Theorem 4.11. *Let $\xi \in E_2$. Then Γ_ξ is isomorphic to $\Gamma(\xi)$.*

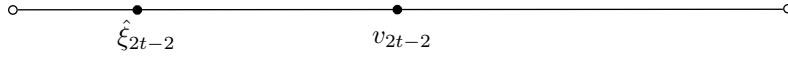
Proof. We again only consider the case where $\xi \in E_{2,even}$ (the case $\xi \in E_{2,odd}$ can be treated analogously). For any $t \geq 1$, consider the diagram \overline{D}_{2t} encoding Γ_{2t} . Consider also $\hat{\xi} := 010\beta_20\beta_3\dots$. Note that $\hat{\xi}$ and ξ are cofinal, $\hat{\xi} \sim \xi$, so that, by Proposition 2.4, $\Gamma_\xi = \Gamma_{\hat{\xi}}$. Arguments as in Proof of Proposition 3.9 imply that $\hat{\xi}_{2t} \in V(\Gamma_{2t})$ is situated on the central cycle of this graph, and therefore can be identified with a vertex of \overline{D}_{2t} (also denoted by $\hat{\xi}_{2t}$). Let v_{2t} be the middle vertex in \overline{D}_{2t} ; it is labeled by $2t$ (see the figures below).

It is convenient to embed the diagram \overline{D}_{2t} in \mathbb{Z} , so that every vertex of \overline{D}_{2t} has a *coordinate* (in addition to its label). Suppose that we embed \overline{D}_{2t} in \mathbb{Z} so that the coordinate of $\hat{\xi}_{2t}$ is 0.

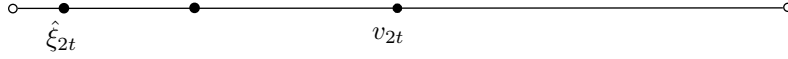
Claim: The coordinate of v_{2t} is $2^{t-1} - 1 - \sum_{i=1}^{t-1} 2^i \beta_{i+1}$.

We prove the claim by induction on t . For $t = 2$, v_4 has coordinate 1 if $\beta_2 = 0$, while it has coordinate -1 if $\beta_2 = 1$, and the claim holds.

Suppose that the assertion is true for $t - 1$ and suppose that $\beta_{t-1} = 0$ (the case $\beta_{t-1} = 1$ is treated similarly). The diagram \overline{D}_{2t-2} is



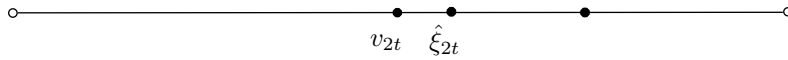
If $\beta_t = 0$, then by RCR, the diagram \overline{D}_{2t} is



Hence, recalling that the length of \overline{D}_{2t} is 2^t , the coordinate of v_{2t} is

$$2^{t-2} + 2^{t-2} - 1 - \sum_{i=1}^{t-2} 2^i \beta_{i+1} = 2^{t-1} - 1 - \sum_{i=1}^{t-1} 2^i \beta_{i+1}.$$

On the other hand, if $\beta_t = 1$, the diagram \overline{D}_{2t} is



Thus, in this case, the coordinate of v_{2t} is

$$2^{t-2} - 1 - \sum_{i=1}^{t-2} 2^i \beta_{i+1} - 2^{t-2} = 2^{t-1} - 1 - \sum_{i=1}^{t-1} 2^i \beta_{i+1},$$

and the claim is proven.

Let $R_t = d_{\Gamma_{2t}}(\hat{\xi}_{2t}, 0^{2t})$; we prove by induction on t , that for any $t \geq 2$, $B_{\Gamma(\xi)}(0, R_t) \simeq B_{\Gamma_{2t}}(\hat{\xi}_{2t}, R_t)$. If $t = 2$, then $R_2 = 1$ and, clearly, $B_{\Gamma(\xi)}(0, 1) \simeq B_{\Gamma_4}(\hat{\xi}_4, 1)$. Suppose that $B_{\Gamma(\xi)}(0, R_{t-1}) \simeq B_{\Gamma_{2t-2}}(\hat{\xi}_{2t-2}, R_{t-1})$. Then, by the RCR and the Claim, the only vertex in D_{2t} labeled by $2t$ is situated at distance $2^{t-1} - 1 - \sum_{i=1}^{t-1} 2^i \beta_{i+1}$ from $\hat{\xi}_{2t}$. Moreover, any two consecutive vertices in \overline{D}_{2t} labeled by $2k + 2$ are situated

at distance 2^{k+1} from each other. Thus, by definition of the graph $\Gamma(\xi)$, $B_{\Gamma(\xi)}(0, R_t) \simeq B_{\Gamma_{2t}}(\hat{\xi}_{2t}, R_t)$. Finally, since $\hat{\xi} \in E_2$, R_t tends to infinity as $t \rightarrow \infty$ and since $\Gamma_{\hat{\xi}} = \Gamma_{\xi}$, the proof of the theorem is completed. \square

5 The typical case: limit graphs with one end

5.1 Classification of infinite Schreier graphs with one end

It follows from Theorem 4.1 that the set of infinite words whose associated infinite orbital Schreier graph has one end is

$$E_1 = \{\alpha_1\beta_1\alpha_2\beta_2\dots, \alpha_i, \beta_j \in \{0, 1\} \mid \{\alpha_i\}_{i \geq 1} \text{ and } \{\beta_j\}_{j \geq 1} \text{ both contain infinitely many 1's}\}.$$

We prove in this subsection Part 3. of Corollary 4.2 by classifying all limit graphs Γ_{ξ} , $\xi \in E_1$, up to isomorphism (of unrooted graphs). We begin with the following lemma:

Lemma 5.1. *Let $\xi \in E_1$ and let $\{P_i\}_{i \geq 1}, \{Q_j\}_{j \geq 1}$ be two infinite cycle-paths in Γ_{ξ} . Then, there exist $i_0, j_0 \in \mathbb{N}$ such that $P_{i_0+k} = Q_{j_0+k}$ for every $k \geq 0$.*

Proof. Let $\mathcal{R}_1 \subset \{P_i\}_{i \geq 1}$ and $\mathcal{R}_2 \subset \{Q_j\}_{j \geq 1}$ be two rays. Since Γ_{ξ} has one end, \mathcal{R}_1 and \mathcal{R}_2 are equivalent. Suppose that $\{P_i\}_{i \geq 1}$ and $\{Q_j\}_{j \geq 1}$ are disjoint. Since Γ_{ξ} is separable with blocks either cycles or single edges, there exists a cut vertex which separates $\{P_i\}_{i \geq 1}$ from $\{Q_j\}_{j \geq 1}$; this contradicts the fact that \mathcal{R}_1 and \mathcal{R}_2 are equivalent.

If there is only a finite number of vertices belonging both to $\{P_i\}_{i \geq 1}$ and $\{Q_j\}_{j \geq 1}$, then there are integers I, J such that $\{P_i\}_{i \geq I}$ and $\{Q_j\}_{j \geq J}$ are disjoint. Then, $V(\{P_i\}_{i=1}^{I-1} \cup \{Q_j\}_{j=1}^{J-1})$ is a finite subset of vertices which separates $\{P_i\}_{i \geq I}$ from $\{Q_j\}_{j \geq J}$ which contradicts the fact that \mathcal{R}_1 and \mathcal{R}_2 are equivalent.

Suppose that $|\{P_i\}_{i \geq 1} \cap \{Q_j\}_{j \geq 1}| = \infty$ and consider the minimal i_0, j_0 such that P_{i_0} and Q_{j_0} have a vertex v in common. Let $w \in P_{i_0+k} \cap Q_{j_0+k'}$ for some $k, k' \geq 0$ be at minimal distance from v . Then, $\{P_i\}_{i=i_0}^{i_0+k}$ and $\{Q_j\}_{j=j_0}^{j_0+k'}$ must coincide. Indeed, suppose that this is not the case: if $k = k' = 0$, then there are four disjoint paths joining v to w which contradicts Part 1. of Proposition 4.3. If $k = 0$ or $k' = 0$, then there are three disjoint paths joining v to w which is again a contradiction. Finally, if $k, k' \neq 0$, then there are two disjoint paths joining v to w which again contradicts Proposition 4.3. \square

Proposition 5.2. *An element $\xi \in \{0, 1\}^{\omega}$, ξ not cofinal to 1^{ω} , belongs to E_1 if and only if there exists a unique triple $(k, \{m_l\}_{l \geq 0}, \{t_l\}_{l \geq 0})$ where $k \geq 1$ and $m_0 \geq 0$ are integers (and m_0 is even); $t_0 = 0$; and $\{m_l\}_{l \geq 1}, \{t_l\}_{l \geq 1}$ are sequences of strictly positive integers (and the m_l 's are even), such that ξ can be written as*

$$\xi = 0^{k-1}1(0x_1^00x_2^0\dots 0x_{\frac{m_0}{2}}^0)1^{t_1}(0x_1^10x_2^1\dots 0x_{\frac{m_1}{2}}^1)1^{t_2}\dots \quad (6)$$

with $x_i^j \in \{0, 1\}$ for all i, j .

If $\xi = w1^{\omega}$ for some $w \in X^*$, then there exists a unique triple $(k, \{m_l\}_{l=0}^{l_0}, \{t_l\}_{l=0}^{l_0})$ where $k \geq 1$ and $m_0 \geq 0$ are integers (and m_0 is even); $t_0 = 0$; and $\{m_l\}_{l=1}^{l_0}, \{t_l\}_{l=1}^{l_0}$ are finite sequences of strictly positive integers (and the m_l 's are even), such that ξ can be written as

$$\xi = 0^{k-1}1(0x_1^00x_2^0\dots 0x_{\frac{m_0}{2}}^0)1^{t_1}(0x_1^10x_2^1\dots 0x_{\frac{m_1}{2}}^1)1^{t_2}\dots 1^{t_{l_0}}(0x_1^{l_0}0x_2^{l_0}\dots 0x_{\frac{m_{l_0}}{2}}^{l_0})1^{\omega}.$$

Corollary 5.3. *Observe that the index of stability of ξ is equal to $k + m_0$ in notation of (6). Hence, the SEC-sequence corresponding to $\xi \in E_1$, $\xi \neq w1^{\omega}$, is*

$$S^{k+m_0}E^{t_1}(EC)^{\frac{m_1}{2}}E^{t_2}(EC)^{\frac{m_2}{2}}\dots \quad (7)$$

while the SEC-sequence corresponding to $\xi = w1^{\omega}$ is

$$S^{k+m_0}E^{t_1}(EC)^{\frac{m_1}{2}}E^{t_2}\dots E^{t_{l_0}}(EC)^{\frac{m_{l_0}}{2}}E^{\omega}.$$

Proof. Corollary follows from the description of the inverse covering map π_{n+1}^{-1} , as in Proof of Proposition 3.9 and from Remark 3.11. \square

Proof of Proposition 5.2. Suppose that $\xi \in \{0, 1\}^\omega$ has the form (6). If we write $\xi = \alpha_1\beta_1\alpha_2\beta_2\dots$ where $\alpha_i, \beta_i \in \{0, 1\}$, then, since $t_l \geq 1$ for all $l \geq 1$, it follows that both sequences $\{\alpha_i\}_{i \geq 1}$ and $\{\beta_i\}_{i \geq 1}$ take infinitely often the value 1. This implies by definition that $\xi \in E_1$.

The proof of the converse is constructive; before writing down the explicit algorithm, introduce the following notation: given two sequences of integers $\{m_l\}_{l \geq 0}$ and $\{t_l\}_{l \geq 0}$, define for all $i, j \geq 0$, $M_i := \sum_{l=0}^i m_l$, $T_j := \sum_{l=0}^j t_l$. Finally, set $M_{-1} := 0$. Let $\xi \in E_1$ and for all $i \geq 1$, denote by $X(i) \in \{0, 1\}$ the i -th letter in ξ . The output of the following algorithm is a triple $(k, \{m_l\}_{l \geq 0}, \{t_l\}_{l \geq 0})$ satisfying the assumptions of the proposition.

procedure(ξ)

$k := \min\{n \geq 1; X(n) = 1\}$

$M_{-1} := 0$

if $X(k+1) = 0$ **then**

for $j = 0$ **to** " ∞ "

$m_j := 2$

$n := k + M_{j-1} + T_j + 3$

while $X(n) = 0$

$m_j := m_j + 2$

$n := n + 2$

end while

$M_j := M_{j-1} + m_j$

output m_j

$t_{j+1} := 0$

while $X(n) = 1$

$t_{j+1} := t_{j+1} + 1$

$n := n + 1$

end while

$T_{j+1} := T_j + t_{j+1}$

output t_{j+1}

end for

if $X(k+1) = 1$ **then**

$m_0 := 0$

$M_0 := 0$

for $j = 1$ **to** " ∞ "

$t_j := 1$

$n := k + M_{j-1} + T_{j-1} + 2$

while $X(n) = 1$

$t_j := t_j + 1$

$n := n + 1$

end while

$T_j := T_{j-1} + t_j$

output t_j

$m_j := 0$

while $X(n) = 0$

$m_j := m_j + 2$

$n := n + 2$

end while

$M_j := M_{j-1} + m_j$

output m_j

end for

end procedure

The second part of the proposition follows identically after we put $t_{l_0+1} = \infty$. \square

The main result of this subsection is the following characterization of words in E_1 with isomorphic unrooted Schreier graphs.

Theorem 5.4. *Let $\xi, \eta \in E_1$.*

1. *If $\xi = u1^\omega$ and $\eta = w1^\omega$ with $u, w \in \{0, 1\}^*$, then $\Gamma_\xi = \Gamma_\eta$. If $\xi = u1^\omega$ but $\eta \neq w1^\omega$ for any $w \in \{0, 1\}^*$, then $\Gamma_\xi \not\cong \Gamma_\eta$.*
2. *Suppose that $\xi \neq u1^\omega$ and $\eta \neq w1^\omega$ for any $u, w \in \{0, 1\}^*$; and consider the respective triples $(k, \{m_l\}_{l \geq 0}, \{t_l\}_{l \geq 0})$ and $(k', \{m'_l\}_{l \geq 0}, \{t'_l\}_{l \geq 0})$ that give the canonical representation of ξ and η*

$$\xi = 0^{k-1}1(0x_1^0 \dots 0x_{m_0/2}^0)1^{t_1}(0x_1^1 \dots 0x_{m_1/2}^1)1^{t_2} \dots,$$

$$\eta = 0^{k'-1} 1(0y_1^0 \dots 0y_{m'_0/2}^0) 1^{t'_1} (0y_1^1 \dots 0y_{m'_1/2}^1) 1^{t'_2} \dots$$

where $x_i^j, y_i^j \in \{0, 1\}$. Then $\Gamma_\eta \simeq \Gamma_\xi$ if and only if there exist $r, s \geq 1$ such that, for each $n \geq 0$,

- (a) $m_{r+n} = m'_{s+n}, t_{r+n+1} = t'_{s+n+1}$;
- (b) $k + T_r + M_{r-1} = k' + T'_s + M'_{s-1}$, where $M_i = \sum_{l=0}^i m_l$ and $T_j = \sum_{l=0}^j t_l$.
- (c) either $y_p^{s+n} = x_p^{r+n}$ or $y_p^{s+n} = 1 - x_p^{r+n}$, for $p = 1, \dots, m_{r+n}/2$.

In order to prove Theorem 5.4, we need to study in more detail the geometry of the limit graphs Γ_ξ . It turns out that it is determined by the infinite cycle-path \mathcal{CP}_ξ in Γ_ξ . In Propositions 5.6 and 5.8 below we describe these infinite cycle-paths completely by listing the lengths of the cycles and finding their relative position, in terms of the infinite ray $\xi \in E_1$ represented in the canonical form (6).

Definition 5.5. Given $\xi \in E_1$, we associate with it the sequence of integers $a_i = a_i^\xi, i \geq 1$, defined as follows:

- if $\xi = 1^\omega$, then $a_i := i$ for all $i \geq 1$;
- if $\xi \neq 1^\omega$, then Proposition 5.2 provides a triple $(k, \{m_l\}, \{t_l\})$ associated with ξ . Define for all $j \geq 1, 0 \leq s < t_j$,

$$a_{T_{j-1}+s+1} := k + M_{j-1} + T_{j-1} + s. \quad (8)$$

Proposition 5.6. Given $\xi \in E_1$, let $\mathcal{CP}_\xi = \{P_i\}_{i \geq 1}$ be the infinite cycle-path associated with ξ . Then, for all $i \geq 1, |P_i| = 2^{\lceil \frac{a_i}{2} \rceil}$.

Proof. By induction on i : if $i = 1$, then by definition, $a_1 = k + m_0$, that is the index of stability of ξ (see (2) and Corollary 5.3). In other words, the prefix ξ_{k+m_0} of ξ is situated on the central cycle of Γ_{k+m_0} . Thus, $\mathcal{CP}_{\xi_{k+m_0}}$ is constituted of a single cycle of length $2^{\lceil \frac{k+m_0}{2} \rceil} = 2^{\lceil \frac{a_1}{2} \rceil}$. Moreover, since $\xi_{a_1+1} = \xi_{a_1} 1$, then this cycle is the first one of the cycle-path \mathcal{CP}_ξ (see Remark 3.11).

Suppose that the statement is true for i ; the cycle P_i of \mathcal{CP}_ξ is then isomorphic to the central cycle of Γ_{a_i} . Consider the prefix ξ_{a_i} of ξ as well as the cycle-path $\mathcal{CP}_{\xi_{a_i}}$ in Γ_{a_i} . Let $j \geq 1, 0 \leq s < t_j$ be such that $T_{j-1} + s + 1 = i$ so that $a_i = a_{T_{j-1}+s+1}$. We distinguish two cases:

- if $s \leq t_j - 2$, then $a_{i+1} = (a_{T_{j-1}+s+2} =) k + M_{j-1} + T_{j-1} + s + 1 = a_i + 1$. On one hand, $\xi_{a_{i+1}} = \xi_{a_i} 1$, so that the central cycle of Γ_{a_i} is isomorphic to the penultimate cycle of the cycle-path $\mathcal{CP}_{\xi_{a_{i+1}}}$ in $\Gamma_{a_{i+1}}$. On the other hand, $\xi_{a_{i+2}} = \xi_{a_{i+1}} 1$ which implies that the central cycle of $\Gamma_{a_{i+1}}$ is isomorphic to the penultimate cycle of the cycle-path $\mathcal{CP}_{\xi_{a_{i+2}}}$ (see Remark 3.11). Hence, by induction hypothesis, the penultimate cycle of $\mathcal{CP}_{\xi_{a_{i+2}}}$ is the $(i+1)$ -st cycle of $\mathcal{CP}_{\xi_{a_{i+2}}}$ which is isomorphic (by Lemma 4.7) to the $(i+1)$ -st cycle of \mathcal{CP}_ξ . Its length is $|P_{i+1}| = 2^{\lceil \frac{a_{i+1}}{2} \rceil} = 2^{\lceil \frac{a_i+1}{2} \rceil}$.
- if $s = t_j - 1$, then $a_{i+1} = (a_{T_{j-1}+s+2} =) k + M_{j-1} + T_j + m_j = a_i + 1 + m_j$. Moreover, $\xi_{a_{i+1}+1} = \xi_{a_{i+1}} 1$. Thus, as in the previous case, the last cycle of $\mathcal{CP}_{\xi_{a_{i+1}}}$ is isomorphic to the $(i+1)$ -st cycle of \mathcal{CP}_ξ . By induction hypothesis, its length is

$$|P_{i+1}| = \begin{cases} |P_i| \cdot 2^{\frac{m_j}{2}+1} = 2^{\frac{a_i}{2} + \frac{m_j}{2} + 1} = 2^{\frac{a_{i+1}+1}{2}} = 2^{\lceil \frac{a_{i+1}}{2} \rceil} & \text{if } a_i \text{ is even,} \\ |P_i| \cdot 2^{\frac{m_j}{2}} = 2^{\frac{a_i+1}{2} + \frac{m_j}{2}} = 2^{\frac{a_{i+1}}{2}} = 2^{\lceil \frac{a_{i+1}}{2} \rceil} & \text{if } a_i \text{ is odd.} \end{cases}$$

□

Given $\xi \in E_1$, for any $n \geq 1$, consider the cycle path $\mathcal{CP}_{\xi_n} = \{P_i^n\}_{i=1}^t$ in Γ_n . Set $v_0^n := \xi_n$ and $\{v_i^n\} := P_i^n \cap P_{i+1}^n$ for every $i = 1, \dots, t-1$. By Lemma 4.7, the limit $v_i := \lim_{n \rightarrow \infty} v_i^n$ exists for each $i \geq 1$; $\{v_i\} = P_i \cap P_{i+1}$ where $P_i, P_{i+1} \in \mathcal{CP}_\xi$. Since we consider the graph Γ_n embedded in the plane in such a way that every P_i^n is a regular polygon, we may introduce the notion of an angle between two consecutive polygons, as follows.

Definition 5.7. Let $\xi \in E_1$. For any $n \geq 1$, consider $\mathcal{CP}_{\xi_n} = \{P_i^n\}_{i=1}^t$. For $1 \leq i \leq t-1$, α_i^n is the (counterclockwise) angle between vectors $\overrightarrow{c_i^n v_{i-1}^n}$ and $\overrightarrow{c_i^n v_i^n}$ (see Fig. 11). The limit $\alpha_i := \lim_{n \rightarrow \infty} \alpha_i^n$ is well defined for each $i \geq 1$; we call α_i the angle between cycles P_{i-1} and P_i (in \mathcal{CP}_{ξ}).

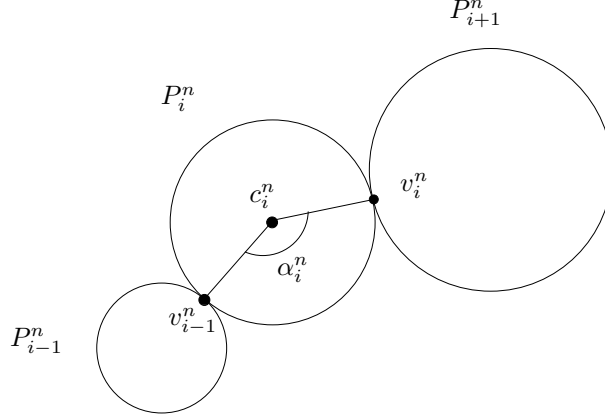


Fig. 11.

The next proposition shows that, given $\xi \in E_1$, the angle α_i can be expressed, for each $i \geq 1$, in terms of the canonical representation (6) of ξ .

Proposition 5.8. Let $\xi \in E_1$ and let $(k, \{m_l\}, \{t_l\})$ be the triple provided by Proposition 5.2. Let $\{a_i\}_{i \geq 1}$ be the sequence associated with ξ as in Definition 5.5. Then,

- if $m_0 = 0$, then $\alpha_1 = \pi$, whereas if $m_0 > 0$, then $\alpha_1 = \pi/2^{m_0/2} + \sum_{h=1}^{m_0/2} x_h \pi/2^{m_0/2-h}$;
- if $i > 1$ and $a_i - a_{i-1} = 1$, then $\alpha_i = \pi$;
- if $i > 1$ and $a_i - a_{i-1} > 1$, then there exists $d \geq 1$ such that $a_i - a_{i-1} = m_d + 1$. In this case $\alpha_i = \pi/2^{m_d/2} + \sum_{h=1}^{m_d/2} x_h^d \pi/2^{m_d/2-h}$.

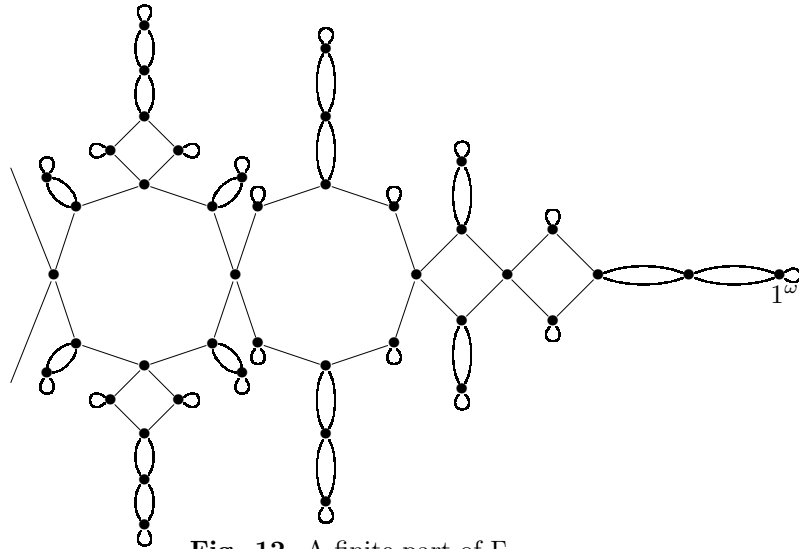


Fig. 12. A finite part of Γ_{1^ω} .

Proof. Suppose that $\xi \in E_1$ is such that $\xi \neq w1^\omega$ for any $w \in \{0, 1\}^*$ (the other case is treated similarly). By Corollary 5.3, the SEC-sequence associated with ξ is $S^{k+m_0} E^{t_1}(EC)^{\frac{m_1}{2}} E^{t_2}(EC)^{\frac{m_2}{2}} \dots E^{t_d}(EC)^{\frac{m_d}{2}} \dots$. If $a_i - a_{i-1} = 1$, then there exists $n \geq 0$ such that the a_i -th and $(a_i + 1)$ -st letter in (7) belong to the block E^{t_n} and $\xi_{a_i+1} = \xi_{a_i}1 = \xi_{a_i-1}11$. Hence, $\alpha_i = \pi$.

If $a_i - a_{i-1} > 1$, then there exists $d \geq 1$ such that $a_i - a_{i-1} = m_d + 1$. More precisely, we have $\xi_{a_i} = \xi_{a_{i-1}}10x_1^d 0x_2^d \dots 0x_{\frac{m_d}{2}}^d$, where $x_j^d \in \{0, 1\}$. Corollary 5.3 tells us that every letter x_j^d corresponds to a contraction. The value of x_j^d determines whether, after the contraction, $\xi_{a_{i-1}+1+2j}$ is in the upper or in the lower part of the diagram in Fig. 6.

Fix $i \geq 2$. We prove by induction on m_d , that

$$\alpha_i = \frac{\pi}{2^{\frac{m_d}{2}}} + \sum_{h=1}^{\frac{m_d}{2}} \frac{\pi}{2^{\frac{m_d}{2}-h}} x_h^d.$$

If $m_d = 2$, then (6) becomes

$$\xi = 0^{k-1}1(0x_1^0 \dots 0x_{\frac{m_0}{2}}^0)1^{t_1}(0x_1^1 \dots 0x_{\frac{m_1}{2}}^1)1^{t_2}(0x_1^2 \dots 0x_{\frac{m_2}{2}}^2) \dots 1^{t_d}(0x_1^d)1^{t_{d+1}} \dots$$

and (7) becomes

$$S^{k+m_0} E^{t_1}(EC)^{\frac{m_1}{2}} E^{t_2}(EC)^{\frac{m_2}{2}} \dots E^{t_d}(EC)E^{t_{d+1}} \dots$$

We look at how P_{i-1} is attached to P_i in \mathcal{CP}_ξ . Consider the finite graph $\Gamma_{a_{i-1}+1}$; since

$$\xi_{a_{i-1}+1} = 0^{k-1}1(0x_1^0 \dots 0x_{\frac{m_0}{2}}^0)1^{t_1}(0x_1^1 \dots 0x_{\frac{m_1}{2}}^1)1^{t_2}(0x_1^2 \dots 0x_{\frac{m_2}{2}}^2) \dots 1^{t_d}$$

$\xi_{a_{i-1}+1}$ is situated in the right part of $\Gamma_{a_{i-1}+1}$:

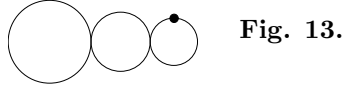


Fig. 13.

Then, since $\xi_{a_{i-1}+2} = \xi_{a_{i-1}+1}0$, we get

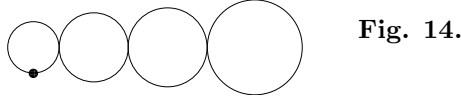


Fig. 14.

Suppose now that $x_1^d = 0$. (If $x_1^d = 1$ we get a symmetric picture with the decoration growing downwards.) We have the following picture for $\mathcal{CP}_{\xi_{a_{i-1}+3}} \subset \Gamma_{a_{i-1}+3}$:

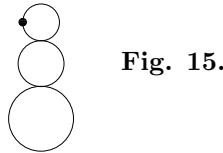


Fig. 15.

Finally, adding 1 (corresponding to the first letter of the block $E^{t_{d+1}}$) to the right of $\xi_{a_{i-1}+3}$, we get

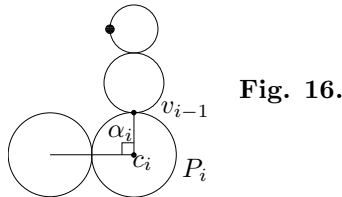


Fig. 16.

It follows that $\alpha_i = \frac{\pi}{2} = \frac{\pi}{2^1} + \sum_{h=1}^1 \frac{\pi}{2^{m_d-h}} \cdot 0$.

Suppose now that the assertion is true for $m_d - 2$; consider the graph Γ_{a_i-2} and

$$\xi_{a_i} = 0^{k-1} 1 (0x_1^0 \dots 0x_{m_0/2}^0) 1^{t_1} (0x_1^1 \dots 0x_{m_1/2}^1) 1^{t_2} (0x_1^2 \dots 0x_{m_d/2}^2) \dots 1^{t_d} (0x_1^d \dots 0x_{(m_d-2)/2}^d) 0x_{m_d/2}^d.$$

For $\mathcal{CP}_{\xi_{a_i-2}} \subset \Gamma_{a_i-2}$, define the angle α' as shown on Fig. 17. Then $\alpha' = \bar{\alpha}_i$ where $\bar{\alpha}_i$ is the i -th angle in the infinite cycle-path (as in Definition 5.7) for any $\bar{\xi} \in E_1$ with $\bar{\xi}_{a_i-1} = \xi_{a_i-2}$ (i.e., $\bar{a}_i - \bar{a}_{i-1} = (m_d - 2) + 1 = m_d - 1$.) Therefore we can apply the induction hypothesis to α' , and we have $\alpha' = \pi/2^{m_d/2-1} + \sum_{h=1}^{m_d/2-1} x_h^d \pi/2^{m_d/2-h-1}$.

The following picture explains how α_i is computed from α' . The two cases correspond to $x_{m_d/2}^d$ being 0 or 1.

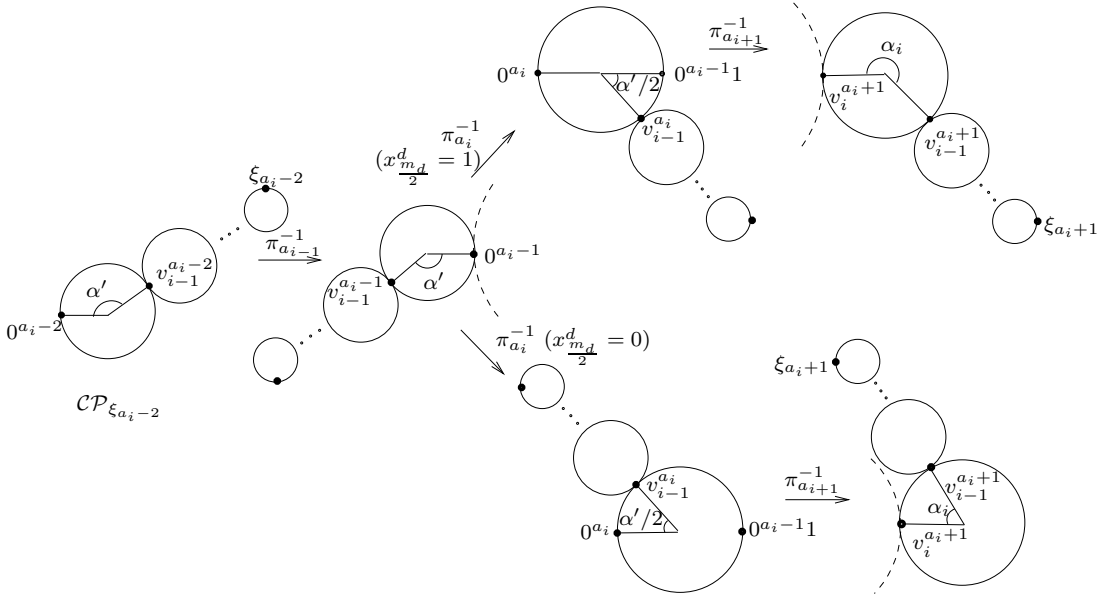


Fig. 17.

We compute, using the induction hypothesis,

$$\begin{aligned} \alpha_i &= \frac{\alpha'}{2} (+\pi) = \frac{\pi}{2^{\frac{m_d}{2}}} + \sum_{h=1}^{\frac{m_d}{2}-1} \frac{\pi}{2^{\frac{m_d}{2}-h}} x_h^d (+\pi) \\ &= \frac{\pi}{2^{\frac{m_d}{2}}} + \sum_{h=1}^{\frac{m_d}{2}} \frac{\pi}{2^{\frac{m_d}{2}-h}} x_h^d. \end{aligned}$$

The case $i = 1$ can be treated similarly. □

The following statement is easily checked by a direct computation:

Lemma 5.9. *Let $\alpha_i = \frac{\pi}{2^{\frac{m_d}{2}}} + \sum_{h=1}^{\frac{m_d}{2}} \frac{\pi}{2^{\frac{m_d}{2}-h}} x_h^d$ and $\beta_i = \frac{\pi}{2^{\frac{m_d}{2}}} + \sum_{h=1}^{\frac{m_d}{2}} \frac{\pi}{2^{\frac{m_d}{2}-h}} y_h^d$ where m_d is even positive and $x_h^d, y_h^d \in \{0, 1\}$. Then $\alpha_i = \beta_i$ if and only if $x_h^d = y_h^d$ for each $h = 1, \dots, \frac{m_d}{2}$ and $\alpha_i + \beta_i = 2\pi$ if and only if $y_h^d = 1 - x_h^d$ for each $h = 1, \dots, \frac{m_d}{2}$.*

Example 5.10. Consider $\xi = 1(1100)^\omega$. The SEC-sequence associated with ξ is $S(E^2(EC))^\omega$ (so that $k = 1$, $m_0 = 0$ and $t_i = m_i = 2$ for every $i \geq 1$). The sequence $\{a_i\}$ satisfy then

$$a_i = \begin{cases} 2i - 1 & \text{if } i \text{ is odd,} \\ 2i - 2 & \text{if } i \text{ is even.} \end{cases}$$

For each $i \geq 1$ one has $x_1^i = 0$. Moreover, $a_{2k} - a_{2k-1} = 1$ and $a_{2k+1} - a_{2k} = 3$ for each $k \geq 1$. This gives, by Proposition 5.8, $\alpha_1 = \pi$, $\alpha_{2k} = \pi$ and $\alpha_{2k+1} = \frac{\pi}{2}$ for every $k \geq 1$ (see Fig. 18).

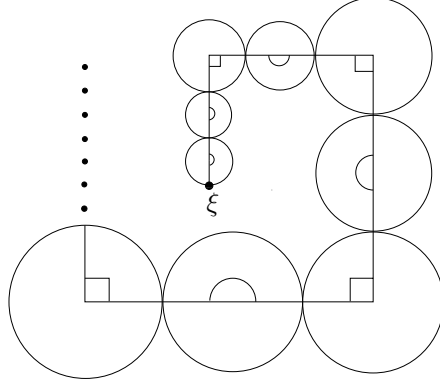


Fig. 18.

Proof of Theorem 5.4. Part 1. Suppose that $\xi = v1^\omega$ for some $v \in \{0, 1\}^*$ and consider the associated sequence $\{a_i^\xi\}_{i \geq 1}$. If $\eta = w1^\omega$, $w \in \{0, 1\}^*$, then Proposition 2.4 implies that $\Gamma_\xi = \Gamma_\eta$. If $\eta \neq w1^\omega$ for any $w \in \{0, 1\}^*$, then there exist infinitely many indices i such that $a_{i+1}^\eta - a_i^\eta > 1$. Indeed, if $\eta \neq w1^\omega$, then by Proposition 5.2, there exist infinitely many indices $l \geq 1$ such that $m_l > 0$ in the sequence $\{m_l\}_{l \geq 1}$ associated with η . By definition of the a_i 's (Definition 5.5), $a_{T_{l+1}} - a_{T_{l-1} + t_l} = m_l + 1 > 1$.

Hence, there are no indices i, j such that $a_{i+n}^\xi = a_{j+n}^\eta$ for all $n \geq 0$. Proposition 5.6 together with Lemma 5.1 imply that the graphs Γ_ξ and Γ_η are not isomorphic.

Part 2. Let $\xi \neq v1^\omega$ and $\eta \neq w1^\omega$ and suppose that there exist r, s such that (a), (b), and (c) hold. It follows from Definition 5.5, that there exist i, j such that $a_{i+n}^\xi = a_{j+n}^\eta$, for all $n \geq 0$. Writing $\Gamma_\xi \supset \mathcal{CP}_\xi = \{P_h\}_{h \geq 1}$ and $\Gamma_\eta \supset \mathcal{CP}_\eta = \{P'_h\}_{h \geq 1}$, let $\{v_i\} = P_i \cap P_{i+1}$ and similarly $\{v'_j\} = P'_j \cap P'_{j+1}$. Consider the decoration X_i (respectively X'_j) attached to v_i (respectively v'_j). Since $a_i^\xi = a_j^\eta$, these decorations are isomorphic via an isomorphism $\phi_0 : X_i \rightarrow X'_j$. We can extend the isomorphism $\phi_0 : X_i \rightarrow X'_j$ to an isomorphism $\phi_1 : X_{i+1} \rightarrow X'_{j+1}$ where X_{i+1} (respectively X'_{j+1}) is the decoration attached to v_{i+1} (respectively v'_{j+1}). Clearly, $X_i \subset X_{i+1}$ (respectively $X'_j \subset X'_{j+1}$).

Indeed, by Proposition 5.8 and Lemma 5.9, writing $\{v_{i+1}\} = P_{i+1} \cap P_{i+2}$ and $\{v'_{j+1}\} = P'_{j+1} \cap P'_{j+2}$, one has $d_{\Gamma_\xi}(v_i, v_{i+1}) = d_{\Gamma_\eta}(v'_j, v'_{j+1})$. Note that decorations of v_{i+1} and v'_{j+1} are both $(a_{i+1}^\xi + 1)$ -decorations which means that they are both isomorphic to the decoration $\mathcal{D}(0^{a_{i+1}^\xi + 1})$ of $0^{a_{i+1}^\xi + 1}$ in $\Gamma_{a_{i+1}^\xi + 1}$. This decoration is encoded by the diagram $\bar{D}_{a_{i+1}^\xi}$ where boundary vertices correspond to v_{i+1} and v'_{j+1} respectively. By Proposition 3.6, any two vertices situated at the same distance from the boundary vertices of the diagram have isomorphic neighbourhoods.

Repeating the above argument, one can construct a sequence of isomorphisms $\{\phi_n\}_{n \geq 0}$, $\phi_n : X_{i+n} \rightarrow X'_{j+n}$, such that for every $n' > n$, the restriction of $\phi_{n'}$ to X_{i+n} is equal to ϕ_n . Thus, there is a well-defined isomorphism $\phi : \Gamma_\xi \rightarrow \Gamma_\eta$ given by $\phi := \lim_{n \rightarrow \infty} \phi_n$.

Conversely, if we suppose that the conditions (a), (b), and (c) are not all satisfied, then three possibilities may occur.

- There are no indices i, j such that $a_{i+n}^\xi = a_{j+n}^\eta$ for all $n \geq 0$. By Proposition 5.6 and Lemma 5.1, the graphs Γ_ξ and Γ_η are not isomorphic.

- There exist indices i, j such that $a_{i+n}^\xi = a_{j+n}^\eta$ for all $n \geq 0$, but there are infinitely many indices d such that the condition $y_h^d = x_h^d$ or $y_h^d = 1 - x_h^d$ for each $h = 1, \dots, m_d/2$ is not satisfied. This implies that there are infinitely many indices n such that $d(v_{i+n-1}, v_{i+n}) \neq d(v'_{j+n-1}, v'_{j+n})$. Indeed, it follows from Proposition 5.8, that for any $d \geq 1$, the choice of the letters x_h^d , for $h = 1, \dots, \frac{m_d}{2}$, in the block $(0x_1^d \dots 0x_{\frac{m_d-2}{2}}^d)$ determines how P_{i-1} is attached to P_i . In particular, the choices x_h^d and $1 - x_h^d$ for each $h = 1, \dots, \frac{m_d}{2}$ correspond to vertices $\{v_{i-1}\} = P_{i-1} \cap P_i$ and $\{v'_{i-1}\} = P_{i-1} \cap P_i$ situated at the same distance from v_i .
On the other hand, an isomorphism ϕ between Γ_ξ and Γ_η must map v_{i+n} onto v'_{j+n} for each $n \geq 0$. This is a contradiction.
- Finally, suppose that (a) and (c) are satisfied but not (b). Let $r_0, s_0 > 2$ such that $m_{r_0+n-1} = m'_{s_0+n-1}$ and $t_{r_0+n} = t'_{s_0+n}$ for each $n \geq 0$. We prove that, for any $i, j \geq 1$, there exists $q \geq 0$ such that $a_{i+q}^\xi \neq a_{j+q}^\eta$. Without loss of generality, we fix i, j sufficiently large, so that there exist $r > r_0$, $0 \leq u < t_r$, $s > s_0$ and $0 \leq u' < t'_s$ such that $a_{i+q}^\xi = a_{T_{r-1}+u+1}^\xi$ and $a_{j+q}^\eta = a_{T'_{s-1}+u'+1}^\eta$. On the other hand, by Definition 5.5, for all $r > r_0$ and for all $0 \leq u < t_r$,

$$a_{T_{r-1}+u+1}^\xi = k + M_{r-1} + T_{r-1} + u = k + M_{r_0-2} + T_{r_0-1} + t_{r_0} + \dots + t_{r-1} + m_{r_0-1} + \dots + m_{r-1} + u;$$

similarly, for all $s > s_0$ and $0 \leq u' < t'_s$,

$$a_{T'_{s-1}+u'+1}^\eta = k' + M'_{s-1} + T'_{s-1} + u' = k' + M'_{s_0-2} + T'_{s_0-1} + t'_{s_0} + \dots + t'_{s-1} + m'_{s_0-1} + \dots + m'_{s-1} + u'.$$

Write $\delta := k' + M'_{s_0-2} + T'_{s_0-1} - k - M_{r_0-2} - T_{r_0-1}$. If $s = r$, then $a_{T'_{s-1}+u'+1}^\eta - a_{T_{r-1}+u+1}^\xi = \delta + u' - u$ whereas if $s > r$, then $a_{T'_{s-1}+u'+1}^\eta - a_{T_{r-1}+u+1}^\xi = \delta + t'_r + \dots + t'_{s-1} + m'_r + \dots + m'_{s-1} + u' - u$ where $\delta \neq 0$ by hypothesis. Observe that the sequences $\{m_l\}$ and $\{t_l\}$ (and hence $\{m'_l\}$ and $\{t'_l\}$) cannot be eventually both constant. Otherwise, we may find r_0, s_0 such that (b) would be satisfied which is a contradiction. Thus, since $u < t_r$, $u' < t'_s$, and $t'_s \geq 1$, $m'_{s-1} \geq 2$ for any $s > 1$, it is easy to check that there exists an index q such that $a_{j+q}^\eta - a_{i+q}^\xi = a_{T'_{s-1}+u'+1}^\eta - a_{T_{r-1}+u+1}^\xi \neq 0$. Hence, it follows from Proposition 5.6 and Lemma 5.1, that the graphs Γ_ξ and Γ_η are not isomorphic. □

5.2 Random weak limit of $\{\Gamma_n\}_{n \geq 1}$

Given a sequence of finite connected graphs $\{\Gamma_n\}_{n \geq 1}$, one can consider it as a sequence of random rooted graphs $\{(\Gamma_n, v_n)\}_{n \geq 1}$ by choosing for each $n \geq 1$ a root $v_n \in V(\Gamma_n)$ uniformly at random. Then, one says that (Γ, v) is the random weak limit (or distributional limit) of the sequence $\{(\Gamma_n, v_n)\}_{n \geq 1}$ as $n \rightarrow \infty$ if, for every $r > 0$ and for every finite rooted graph (H, o) , the probability that (H, o) is isomorphic to the ball in Γ_n centered in ξ_n and of radius r converges to the probability that (H, o) is isomorphic to the ball in Γ centered in ξ and of the same radius. This is equivalent to say that the law of (Γ_n, v_n) weakly converges to the law of (Γ, v) as probability measures on the space \mathcal{X} of connected rooted graphs (see Subsection 2.2).

If $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$ are the Schreier graphs of a spherically transitive action of a finitely generated group of automorphisms of a regular rooted tree T ; and if the root ξ_n is chosen uniformly at random in each Γ_n , then the random weak limit of this sequence is the uniform measure on the set $\{(\Gamma_\xi, \xi), \xi \in \partial T\}$ of rooted orbital Schreier graphs of the group action on ∂T .

Observe that, since the action is spherically transitive, it is ergodic on ∂T . This implies that almost all infinite Schreier graphs of a given group have the same number of ends. We discuss this typical value of the number of ends in more detail in the forthcoming paper [6]. Here, we show that in the case

of the Basilica group, the set $\{(\Gamma_\xi, \xi), \xi \in E_1\}$ is of full measure. Moreover, we also show that if we partition $\{(\Gamma_\xi, \xi), \xi \in E_1\}$ into classes of isomorphisms of *unrooted* graphs, then each isomorphism class is of measure 0.

So, we consider again the Basilica group B acting by automorphisms on the binary tree, and we denote by Γ_ξ , $\xi \in \{0, 1\}^\omega$, infinite Schreier graphs of the induced action of B on the boundary of the binary tree identified with the set of infinite binary words and equipped with the uniform measure λ . Recall that E_4 is the set of infinite words defining Schreier graphs with four ends, $E_{2,even}$ and $E_{2,odd}$ are the sets defining Schreier graphs with two ends and, respectively, with odd or even decorations. Finally, E_1 denotes the set of infinite words defining Schreier graphs with one end.

Proposition 5.11. *Almost every graph Γ_ξ has one end with respect to the uniform measure on the set $\{(\Gamma_\xi, \xi), \xi \in \{0, 1\}^\omega\}$ of rooted orbital Schreier graphs of the action of the Basilica group B on $\{0, 1\}^\omega$.*

Proof. We will show that $\lambda(E_1) = 1$, where λ is the uniform measure on $\{0, 1\}^\omega$. We have proven that $\{0, 1\}^\omega = E_4 \sqcup E_{2,even} \sqcup E_{2,odd} \sqcup E_1$. Each part is clearly invariant under B .

An infinite word $\xi \in \{0, 1\}^\omega$ is periodic if there exist w, u in $\{0, 1\}^*$ such that $\xi = wu^\omega$, where w is possibly empty. We denote by $\{0, 1\}^\omega_{per}$ the set of all periodic infinite binary words. One checks easily that $\lambda(\{0, 1\}^\omega_{per}) = 0$; indeed, $\{0, 1\}^\omega_{per}$ can be written as the union $\bigcup_{k=0}^\infty Y_k$ where $Y_k := \bigcup_{n=1}^\infty Y_{k,n}$ with $Y_{k,n} := \{wu^\omega \mid |w| = k, |u| = n\}$. Clearly, $\lambda(Y_{k,n}) = 0$, since this set contains 2^{k+n} words.

We have that $E_4 \subset \{0, 1\}^\omega_{per}$ and so $\lambda(E_4) = 0$. It is clear that $\lambda(E_{2,even}) = \lambda(E_{2,odd})$, and so, by ergodicity, $\lambda(E_{2,even}) = \lambda(E_{2,odd}) = 0$. It follows that $\lambda(E_1) = 1$. \square

Given $\xi \in \{0, 1\}^\omega$, write

$$I_\xi := \{\eta \in \{0, 1\}^\omega \mid \Gamma_\xi \text{ and } \Gamma_\eta \text{ are isomorphic as unrooted graphs}\}.$$

Proposition 5.12. *For any $\xi \in \{0, 1\}^\omega$, $\lambda(I_\xi) = 0$.*

Proof. If $\xi \in E_4 \sqcup E_2$, then the statement is obvious by the previous proposition. If $\xi \in E_1$, let $(k, \{m_i\}, \{t_i\})$ be the triple provided by Proposition 5.2. Denote by $W_{z, \{t_i\}, \{m_i\}} \subset E_1$ the set of words with triples satisfying $k + m_0 = z$. We show that $\lambda(W_{z, \{t_i\}, \{m_i\}}) = 0$. By Proposition 5.2, $W_{z, \{t_i\}, \{m_i\}}$ is constituted of words of the form

$$0^{k-1} 1 (0x_1^0 \dots 0x_{m_0/2}^0) 1^{t_1} (0x_1^1 \dots 0x_{m_1/2}^1) \dots 1^{t_l} (0x_1^l \dots 0x_{m_l/2}^l) \dots,$$

with $x_i^j \in \{0, 1\}$. From this, one has

$$\begin{aligned} \lambda(W_{z, \{t_i\}, \{m_i\}}) &= 1 - \sum_{i=1}^k \frac{1}{2^i} - \sum_{i=1}^{\frac{m_0}{2}} \frac{2^{i-1}}{2^{k+2i-1}} - \sum_{i=1}^{t_1} \frac{2^{\frac{m_0}{2}}}{2^{k+m_0+i}} - \sum_{i=1}^{\frac{m_1}{2}} \frac{2^{\frac{m_0}{2}+i-1}}{2^{k+m_0+t_1+2i-1}} \\ &\quad - \sum_{i=2}^{\infty} \left(\sum_{j=1}^{t_i} \frac{2^{\frac{M_i-1}{2}}}{2^{T_{i-1}+M_{i-1}+k+j}} + \sum_{h=1}^{\frac{m_i}{2}} \frac{2^{\frac{M_i-1}{2}+h-1}}{2^{T_i+M_{i-1}+k+2h-1}} \right). \end{aligned}$$

Setting $s_i := T_i + M_i/2 + k$, we get

$$\lambda(W_{z, \{t_i\}, \{m_i\}}) = \sum_{i=1}^k \frac{1}{2^i} - \sum_{i=1}^{\frac{m_0}{2}} \frac{1}{2^{k+i}} - \sum_{i=1}^{t_1} \frac{1}{2^{k+\frac{m_0}{2}+i}} - \sum_{i=1}^{\frac{m_1}{2}} \frac{1}{2^{k+\frac{m_0}{2}+t_1+i}} - \sum_{i=2}^{\infty} \left(\sum_{j=s_{i-1}+1}^{s_i} \frac{1}{2^j} \right) \quad (9)$$

$$= 1 - \sum_{i=1}^{\infty} \frac{1}{2^i} = 0. \quad (10)$$

For any $i \geq 1$, consider the set $I_i^\xi \subset E_1$ defined as follows: $\eta \in I_i^\xi$ if and only if i is the smallest integer such that the triple $(k', \{m'_i\}, \{t'_i\})$ associated with η satisfies $m'_{i+k} = m_{j+k}$, $t'_{i+k+1} = t_{j+k+1}$ and $z' + T'_i + M'_{i-1} = z + T_j + M_{j-1}$ for some $j \geq 1$ and each $k \geq 0$. By Theorem 5.4, $I_\xi \subset \bigcup_{i=1}^\infty I_i^\xi$. Since for any $\xi \in \{0, 1\}^\omega$, $\lambda(\{w\xi \mid w \in \{0, 1\}^i\}) = 0$, (9) implies that $\lambda(I_i^\xi) = 0$ for every $i \geq 1$. \square

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