

ON THE q -EXTENSION OF HIGHER-ORDER EULER POLYNOMIALS

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ABSTRACT. The purpose of this paper is to present a systemic study of some families of the generalized q -Euler numbers and polynomials of higher-order. In particular, by using multivariate p -adic invariant integral on \mathbb{Z}_p , we construct the generalized q -Euler numbers and polynomials of higher-order.

1. Introduction

Let p be a fixed odd prime and let \mathbb{Z}_p , \mathbb{Q}_p , \mathbb{C} and \mathbb{C}_p denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field and the completion of algebraic closure of \mathbb{Q}_p . The p -adic absolute value in \mathbb{C}_p is normalized so that $|p|_p = \frac{1}{p}$. For d a fixed positive odd integer with $(p, d) = 1$, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [3-19]).

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When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < 1$. In this paper we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Let χ be the Dirichlet's character with conductor $d (= odd) \in \mathbb{N}$. Then the generalized Euler polynomials, $E_{n,\chi}(x)$, are defined as

$$(1) \quad F_\chi(x, t) = \frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{e^{dt} + 1} e^{xt} = \sum_{l=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}, \text{ (see [3, 6])}.$$

We note that, by substituting $x = 0$ in (1), $E_{n,\chi}(0) = E_{n,\chi}$ is the familiar n -th Euler number defined by

$$F_\chi(0, t) = \frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{e^{dt} + 1} = \sum_{l=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.$$

For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\sum_{0 \leq j < p^N} (-1)^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu(j + p^N \mathbb{Z}_p)$$

representing analogue of Riemann's sums for f , cf.[1-10].

The fermionic p -adic invariant integral of f on \mathbb{Z}_p will be defined as the limit ($N \rightarrow \infty$) of these sums, which it exists. The fermionic p -adic invariant integral of a function $f \in UD(\mathbb{Z}_p)$ is defined in [1, 3, 5, 7, 10] as follows:

$$(2) \quad I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \rightarrow \infty} \sum_{0 \leq j < p^N} f(j) \mu(j + p^N \mathbb{Z}_p) = \lim_{N \rightarrow \infty} \sum_{0 \leq j < p^N} f(j) (-1)^j.$$

Thus, we have

$$I(f_1) + I(f) = 2f(0), \text{ where } f_1(x) = f(x + 1).$$

By using integral iterative method, we also easily see that

$$(3) \quad I(f_n) + (-1)^{n-1} I(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \text{ where } f_n(x) = f(x + n) \text{ for } n \in \mathbb{N}.$$

From (3), we note that

$$(4) \quad \int_X \chi(x) e^{xt} d\mu(x) = \frac{2 \sum_{l=0}^{d-1} (-1)^l e^{lt} \chi(l)}{e^{dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.$$

By (4), we see that

$$(5) \quad \int_X \chi(x) x^n d\mu(x) = E_{n,\chi}, \text{ and } \int_X \chi(y) (x+y)^n d\mu(y) = E_{n,\chi}(x), \text{ (see [6])}.$$

The n -th generalized Euler polynomials of order k , $E_{n,\chi}^{(k)}(x)$, are defined as

$$(6) \quad \left(\frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{e^{dt} + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}^{(k)}(x) \frac{t^n}{n!}, \text{ (see [6, 7])}.$$

In the special case $x = 0$, $E_{n,\chi}^{(k)}(0) = E_{n,\chi}^{(k)}$ are called the n -th generalized Euler numbers of order k . Now, we consider the multivariate p -adic invariant integral on \mathbb{Z}_p as follows:

$$(7) \quad \begin{aligned} & \int_X \cdots \int_X \chi(x_1) \cdots \chi(x_k) e^{(x_1 + \cdots + x_k + x)t} d\mu(x_1) \cdots d\mu(x_k) \\ &= \left(\frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{e^{dt} + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}^{(k)}(x) \frac{t^n}{n!}. \end{aligned}$$

By (6) and (7), we obtain the Witt's formula for the n -th generalized Euler polynomials of order k as follows:

$$(8) \quad \int_X \cdots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) (x_1 + \cdots + x_k + x)^n d\mu(x_1) \cdots d\mu(x_k) = E_{n,\chi}^{(k)}(x).$$

In the viewpoint of the q -extension of (8), we will consider the q -extension of generalized Euler numbers and polynomials of order k . The purpose of this paper is to present a systemic study of some families of the generalized q -Euler numbers and polynomials of higher-order. In particular, by using multivariate p -adic invariant integral on \mathbb{Z}_p , we construct the generalized q -Euler numbers and polynomials of higher-order.

2. On the q -extension of higher-order Euler numbers and polynomials

In this section we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let χ be the Dirichlet's character with conductor d . For $h \in \mathbb{Z}, k \in \mathbb{N}$, let us consider the generalized q -Euler numbers and polynomials of order k in the viewpoint of the q -extension of (8). First, we consider the q -extension of (1) as follows:

$$(9) \quad \sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!} = \int_X e^{[x+y]_q t} \chi(y) d\mu(y) = 2 \sum_{m=0}^{\infty} \chi(m) (-1)^m e^{[m]_q t}, \quad (\text{cf. [1, 4]}).$$

By (9), we have

$$(10) \quad \begin{aligned} \int_X [x+y]_q^n \chi(y) d\mu(y) &= 2 \sum_{m=0}^{\infty} \chi(m) (-1)^m [m]_q^n \\ &= 2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{l(a+x)}}{1+q^{ld}}. \end{aligned}$$

From the multivariate p -adic invariant integral on \mathbb{Z}_p , we can also derive the q -extension of the generalized Euler polynomials of order k as follows:

$$(11) \quad \begin{aligned} E_{n,\chi,q}^{(k)}(x) &= \int_X \cdots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) [x_1 + \cdots + x_k + x]_q^n d\mu(x_1) \cdots d\mu(x_k) \\ &= \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \frac{2^k}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^k a_j)}}{(1+q^{dl})^k} \\ &= 2^k \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m [x + \sum_{j=1}^k a_j + dm]_q^n \\ &= 2^k \sum_{a_1, \dots, a_{k-1}=0}^{d-1} \left(\prod_{i=1}^{k-1} \chi(a_i) \right) (-1)^{\sum_{j=1}^{k-1} a_j} \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \chi(m) [x + \sum_{j=1}^{k-1} a_j + m]_q^n. \end{aligned}$$

Let $F_{q,\chi}^{(k)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(k)}(x) \frac{t^n}{n!}$. Then we have

$$(12) \quad \begin{aligned} F_{q,\chi}^{(k)}(t, x) &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(k)}(x) \frac{t^n}{n!} \\ &= 2^k \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m e^{t[x+\sum_{j=1}^k a_j + dm]_q}. \end{aligned}$$

From (12), we obtain the following theorem.

Theorem 1. *For $k \in \mathbb{N}, n \geq 0$, we have*

$$\begin{aligned} E_{n,\chi,q}^{(k)} &= \frac{2^k}{(1-q)^n} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^k a_j)}}{(1+q^{ld})^k} \\ &= 2^k \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m [x + \sum_{j=1}^k a_j + md]_q^n. \end{aligned}$$

For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, let us consider the extension of $E_{n,\chi,q}^{(k)}(x)$ as follows:

$$\begin{aligned} (13) \quad E_{n,\chi,q}^{(h,k)}(x) &= \int_X \cdots \int_X q^{\sum_{j=1}^k (h-j)x_j} \left(\prod_{j=1}^k \chi(x_j) \right) [x + \sum_{j=1}^k x_j]_q^n d\mu(x_1) \cdots d\mu(x_k) \\ &= \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} q^{\sum_{j=1}^k a_j(h-j)} \int_X \cdots \int_X q^{d \sum_{j=1}^k (h-j)x_j} \\ &\quad [x + \sum_{j=1}^k (dx_j + a_j)]_q^n d\mu(x_1) \cdots d\mu(x_k) \\ &= \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} q^{\sum_{j=1}^k a_j(h-j)} \frac{2^k}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^k a_j)}}{(-q^{d(h-k+l)} : q^d)_k}, \end{aligned}$$

where $(a : q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1})$, (see [1, 4]).

It is well known that the Gaussian binomial coefficient is defined as

$$(14) \quad \binom{n}{k}_q = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [2]_q \cdot [1]_q}, \text{ (see [1, 4])}.$$

By (13) and (14), we easily see that

$$\begin{aligned}
 (15) \quad E_{n,\chi,q}^{(h,k)}(x) &= \frac{2^k}{(1-q)^n} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{i=1}^k a_i} q^{\sum_{j=1}^k (h-j)a_j} \sum_{l=0}^n \binom{n}{l} (-1)^l \\
 &\quad q^{l(x+\sum_{j=1}^k a_j)} \sum_{m=0}^{\infty} \binom{m+k-1}{m}_{q^d} (-1)^m q^{d(h-k)m} q^{dlm} \\
 &= 2^k [d]_q^n \sum_{m=0}^{\infty} \binom{m+k-1}{m}_{q^d} (-1)^m q^{d(h-k)m} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \\
 &\quad q^{\sum_{j=1}^k (h-j)a_j} [m + \frac{x + \sum_{j=1}^k a_j}{d}]_{q^d}^n.
 \end{aligned}$$

Let $F_{\chi,q}^{(h,k)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,k)}(x) \frac{t^n}{n!}$. From (15), we note that

$$\begin{aligned}
 (16) \quad F_{\chi,q}^{(h,k)}(t, x) &= 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q (-1)^m q^{d(h-k)m} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} \\
 &\quad q^{\sum_{j=1}^k (h-j)a_j} e^{t[m d + x + \sum_{j=1}^k a_j]_q}.
 \end{aligned}$$

By (16), we obtain the following theorem.

Theorem 2. *For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, we have*

$$\begin{aligned}
 E_{n,\chi,q}^{(h,k)}(x) &= 2^k [d]_q^n \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q (-1)^m q^{d(h-k)m} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) \\
 &\quad (-1)^{\sum_{j=1}^k a_j} q^{\sum_{j=1}^k (h-j)a_j - j} [m + \frac{x + a_1 + a_2 + \dots + a_k}{d}]_{q^d}^n \\
 &= \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} q^{\sum_{j=1}^k (h-j)a_j} \frac{2^k}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^k a_j)}}{(-q^{d(h-k+l)} : q^d)_k}.
 \end{aligned}$$

For $h = k$, we have

$$\begin{aligned}
(17) \quad & E_{n,\chi,q}^{(k,k)}(x) \\
&= \frac{2^k}{(1-q)^n} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} q^{\sum_{j=1}^k (h-j)a_j} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(\sum_{j=1}^k a_j + x)}}{(-q^{ld} : q^d)_k} \\
&= 2^k [d]_q^n \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q (-1)^m \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-1)^{\sum_{j=1}^k a_j} q^{\sum_{j=1}^k (k-j)a_j} \\
&\cdot [m + \frac{x+a_1+a_2+\dots+a_k}{d}]_{q^d}^n.
\end{aligned}$$

It is not difficult to show that

$$\begin{aligned}
(18) \quad & \int_X \cdots \int_X \left(\prod_{j=1}^k \chi(x_j) \right) q^{\sum_{j=1}^k (m-j)x_j + mx} d\mu(x_1) \cdots d\mu(x_k) = \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{j=1}^k \chi(a_j) \right) \\
& q^{mx + \sum_{j=1}^k (m-j)a_j} (-1)^{\sum_{j=1}^k a_j} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{d \sum_{j=1}^k (m-j)x_j} d\mu(x_1) \cdots d\mu(x_k) \\
&= \frac{2^k q^{mx} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{j=1}^k \chi(a_j) \right) q^{\sum_{j=1}^k (m-j)a_j} (-1)^{\sum_{j=1}^k a_j}}{(-q^{d(m-k)} : q^d)_k}
\end{aligned}$$

From (18), we can derive the following equation (19).

$$\begin{aligned}
(19) \quad & \frac{2^k q^{mx} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{j=1}^k \chi(a_j) \right) q^{\sum_{j=1}^k (m-j)a_j} (-1)^{\sum_{j=1}^k a_j}}{(-q^{d(m-k)} : q^d)_k} \\
&= \int_X \cdots \int_X ([x+x_1+\dots+x_k]_q (q-1)+1)^m q^{-\sum_{j=1}^k j x_j} \left(\prod_{j=1}^k \chi(x_j) \right) d\mu(x_1) \cdots d\mu(x_k) \\
&= \sum_{l=0}^m \binom{m}{l} (q-1)^l \int_X \cdots \int_X \left(\prod_{j=1}^k \chi(x_j) \right) [x+x_1+\dots+x_k]_q^l q^{-\sum_{j=1}^k j x_j} d\mu(x_1) \cdots d\mu(x_k) \\
&= \sum_{l=0}^m \binom{m}{l} (q-1)^l E_{l,\chi,q}^{(0,k)}(x).
\end{aligned}$$

By (19), we obtain the following theorem.

Theorem 3. For $d, k \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\frac{2^k q^{mx} \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{j=1}^k \chi(a_j) \right) q^{\sum_{j=1}^k (m-j)a_j} (-1)^{\sum_{j=1}^k a_j}}{(-q^{d(m-k)} : q^d)_k} = \sum_{l=0}^m \binom{m}{l} (q-1)^l E_{l,\chi,q}^{(0,k)}(x).$$

From the definition of p -adic invariant integral on \mathbb{Z}_p , we note that
(20)

$$\begin{aligned} & q^{d(h-1)} \int_X \cdots \int_X [x + d + x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^k (k-j)x_j} \left(\prod_{j=1}^k \chi(x_j) \right) d\mu(x_1) \cdots d\mu(x_k) \\ &= - \int_X \cdots \int_X [x + x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^k (k-j)x_j} \left(\prod_{j=1}^k \chi(x_j) \right) d\mu(x_1) \cdots d\mu(x_k) + 2 \sum_{l=0}^{d-1} \chi(l) \\ & (-1)^l \int_X \cdots \int_X [x + \sum_{j=1}^{k-1} x_{j+1}]_q^n \left(\prod_{j=1}^{k-1} \chi(x_{j+1}) \right) q^{\sum_{j=1}^{k-1} (h-1-j)x_{j+1}} d\mu(x_2) \cdots d\mu(x_k). \end{aligned}$$

By (20), we obtain the following theorem.

Theorem 4. For $h \in \mathbb{Z}$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$(21) \quad q^{d(h-1)} E_{n,\chi,q}^{(h,k)}(x+d) + E_{n,\chi,q}^{(h,k)}(x) = 2 \sum_{l=0}^{d-1} \chi(l) (-1)^l E_{n,q}^{(h-1,k-1)}(x).$$

Moreover,

$$q^x E_{n,\chi,q}^{(h+1,k)}(x) = (q-1) E_{n+1,\chi,q}^{(h,k)}(x) + E_{n,\chi,q}^{(h,k)}(x).$$

Let

$$F_{\chi,q}^{(h,1)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,1)}(x) \frac{t^n}{n!}.$$

Then we have

$$(22) \quad F_{\chi,q}^{(h,1)}(t, x) = 2 \sum_{n=0}^{\infty} \chi(n) q^{(h-1)n} (-1)^n e^{[n+x]_q t}.$$

By (22), we see that

$$\begin{aligned} E_{n,\chi,q}^{(h,1)}(x) &= 2 \sum_{m=0}^{\infty} \chi(m) q^{(h-1)m} (-1)^m [m+x]_q^n \\ &= \frac{2}{(1-q)^n} \sum_{a_1=0}^{d-1} \chi(a_1) (-1)^{a_1} \sum_{l=0}^{d-1} \frac{\binom{n}{l} (-1)^l q^{l(x+a_1)}}{(1+q^{ld})}. \end{aligned}$$

REFERENCES

1. A. Aral, V. Gupta, *On the Durrmeyer type modification of the q -Baskakov type operators*, Non-linear Analysis (2009), doi:10.1016/j.na.2009.07.052.
2. M. Acikgoz, Y. Simsek, *On multiple interpolation functions of Nörlund-type q -Euler polynomials*, Abst. Appl. Anal. **2009** (2009), Art. ID 382574, P.14.
3. M. Can, M. Cenkci, V. Kurt, Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l -functions*, Adv. Stud. Contemp. Math. **18** (2009), 135-160.
4. M. Cenkci, *The p -adic generalized twisted (h, q) -Euler- l -function and its applications*, Adv. Stud. Contemp. Math. **14** (2007), 49-68.
5. N. K. Govil, V. Gupta, *Convergence of q -Meyer-König-Zeller-Durrmeyer operators*, Adv. Stud. Contemp. Math. **19** (2009), 97-108.
6. T. Kim, *q -Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), 288-299.
7. T. Kim, *Symmetry identities for the twisted generalized Euler polynomials*, Adv. Stud. Contemp. Math. **19** (2009), 151-155.
8. T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p -adic integral on \mathbb{Z}_p* , Russian J. Math. Phys. **16** (2009), 93-96.
9. V. Kurt, *A further symmetric relation on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials*, Appl. Math. Sciences. **3**, no.56 (2009), 2757-2764.
10. Y. H. Kim, K.-W. Hwang, *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math. **18** (2009), 127-133.
11. M. Cenkci, *The p -adic generalized twisted (h, q) -Euler- l -function and its applications*, Adv. Stud. Contemp. Math. **15** (2007), 37-47.
12. Z. Zhang, H. Yang, *Some closed formulas for generalizations of Bernoulli and Euler numbers and polynomials*, Proceedings of the Jangjeon Mathematical Society **11** (2008), 191-198.
13. H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on q -Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. **18** (2009), 41-48.
14. C.-P. Chen, L. Lin, *An inequality for the generalized Euler constant function*, Adv. Stud. Contemp. Math. **17** (2008), 105-107.
15. C. S. Ryoo, *Calculating zeros of the twisted Genocchi polynomials*, Adv. Stud. Contemp. Math. **17** (2008), 147-159.
16. M. Cenkci, Y. Simsek, V. Kurt, *Multiple two-variable p -adic q -L-function and its behavior at $s = 0$* , Russ. J. Math. Phys. **15** (2008), 447-459.
17. Y. Simsek, *On p -adic twisted q -L-functions related to generalized twisted Bernoulli numbers*, Russian J. Math. Phys. **13** (2006), 340-348.
18. M. Cenkci, M. Can, *Some results on q -analogue of the Lerch zeta function*, Adv. Stud. Contemp. Math. **12** (2006), 213-223.

19. M. Cenkci, M. Can, V. Kurt, *p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers*, Adv. Stud. Contemp. Math. **9** (2004), 203-216.