Accurate calculation of thermal noise in multilayer coating

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We derive accurate formulas for thermal fluctuations in multilayer interferometric coating taking into account light propagation inside the coating. In particular, we calculate the reflected wave phase as a function of small displacements of the boundaries between the layers using transmission line model for interferometric coating and derive formula for spectral density of reflected phase in accordance with Fluctuation-Dissipation Theorem. We apply the developed approach for calculation of the spectral density of coating Brownian noise.

I. INTRODUCTION

Thermal fluctuations in the mirror are becoming significant noise sources in the second generation gravitational-wave antennae (Advanced LIGO, Advanced VIRGO, HF GEO, TAMA) [1]. The pioneering articles on this issue dealt with Brownian fluctuations in *the body* of mirror [2–4]. Later the importance of thermoelastic noise [5] was realized especially for mirrors manufactured from sapphire. The physical reasons behind thermoelastic noise are provided by fundamental thermodynamic fluctuations of temperature manifesting through thermal expansion. The same reason produces the thermorefractive noise [6] through the mechanism of relation between the refractive index and temperature. These results were obtained for model of infinite test mass (the mirror was considered occupying semi-infinite elastic space). Later these results were generalized for the finite-size mirror [4, 7].

Soon the importance of thermal noise in the mirror *coating* was realized as the parameters of coating may differ considerably from the mirror parameters bulk. Thermoelastic noise in interferometric coating was calculated in [8, 9]. Later, following direction of H.J. Kimble [10] the potential of partial compensation of thermoelastic and thermorefractive noise in coatings was explored [11, 12]. However, today the Brownian noise in interferometric coating produces the main contribution into the noise spectrum [13–15] of gravitational wave antennae because the loss angles of substances used in coating (as usual, pair Ta_2O_5, SiO_2) are much greater than the loss angles in the same bulk materials.

As a rule the calculation of Brownian and thermoelastic noise in coating is reduced to calculation of fluctuations of the total coating thickness. Strictly speaking, the light partially travels inside the coating, however, its power exponentially decreases with the depth when the light travels inside the coating. It allows considering the light to be perfectly reflected from the outer surface of the coating. Usually, the calculation of thermal noise in coating is provided under assumption that light is reflected from the front surface of the coating *without* detailed analysis of the light propagating inside the coating [8, 9, 11–15].
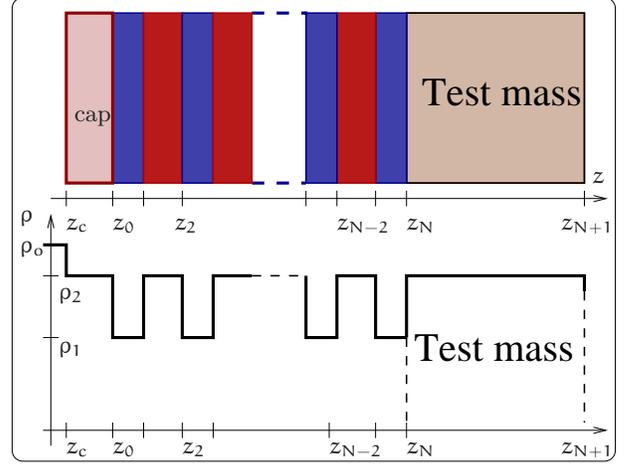
The purpose of this article is to make accurate calcu-

FIG. 1: Top: Conventional mirror with interferometric coating deposited on the face side. The coating consists of alternating quarter wavelength layers plus the outer half wavelength cap layer. Bottom: transmission line model with alternating quarter wavelength parts, $\rho_1 = \frac{1}{n_1}$ $\rho_2 = \frac{1}{n_2}$, $\rho_o = 1$.

lations of reflected wave phase taking into account propagation of light inside the coating. In particular, we calculate the reflected wave phase as a function of small displacements of the boundaries between the layers. In our analysis we use transmission line model for interferometric coating [16]. In Sec. II we calculate the dependence of the phase of the wave reflected from the mirror as function of small displacements of boundaries between interferometric layers of coating. In Sec. III we apply the formula obtained to calculate the spectral density of Brownian noise inside the coating to obtain numerical estimates. In Sec. IV we discuss obtained results.

II. CALCULATION OF REFLECTED WAVE PHASE

It is known that the interferometric coating consists of alternating quarter wave length layers with different refraction indices n_1, n_2 ($n_1 > n_2$) as shown on Fig. 1. Specifically, below we assume that coating consists of Ta_2O_5 (n_1) and fused silica (n_2) layers, the outer layer

(with positions z_0, z_1) is made from Ta_2O_5 and it is covered by additional half wave length layer (cap) which outer position is z_c. Such a coating is similar to transmission line consisting of alternating quarter wave length pieces with characteristic (wave) impedances $\rho_1 = 1/n_1$ and $\rho_2 = 1/n_2$ [16].

We assume that the test mass (the body of the mirror) is manufactured from fused silica, hence, the wave impedance between z_N and z_{N+1} is equal to $\rho_2 = 1/n_2$. The mean position z_i of the surface of each layer (see Fig. 1) corresponds to the quarter wave length thickness of each layer, however, small position fluctuations ζ_i (caused by thermal fluctuations) produce fluctuations in reflected wave phase. In this section we calculate the reflected wave phase as a function of small displacements ζ_i using the successive approximation technique.

A. Main formulas and the zeroth approximation

In our consideration we ignore the optical losses in materials of coating. We start with considering the last layer assuming that all fluctuation displacements ζ_i are equal to zero. We assume that the wave propagating in the positive direction of axis z is described as $\sim e^{-i\omega(t-z/c)}$. The equivalent impedance at position $z = z_{N+1}$ is equal to $Z_{N+1} = \rho_o = 1$ (vacuum). It is convenient to define the amplitude reflectivy R as

$$U_{refl} = R\, U_{incident}$$

Obviously, the amplitude reflectivity R_{N+1} at $z = z_{N+1}$ is equal to:

$$Z_{N+1} = \rho_o, \quad R_{N+1} = \frac{\rho_o - \rho_2}{\rho_2 + \rho_o} \quad (2.1)$$

Now we can calculate the effective impedance Z_N at position z_N — it allows considering the piece of transmission line between z_N and z_{N+1} as a single impedance Z_N

$$Z_N = \rho_2 \frac{1 + R_{N+1}\theta_{N+1}^2}{1 - R_{N+1}\theta_{N+1}^2}, \quad Z_N^{(0)} = \frac{\rho_2^2}{\rho_o}, \quad (2.2)$$

$$\theta_{N+1}^2 \equiv \exp\left[2ik_2(z_{N+1}-z_N)\right] = -\exp\left[2ik_2(\zeta_{N+1}-\zeta_N)\right],$$

$$k_{1,2} = k\frac{\rho_o}{\rho_{1,2}}, \quad k = \frac{2\pi}{\lambda}. \quad (2.3)$$

Here θ_{N+1} is the exponent describing phase advance of wave traveling between positions z_N and z_{N+1}, $k_{1,2}$ are the wave vectors, λ is the optical wavelength in vacuum. Here and below superscript $^{(0)}$ refers to the particular case of the zeroth approximation when fluctuations ζ_N, ζ_{N+1} are absent. We consider the test mass as one fused silica layer ("slightly" thicker). For convenience we assume its thickness being a fold to the quarter wave length (to meet the condition of anti resonance) — it means $\theta_{N+1}^2 = -1$. We make this assumption to make final formula more compact, without it the final result

does not change practically but formulas looks more akward.

In the similar manner we successively calculate reflectivity R_N, impedance Z_{N-1} and so on:

$$R_N = \frac{Z_N - \rho_1}{Z_N + \rho_1}, \quad R_N^{(0)} = \frac{\frac{\rho_2^2}{\rho_1\rho_o}-1}{1 + \frac{\rho_2^2}{\rho_1\rho_o}} \quad (2.4a)$$

$$Z_{N-1} = \rho_1\frac{1 + R_N\theta_N^2}{1 - R_N\theta_N^2}, \quad Z_{N-1}^{(0)} = \frac{\rho_2^2}{\rho_o} = \rho_o\frac{\rho_1^2}{\rho_2^2}, \quad (2.4b)$$

$$\theta_N^2 \equiv -\exp\left[2ik_1(\zeta_N - \zeta_{N-1})\right], \quad (2.4c)$$

$$R_{N-1} = \frac{Z_{N-1}-\rho_2}{Z_{N-1}+\rho_2}, \quad R_{N-1}^{(0)} = \frac{\rho_1^2\rho_o/\rho_2^3-1}{1+\rho_1^2\rho_o/\rho_2^3}, \quad (2.4d)$$

$$Z_{N-2} = \rho_2\frac{1+R_{N-1}\theta_{N-1}^2}{1-R_{N-1}\theta_{N-1}^2}, \quad Z_{N-2}^{(0)} = \frac{\rho_2^2}{\rho_o}\left(\frac{\rho_2^2}{\rho_1^2}\right), \quad (2.4e)$$

$$R_{N-2} = \frac{Z_{N-2}-\rho_1}{Z_{N-2}+\rho_1}, \quad R_{N-2}^{(0)} = \frac{\frac{\rho_2^4}{\rho_1^3\rho_o}-1}{1+\frac{\rho_2^4}{\rho_1^3\rho_o}}, \quad \ldots \quad (2.4f)$$

The total number N of layers is assumed to be an odd number. Hence, we can calculate the impedance at point $z = z_0$ in the zero approximation:

$$Z_0^{(0)} = \rho_o\left(\frac{\rho_1^2}{\rho_2^2}\right)^{(N+1)/2} = \rho_o\left(\frac{\rho_1}{\rho_2}\right)^{N+1} \ll \rho_o, \quad (2.5)$$

$$Z_{2m}^{(0)} = \rho_o\left(\frac{\rho_1}{\rho_2}\right)^{N+1-2m}, \quad Z_{2m-1}^{(0)} = \frac{\rho_2^2}{Z_{2m}^{(0)}} \quad (2.6)$$

Accounting for the half wave length cap layer gives that the effective impedance $Z_c^{(0)}$ at position z_c is equal to $Z_c^{(0)} = Z_0^{(0)}$.

Reflectivity and transparency. In case of small impedance $Z_0 \ll \rho_o$ one may easily calculate the effective amplitude reflectivity R_0 and transparency $T_0 = \sqrt{1 - R_0^2}$

$$R_0 = \frac{Z_0 - \rho_o}{\rho_o + Z_0} \simeq -1 + \frac{2Z_0}{\rho_o}, \quad T_0 \simeq \frac{4Z_0}{\rho_o} \quad (2.7)$$

In the zeroth approximation we have:

$$R_0^{(0)} = \frac{Z_0^{(0)} - \rho_o}{\rho_o + Z_0^{(0)}} = -\frac{1 - \left(\frac{\rho_1}{\rho_2}\right)^{N+1}}{1 + \left(\frac{\rho_1}{\rho_2}\right)^{N+1}}, \quad (2.8)$$

$$\left(T_0^{(0)}\right)^2 = \frac{4\left(\frac{n_2}{n_1}\right)^{N+1}}{\left(1 + \left(\frac{n_2}{n_1}\right)^{N+1}\right)^2} \simeq 4\left(\frac{n_2}{n_1}\right)^{N+1} \quad (2.9)$$

B. Fluctuations of layer positions in coating

Now we take into account displacements ζ_j in position of each layer. We are interested in only linear terms of

expansion Z_0 over ζ_j. Hence, we can calculate each term separately, i.e. we can calculate a term, for example, proportional to a certain fluctuation ζ_m putting all other positions to be equal to zero: $\zeta_{i \neq m} = 0$.
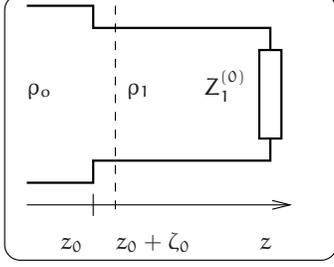

FIG. 2: For calculation of the dependence on displacement ζ_0.

First we assume that the cap layer is absent (later we will include it into considerations). We start by taking into account a small displacement ζ_0 assuming that all other displacements are equal to zero: $\zeta_{j \neq 0} = 0$. Hence, we can account for the unperturbed piece of transmission line between $z_1 \ldots z_{N+1}$ as a single impedance $Z_1^{(0)}$ — see Fig. 2.

The displacement ζ_0 influences the phase of reflected wave in two ways: a) through variation of effective impedance which, in turn, changes the reflectivity and b) through direct displacement of front surface which reflects the wave (see formula (2.15) below).

a). We calculate the perturbed impedance Z_0:

$$Z_1^{(0)} = \frac{\rho_2^2}{Z_2^{(0)}} = \frac{\rho_2^2}{\rho_o}\left(\frac{\rho_2}{\rho_1}\right)^{N-1} \gg \rho_o, \qquad (2.10)$$

$$R_1^{(0)} = \frac{Z_1^{(0)} - \rho_1}{Z_1^{(0)} + \rho_1} = \frac{1 - \alpha_N}{1 + \alpha_N}, \quad \alpha_N \equiv \frac{\rho_o}{\rho_2}\left(\frac{\rho_1}{\rho_2}\right)^N \ll 1,$$

$$\theta_1^2 \simeq -\left[1 - 2ik_1\zeta_0\right], \qquad (2.11)$$

$$Z_0 = \rho_1 \frac{1 + R_1^{(0)}\theta_1^2}{1 - R_1^{(0)}\theta_1^2} = \rho_1 \frac{\alpha_N - (1 - \alpha_N)2ik_1\zeta_0}{1 + (1 - \alpha_N)2ik_1\zeta_0} \simeq$$

$$= Z_0^{(0)} + ik_1\zeta_0\rho_1(1 - \alpha_N^2) =$$

$$= Z_0^{(0)} + ik\zeta_0\rho_o\left(1 - \left[\frac{\rho_o}{\rho_2}\left(\frac{\rho_1}{\rho_2}\right)^N\right]^2\right). \qquad (2.12)$$

Now we can estimate the perturbed reflectivity R_0:

$$R_0 = \frac{Z_0 - \rho_o}{Z_0 + \rho_o} \simeq R_0^{(0)}\left(1 - \frac{2ik\zeta_0(1 - \alpha_N^2)}{1 - \alpha_{N+1}^2}\right) \qquad (2.13)$$

$$\alpha_{N+1} \equiv \left(\frac{\rho_1}{\rho_2}\right)^{N+1} \ll 1 \qquad (2.14)$$

b). Let the amplitude of the incident wave to be A then the complex amplitude B of the reflected wave is provided as

$$B = R_0 A\, e^{2ik\zeta_0} \qquad (2.15)$$

Now expanding exponent in (2.15) in series over ζ_0 and substituting (2.13) we obtain

$$B \simeq R_0^{(0)}A\left(1 - \frac{2ik\xi_0(1 - \alpha_N^2)}{1 - \alpha_{N+1}^2} + 2ik\zeta_0\right) \simeq \qquad (2.16)$$

$$\simeq R_0^{(0)}A\left(1 - 2ik\zeta_0\left(\frac{\rho_1}{\rho_2}\right)^{2N}\frac{\rho_2^2 - \rho_1^2}{\rho_2^2}\right).$$

We see that contribution of displacement ζ_0 into the phase of reflected wave is depressed by a small factor $\sim (\rho_1/\rho_2)^N$.

In the same manner we can calculate contribution of fluctuation displacement in each layer ζ_i into the phase of reflected wave. In addition we can take into account fluctuations of position ζ_c of additional half wavelength layer (cap) made from fused silica. Then we can calculate the phase of the reflected wave expressing it as a function of small displacements ζ_i with account of the cap — see detailed calculations in Appendix A
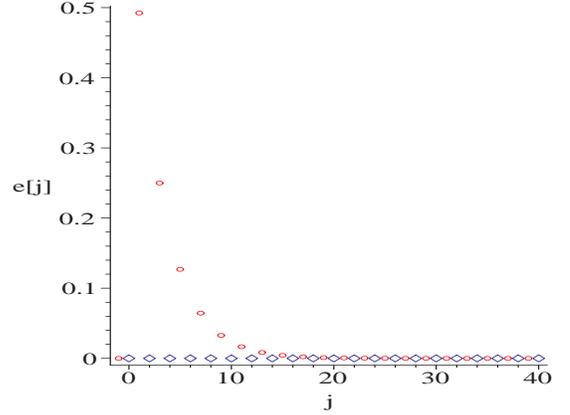

FIG. 3: The coefficients ϵ_j from formula (2.17) for a coating with a cap. Red circles correspond to odd subscript and blue diamonds – to even ones. $n_1 = 2.035$ (Ta_2O_5), $n_2 = 1.45$ (SiO_2). The number of Ta_2O_5 layers is assumed to be $N_1 = 20$, and the number of SiO_2 layers — $N_2 = 19$ plus cap. Subscript $_{-1}$ on the plot corresponds to ϵ_c.

The formula for phase ϕ of reflected wave is convenient to write through refractive indices ($\rho_1 = 1/n_1$, $\rho_2 = 1/n_2$, $n_1 > n_2$):

$$i\phi = 2ik\left(\epsilon_c\zeta_c + \sum_{j=0}^{N+1}\epsilon_j\,\zeta_j\right), \qquad (2.17a)$$

$$\epsilon_c = n_2\left(\frac{n_2}{n_1}\right)^{2N+2}, \qquad (2.17b)$$

$$\epsilon_0 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)n_2^2\left(\frac{n_2}{n_1}\right)^{2N}, \qquad (2.17c)$$

$$\epsilon_1 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right), \qquad (2.17d)$$

$$\epsilon_2 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)n_2^2\left(\frac{n_2}{n_1}\right)^{2N-2}, \qquad (2.17e)$$

$$\epsilon_3 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)\left(\frac{n_2}{n_1}\right)^2, \qquad (2.17f)$$

$$\epsilon_4 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)n_2^2\left(\frac{n_2}{n_1}\right)^{2N-4}, \ldots \qquad (2.17g)$$

$$\epsilon_{2m-1} = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)\left(\frac{n_2}{n_1}\right)^{2m-2}, \qquad (2.17h)$$

$$\epsilon_{2m} = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)n_2^2\left(\frac{n_2}{n_1}\right)^{2N-2m}, \ldots \qquad (2.17i)$$

$$\epsilon_N = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)\left(\frac{n_2}{n_1}\right)^{N-1}, \qquad (2.17j)$$

$$\epsilon_{N+1} = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right)n_2^2\left(\frac{n_2}{n_1}\right)^{N-1}. \qquad (2.17k)$$

We see that fluctuation displacements ζ_1, ζ_3,... (odd numbers) provide the main contribution into sum (2.17) whereas the input of displacements ζ_c, ζ_0, ζ_2, ζ_4... (even numbers) is negligible. The plot of coefficients ϵ_i is presented in Fig.3.

Note that formulas for coefficients ϵ_i obtained in the cases with and without half wavelength cap are practically the same — compare formulas (A12) and plots in Fig. 8 in Appendix A with formulas (2.17) and plot in Fig. 3 correspondingly.

III.   CALCULATION OF COATING BROWNIAN NOISE

We can apply the exact formulas (2.17) to calculate Brownian noise to coating. In accordance with Fluctuation Dissipation Theorem (FDT) [3, 17, 18], in order to calculate the spectral density $S_X(\omega)$ of fluctuations of variable $X = \sum \epsilon_i \zeta_i$ at the frequency ω, we have to apply force $\sum_i \epsilon_i F_0 e^{i\omega t}$ acting at frequency ω so that the force $\epsilon_i F_0$ is to be applied to position z_i. Then we have to calculate the total dissipated power W and to find the spectral density using formula

$$S_X(\omega) = \frac{8 k_B T W}{F_0^2 \omega^2} \qquad (3.1)$$

In so doing we just calculate the spectral density of fluctuating variable X as there is no need to calculate additional correlations between thickness fluctuations in different layers.

In contrast to previous approximate approaches [8, 9, 11–14] the formulas (2.17) provide the option of direct explicit calculation of the thermal noise in coating. Recall that in above articles the following approximate formula was used for reflected wave phase

$$\phi_{approx} = 2ik\zeta_c \qquad (3.2)$$

In this case we have to apply force $F_0 e^{i\omega t}$ to position z_c in order to calculate spectral density of variable ζ_c.

Obviously, the use of explicit formula (2.17) will give *smaller* value of dissipated power relative to result of approximate approach (3.2). Hence, the spectral density should be also smaller.

A.   Mirror as an infinite half space

We compare the explicit and approximate calculations of Brownian noise of coating using the model of the mirror as a semi-infinite half space using approach developed for structural losses in [13, 14]. In accordance with FDT [3, 17, 18] we have to apply pressure $\epsilon_i p_0$ to the i-th boundary between layers:

$\epsilon_c p_0$ – to outer surface of cap, $\quad p_0 = \dfrac{2F}{\pi w^2} e^{-2r^2/w^2}$,

$\epsilon_0 p_0$ – to position z_0, $\hfill (3.3)$

$\epsilon_1 p_0$ – to position z_1, ... $\hfill (3.4)$

and so on. We are interested in elastic energy U_i stored in each layer.

First we write down formulas for strains u_{ij} and stress σ_{zz} in outer layer of the mirror body (substrate) [4, 13, 19]

$$u_{rr} = \frac{F\Sigma}{4\pi(\lambda+\mu)}\left(\frac{1}{r^2}\left[1 - e^{-2r^2/w^2}\right] - \frac{4}{w^2}e^{-2r^2/w^2}\right), \qquad (3.5a)$$

$$u_{\phi\phi} = \frac{-F\Sigma}{4\pi(\lambda+\mu)}\frac{1}{r^2}\left[1 - e^{-2r^2/w^2}\right], \qquad (3.5b)$$

$$u_{zz} = \frac{-F\Sigma}{4\pi(\lambda+\mu)}\frac{4}{w^2}e^{-2r^2/w^2}, \qquad (3.5c)$$

$$\Rightarrow u_{zz} = u_{rr} + u_{\phi\phi},$$

$$u_{rz} = 0, \quad \Sigma \equiv \epsilon_c + \sum_{i=0}^{N}\epsilon_i, \qquad (3.5d)$$

$$\sigma_{zz} = \frac{2F\Sigma}{\pi w^2}e^{-2r^2/w^2}. \qquad (3.5e)$$

Here λ, μ are the Lamé coefficients of substrate, which are known to be expressed through the Young modulus Y and the Poisson ratio ν as following

$$\lambda \equiv \frac{\nu Y}{(1+\nu)(1-2\nu)}, \quad \mu \equiv \frac{Y}{2(1+\nu)}, \qquad (3.6)$$

We assume that the number $i = N$ corresponds to the surface of substrate (as shown in Fig. 1)..

Now we calculate strains and stresses in coating layers. As usual we assume that the tangent strains in the i-th layer of coating (the positions of its boundaries are z_{i-1}, z_i) are equal to corresponding strains in substrate:

$$u_{rr}^i = u_{rr}, \quad u_{\phi\phi}^i = u_{\phi\phi}, \quad u_{rz}^i = u_{rz} = 0, \qquad (3.7)$$

The last components u_{zz}^i of the normal strain in the i-th layer can be found from the known formula binding stress

and strain tensors [19]:

$$\sigma_{zz}^i = (\lambda_i + 2\mu_i)u_{zz}^i + \lambda_i(u_{rr}^i + u_{\phi\phi}^i) \qquad (3.8)$$

The normal stress $\sigma_{zz, i}^i$ can be easily calculated as following

$$\sigma_{zz}^i = \frac{F\Sigma_i}{\pi w^2} e^{-2r^2/w^2}, \quad \Sigma_i \equiv \epsilon_c + \sum_{j=0}^{i-1} \epsilon_j. \qquad (3.9)$$
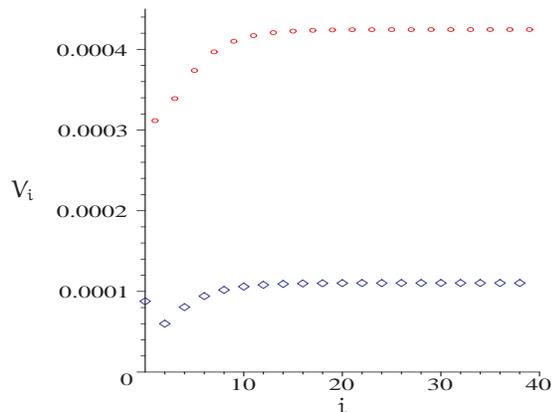

FIG. 4: The coefficient V_i proportional to energy stored in the i-th layer from formula (2.17) for coating with a cap. Red circles correspond to odd subscript ($n_{odd} = 2.035$ (Ta_2O_5)) and blue diamonds ($n_{even} = 1.45$ (SiO_2)) – to even ones. Number of Ta_2O_5 layers is assumed to be $N_1 = 20$, and the number of SiO_2 layers — $N_2 = 19$. Coeffcent V_0 on the plot corresponds to the cap layer.

Manipulating with (3.8, 3.9) we write down a useful formula:

$$u_{zz}^i = A_i u_{zz}, \quad A_i = \frac{2(\lambda + \mu)\Sigma_i - \lambda_i \Sigma}{(\lambda_i + 2\mu_i)\Sigma} \qquad (3.10)$$

and to calculate the elastic energy U_i in the i-th layer with thickness d_i through its volume density w_i using the known formulas:

$$w_i = \mu_i\left(u_{rr}^2 + u_{\phi\phi}^2 + (u_{zz}^i)^2\right) + \frac{\lambda_i}{2}\left(u_{zz}^i + u_{rr} + u_{\phi\phi}\right)^2,$$

$$U_i = d_i\, 2\pi \int_0^\infty w_i\, r\, dr =$$

$$= \frac{d_i}{8\pi}\left(\frac{F\Sigma_N}{w(\lambda + \mu)}\right)^2\left[2\mu_i\left(1 + A_i^2\right) + \lambda_i\left(1 + A_i\right)^2\right]. \qquad (3.11)$$

Now we calculate the spectral density of variable X (3.1) using the fact that for the structural losses the power dissipated in the i-th layer can be expressed as $W_i = U_i \phi_i \omega$:

$$S_X(\omega) = \frac{k_B T}{\omega} \frac{\Lambda}{\pi(\mu + \lambda)} \left(\frac{F\Sigma_N}{2w}\right)^2 \sum_{i=0}^{N} V_i, \qquad (3.12)$$

$$V_i = [1 + \delta_{0i}] \frac{2\mu_i\left(1 + A_i^2\right) + \lambda_i\left(1 + A_i\right)^2}{n_i(\mu + \lambda)} \phi_i \qquad (3.13)$$

Here ϕ_i is the loss angle of structural losses in the i-th layer, Λ denotes the optical wavelength in vacuum, thickness d_i of i-th layer is expressed through its refractive index n_i as $d_i = \Lambda/4n_i$ for each layer except cap, the multiplier $[1 + \delta_{0i}]$ is introduced to account for the fact that the cap thickness is two times larger (as its width is equal to *half* wavelength). The plot of dimensionless coefficients V_i is presented on Fig. 4, the used numerical parameters are presented in Table I.

TABLE I: Parameters used for numerical calculations.

Parameter	substrate	Ta_2O_5 layer	SiO_2 layer
T, K		290	
Λ, m		1.064×10^{-6}	
N_{CM}	-	20	$19 + cap$
n	1.45	2.035	1.45
Y, Pa	72×10^9	140×10^9	72×10^9
ν	0.17	0.23	0.17
ϕ	4×10^{-10}	2×10^{-4}	4×10^{-5}

Now we may compare the estimate of accurate formula for spectral density S_X of Brownian noise with approximation S_{ζ_c} used previously [13, 14]. Numerical calculations for parameters presented in Table I gives:

$$\frac{S_{\zeta_c} - S_X}{S_X} \simeq 0.0539 \qquad (3.14)$$

We see that accurate calculation gives slightly less value of spectral density. However, the difference is quite modest — about 5 percents only. We may qualitatively explain it by the fact that the tagent strains u_{rr}, $u_{\phi\phi}$ are the same for all layers and they make considerable contribution into elastic energy. The account of different coefficients ϵ_i change only strain u_{zz}, which is smaller than tagent strains for the majority layers.

B. Mirror as a finite cylinder

We repeated our calculations of coating Brownian noise for mirror as a finite cylinder using results of [4, 7, 15]. For estimates we used the parameters listed in Table I. For fused silica cylinder planned in third generation laser gravitational antenna (Einstein Telescope, radius $R = 0.30$ m, height $H = 0.31$ m, $w = 0.12$ m) we have got

$$\left.\frac{S_{\zeta_c} - S_X}{S_X}\right|_{ET} \simeq 0.0524 \qquad (3.15)$$

For cylindric test mass planned in Advanced LIGO (radius $R = 0.17$ m, height $H = 0.20$ m, $w = 0.06$ m) we have got

$$\left. \frac{S_{\zeta_c} - S_X}{S_X} \right|_{\text{aLIGO}} \simeq 0.0569 \qquad (3.16)$$

We see that the difference is practically the same as for the model of mirror as an infinite half space.

C. Double mirror (Khalili etalon)

The number of layers used in conventional coating is large enough (about 40) — it is the reason why explicit and approximate formulas give close numerical results as estimates (3.14, 3.15, 3.16) show. However, explicit formulas may give considerably different results in double mirror [20]. Recall, in conventional mirror the fluctuations of thickness of each layer in the coating are transformed into phase fluctuations of reflected wave in two ways. First, each layer makes contribution into variation of front position of coating (position of first layer) and each layer makes approximately equal contribution. (It is worth underlying that we are interested in position of front surface of mirror relative to its center of the mass.) Second, the fluctuations of the layer thickness vary the pathlength of light traveling inside it and only several first layers make the main contribution whereas inner layers make exponentially small contribution. The first effect is much bigger than the second and usually only it is taken into account.

The idea of double mirror, put forward by F. Khalili [20] (now the term "Khalili etalon" is frequently used), is to displace the part of layers from the front surface to the rear surface of the mirror. In this case the thickness fluctuations of layers on the rear surface do not make contribution into fluctuations of front surface relative to mirror's center of the mass. And number of layers on the front surface may be smaller than total number of layers. Obviously, the explicit calculation of Brownian noise of the front layers have to give considerably less value of the spectral density as compared to approximate one due to relatively small number of layers. Obviously, there are precisely layers on front surface which make the main contribution into the coating Brownian noise of double mirror.

As an example, we have applied approach presented in this paper for Khalili etalon in order to estimate difference between two approaches. We assume that the coating on front surface consists of 3 layers of Ta_2O_5 and 2 layers of SiO_2 plus cap (also manufactured from fused silica) and coating on rear surface — 17 layers of Ta_2O_5 and 17 layers of SiO_2. The other parameters were taken the same. One may calculate coating Brownian noise in Khalili etalon by two ways. In traditional (approximate) approach one has to apply corresponding forces $\varepsilon_f F_0$ and $\varepsilon_r F_0$ forces to front and rear surfaces of cylindric mirror (the coefficients ε_f and ε_r are calculated from

effective tansmittances of corresponding coatings), to calculate the power dissipated in coating and, finally, to calculate spectral density $S_{\text{app}}(\omega)$ using formula (3.1). We have also calculated spectral density $S_{\text{acc}}(\omega)$ using our approach through coefficients ϵ_i. We have found that $S_{\text{acc}}(\omega)$ is smaller than $S_{\text{app}}(\omega)$ by about 17%:

$$\frac{S_{\text{app}}(\omega) - S_{\text{acc}}(\omega)}{S_{\text{acc}}(\omega)} \simeq 0.172 \qquad (3.17)$$

Here we used the parameters from Table I and Advanced LIGO (radius $R = 0.17$ m, height $H = 0.20$ m, $w = 0.06$ m).

IV. CONCLUSION

The formulas (2.17) may be applied to the *explicit* calculation of Brownian, thermoelastic, thermorefractive noise in the coating. These formulas are especially important for the thermal noise *compensation* first proposed by J. Kimble [10] and then demonstarted for thermoelastic and thermorefractive noises [11, 12]. Recall that in [11, 12] consideration was based on approximation (3.2) but we hope that formulas (2.17) will allow to formulate *explicit* recommendation for thickness of each layer.

In order to do it we have to rewrite formula for reflected wave phase ϕ in form useful for the calculation of thermorefractive noise. For this case we have to assume that layers positions are not fluctuated $\zeta_j = 0$ but there are fluctuations of light path lengths due to variation of refractive index in each layer — see corresponding formulas (A14) in Appendix A. This formula does not account influence of cap because in accordance to recommendation of [12] the thickness of additional cap layer is a subject for optimization.

Recently non-quarter wavelength coating was suggested by I. Pinto [21]. As loss angle in Ta_2O_5 is much bigger than in SiO_2 one can use, for example, 1/8 wavelength layer of Ta_2O_5 and 3/8 wavelength layer of SiO_2 in order to reduce thermal coating noise. We plan to apply our approach for detail analysis of such coatings.

We hope that proposed approach will be useful for detailed analysis of coating noise.

Acknowledgments

FIG. 5: For calculation of dependence on displacement ζ_1.
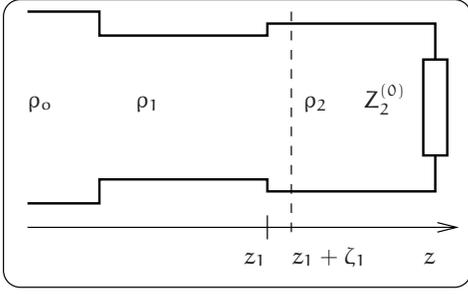
FIG. 6: For calculation of dependence on displacement ζ_2.

Appendix A: Calculations of reflected wave phase

Here we present detailed derivation of formulas (2.17) starting with taking into account of displacement ζ_1 assuming that all others are zero: $\zeta_{j\neq 1} = 0$. Hence, we may consider the unperturbed piece of transmission line between $z_2 \ldots z_{N+1}$ as a single impedance $Z_2^{(0)}$ — see Fig. 5. Again, the displacement ζ_1 influences in two ways: a) variation of optical pathlength between z_0 and z_1 and b) between z_1 and z_2. We consider each of them separately.

$a)$. Note that influence of ζ_1 on pathlength between z_0 and z_1 we have almost calculated above. Indeed, calculating influence of ζ_0 we have taken into account variation of exponent θ_1 and expanded $\theta_1^2 \simeq -\left(1 + 2ik_1(\zeta_1 - \zeta_0)\right)$ keeping only term $\sim \zeta_0$. Now we may account displacement ζ_1 just making substitution $\zeta_0 \to -\zeta_1$ in (2.12)

$$Z_0 = Z_0^{(0)} - ik\zeta_1 \rho_o \left(1 - \alpha_N^2\right). \qquad (A1)$$

$b)$. We calculate variation of Z_1 by ζ_1 and then recalculate it into Z_0:

$$R_2^{(0)} = \frac{Z_2^{(0)} - \rho_2}{Z_2^{(0)} + \rho_2} = -\frac{1 - \alpha_{N-1}}{1 + \alpha_{N-1}}, \quad \alpha_{N-1} \equiv \frac{\rho_o}{\rho_2}\left(\frac{\rho_1}{\rho_2}\right)^{N-1},$$

$$Z_1 = \rho_2 \frac{1 + R_2\theta_2^2}{1 - R_2\theta_2^2}, \quad \theta_2^2 \simeq -(1 - 2ik_2\zeta_1),$$

$$Z_1 \simeq \frac{\rho_2}{\alpha_{N-1}}\left(1 - \frac{ik_2\zeta_1(1 - \alpha_{N-1}^2)}{\alpha_{N-1}}\right),$$

$$Z_0 = \frac{\rho_1^2}{Z_1} = Z_0^{(0)} + ik_2\zeta_1(1 - \alpha_{N-1}^2)\frac{\rho_1^2}{\rho_2} =$$
$$= Z_0^{(0)} + ik\rho_o\zeta_1(1 - \alpha_{N-1}^2)\frac{\rho_1^2}{\rho_2^2}. \qquad (A2)$$

Collecting (A1) and (A2) we obtain the total contribution of ζ_1 in impedance Z_0

$$Z_0 = Z_0^{(0)} + \rho_o\, ik\zeta_1 \left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) \qquad (A3)$$

Now we take into account only displacement ζ_2 assuming that all others are zero: $\zeta_{j\neq 2} = 0$. Hence, we may consider the unperturbed piece of transmission line between $z_3 \ldots z_{N+1}$ as a single impedance $Z_3^{(0)}$ — see
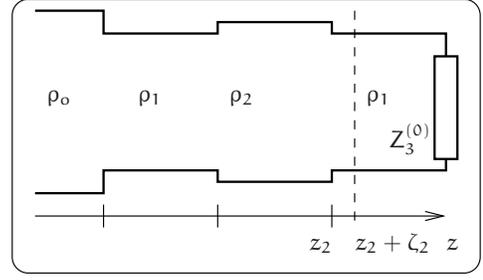
Fig. 6. Again we note that displacement ζ_2 influences in two ways: variation of optical pathlength a) between z_1 and z_2 and b) between z_2 and z_3.

$a)$. We may account displacement ζ_2 just making substitution $\zeta_1 \to -\zeta_2$ in (A2):

$$Z_0 = Z_0^{(0)} - ik\rho_o\zeta_2(1 - \alpha_{N-1}^2)\frac{\rho_1^2}{\rho_2^2}. \qquad (A4)$$

$b)$. We calculate variation of Z_2 by ζ_2 and then recalculate it into Z_0:

$$Z_3^{(0)} = \frac{\rho_2^2}{Z_4^{(0)}} = \frac{\rho_2^2}{\rho_o}\left(\frac{\rho_1}{\rho_2}\right)^{N-3} \gg \rho_o,$$

$$R_3^{(0)} = \frac{Z_3^{(0)} - \rho_1}{Z_3^{(0)} + \rho_1} = \frac{1 - \alpha_{N-2}}{1 + \alpha_{N-2}}, \quad \alpha_{N-2} \equiv \frac{\rho_o}{\rho_2}\left(\frac{\rho_1}{\rho_2}\right)^{N-2},$$

$$Z_2 = \rho_1 \frac{1 + R_3\theta_3^2}{1 - R_3\theta_3^2}, \quad \theta_3^2 \simeq -(1 - 2ik_1\zeta_3),$$

$$Z_2 \simeq \rho_1 \frac{\alpha_{N-2} + (1 - \alpha_{N-2})ik_1\zeta_2}{1 + (1 - \alpha_{N-2})ik_1\zeta_2} \simeq$$
$$\simeq \rho_1\alpha_{N-2} + \rho_o\, ik\zeta_2(1 - \alpha_{N-2}^2), \qquad (A5)$$

$$Z_0 = \frac{\rho_1^2}{\rho_2^2}Z_2 = Z_0^{(0)} + \rho_o\, ik\zeta_2(1 - \alpha_{N-2}^2)\frac{\rho_1^2}{\rho_2^2}. \qquad (A6)$$

Collecting (A4) and (A6) we obtain the total contribution of ζ_2 in impedance Z_0

$$Z_0 = Z_0^{(0)} + \rho_o\, ik\zeta_2\,(\alpha_{N-1}^2 - \alpha_{N-2}^2)\left(\frac{\rho_1^2}{\rho_2^2}\right) =$$
$$= Z_0^{(0)} + \rho_o\, ik\zeta_2\left(\frac{\rho_o^2}{\rho_2^2}\right)\left(\frac{\rho_1}{\rho_2}\right)^{2N-2}\frac{\rho_1^2 - \rho_2^2}{\rho_2^2} \qquad (A7)$$

By the same consideration we may take into account displacements ζ_3 and ζ_4:

$$Z_0 = Z_0^{(0)} + \rho_o\, ik\zeta_3\left(\frac{\rho_1}{\rho_2}\right)^2\left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \qquad (A8)$$

$$+ \rho_o\, ik\zeta_4 \frac{(\rho_1^2 - \rho_2^2)}{\rho_2^2}\left(\frac{\rho_o}{\rho_2}\right)^2\left(\frac{\rho_1}{\rho_2}\right)^{2N-4} \qquad (A9)$$

and write down the final formula for effective impedance of transmission line as a function of small displacements

ζ_i:

$$Z_0 = \rho_0 \left(\frac{\rho_1}{\rho_2}\right)^{N+1} + ik\zeta_0\rho_0 \left(1 - \left[\frac{\rho_o}{\rho_2}\left(\frac{\rho_1}{\rho_2}\right)^N\right]^2\right) +$$

$$+ \rho_o \, ik\zeta_1 \left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \qquad \text{(A10a)}$$

$$+ \rho_o \, ik\zeta_2 \, \frac{\rho_1^2 - \rho_2^2}{\rho_2^2} \left(\frac{\rho_o^2}{\rho_2^2}\right) \left(\frac{\rho_1}{\rho_2}\right)^{2N-2} + \quad \text{(A10b)}$$

$$+ \rho_o \, ik\zeta_3 \left(\frac{\rho_1}{\rho_2}\right)^2 \left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \qquad \text{(A10c)}$$

$$+ \rho_o \, ik\zeta_4 \, \frac{(\rho_1^2 - \rho_2^2)}{\rho_2^2} \left(\frac{\rho_o}{\rho_2}\right)^2 \left(\frac{\rho_1}{\rho_2}\right)^{2N-4} + \dots$$
$$\text{(A10d)}$$

$$+ \rho_o \, ik\zeta_{2m-1} \left(\frac{\rho_1}{\rho_2}\right)^{2m-2} \left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \quad \text{(A10e)}$$

$$+ \rho_o \, ik\zeta_{2m} \, \frac{(\rho_1^2 - \rho_2^2)}{\rho_2^2} \left(\frac{\rho_o}{\rho_2}\right)^2 \left(\frac{\rho_1}{\rho_2}\right)^{2N-2m} + \dots$$
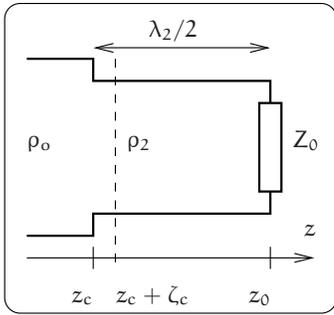
FIG. 7: For calculation of equivalent impedance of the coating with a half wavelength cap.

The formula for the phase ϕ of reflected wave complex amplitude B (analog of (2.15)) is more informative one:

$$B = \frac{Z_0 - \rho_o}{Z_0 + \rho_o} A e^{2ik\zeta_0} \simeq R_0^{(0)} A(1 + i\phi), \qquad \text{(A11a)}$$

$$i\phi = 2ik \sum_{j=0}^{N+1} \epsilon_j \, \zeta_j. \qquad \text{(A11b)}$$

One may rewrite these formulas through refractive indices ($\rho_1 = 1/n_1$, $\rho_2 = 1/n_2$, $n_1 > n_2$):

$$i\phi = 2ik \sum_{j=0}^{N+1} \epsilon_j \, \zeta_j, \qquad \text{(A12a)}$$

$$\epsilon_0 = n_2^2 \left(\frac{n_2}{n_1}\right)^{2N}, \qquad \text{(A12b)}$$

$$\epsilon_1 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right), \qquad \text{(A12c)}$$

$$\epsilon_2 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right) n_2^2 \left(\frac{n_2}{n_1}\right)^{2N-2}, \qquad \text{(A12d)}$$

$$\epsilon_3 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right) \left(\frac{n_2}{n_1}\right)^2, \qquad \text{(A12e)}$$

$$\epsilon_4 = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right) n_2^2 \left(\frac{n_2}{n_1}\right)^{2N-4}, \dots \qquad \text{(A12f)}$$

$$\epsilon_{2m-1} = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right) \left(\frac{n_2}{n_1}\right)^{2m-2},$$

$$\epsilon_{2m} = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right) n_2^2 \left(\frac{n_2}{n_1}\right)^{2N-2m}, \dots \qquad \text{(A12g)}$$

$$\epsilon_{N+1} = \left(\frac{n_1^2 - n_2^2}{n_1^2}\right) n_2^2 \left(\frac{n_2}{n_1}\right)^{N-1}. \qquad \text{(A12h)}$$

We see that coefficients ϵ_{2m} with *even* index are relatively small and their contribution may be omitted. The main contribution is made by coefficients ϵ_{2m-1} with *odd* subscripts — see plot on Fig. 8.
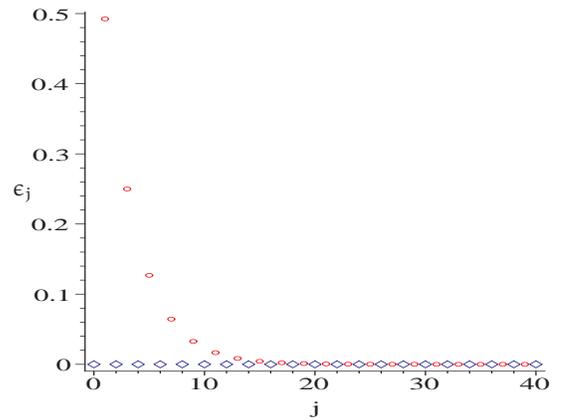
FIG. 8: The coefficients ϵ_j from formulas (A12). Red circles correspond to odd subscript and blue diamonds – to even ones. $n_1 = 2.035$ (Ta$_2$O$_5$), $n_2 = 1.45$ (SiO$_2$). Number of Ta$_2$O$_5$ layers is assumed to be $N_1 = 20$, and number of SiO$_2$ layers — $N_2 = 19$.

We may rewrite formulas (A12) for the reflected wave phase ϕ in the form useful for the calculation of thermorefractive noise. For this case we have to assume that layers positions are not fluctuated $\zeta_j = 0$ but there are fluctuations of path lengths due to variation of refractive index in each layer:

$$\theta_j^2 = \exp\left[2ik(n_{1,2} + \delta n_{1,2})(z_{j+1} - z_j)\right] \simeq$$

$$\simeq -[1 + \pi\Delta_j], \quad \Delta_j = \frac{\delta n_j}{n_j}. \qquad \text{(A13)}$$

Here Δ_j is the *relative* fluctuation of refraction index in j-th layer. Finally the phase of reflected wave has the following dependence on fluctuations of refractive indexes:

$$i\phi = i\pi \sum_{j=1}^{N+1} \varepsilon_j \, \Delta_j, \quad \Delta_j \equiv \frac{\delta n_j}{n_j}, \qquad \text{(A14a)}$$
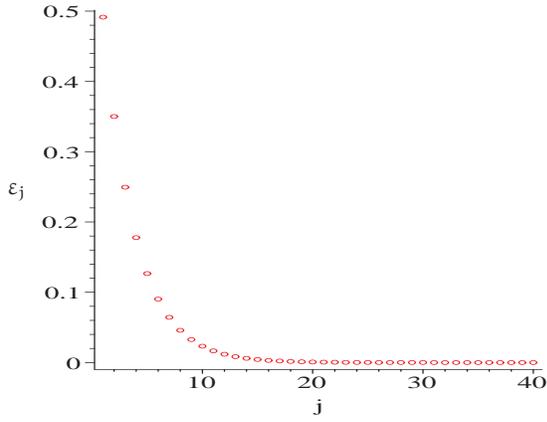
FIG. 9: The coefficients ε_j from formulas (A14). $n_1 = 2.035$ (Ta_2O_5), $n_2 = 1.45$ (SiO_2). Number of Ta_2O_5 layers is assumed to be $N_1 = 20$, and number of SiO_2 layers — $N_2 = 19$.

$$\varepsilon_1 = \frac{1}{n_1}\left(1 - n_2^2\left(\frac{n_2}{n_1}\right)^{2N}\right), \tag{A14b}$$

$$\varepsilon_2 = \frac{1}{n_1}\left(\frac{n_2}{n_1}\right)\left(1 - n_2^2\left(\frac{n_2}{n_1}\right)^{2(N-1)}\right), \tag{A14c}$$

$$\varepsilon_3 = \frac{1}{n_1}\left(\frac{n_2}{n_1}\right)^2\left(1 - n_2^2\left(\frac{n_2}{n_1}\right)^{2(N-2)}\right), \tag{A14d}$$

$$\varepsilon_4 = \frac{1}{n_1}\left(\frac{n_2}{n_1}\right)^3\left(1 - n_2^2\left(\frac{n_2}{n_1}\right)^{2(N-3)}\right), \ldots , \tag{A14e}$$

$$\varepsilon_N = \frac{1}{n_1}\left(\frac{n_2}{n_1}\right)^{N-1}\left(1 - n_2^2\left(\frac{n_2}{n_1}\right)^{2}\right), \tag{A14f}$$

$$\varepsilon_{N+1} = \frac{1}{n_1}\left(\frac{n_2}{n_1}\right)^{N}\left(1 - n_2^2\left(\frac{n_2}{n_1}\right)^{0}\right). \tag{A14g}$$

We present plot ε_j in Fig. 9.

Account of a cap. To account for the outer half wavelength layer (cap) we have to consider the transmission line model shown in Fig. 7. In the zeroth approximation we put $\zeta_c = \zeta_0 = 0$ and obtain $Z_c^{(0)} = Z_0^{(0)}$. In the first order approximation we use the expansion in series for

θ_0 to account for layer positions fluctuations:

$$\theta_0^2 \simeq 1 + 2ik_2(\zeta_0 - \zeta_c), \tag{A15}$$

$$Z_c \simeq \rho_2 \frac{2Z_0 + (Z_0^{(0)} - \rho_2)2ik_2(\zeta_0 - \zeta_c)}{2\rho_2 - (Z_0^{(0)} - \rho_2)2ik_2(\zeta_0 - \zeta_c)} =$$

$$= Z_0 + ik_2(\xi_0 - \xi_c)\left(\frac{(Z_0^{(0)})^2 - \rho_2^2}{\rho_2}\right) \tag{A16}$$

Now using (A10) and $k_{1,2} = k\rho_o/\rho_{1,2}$ we rewrite:

$$Z_c = \rho_0\left(\frac{\rho_1}{\rho_2}\right)^{N+1} + \tag{A17a}$$

$$+ \rho_0 ik\zeta_c\left(1 - \left[\frac{\rho_o}{\rho_2}\left(\frac{\rho_1}{\rho_2}\right)^{N+1}\right]^2\right) + \tag{A17b}$$

$$+ \rho_o\, ik\zeta_0\, \frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\left(\frac{\rho_o^2}{\rho_2^2}\right)\left(\frac{\rho_1}{\rho_2}\right)^{2N} + \tag{A17c}$$

$$+ \rho_o\, ik\zeta_1\left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \tag{A17d}$$

$$+ \rho_o\, ik\zeta_2\, \frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\left(\frac{\rho_o^2}{\rho_2^2}\right)\left(\frac{\rho_1}{\rho_2}\right)^{2N-2} + \tag{A17e}$$

$$+ \rho_o\, ik\zeta_3\left(\frac{\rho_1}{\rho_2}\right)^2\left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \tag{A17f}$$

$$+ \rho_o\, ik\zeta_4\, \frac{(\rho_1^2 - \rho_2^2)}{\rho_2^2}\left(\frac{\rho_o}{\rho_2}\right)^2\left(\frac{\rho_1}{\rho_2}\right)^{2N-4} + \ldots \tag{A17g}$$

$$+ \rho_o\, ik\zeta_{2m-1}\left(\frac{\rho_1}{\rho_2}\right)^{2m-2}\left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) + \tag{A17h}$$

$$+ \rho_o\, ik\zeta_{2m}\, \frac{(\rho_1^2 - \rho_2^2)}{\rho_2^2}\left(\frac{\rho_o}{\rho_2}\right)^2\left(\frac{\rho_1}{\rho_2}\right)^{2N-2m} + \ldots$$

From these formulas one can obtain expressions (2.17) to expand into series the reflected wave B

$$B = \frac{Z_c - \rho_o}{Z_c + \rho_o}\, A e^{2ik\zeta_0} \simeq R_0^{(0)}A(1 + i\phi) \tag{A18}$$

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