Ghost points in inverse scattering constructions of stationary Einstein metrics

Piotr T. Chrusµciel[∗] and Luc Nguyen[†]

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Abstract

We prove a removable singularities theorem for stationary Einstein equations, with useful implications for constructions of stationary solutions using soliton methods.

1 Introduction

The soliton technique has proved to be a very effective tool for constructing stationary black holes in five dimensions, see e.g. [\[4,](#page-10-0) [6,](#page-10-1) [7,](#page-10-2) [16,](#page-11-0) [17\]](#page-11-1). The method is used to construct singular solutions of harmonic-map type equations. One then needs to make sure that the singularity structure of the resulting harmonic map is compatible with a smooth geometry of the associated space-time.

Proceeding in this way, in their ingenious construction of Black Saturns [\[4\]](#page-10-0), Elvang and Figueras introduce a singular point

$$
\alpha_1 := (\rho = 0, z = a_1)
$$

on the boundary $\{\rho = 0\}$ of the Weyl coordinates domain $\{\rho \geq 0, z \in \mathbb{R}\},\$ and fine-tune certain constants to ensure that the metric functions remain uniformly bounded near α_1 . Now, the resulting metric ends up being a rational function of

$$
R_1 := \sqrt{\rho^2 + (z - a_1)^2} \ .
$$

[∗]University of Vienna

[†]OxPDE, Mathematical Institute, Oxford

This leads to a potential problem because R_1 is not differentiable at α_1 , and therefore the differentiability of the metric functions at α_1 is not apparent.

We will refer to such points as *ghost points*, as their occurrence does not seem to be related in any obvious way to desirable geometric properties of the resulting space-time such as end points of horizons or fixed points of the isometry action, compare [\[10–](#page-11-2)[12\]](#page-11-3).

Closer inspection [\[2\]](#page-10-3) of the Black Saturn metric near α_1 shows that, with the choice of free constants that makes the metric functions bounded, all the metric functions can be rewritten as rational functions of R_1^2 ; since this last function is smooth, smoothness of the metric near α_1 becomes obvious. The calculations required to establish this fact turn out to be rather heavy, requiring quite a bit of effort to coerce Mathematica to produce the result. We emphasize that the result is non-trivial and requires non-obvious factorisations and cancellations of odd-order polynomials in R_1 .

A similar trick of introducing ghost points has been used in other related constructions [\[5,](#page-10-4)[8,](#page-11-4)[13,](#page-11-5)[18,](#page-11-6)[19\]](#page-11-7). The question then arises, whether one needs to redo the calculations of [\[2\]](#page-10-3) case by case, or there is a general mechanism which guarantees that ghost points are smooth points for the resulting metric.

The object of this note is to show that smoothness of the metric at such points is a consequence of the stationary Einstein equations with matter fields, without the need to assume more Killing vectors. This can be roughly stated as follows, see Theorem [2.1](#page-3-0) for a precise version:

Theorem 1.1 Singularities of Lipschitz continuous stationary Einstein metrics located on timelike submanifolds of codimension $m \geq 2$ are removable.

We discuss in somewhat more detail in Section [3](#page-8-0) how this theorem takes care of the ghost point problem.

2 Stationary Einstein equations

We consider a time-independent metric in a space-time of dimension $n + 1$. Since the problem we address is purely local, we assume that the space-time metric functions $\mathfrak{g}_{\mu\nu}$ are given in local spatial coordinates ranging over a ball $B(R) \subset \mathbb{R}^n$, $n \geq 2$, of radius R centred at the origin, and the prospective singularities lie along a smooth submanifold

$$
\Sigma \equiv \Sigma^{n-m}
$$

of $B(R)$ of codimension $2 \le m \le n$, with either $\partial \Sigma = \emptyset$ or $\partial \Sigma \subset \partial B(R)$. We set

$$
B^*(R) := B(R) \setminus \Sigma.
$$

In adapted coordinates the metric can be written as

$$
\mathfrak{g} = -V^2 (dt + \underbrace{\theta_i dy^i}_{=\theta})^2 + \underbrace{g_{ij} dy^i dy^j}_{=\theta}, \qquad (2.1)
$$

where ∂_t is (stationary) Killing, i.e.

$$
\partial_t V = \partial_t \theta = \partial_t g = 0.
$$
\n(2.2)

We allow matter fields $\varphi = (\varphi^A)$ with energy-momentum tensor that depends upon $\mathfrak{g}, \partial \mathfrak{g}, \varphi$ and $\partial \varphi$. For simplicity we assume that the φ^{A} 's transform as scalars or tensors under coordinate changes, and that the stationary matter field equations constitute a tensorial system of the form

$$
\Delta_g \varphi = F(\mathfrak{g}, \partial \mathfrak{g}, \varphi, \partial \varphi) \text{ in } B^*(R) , \qquad (2.3)
$$

though a wider class of more general elliptic systems can be easily incorporated in our analysis. We note that (linear) electromagnetic fields, for example, satisfy this assumption in Lorenz gauge.

The Einstein equations with (possibly zero) cosmological constant Λ for a metric satisfying $(2.1)-(2.2)$ $(2.1)-(2.2)$ read (see, e.g., $[3]$ or $[1]$)

$$
\begin{cases}\nV \Delta_g V = -\frac{1}{4} |\lambda|_g^2 + T_{00} - \left(\frac{n+1}{n-1} \Lambda - \frac{\text{tr}_g(T)}{n-1}\right) V^2, \\
\text{div}_g(V\lambda) = 2V \Big[T_0 - \left(\frac{n+1}{n-1} \Lambda - \frac{\text{tr}_g(T)}{n-1}\right) V^2 \theta \Big], \\
\text{Ric}(g) - V^{-1} \text{Hess}_g V = \frac{1}{2V^2} \lambda \circ \lambda + T_g + \left(\frac{n+1}{n-1} \Lambda - \frac{\text{tr}_g(T_g)}{n-1}\right) g,\n\end{cases} (2.4)
$$

where $T_g := T_{ij} dx^i dx^j$, $T_0 = T_{0i} dx^i$, and

$$
\lambda_{ij} = -V^2(\partial_i \theta_j - \partial_j \theta_i) , \quad (\lambda \circ \lambda)_{ij} = \lambda_i^k \lambda_{kj} .
$$

Altogether the Einstein-matter field equations, which are supposed to hold in $B^*(R)$, can therefore be written as

$$
\begin{cases}\n\Delta_g \varphi = F(\mathfrak{g}, \partial \mathfrak{g}, \varphi, \partial \varphi) ,\n\Delta_g V = F_1(\mathfrak{g}, \partial \mathfrak{g}) + \tilde{T}_V ,\ndiv_g(d\theta) - d(\text{div}_g(\theta)) = F_2(\mathfrak{g}, \partial \mathfrak{g}) + \tilde{T}_\theta ,\n\text{Ric}(g) - V^{-1} \text{Hess}_g V = F_3(\mathfrak{g}, \partial \mathfrak{g}) + \tilde{T}_g ,\n\end{cases}
$$
\n(2.5)

for some (explicitly computable) F_1 , F_2 and F_3 which are polynomials in \mathfrak{g} , \mathfrak{g}^{-1} , $\partial \mathfrak{g}$ and quadratic in $\partial \mathfrak{g}$ and some \tilde{T}_V , \tilde{T}_θ and \tilde{T}_g which arise from $T_{\mu\nu}$.

We have:

THEOREM 2.1 Under the conditions above, suppose that $(\mathfrak{g}, \varphi) \in C^{0,\alpha}(B(R)) \cap$ $C^2(B^*(R))$ and $V > 0$ in $B(R)$. Assume further that

- 1. either $\alpha = 1$,
- 2. or $\frac{n-m}{n} < \alpha < 1$ and there exists a constant C, possibly depending upon (\mathfrak{g}, φ) , such that

$$
|\partial \mathfrak{g}| + |\partial \varphi| \le C \operatorname{dist}_{\mathbb{R}^n}(\cdot, \Sigma)^{\alpha - 1}, \qquad (2.6)
$$

and

$$
|T_{\mu\nu}|+|F| \leq C(1+|\partial \mathfrak{g}|^2+|\partial \varphi|^2).
$$

Then

$$
\mathfrak{g}\, ,\, \varphi\in C^\omega(B(R/2))\ .
$$

The proof will use the following simple lemma, whose proof is deferred until after the proof of Theorem [2.1.](#page-3-0)

LEMMA 2.2 Let Ω be an open subset of \mathbb{R}^n and $\Sigma \subset \Omega$ be a smooth submanifold of codimension $m \geq 2$ which either has no boundary or has boundary contained in $\partial\Omega$. Assume that $u \in W^{1,2}(\Omega)$ satisfies

$$
\partial_i(a^{ij}\partial_j u) = \partial_i g^i + f \ \text{in} \ \Omega \setminus \Sigma
$$

in the sense of distributions for some $a^{ij} \in L^{\infty}(\Omega)$, $f \in L^{1}(\Omega)$ and $g^{i} \in L^{2}(\Omega)$ $L^2(\Omega)$. Then u satisfies the above equation in Ω in the sense of distributions.

PROOF OF THEOREM [2.1:](#page-3-0) By hypothesis there exist coordinates y^i in which the metric coefficients and the fields satisfy

$$
V, \theta_i, g_{ij}, \varphi \in C^{0,\alpha}(B(R)) \cap C^2(B^*(R)).
$$
\n(2.7)

Standard arguments (compare [\[15\]](#page-11-8)) show that the metric is in fact smooth away from $Σ$:

$$
V, \theta_i, g_{ij}, \varphi \in C^{0,\alpha}(B(R)) \cap C^{\infty}(B^*(R)).
$$
\n(2.8)

Since the problem is local, it suffices to establish the desired regularity in a small ball $B(\epsilon)$ centered at a point on $\Sigma \cap B(R/2)$, which is assumed to be the origin. By a linear change of coordinates we can without loss of generality assume that $g_{ij}(0) = \delta_i^j$ i^j . Furthermore, we can also assume that

$$
\Sigma \cap B(10\epsilon) = \{ y = (y', 0) : y' \in \pi(\Sigma) \}
$$
 (2.9)

for some open set $\pi(\Sigma) \subset B(10\epsilon) \cap \mathbb{R}^{n-m}$.

Most of our arguments rely on elliptic estimates. Of the four equations in [\(2.5\)](#page-2-2), the last two are not manifestly elliptic. As is well known, this issue can be cured by passing to harmonic space-coordinates and using an appropriately chosen time function.

Suppose, first, that $\frac{n-m}{n} < \alpha < 1$.

For $\epsilon > 0$ let x^i be solutions of the problem

$$
\Delta_g x^i = 0 , \quad x^i |_{S(\epsilon)} = y^i .
$$

where $S(\epsilon) := \partial B(\epsilon)$ is a yⁱ-coordinate sphere of radius ϵ .

By [\[9,](#page-11-9) Theorem 8.34], and elliptic regularity away from the origin, we have $x^i \in C^{1,\alpha}(\bar{B}(\epsilon)) \cap C^{\infty}(B(\epsilon) \setminus \Sigma)$. If we write

$$
x^i = y^i + f^i \;,
$$

then f^i solves the divergence-type equation

$$
\partial_i(\sqrt{\det g}g^{ij}\partial_j f^{\ell}) = -\partial_i(\sqrt{\det g}g^{i\ell}) = -\partial_i(\sqrt{\det g}g^{i\ell} - \delta_i^{\ell}).
$$

Let

$$
f_{\epsilon}^{\ell}(x) := f^{\ell}(\epsilon x) , \quad \psi_{\epsilon}^{i\ell}(x) := \sqrt{\det g(\epsilon x)} g^{i\ell}(\epsilon x) , \quad \bar{\psi}_{\epsilon}^{i\ell}(x) := \psi_{\epsilon}^{i\ell}(x) - \psi_{\epsilon}^{i\ell}(0) .
$$

Then

$$
\partial_i(\psi_{\epsilon}^{ij}\,\partial_j f_{\epsilon}^{\ell}) = -\epsilon\,\partial_i\,\bar{\psi}_{\epsilon}^{i\ell} \text{ in } B(\epsilon) .
$$

Applying [\[9,](#page-11-9) Theorem 8.33] to the equation satisfied by f_{ϵ}^{ℓ} on a ball of radius two we obtain (note that the first term $|u|_0$ there can be discarded by the usual argument that exploits injectivity of the Laplace equation)

$$
||f_{\epsilon}^{\ell}||_{C^{1,\alpha}(B(1))} \leq C\epsilon ||\bar{\psi}_{\epsilon}^{i\ell}||_{C^{0,\alpha}(B(1))} \leq C\epsilon^{1+\alpha} . \tag{2.10}
$$

It thus follows that

$$
\frac{\partial x^i}{\partial y^j} = \delta_j^i + O(\epsilon^{\alpha}) \text{ in } B(\epsilon) . \tag{2.11}
$$

The implicit function theorem shows the x^i 's can be used as a coordinate system near $y^i = 0$ for all ϵ small enough. In what follows we choose some such value of ϵ .

As for the choice of a spacelike slice, we will now show that we can assume without loss of generality that

$$
\operatorname{div}_g(\theta) = 0 \text{ in } B(\epsilon) \text{ for sufficiently small } \epsilon. \tag{2.12}
$$

To use the freedom of defining t (equivalently, to fix the gauge freedom of θ), we make a coordinate change of the form $\tilde{t} = t + h(y)$. The metric **g** then takes the form

$$
\mathfrak{g} = -V(d\tilde{t} + \underbrace{(\theta_i - h_{,i}) dy^i}_{=: \tilde{\theta}_i dy^i \equiv \tilde{\theta}})^2 + g_{ij} dy^i dy^j.
$$

To obtain [\(2.12\)](#page-5-0), we pick $h \in W^{1,2}(B(\epsilon))$ to be a solution to

 $\Delta_g h = \text{div}_g(\theta)$ in $B(\epsilon)$, and $h|_{S(\epsilon)} = 0$.

By [\[9,](#page-11-9) Theorems 8.12 and 8.34], we have

$$
h \in C^{1,\alpha}(\bar{B}(\epsilon)) \cap W^{2,2}(B(\epsilon)) \cap C^{\infty}(B(\epsilon) \setminus \Sigma).
$$

Recalling (2.6) and rewriting the above equation for h as

$$
g^{k\ell}\,\partial_k\,\partial_\ell h = -\partial_\ell h\,\partial_k(\sqrt{\det g}\,g^{k\ell}) + \frac{1}{\sqrt{\det g}}\partial_k(\sqrt{\det g}\,g^{k\ell}\theta_\ell) ,
$$

we can apply [\[9,](#page-11-9) Theorem 9.11 and Lemma 9.16] to get,

$$
\partial^2 h \in L^q(B(\epsilon)) \text{ for any } 1 < q < \frac{m}{1-\alpha} ;
$$

here and in what follows the norm might depend upon ϵ , but this is irrelevant since a small epsilon has been now fixed. We have thus achieved [\(2.12\)](#page-5-0) with a penalty that the derivatives $\partial\theta$ no longer satisfy a pointwise estimate given by [\(2.6\)](#page-3-1) but the weaker estimate

$$
\partial \theta \in L^q(B(\epsilon)) \text{ for any } 1 < q < \frac{m}{1 - \alpha} \,. \tag{2.13}
$$

The above suffices for our purposes. We emphasize that V and g remain unchanged under the redefinition of θ , equivalently of time, as above.

To prepare for our passing to the coordinates x^i , we need some bound for the Hessian of f^i . For $y \in \mathbb{R}^n$, we will write $y = (y', y'')$ where $y' \in \mathbb{R}^{n-m}$ and $y'' \in \mathbb{R}^m$. In view of (2.9) we have

$$
d(y) := \text{dist}_{\mathbb{R}^n}(y, \Sigma) = |y''| \text{ for } y \in B(5\epsilon) \setminus \Sigma.
$$
 (2.14)

Define

$$
Q'_{s,S} = \{ y : s/4 \le |y''| \le 5s/4, |y| \le S \},
$$

$$
Q_{s,S} = \{ y : s/2 \le |y''| \le s, |y| \le S \}.
$$

Then, by (2.6) , f_s^{ℓ} satisfies

$$
|\psi_s^{ij} \, \partial_i \partial_j f_s^{\ell}| \leq C \, s^{\alpha+1} \, \text{ in } \, Q'_{1,s^{-1} \, \epsilon} \, .
$$

Thus, by [\[9,](#page-11-9) Theorem 9.11] and [\(2.10\)](#page-4-1),

$$
\|\partial^2 f_s^{\ell}\|_{L^q(Q_{1,s^{-1}\epsilon/2})} \leq C s^{\alpha+1} \left[\mathcal{H}^{n-m}(\Sigma_s) \right]^{1/q} \text{ for any } 1 < q < \infty,
$$

where \mathcal{H}^{n-m} denotes the $(n-m)$ -dimensional Hausdorff measure: More pre-cisely, we first apply [\[9,](#page-11-9) Theorem 9.11] to cubes of unit size and $f_s^{\ell} - L(f_s^{\ell})$ with $L(f_s^{\ell})$ being the linearization of f_s^{ℓ} at the center of those cubes, and then sum the acquired estimates over a collection of non-overlapping cubes covering the desired region. Because of the simple geometry of Σ_s (compare (2.9) , the number of cubes in each such collection is proportional to s^{m-n} , which is itself proportional to the Hausdorff dimension above. Scaling back, it follows that

$$
\|\partial^2 f^\ell\|_{L^q(Q_{s,\epsilon/2})} \leq C s^{\alpha-1+m/q} \text{ for any } 1 < q < \infty.
$$

Now if we pick q such that $\alpha - 1 + m/q > 0$, we can sum the above over dyadic rings in the tranverse direction to get

$$
\|\partial^2 f^\ell\|_{L^q(B(\epsilon/2))} \le C \text{ for any } 1 < q < \frac{m}{1-\alpha}. \tag{2.15}
$$

We pass now to the coordinates $x^i = y^i + f^i$, and still use the symbol \mathfrak{g} , g, θ and φ for the space-time metric, the spatial metric, the shift one-form and the matter fields in the new coordinates. Shifting the x^i 's by a constant

vector if necessary, we can assume that $x^{i}(0) = 0$. Furthermore, one has in the new coordinates

$$
V, \theta_i, g_{ij}, \varphi \in C^{0,\alpha}(B(\epsilon)) \cap C^{\infty}(B^*(\epsilon)) .
$$

Estimate [\(2.15\)](#page-6-0) shows that $x^i \in W^{2,q}(B(\epsilon/2))$, and by [\(2.6\)](#page-3-1), [\(2.13\)](#page-5-1), the chain rule and the transformation law for tensors one deduces that

$$
|\partial_x \mathfrak{g}| + |\partial_x \varphi| \in L^q(B(\epsilon/2)) \text{ for any } 1 < q < \frac{m}{1-\alpha}. \tag{2.16}
$$

In the coordinates x^i the Einstein-field equations [\(2.5\)](#page-2-2) can be rewritten in the following form

$$
\begin{cases}\n\Delta_g \varphi^A = F(\mathfrak{g}, \partial \mathfrak{g}, \varphi, \partial \varphi) ,\\ \n\Delta_g V = F_1(\mathfrak{g}, \partial \mathfrak{g}) + \tilde{T}_V ,\\ \n\Delta_g \theta_i = F_{(i)}(\mathfrak{g}, \partial \mathfrak{g}) - (\tilde{T}_\theta)_i ,\\ \n\Delta_g g_{ij} - \partial_i (\partial_j \log V) = F_{(i)(j)}(\mathfrak{g}, \partial \mathfrak{g}) + (\tilde{T}_g)_{ij} ,\n\end{cases} (2.17)
$$

where we have used (2.12) . Using (2.16) together with the given growth rate of F and T, one sees that $(V, \theta, g, \varphi) \in W^{1,2}(B(\epsilon/2))$ while the right side of [\(2.17\)](#page-7-1) belongs to $L^p(B(\epsilon/2))$ for any $p < \frac{m}{2(1-\alpha)}$. Also, by Lemma [2.2,](#page-3-2) [\(2.17\)](#page-7-1) is satisfied across Σ in the sense of distribution. It is useful to write the last equation in [\(2.17\)](#page-7-1) as

$$
\Delta_g g_{ij} = \partial_i (\partial_j \log V) + \tilde{F}_3(\mathfrak{g}, \partial \mathfrak{g}) + (\tilde{T}_g)_{ij} . \qquad (2.18)
$$

To proceed, we distinguish two cases according to whether $\alpha > 1 - \frac{m}{2n}$ $2n$ or $\alpha \leq 1 - \frac{m}{2n}$ $\frac{m}{2n}$. In the former case, we apply [\[14,](#page-11-10) Theorem 5.5.3(b)] to the first three equations of [\(2.17\)](#page-7-1) to assert that $(\varphi, V, \theta) \in C^{1,\sigma}(B(\epsilon/3))$ for some $\sigma > 0$. In particular, $\partial \log V \in C^{0,\sigma}(B(\epsilon/3))$. Applying [\[14,](#page-11-10) Theorem 5.5.3(b)] again to [\(2.18\)](#page-7-2), we get $g \in C^{1,\sigma}(B(\epsilon/4))$.

In the latter case, we use $[14,$ Theorem 5.5.3(a). Applying this result to the first three equations in (2.17) and then to (2.18) as in the previous paragraph we get $(\varphi, V, \theta, g) \in W^{1,q}(B(\epsilon/4))$ for any $1 < q < \frac{m}{2(1-\alpha)-\frac{m}{n}}$. In other words, in $B(\epsilon/4)$, [\(2.16\)](#page-7-0) is improved with α replaced by $\alpha + (\alpha - \frac{n-m}{n})$ $\frac{-m}{n}$). Repeating this process for a finite number of time, we arrive at a situation when the argument in the previous paragraph applies.

In any event, one obtains $(g, \varphi) \in C^{1,\sigma}(B(\epsilon/100))$. A standard bootstrap argument based on Schauder estimates proves smoothness; analyticity readily follows.

When $\alpha = 1$ we replace α by any number in $(1 - \frac{m}{2n})$ $\frac{m}{2n}, 1)$ and arrive to (2.16) as before. Since $T_{\mu\nu}$ is a tensor, it gives a bounded contribution to [\(2.17\)](#page-7-1), leading to a metric with improved regularity as before. Similarly F is a tensor giving a bounded contribution to [\(2.3\)](#page-2-3), and we obtain $(g, \varphi) \in C^{1,\sigma}(B(\epsilon/2))$ by the same method as above. The result follows.

To finish this section, we provide the

PROOF OF LEMMA [2.2:](#page-3-2) Let $\xi \in C_c^{\infty}$ $_{c}^{\infty}(\Omega)$, we need to show that

$$
\int_{\Omega} a^{ij} \, \partial_i u \, \partial_j \xi \, dx = \int_{\Omega} (g^i \, \partial_i \xi + f \, \xi) \, dx \,. \tag{2.19}
$$

Let η be a smooth cut-off function on R such that $\eta(t) = 0$ for $t \leq 1$ and $\eta(t) = 1$ for $t \geq 2$. For δ sufficiently small, define

$$
\varrho_{\delta}(x) = \begin{cases} \eta\left(\frac{d(x,\Sigma)}{\delta}\right) \text{ for } m \geq 3, \\ \eta\left(\frac{\log(-\log d(x,\Sigma))}{\log(-\log \delta)}\right) \text{ for } m = 2; . \end{cases}
$$

By hypothesis we have

$$
\int_{\Omega} a^{ij} \, \partial_i u \, \partial_j (\xi \, \varrho_\delta) \, dx = \int_{\Omega} (g^i \, \partial_i (\xi \, \varrho_\delta) + f \, \xi \, \varrho_\delta) \, dx \; .
$$

[\(2.19\)](#page-8-1) can be reached by passing $\delta \rightarrow 0$ using Lebesgue's dominated convergence theorem, Cauchy-Schwarz's inequality and the explicit form of ϱ_{δ} . Note that when $m = 2$, we need to use

$$
\int_0^1 \frac{1}{t (\log t)^2} dt < \infty .
$$

We omit the details. \Box

3 Ghost points

We show how Theorem [2.1](#page-3-0) applies to stationary solutions obtained by introducing ghost points in the solitonic solution-generating technique. Here the space-time metric that one wishes to construct is invariant under an abelian isometry group $\mathbb{R} \times \mathbb{T}^{n-2}$, where the $\mathbb R$ factor represents t–translations. The

metric depends only upon two coordinates (ρ, z) , which can be thought of as cylindrical coordinates on \mathbb{R}^3 . In this construction, if the ghost point is placed at $(\rho = 0, z = 0)$, one obtains a solution of the vacuum Einstein equations $\mathfrak{g}_{\mu\nu}$ defined on

$$
\{t\in\mathbb{R}, (x, y, z)\in B(\delta)\subset\mathbb{R}^3\}\times\mathbb{T}^{n-3},
$$

where

$$
x = \rho \cos \varphi \;, \quad y = \rho \sin \varphi \;.
$$

Note that one S^1 factor from \mathbb{T}^{n-2} has been interpreted as a rotation around the z -axis of \mathbb{R}^3 . Furthermore, there exists a neighborhood of the origin in which the metric functions are analytic functions of (ρ^2, z, d) , where d is the Euclidean distance to the origin in \mathbb{R}^3 :

$$
d:=\sqrt{\rho^2+z^2}.
$$

So the singular set

$$
\Sigma = \{x = y = z = 0\} \times \mathbb{T}^{n-3}
$$

has dimension $n-3$ within each slice $t = \text{const}$, hence codimension three. To apply Theorem [2.1](#page-3-0) we need to verify that, reducing δ if necessary,

there exists
$$
\varepsilon > 0
$$
 such that det $\mathfrak{g}^{ij} > \varepsilon$ and $\mathfrak{g}_{tt} < -\varepsilon$. (3.1)

Note that the function d is Lipschitz-continuous, but not differentiable. This implies that the metric functions are in $C^{\infty}(B^*(R)) \cap C^{0,1}(B(R))$. Theo-rem [2.1](#page-3-0) with $\alpha = 1$ shows then that the metric functions are real-analytic in a whole neighborhood of the origin of \mathbb{R}^n , as desired.

In the case of the Black Saturn metric we have $n = 4$, and the spacedimension of the singular set is one. To verify that this metric is analytic near its ghost point $\alpha_1 = (\rho = 0, z = a_1)$, one needs to verify [\(3.1\)](#page-9-0). A direct verification in the coordinate system used in [\[4\]](#page-10-0) fails, because at this point \mathfrak{g}_{tt} becomes null for all Black Saturn metrics (indeed, α_1 always lies on the ergosurface for those metrics). This can be bypassed by checking that the limit of the metric at α_1 is Lorentzian, and that the determinant of the matrix of scalar products of all Killing vectors there has a strictly negative value. This guarantees that some linear combination of Killing vectors is timelike at α_1 , and our theorem applies.

As another application, our analysis reduces the question of regularity of the metrics of [\[5,](#page-10-4) [8,](#page-11-4) [13,](#page-11-5) [18,](#page-11-6) [19\]](#page-11-7) near their ghost points to showing that the components of the metric tensor in coordinates (x, y, z) have a finite limit at the ghost points, and verifying [\(3.1\)](#page-9-0) there (after perhaps replacing ∂_t by a different, timelike Killing vector if necessary). We note that this might require tedious symbolic algebra calculations, and our experience with the Black Saturn metrics suggests that the checking of the timelike character of the orbit of the isometry group through the ghost point might be non-trivial. In any case we have not attempted to carry this out.

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