Quantum Gravity Model in the framework of Weyl-Cartan geometry

B. S. Park*

College of Natural Sciences, Seoul National University, Seoul 151-742, Korea

D. G. Pak^{\dagger}

Institute of Applied Physics, Uzbekistan National University, Vuzgorodok, Tashkent, 100-174, Uzbekistan

We study the Weyl vector fields which can play an important role in quantum gravity. The metric obtains its dynamical content after dynamical symmetry breaking in the phase of the effective Einstein gravity which is induced by quantum Weyl corrections. In low energy regime with scalar field there is a relation between the Weyl vector fields and the torsion fields. If this condition is given to Weyl vector fields and torsions, then the Lagrangian becomes like Maxwell type.

PACS numbers: 04.60.-m, 04.62.+v, 11.30.Qc, 11.30.Cp

Keywords: effective action, Weyl gravity, conformal gravity, general relativity, quantum gravitation, Riemann normal coordinate, Unified field theory

I. INTRODUCTION

In the former paper[1] we propose a special R^2 -type model of Lorentz gauge gravity which admits a topological phase at classical level and has non-trivial quantum dynamics of torsion. The proposed model is minimal in a sense that only the contortion possesses dynamic degrees of freedom whereas the metric does not. We demonstrate that the contortion has six propagating modes with spins J = (2; 1; 0; 0), exactly the same number of physical degrees of freedom the metric tensor has in general. In the present paper we will study further in the case of Weyl-Cartan geometry. In torsion free case the Gauss-Bonnet Lagrangian with the Weyl vector fields has negative kinetic energy term. But the Yang-Mills type Lagrangian has positive kinetic energy term. And we will show that Weyl vector field also can be a candidate for quantum gravity, and Einstein-Hilbert term can be induced by quantum corrections due to the condensation of Weyl vector fields. Next we consider the Palatini formalism with Einstein-Hilbert action. And in this case we will show that the Weyl vector fields become source of the torsions. But in this case Weyl vector fields and torsions are not dynamical fields. And if we give this special relation between torsions and Weyl vector fields to the general quadratic curvature Lagrangian, the Weyl vector fields become U(1) vector fields.

II. THE YANG-MILLS TYPE WEYL GRAVITY

In Weyl's geometry, besides the general coordinate transformation of Einstein, we deals with the (Weyl) gauge transformation under which any length gets multiplied by a factor $e^{\Lambda(x)}$. So $ds' = e^{\Lambda(x)}ds$. A local tensor T(x) which is transformed as $T'(x) = e^{J\Lambda(x)}T(x)$ is called a co-tensor of weight J. Since $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, $g_{\mu\nu}$ is a co-tensor of weight 2 under understanding that the dx^{μ} are not affected by Weyl transformation : $g'_{\mu\nu} = e^{2\Lambda(x)}g_{\mu\nu}$.¹

^{*}Electronic address: psychist@phya.snu.ac.kr

 $^{^{\}dagger} Electronic address: dmipak@gmail.com$

¹ The signature of the metric is (-,+,+,+).

Now with above settings let us define following co-covariant derivative

$$D_{\mu} = \partial_{\mu} - \Gamma_{\mu} + f J W_{\mu} \equiv \nabla_{\mu} + f J W_{\mu} \tag{1}$$

where Γ_{μ} is the Weyl gauge invariant connections, f is coupling constant, J is the weight of tensor density and W_{μ} is the Weyl vector fields. The metric and the Weyl vector field transform like following under the Weyl gauge transformation.

$$g'_{\mu\nu} = e^{2\Lambda(x)}g_{\mu\nu} \tag{2}$$

$$W'_{\mu} = W_{\mu} - \frac{1}{f} \partial_{\mu} \Lambda \tag{3}$$

The co-covariant derivative acts on co-scalar S of weight J like

$$D_{\mu}S = \partial_{\mu}S + fJW_{\mu}S \equiv \tilde{\partial}_{\mu}S. \tag{4}$$

For co-vector V_{μ} ,

$$D_{\mu}V_{\nu} = \tilde{\partial}_{\mu}V_{\nu} - \Gamma_{\mu\nu}^{\ \rho}V_{\rho} = \partial_{\mu}V_{\nu} - \Gamma_{\mu\nu}^{\ \rho}V_{\rho} + JfW_{\mu}V_{\nu}$$
$$= \partial_{\mu}V_{\nu} - \hat{\Gamma}_{\mu\nu}^{\ \rho}V_{\rho} - f\left(\delta_{\nu}^{\rho}W_{\mu} + \delta_{\mu}^{\rho}W_{\nu} - g_{\mu\nu}W^{\rho}\right)V_{\rho} + JfW_{\mu}V_{\nu}$$
(5)

where $\hat{\Gamma}_{\mu\nu}^{\ \rho} = \bar{\Gamma}_{\mu\nu}^{\ \rho} + K_{\mu\nu}^{\ \rho}$ and $\bar{\Gamma}_{\mu\nu}^{\ \rho}$ is the Christoffel symbol and $K_{\mu\nu}^{\ \rho}$ is the contortion tensor. Here we note that

$$D_{\mu}g_{\nu\lambda} = \nabla_{\mu}g_{\nu\lambda} + fJW_{\mu}g_{\nu\lambda} = \nabla_{\mu}g_{\nu\lambda} + 2fW_{\mu}g_{\nu\lambda} = 0.$$
(6)

So $\nabla_\mu g_{\nu\lambda}=-2fW_\mu g_{\nu\lambda}\equiv -Q_{\mu\nu\lambda}$ which means the non-metricity.

Now let $V^{(0)\mu}$ be a weight 0 vector and let $V^{\mu} = \sqrt{-g}^{J/4} V^{(0)\mu}$. Then we have

$$\begin{split} [D_{\mu}, D_{\nu}] V^{\rho} &= [D_{\mu}, D_{\nu}] (V^{(0)\rho} (\sqrt{-g})^{J/4})^{\cdots D_{\mu} g_{\nu\lambda} = 0} (\sqrt{-g})^{J/4} [D_{\mu}, D_{\nu}] V^{(0)\rho} \\ & \swarrow^{\cdots \Gamma_{\mu\nu}^{\lambda}, W_{\mu} \text{ has no weight!}} \\ &= (\sqrt{-g})^{J/4} [\nabla_{\mu}, \nabla_{\nu}] V^{(0)\rho} = (\sqrt{-g})^{J/4} \left\{ R_{\mu\nu\lambda}^{\ \rho} V^{(0)\lambda} - t_{\mu\nu}^{\ \sigma} \nabla_{\sigma} V^{(0)\rho} \right\} \\ &= \left\{ R_{\mu\nu\lambda}^{\ \rho} (\sqrt{-g})^{J/4} V^{(0)\lambda} - t_{\mu\nu}^{\ \sigma} D_{\sigma} \left(\sqrt{-g}^{J/4} V^{(0)\rho} \right) \right\} \\ &= R_{\mu\nu\lambda}^{\ \rho} V^{\lambda} - t_{\mu\nu}^{\ \sigma} D_{\sigma} V^{\rho} \end{split}$$
(7)

Therefore D_{μ} and ∇_{μ} have the same curvature and torsion. Now under the decomposition (5) the curvature tensor is split into three parts

$$R_{\mu\nu\lambda}^{\ \rho} = \bar{R}_{\mu\nu\lambda}^{\ \rho} + \tilde{R}_{\mu\nu\lambda}^{\ \rho} + Q_{\mu\nu\lambda}^{\ \rho} \tag{8}$$

where

$$\bar{R}_{\mu\nu\lambda}^{\ \rho} = 2 \left\{ \partial_{[\mu}\bar{\Gamma}_{\nu]\lambda}^{\ \rho} + \bar{\Gamma}_{[\mu]\sigma]}^{\ \rho}\bar{\Gamma}_{\nu]\lambda}^{\ \sigma} \right\} \tag{9}$$

$$\tilde{R}^{\ \rho}_{\mu\nu\lambda} = 2\left\{\bar{\nabla}_{[\mu}K^{\ \rho}_{\nu]\lambda} + K^{\ \rho}_{[\mu|\sigma|}K^{\ \sigma}_{\nu]\lambda}\right\}$$
(10)

$$Q_{\mu\nu\lambda}^{\ \rho} = 2 \left\{ \hat{\nabla}_{[\mu}\hat{Q}_{\nu]\lambda}^{\ \rho} + \hat{Q}_{[\mu|\sigma|}^{\ \rho}\hat{Q}_{\nu]\lambda}^{\ \sigma} + \hat{\Gamma}_{[\mu\nu]}^{\ \sigma}\hat{Q}_{\sigma\lambda}^{\ \rho} \right\}$$

$$= 2 \left\{ \bar{\nabla}_{[\mu}\hat{Q}_{\nu]\lambda}^{\ \rho} + \hat{Q}_{[\mu|\sigma|}^{\ \rho}\hat{Q}_{\nu]\lambda}^{\ \sigma} + K_{[\mu|\sigma|}^{\ \rho}\hat{Q}_{\nu]\lambda}^{\ \sigma} + \hat{Q}_{[\mu|\sigma|}^{\ \rho}K_{\nu]\lambda}^{\ \sigma} \right\},$$
(11)

and

$$K_{\mu\nu}^{\ \rho} = K_{[\mu\nu]}^{\ \rho} + K_{(\mu\nu)}^{\ \rho} = \frac{1}{2} t_{\mu\nu}^{\ \rho} + t_{(\mu\nu)}^{\rho}$$
(12)

$$\hat{Q}_{\mu\nu}^{\ \rho} = f\left(\delta^{\rho}_{\nu}W_{\mu} + \delta^{\rho}_{\mu}W_{\nu} - g_{\mu\nu}W^{\rho}\right).$$
(13)

where $\bar{\nabla}_{\mu}$ is the covariant derivative containing only the Christoffel symbol part and $\hat{\nabla}_{\mu}$ is that containing both the Christoffel symbol part and the contortion tensor.

We can express the Weyl part curvature in terms of the Weyl vector fields and the torsion. Then,

$$Q_{\mu\nu\lambda\alpha} = 2fg_{\lambda\alpha}\hat{\nabla}_{[\mu}W_{\nu]} - 4fg_{[\alpha|[\mu}\hat{\nabla}_{\nu]]}W_{\lambda]} + 4f^2W_{[\mu}g_{\nu][\lambda}W_{\alpha]} - 2f^2W^{\sigma}W_{\sigma}g_{\alpha[\mu}g_{\nu]\lambda} + f\left(t_{\mu\nu}^{\sigma}W_{\sigma}g_{\lambda\alpha} + t_{\mu\nu\alpha}W_{\lambda} - t_{\mu\nu\lambda}W_{\alpha}\right) = 2fg_{\lambda\alpha}\bar{\nabla}_{[\mu}W_{\nu]} - 4fg_{[\alpha|[\mu}\bar{\nabla}_{\nu]]}W_{\lambda]} + 4f^2W_{[\mu}g_{\nu][\lambda}W_{\alpha]} - 2f^2W^{\sigma}W_{\sigma}g_{\alpha[\mu}g_{\nu]\lambda}$$
(14)

$$+f\left(4g_{[\alpha|[\mu}K_{\nu]]\lambda]}^{\sigma}W_{\sigma} + t_{\mu\nu\alpha}W_{\lambda} - t_{\mu\nu\lambda}W_{\alpha}\right)$$
(15)

And

$$Q_{\mu\lambda} = g^{\nu\alpha}Q_{\mu\nu\lambda\alpha}$$

= $2f\bar{\nabla}_{\mu}W_{\lambda} + 2f\bar{\nabla}_{[\mu}W_{\lambda]} + fg_{\mu\lambda}\bar{\nabla}_{\alpha}W^{\alpha} - 2f^{2}W_{\mu}W_{\lambda} + 2f^{2}g_{\mu\lambda}W^{\sigma}W_{\sigma}$
 $-f\left(2K_{\mu\lambda}^{\sigma}W_{\sigma} + t_{\mu}^{\alpha}{}_{\lambda}W_{\alpha} - t_{\mu\alpha}^{\alpha}W_{\lambda} + g_{\mu\lambda}t_{\alpha}^{\sigma}{}^{\alpha}W_{\sigma}\right)$ (16)

$$\begin{split}
\check{Q}_{\mu\lambda} &\equiv g^{\nu\alpha}Q_{\nu\mu\alpha\lambda} \\
&= 2f\bar{\nabla}_{\mu}W_{\lambda} - 2f\bar{\nabla}_{[\mu}W_{\lambda]} + fg_{\mu\lambda}\bar{\nabla}_{\alpha}W^{\alpha} - 2f^{2}W_{\mu}W_{\lambda} + 2f^{2}g_{\mu\lambda}W^{\sigma}W_{\sigma} \\
&- f\left(2K_{\mu\lambda}^{\sigma}W_{\sigma} + t_{\mu}^{\alpha}{}_{\lambda}W_{\alpha} + t_{\mu\alpha}^{\alpha}W_{\lambda} + g_{\mu\lambda}t_{\alpha}^{\sigma}{}^{\alpha}W_{\sigma}\right)
\end{split}$$
(17)

$$Q = g^{\mu\lambda}Q_{\mu\lambda}$$

= $6f\hat{\nabla}_{\mu}W^{\mu} + 6f^{2}W^{\mu}W_{\mu} - 2ft_{\mu\nu}^{\ \mu}W^{\nu}$
= $6f\bar{\nabla}_{\mu}W^{\mu} + 6fK_{\mu\nu}^{\ \mu}W^{\nu} + 6f^{2}W^{\mu}W_{\mu} - 2ft_{\mu\nu}^{\ \mu}W^{\nu}$
= $6f\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}W^{\mu}\right) + 6f^{2}W^{\mu}W_{\mu} + 4ft_{\mu\nu}^{\ \mu}W^{\nu}$ (18)
 $\check{Q} = g^{\mu\lambda}\check{Q}_{\mu\lambda}$

$$= 6f\bar{\nabla}_{\mu}W^{\mu} + 6f^{2}W^{\mu}W_{\mu} + 6ft_{\mu\nu}^{\ \mu}W^{\nu}.$$
(19)

Now for simplicity let us concentrate on the torsion free case. And we are going on computing the square terms of curvature. First the square of curvature tensors are

$$(R_{\mu\nu\lambda\alpha})^{2} = (\bar{R}_{\mu\nu\lambda\alpha} + Q_{\mu\nu\lambda\alpha})(\bar{R}^{\mu\nu\lambda\alpha} + Q^{\mu\nu\lambda\alpha})$$

$$= \bar{R}_{\mu\nu\lambda\alpha}\bar{R}^{\mu\nu\lambda\alpha} + 2Q_{\mu\nu\lambda\alpha}\bar{R}^{\mu\nu\lambda\alpha} + Q_{\mu\nu\lambda\alpha}Q^{\mu\nu\lambda\alpha}$$

$$= (\bar{R}_{\mu\nu\lambda\alpha})^{2} + 8f\bar{R}^{\mu\alpha}\bar{\nabla}_{\mu}W_{\alpha}$$

$$+4f^{2}\left\{2(\bar{\nabla}_{\mu}W_{\nu})^{2} + 4(\bar{\nabla}_{[\mu}W_{\nu]})^{2} + (\bar{\nabla}_{\mu}W^{\mu})^{2} - 2\bar{R}^{\mu\alpha}W_{\mu}W_{\alpha} + \bar{R}W^{\sigma}W_{\sigma}\right\}$$

$$-16f^{3}\left\{W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} - W_{\mu}W^{\mu}\bar{\nabla}_{\alpha}W^{\alpha}\right\} + 12f^{4}(W^{\sigma}W_{\sigma})^{2},$$

$$(20)$$

$$R_{\mu\nu\lambda\alpha}R^{\lambda\alpha\mu\nu} = (\bar{R}_{\mu\nu\lambda\alpha} + Q_{\mu\nu\lambda\alpha})(\bar{R}^{\mu\nu\lambda\alpha} + Q^{\lambda\alpha\mu\nu})$$

$$= \bar{R}_{\mu\nu\lambda\alpha}\bar{R}^{\mu\nu\lambda\alpha} + 2Q_{\mu\nu\lambda\alpha}\bar{R}^{\mu\nu\lambda\alpha} + Q_{\mu\nu\lambda\alpha}Q^{\lambda\alpha\mu\nu}$$

$$= (\bar{R}_{\mu\nu\lambda\alpha})^2 + 8f\bar{R}^{\mu\alpha}\bar{\nabla}_{\mu}W_{\alpha} \qquad (21)$$

$$+4f^2 \left\{ 2(\bar{\nabla}_{\mu}W_{\nu})(\bar{\nabla}^{\nu}W^{\mu}) + (\bar{\nabla}_{\mu}W^{\mu})^2 - 2\bar{R}^{\mu\alpha}W_{\mu}W_{\alpha} + \bar{R}W^{\sigma}W_{\sigma} \right\}$$

$$-16f^3 \left\{ W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} - W_{\mu}W^{\mu}\bar{\nabla}_{\alpha}W^{\alpha} \right\} + 12f^4(W^{\sigma}W_{\sigma})^2$$

$$= (\bar{R}_{\mu\nu\lambda\alpha})^2 + 8f\bar{R}^{\mu\alpha}\bar{\nabla}_{\mu}W_{\alpha} \qquad (22)$$

$$+4f^2 \left\{ 2\bar{\nabla}_{\mu}(W_{\nu}\bar{\nabla}^{\nu}W^{\mu}) + (\bar{\nabla}_{\mu}W^{\mu})^2 - 2W^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}W^{\mu} + \bar{R}W^{\sigma}W_{\sigma} \right\}$$

$$-16f^3 \left\{ W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} - W_{\mu}W^{\mu}\bar{\nabla}_{\alpha}W^{\alpha} \right\} + 12f^4(W^{\sigma}W_{\sigma})^2,$$

and the square of the Ricci tensors are

$$(R_{\mu\lambda})^{2} = (\bar{R}_{\mu\lambda} + Q_{\mu\lambda})(\bar{R}^{\mu\lambda} + Q^{\mu\lambda}) = \bar{R}_{\mu\lambda}\bar{R}^{\mu\lambda} + 2Q_{\mu\lambda}\bar{R}^{\mu\lambda} + Q_{\mu\lambda}Q^{\mu\lambda}$$

$$= (\bar{R}_{\mu\lambda})^{2} + 2f\left(2\bar{R}^{\mu\lambda}\bar{\nabla}_{\mu}W_{\lambda} + \bar{R}\bar{\nabla}_{\alpha}W^{\alpha}\right)$$

$$+ f^{2}\left\{-4\bar{R}^{\mu\lambda}W_{\mu}W_{\lambda} + 4\bar{R}W^{\sigma}W_{\sigma} + 4(\bar{\nabla}_{\mu}W_{\lambda})^{2} + 12(\bar{\nabla}_{[\mu}W_{\lambda]})^{2} + 6(\bar{\nabla}_{\alpha}W^{\alpha})^{2}\right\} - f^{3}\left\{8W_{\mu}W_{\lambda}\bar{\nabla}^{\mu}W^{\lambda} - 20(W_{\sigma}W^{\sigma})(\bar{\nabla}_{\alpha}W^{\alpha})\right\}$$

$$+ 12f^{4}(W^{\alpha}W_{\alpha})^{2}, \qquad (23)$$

$$(R_{\mu\lambda} + \breve{R}_{\mu\lambda})(R^{\lambda\mu} + \breve{R}^{\lambda\mu}) = (2\bar{R}_{\mu\lambda} + Q_{\mu\lambda} + \breve{Q}_{\mu\lambda})(2\bar{R}^{\mu\lambda} + Q^{\lambda\mu} + \breve{Q}^{\lambda\mu})$$
(24)

$$= 4(\bar{R}_{\mu\lambda})^2 + 8f\left(2\bar{R}^{\mu\lambda}\bar{\nabla}_{\mu}W_{\lambda} + \bar{R}\bar{\nabla}_{\alpha}W^{\alpha}\right)$$
(25)

$$+16f^{2}\left\{-\bar{R}^{\mu\lambda}W_{\mu}W_{\lambda}+\bar{R}W^{\sigma}W_{\sigma}+(\bar{\nabla}_{\mu}W_{\lambda})(\bar{\nabla}^{\lambda}W^{\mu})+2(\bar{\nabla}_{\alpha}W^{\alpha})^{2}\right\}\\-16f^{3}\left\{2W_{\mu}W_{\lambda}\bar{\nabla}^{\mu}W^{\lambda}-5(W_{\sigma}W^{\sigma})(\bar{\nabla}_{\alpha}W^{\alpha})\right\}+48f^{4}(W^{\alpha}W_{\alpha})^{2},$$

and the square of Ricci scalar is

$$R^{2} = (\bar{R} + Q)^{2} = \bar{R}^{2} + 2\bar{R}Q + Q^{2}$$

= $\bar{R}^{2} + 12f\bar{R}\bar{\nabla}_{\mu}W^{\mu} + 12f^{2}(\bar{R}W^{\mu}W_{\mu} + 3(\bar{\nabla}_{\mu}W^{\mu})^{2})$
+ $72f^{3}W^{\mu}W_{\mu}\bar{\nabla}_{\sigma}W^{\sigma} + 36f^{4}(W^{\mu}W_{\mu})^{2}.$ (26)

Now with these terms, let us think of Gauss-Bonnet like Lagrangian. This Lagrangian has the Weyl symmetry and reduces the topological invariant in Riemann space-time.

$$\mathcal{L}_{GB} = -\frac{1}{4} \left\{ (R_{\mu\nu\lambda\alpha})^2 - 4(R_{\mu\lambda})^2 + R^2 \right\} = -\frac{1}{4} \left\{ (\bar{R}_{\mu\nu\lambda\alpha})^2 - 4(\bar{R}_{\mu\lambda})^2 + \bar{R}^2 - f \left(8\bar{R}^{\mu\nu}\bar{\nabla}_{\mu}W_{\nu} - 4\bar{R}\bar{\nabla}_{\mu}W^{\mu} \right) + f^2 \left(8\bar{R}^{\mu\nu}W_{\mu}W_{\nu} - 8(\bar{\nabla}_{\mu}W_{\nu})^2 - 32(\bar{\nabla}_{[\mu}W_{\nu]})^2 + 16(\bar{\nabla}_{\mu}W^{\mu})^2 \right) + f^3 \left(16W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} + 8W^{\mu}W_{\mu}\bar{\nabla}_{\alpha}W^{\alpha} \right) \right\}$$
(27)

Using $(\bar{\nabla}_{\mu}W_{\nu})^2 = 2(\bar{\nabla}_{[\mu}W_{\nu]})^2 + R_{\mu\nu}W^{\mu}W^{\nu} + \bar{\nabla}_{\mu}(W^{\nu}\bar{\nabla}_{\nu}W^{\mu}) - W^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}W^{\mu}$, we can express the square terms like following

$$\mathcal{L}_{GB}^{(2)} = -\frac{1}{4}f^{2} \Big\{ -48(\bar{\nabla}_{[\mu}W_{\nu]})^{2} + 16(\bar{\nabla}_{\mu}W^{\mu})^{2} - 8\bar{\nabla}_{\mu}(W^{\nu}\bar{\nabla}_{\nu}W^{\mu}) + 8W^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}W^{\mu} \Big\} \\ = -\frac{1}{4}f^{2} \Big\{ -12F_{\mu\nu}F^{\mu\nu} + 16(\bar{\nabla}_{\mu}W^{\mu})^{2} - 8\bar{\nabla}_{\mu}(W^{\nu}\bar{\nabla}_{\nu}W^{\mu}) + 8W^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}W^{\mu} \Big\}$$

where $F_{\mu\nu} \equiv \bar{\nabla}_{\mu} W_{\nu} - \bar{\nabla}_{\nu} W_{\mu}$. Unfortunately this Lagrangian has the negative kinetic energy terms of the Weyl vector fields. So it is not interesting. Of course this Lagrangian is not the topological invariant in Weyl geometry. In general the Weyl-Cartan geometry has the topological invariant quantity slightly different from the Gauss-Bonnet identity[2]. It has the form like following

$$I_{BF} \equiv R^2 - (R_{\mu\nu} + \breve{R}_{\mu\nu})(R^{\nu\mu} + \breve{R}^{\nu\mu}) + R_{\mu\nu\alpha\beta}R^{\alpha\beta\mu\nu}.$$
 (28)

So the topological invariant Lagrangian is

$$\mathcal{L}_{BF} = -\frac{1}{4} \left\{ (R_{\mu\nu\lambda\alpha})^{2} - (R_{\mu\nu} + \breve{R}_{\mu\nu})(R^{\nu\mu} + \breve{R}^{\nu\mu}) + R^{2} \right\} \\
= -\frac{1}{4} \left\{ (\bar{R}_{\mu\nu\lambda\alpha})^{2} - 4(\bar{R}_{\mu\lambda})^{2} + \bar{R}^{2} - f\left(8\bar{R}^{\mu\nu}\bar{\nabla}_{\mu}W_{\nu} - 4\bar{R}\bar{\nabla}_{\mu}W^{\mu}\right) \\
+ f^{2}\left(8\bar{R}^{\mu\nu}W_{\mu}W_{\nu} - 8(\bar{\nabla}_{\mu}W_{\nu})(\bar{\nabla}^{\nu}W^{\mu}) + 8(\bar{\nabla}_{\mu}W^{\mu})^{2}\right) \\
+ f^{3}\left(16W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} + 8W^{\mu}W_{\mu}\bar{\nabla}_{\alpha}W^{\alpha}\right) \right\}$$

$$(29)$$

$$= -\frac{1}{4} \left\{ (\bar{R}_{\mu\nu\lambda\alpha})^{2} - 4(\bar{R}_{\mu\lambda})^{2} + \bar{R}^{2} - f\left(8\bar{R}^{\mu\nu}\bar{\nabla}_{\mu}W_{\nu} - 4\bar{R}\bar{\nabla}_{\mu}W^{\mu}\right) \\
+ f^{2}\left(8W_{\nu}\bar{\nabla}^{\nu}\bar{\nabla}_{\mu}W^{\mu} - 8\bar{\nabla}_{\mu}(W_{\nu}\bar{\nabla}^{\nu}W^{\mu}) + 8(\bar{\nabla}_{\mu}W^{\mu})^{2}\right) \\
+ f^{3}\left(16W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} + 8W^{\mu}W_{\mu}\bar{\nabla}_{\alpha}W^{\alpha}\right) \right\}$$

But if we apply the gauge fixing condition $\overline{\nabla}_{\mu}W^{\mu} = 0$ to this Lagrangian, the square terms has no dynamics. So this is also out of interesting. Now let us think of another the Yang-Mills type Lagrangian.

$$\mathcal{L} = -\frac{1}{4} (R_{\mu\nu\lambda\alpha})^{2}
= -\frac{1}{4} \Big\{ (\bar{R}_{\mu\nu\lambda\alpha})^{2} + 8f\bar{R}^{\mu\alpha}\bar{\nabla}_{\mu}W_{\alpha}
+ 4f^{2} \Big\{ 8(\bar{\nabla}_{[\mu}W_{\nu]})^{2} + (\bar{\nabla}_{\mu}W^{\mu})^{2} + \bar{R}W^{\sigma}W_{\sigma} + 2\bar{\nabla}_{\mu}(W^{\nu}\bar{\nabla}_{\nu}W^{\mu})
- 2W^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}W^{\mu} \Big) - 16f^{3} \Big(W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} - W_{\mu}W^{\mu}\bar{\nabla}_{\alpha}W^{\alpha} \Big) + 12f^{4}(W^{\sigma}W_{\sigma})^{2} \Big\}$$
(30)

And this Lagrangian has the positive kinetic energy terms. If we omit the total divergence terms and $\bar{\nabla}_{\mu}W^{\mu}$ terms which vanish under the gauge fixing condition $\bar{\nabla}_{\mu}W^{\mu} = 0$. Then the Lagrangian becomes

$$\mathcal{L}' = -\frac{1}{4} \Big\{ (\bar{R}_{\mu\nu\lambda\alpha})^2 + 8f\bar{R}^{\mu\alpha}\bar{\nabla}_{\mu}W_{\alpha} + 4f^2 \left(4(\bar{\nabla}_{\mu}W_{\nu})^2 - 4\bar{R}_{\mu\nu}W^{\mu}W^{\nu} + \bar{R}W^{\sigma}W_{\sigma} \right) \\ -16f^3W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} + 12f^4(W^{\sigma}W_{\sigma})^2 \Big\} \\ = -\frac{1}{4}(\bar{R}_{\mu\nu\lambda\alpha})^2 - 2f\bar{R}^{\mu\alpha}\bar{\nabla}_{\mu}W_{\alpha} - 4f^2 \left((\bar{\nabla}_{\mu}W_{\nu})^2 - \bar{R}_{\mu\nu}W^{\mu}W^{\nu} + \frac{1}{4}\bar{R}W^{\sigma}W_{\sigma} \right) \\ +4f^3W^{\mu}W^{\nu}\bar{\nabla}_{\mu}W_{\nu} - 3f^4(W^{\sigma}W_{\sigma})^2.$$
(31)

So in torsion free case, it is desirable to use the Yang-Mills type Lagrangian rather than the Gauss-Bonnet type.

III. ONE-LOOP EFFECTIVE ACTIONS

Now let us calculate the one-loop effective actions. Exact calculation of the effective action for an arbitrary curvature is very hard to solve, so we will consider the constant curvature cases with Riemann normal coordinates[4].

First let us consider the Lagrangian (31) which is the torsion free Yang-Mills type Lagrangian. To calculate the effective action we should split the Weyl vector field into the "classical" part W^0_{μ} and the quantum fluctuating part W^q_{μ} . But the vacuum Weyl field condensate $\langle 0|Q_{\mu\nu\lambda}{}^{\kappa}|0\rangle$ should be the form of a gauge covariant additive combination $\bar{R}_{\mu\nu\lambda}{}^{\kappa}_{+} \langle Q_{\mu\nu\lambda}{}^{\kappa}_{+}\rangle$. So, to find the functional dependence of the effective potential $V_{eff}(\bar{R}+\langle Q \rangle)$ on $\langle Q \rangle$ we calculate first the effective potential $V_{eff}(\bar{R})$ by setting $\tilde{W}^0_{\mu} = 0$, i.e., $W_{\mu} = W^q_{\mu}$. Then, after completing the calculation we will restore the dependence on Weyl field condensate $\langle Q \rangle$ by simple adding this term to \bar{R} in the final expression for $V_{eff}(\bar{R})[3]$. With the Weyl gauge fixing condition $\bar{\nabla}_{\mu}W^{\mu} = 0$ and $\delta(\bar{\nabla}_{\mu}W^{\mu}) = -\frac{1}{f}(\bar{\nabla}^{\mu} - 4fW^{\mu})\partial_{\mu}\Lambda$, one can find the gauge fixing term \mathcal{L}_{GF} and Faddeev-Popov ghost term \mathcal{L}_{FP}

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} (\bar{\nabla}_{\mu} W^{\mu})^2, \qquad (32)$$

$$\mathcal{L}_{FP} = \bar{c}\bar{\nabla}^{\mu}(\bar{\nabla}_{\mu}c) - 4f\bar{c}W^{\mu}(\bar{\nabla}_{\mu}c).$$
(33)

So the total quadratic Lagrangian becomes

$$\mathcal{L}' = -\frac{1}{4} (\bar{R}_{\mu\nu\lambda\alpha})^2 - 4f^2 \left((\bar{\nabla}_{\mu}W_{\nu})^2 - \bar{R}_{\mu\nu}W^{\mu}W^{\nu} + \frac{1}{4}\bar{R}W^{\sigma}W_{\sigma} \right) -\frac{1}{2\xi} (\bar{\nabla}_{\mu}W^{\mu})^2 + \bar{c}\bar{\nabla}^{\mu}(\bar{\nabla}_{\mu}c) - 4f\bar{c}W^{\mu}(\bar{\nabla}_{\mu}c)$$
(34)

Now let us think of the Riemann normal coordinates in constant curvature. The curvature tensor becomes

$$\bar{R}_{\mu\nu\alpha\beta} = \frac{1}{12}\bar{R}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}).$$
(35)

Let the Riemann normal coordinates and metric be x^{μ} and $g_{\mu\nu}$ for the point *P* respectively. And let the metric at origin be $\eta_{\mu\nu}$. Then in the case of $\bar{\nabla}_{\alpha}\bar{R}_{\mu\nu\lambda\kappa} = 0$ the metric at the point *P* may be expanded by [4]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^{2k+2}}{(2k+2)!} f_{\mu\sigma_1} f_{\sigma_2}^{\sigma_1} \cdots f_{\nu}^{\sigma_{k-1}} = \eta_{\mu\nu} + \frac{1}{3} \bar{R}_{\mu\alpha\nu\beta} x^{\alpha} x^{\beta} + \frac{2}{45} \bar{R}_{\alpha\mu\beta\sigma} \bar{R}_{\gamma\nu\delta}^{\sigma} x^{\alpha} x^{\beta} x^{\gamma} x^{\delta} + \cdots$$
(36)

where $f^{\sigma}_{\mu} = \bar{R}^{\sigma}_{\alpha\mu\beta} x^{\alpha} x^{\beta}$ and $f_{\sigma\nu} = \bar{R}_{\alpha\sigma\beta\nu} x^{\alpha} x^{\beta}$.

Now we want to put eq.(35) into eq.(36). Since

$$f^{\sigma}_{\mu} = \bar{R}_{\alpha\mu\beta}{}^{\sigma}_{\alpha}x^{\alpha}x^{\beta} = \frac{1}{12}\bar{R}(g^{\sigma}_{\mu}x_{\alpha}x^{\alpha} - x_{\mu}x^{\sigma}) = \frac{1}{12}\bar{R}(g^{\sigma}_{\mu}x^{2} - x_{\mu}x^{\sigma})$$
$$f_{\mu\sigma} = \bar{R}_{\alpha\sigma\beta\nu}x^{\alpha}x^{\beta} = \frac{1}{12}\bar{R}(g_{\mu\sigma}x^{2} - x_{\mu}x_{\sigma}) = g_{\sigma\alpha}f^{\alpha}_{\mu},$$

then $f^{\sigma}_{\mu}x^{\mu} = \frac{1}{12}\bar{R}(x^{\sigma}x^2 - x^2x^{\sigma}) = 0$. So $x_{\mu} = g_{\mu\nu}x^{\nu} = \eta_{\mu\nu}x^{\nu}$ and $x^2 = g_{\mu\nu}x^{\mu}x^{\nu} = \eta_{\mu\nu}x^{\mu}x^{\nu}$. And

$$f_{\mu\sigma}f_{\nu}^{\sigma} = \left(\frac{R}{12}\right)^{2}(g_{\mu\sigma}x^{2} - x_{\mu}x_{\sigma})(g_{\nu}^{\sigma}x^{2} - x_{\nu}x^{\sigma})$$

$$= \left(\frac{\bar{R}}{12}\right)^{2}x^{2}(g_{\mu\nu}x^{2} - x_{\mu}x_{\nu}) = \frac{\bar{R}}{12}x^{2}f_{\mu\nu}$$

$$f_{\mu\sigma_{1}}f_{\sigma_{2}}^{\sigma_{1}}f_{\nu}^{\sigma_{2}} = \frac{\bar{R}}{12}x^{2}f_{\mu\sigma_{2}}f_{\nu}^{\sigma_{2}} = \frac{\bar{R}}{12}x^{2}\cdot\frac{\bar{R}}{12}x^{2}f_{\mu\nu} = \left(\frac{\bar{R}}{12}x^{2}\right)^{2}f_{\mu\nu}$$

$$\vdots$$

$$f_{\mu\sigma_{1}}f_{\sigma_{2}}^{\sigma_{1}}\cdots f_{\nu}^{\sigma_{k-1}} = \left(\frac{\bar{R}}{12}x^{2}\right)^{k-1}f_{\mu\nu}$$

And putting these results into eq.(36), we get

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^{2k+2}}{(2k+2)!} f_{\mu\sigma_1} f_{\sigma_2}^{\sigma_1} \cdots f_{\nu}^{\sigma_{k-1}}$$
$$= \eta_{\mu\nu} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^{2k+2}}{(2k+2)!} \left(\frac{\bar{R}}{12} x^2\right)^{k-1} f_{\mu\nu}$$
(37)

$$= \eta_{\mu\nu} + \frac{72\cosh\sqrt{\frac{\bar{R}x^2}{3}} - 12\bar{R}x^2 - 72}{\bar{R}^2x^4} f_{\mu\nu}$$
(38)

$$= \eta_{\mu\nu} + \frac{72\cosh\sqrt{\frac{\bar{R}x^2}{3}} - 12\bar{R}x^2 - 72}{\bar{R}^2x^4} \cdot \frac{1}{12}\bar{R}(g_{\mu\nu}x^2 - x_{\mu}x_{\nu})$$
(39)

$$= \eta_{\mu\nu} + \frac{6\cosh\sqrt{\frac{\bar{R}x^2}{3}} - \bar{R}x^2 - 6}{\bar{R}x^4} (g_{\mu\nu}x^2 - x_{\mu}x_{\nu})$$
(40)

where we have used $\frac{1}{2} \sum_{k=1}^{\infty} \frac{2^{2k+2}}{(2k+2)!} \left(\frac{\bar{R}}{12}x^2\right)^{k-1} = \frac{72\cosh\sqrt{\frac{\bar{R}x^2}{3}} - 12\bar{R}x^2 - 72}{\bar{R}^2x^4}.$

Now we can solve the eq.(40) for $g_{\mu\nu}$ and the final expression of the metric in terms of Riemann normal coordinate with eq.(35) becomes

$$g_{\mu\nu}(x) = \frac{\bar{R}x^2}{2\bar{R}x^2 - 12\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}} \left(\eta_{\mu\nu} - \frac{12\sinh^2\sqrt{\frac{\bar{R}x^2}{12}} - \bar{R}x^2}{\bar{R}x^4} x_{\mu}x_{\nu}\right).$$
(41)

And the inverse of this metric is

$$g^{\mu\nu}(x) = \eta^{\mu\nu} + \frac{12\sinh^2\sqrt{\frac{\bar{R}x^2}{12}} - \bar{R}x^2}{\bar{R}x^4} (x^{\mu}x^{\nu} - \eta^{\mu\nu}x^2).$$
(42)

The determinant of $g_{\mu\nu}$ is $\det(g_{\mu\nu}) = \left(\frac{\bar{R}x^2}{2\bar{R}x^2 - 12\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}}\right)^3 \det(\eta_{\mu\nu}).$

Next calculating the Christoffel symbols, they are

$$\begin{split} \bar{\Gamma}_{\mu\nu\lambda}(x) &= \frac{1}{2} (\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}) \\ &= \left(\frac{1}{x^2} - \frac{\bar{R}}{\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}} + \frac{2\bar{R}^2x^2 - \sqrt{3\bar{R}^3x^2}\sinh\sqrt{\frac{\bar{R}x^2}{3}}}{4\left(\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)^2} \right) \eta_{\mu\nu}x_{\lambda} \\ &+ \left(-\frac{1}{x^4} + \frac{2\bar{R}^2\sqrt{x^2} - \sqrt{3\bar{R}^3}\sinh\sqrt{\frac{\bar{R}x^2}{3}}}{4\sqrt{x^2}\left(\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)^2} \right) x_{\mu}x_{\nu}x_{\lambda} \\ &+ \left(\frac{\bar{R}\left(\sqrt{3\bar{R}x^2}\sinh\sqrt{\frac{\bar{R}x^2}{3}} - 12\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)}{4\sqrt{x^2}\left(\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)^2} \right) (x_{\mu}\eta_{\nu\lambda} + x_{\nu}\eta_{\mu\lambda}) \end{split}$$
(43)

and

$$\begin{split} \bar{\Gamma}_{\mu\nu}^{\kappa}(x) &= g^{\kappa\lambda}\bar{\Gamma}_{\mu\nu\lambda}(x) \\ &= \left(\frac{1}{x^2} - \frac{\bar{R}}{\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}} + \frac{2\bar{R}^2x^2 - \sqrt{3\bar{R}^3x^2}\sinh\sqrt{\frac{\bar{R}x^2}{3}}}{4\left(\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)^2}\right)\eta_{\mu\nu}x^{\kappa} \\ &+ \left(\frac{\bar{R}\left(\sqrt{3\bar{R}x^2}\cosh\sqrt{\frac{\bar{R}x^2}{12}} - 6\sinh\sqrt{\frac{\bar{R}x^2}{12}}\right)}{2x^4\left(\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)^2}\right)(x_{\mu}\delta_{\nu}^{\kappa} + x_{\nu}\delta_{\mu}^{\kappa}) \\ &\quad \cdot \left\{(9\bar{R} + \bar{R}^2x^2 + x^4)\sinh\sqrt{\frac{\bar{R}x^2}{12}} - 3\bar{R}\sinh\sqrt{\frac{\bar{R}x^2}{4}}\right\} \\ &- \frac{1}{4x^4\left(\bar{R}x^2 - 6\sinh^2\sqrt{\frac{\bar{R}x^2}{12}}\right)^2} \left\{2(9 + \bar{R}^2x^2)^2 \\ &- 36(6 + \bar{R}x^2)\cosh\sqrt{\frac{\bar{R}x^2}{3}} + 54\cosh\sqrt{\frac{4\bar{R}x^2}{3}} \\ &+ 3\sqrt{3\bar{R}x^2}\left(\bar{R}x^2 - 8\sinh\sqrt{\frac{\bar{R}x^2}{12}}\right)\sinh\sqrt{\frac{\bar{R}x^2}{3}}\right\}x_{\mu}x_{\nu}x^{\kappa} \end{split}$$

Now with above equipments, let us compute the effective action. With constant curvature the effective action can be written in the form

$$\exp\left(i\Gamma_{eff}\right) = \int \mathcal{D}W_{\mu}\mathcal{D}c\mathcal{D}\bar{c} \qquad (45)$$
$$\exp\left\{i\int d^{4}x\sqrt{-g}\left[-\frac{1}{4}(\bar{R}_{\mu\nu\lambda\alpha})^{2} - 4f^{2}(\bar{\nabla}_{\mu}W_{\nu})^{2} + \bar{c}\bar{\nabla}^{\mu}(\bar{\nabla}_{\mu}c)\right]\right\}.$$

In flat space-time, to calculate functional integration we can use following

$$\int \mathcal{D}\phi e^{-\int d^4x d^4y \phi(x) \cdot A(x,y) \cdot \phi(y)} = (\det A)^{-1/2}$$
(46)

for any real *symmetric*, positive, non-singular matrix. But in curved space-time it is not trivial to make the symmetric matrix. For the scalar fields,

$$\int d^{4}x \sqrt{-g(x)} \partial_{\mu}\phi(x) \partial_{\nu}\phi(x)g^{\mu\nu}(x)
= \int d^{4}x \sqrt{-g(x)} \int d^{4}y \sqrt{-g(y)}
\cdot \partial_{\mu}^{x}\phi(x) \partial_{\nu}^{y}\phi(y)g^{\mu\nu}(x,y)\delta^{4}(x,y)(-g(x))^{-\frac{1}{4}}(-g(y))^{-\frac{1}{4}}
= -\int d^{4}x \int d^{4}y\phi(x)\partial_{\mu}^{x}\{(-g(x))^{\frac{1}{4}}(-g(y))^{\frac{1}{4}}\partial_{\nu}^{y}\phi(y)g^{\mu\nu}(x,y)\delta^{4}(x,y)\}
= \int d^{4}x \int d^{4}y\phi(x)\phi(y)\partial_{\mu}^{x}\partial_{\nu}^{y}\{(-g(x))^{\frac{1}{4}}(-g(y))^{\frac{1}{4}}g^{\mu\nu}(x,y)\delta^{4}(x,y)\}
= \int d^{4}x \int d^{4}y \sqrt{-g(x)} \sqrt{-g(y)}\phi(x)\phi(y)
\cdot \frac{1}{\sqrt{-g(x)}\sqrt{-g(y)}}\partial_{\mu}^{x}\partial_{\nu}^{y}\{(-g(x))^{\frac{1}{4}}(-g(y))^{\frac{1}{4}}g^{\mu\nu}(x,y)\delta^{4}(x,y)\}$$
(47)

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where in $g^{\mu\nu}(x,y)$, the variables of $g^{\mu\nu}(x)$ are changed symmetrically by the rule $x^{\mu}x^{\nu} \to \frac{x^{\mu}y^{\nu}+y^{\mu}x^{\nu}}{2}$. So we have got the symmetric matrix which is defined by

$$A_{scalar}(x,y) \equiv \frac{1}{\sqrt{-g(x)}\sqrt{-g(y)}} \partial^x_\mu \partial^y_\nu \left\{ (-g(x))^{\frac{1}{4}} (-g(y))^{\frac{1}{4}} g^{\mu\nu}(x,y) \delta^4(x,y) \right\}$$
(48)

After some integration by parts of Dirac delta function with Riemann normal coordinates and neglecting the total divergence, we get following in the limit of $x^{\mu}, y^{\mu} \rightarrow 0$

$$A_{scalar}(x,y) \simeq (\partial^x_{\mu}\partial^{\mu}_{y} + \frac{1}{6}\bar{R})\delta^4(x,y)$$

$$= \int \frac{d^4p}{(2\pi)^4} (\partial^x_{\mu}\partial^{\mu}_{y} + \frac{1}{6}\bar{R})e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} (p_{\mu}p^{\mu} + \frac{1}{6}\bar{R})e^{-ip\cdot(x-y)}$$
(49)

Therefore

$$\operatorname{Tr}\ln A_{scalar}(x,y) = \int d^4x d^4y \delta^4(x-y) \ln A_{scalar}(x,y)$$
$$\simeq \int d^4x \int \frac{d^4p}{(2\pi)^4} \ln(p^2 + \frac{1}{6}\bar{R})$$
(50)

For the vector fields,

$$\int d^4x \sqrt{-g(x)} g^{\mu\nu}(x) g_{\alpha\beta}(x) \bar{\nabla}_{\mu} W^{\alpha}(x) \bar{\nabla}_{\nu} W^{\beta}(x)$$

$$= \int d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)}$$

$$g^{\mu\nu}(x,y) g_{\alpha\beta}(x,y) \bar{\nabla}^x_{\mu} W^{\alpha}(x) \bar{\nabla}^y_{\nu} W^{\beta}(y) \frac{\delta^4(x,y)}{(-g(x))^{\frac{1}{4}} (-g(y))^{\frac{1}{4}}}$$

$$\simeq \int d^4x \int d^4y W^{\alpha}(x) W^{\beta}(y) \eta_{\alpha\beta} (\partial^x_{\mu} \partial^{\mu}_y + \frac{1}{12}\bar{R}) \delta^4(x,y)$$
(51)

Therefore,

$$A_{\alpha\beta}(x,y) \simeq \eta_{\alpha\beta}(\partial^x_{\mu}\partial^{\mu}_{y} + \frac{1}{12}\bar{R})\delta^4(x,y)$$

=
$$\int \frac{d^4p}{(2\pi)^4}\eta_{\alpha\beta}(p^2 + \frac{1}{12}\bar{R})e^{-ip\cdot(x-y)}.$$
 (52)

and

$$\operatorname{Tr} \ln A_{\alpha\beta}(x,y) = \int d^4x d^4y \delta^4(x-y) \operatorname{tr} \ln A_{\alpha\beta}(x,y) \simeq 4 \int d^4x \int \frac{d^4p}{(2\pi)^4} \ln(p^2 + \frac{1}{12}\bar{R})$$
(53)

Now the effective action is given by

$$\Gamma_{eff} = \frac{i}{2} \operatorname{Tr} \ln A_{\alpha\beta}(x, y) - i \operatorname{Tr} \ln A_{scalar}(x, y)$$
(54)

Neglecting overall constant we have one-loop effective potential

$$\Gamma_{(1)} = \int d^4x \int \frac{d^4p}{(2\pi)^4} \{ 2i \ln(1 + \frac{\bar{R}}{12p^2}) - i \ln(1 + \frac{\bar{R}}{6p^2}) \} \equiv i \int d^4x \mathcal{L}_{eff}^{(1)}$$
(55)

Then,

$$\mathcal{L}_{eff}^{(1)} = 2 \int \frac{d^4 p}{(2\pi)^4} \ln(1 + \frac{\bar{R}}{12p^2}) - \int \frac{d^4 p}{(2\pi)^4} \ln(1 + \frac{\bar{R}}{6p^2})$$
(56)

Since

$$\int \frac{d^4 p}{(2\pi)^4} \ln(1 + \frac{a}{p^2}) = \frac{2\pi^2}{(2\pi)^4} \int_0^{\Lambda} dp \ p^3 \ln(1 + \frac{a}{p^2})$$
$$= 2 \cdot \frac{\Lambda^2}{32\pi^2} a + \frac{a^2}{32\pi^2} (\ln \frac{a}{\Lambda^2} - \frac{1}{2}) + (\cdots)$$
(57)

where the last term vanishes when $\Lambda \to \infty$,

$$\mathcal{L}_{eff}^{(1)} \simeq 2 \cdot 2 \cdot \frac{\Lambda^2}{32\pi^2} (\frac{\bar{R}}{12}) + 2(\frac{\bar{R}}{12})^2 \frac{1}{32\pi^2} (\ln \frac{1}{\Lambda^2} \cdot \frac{\bar{R}}{12} - \frac{1}{2}) -2 \cdot \frac{\Lambda^2}{32\pi^2} (\frac{\bar{R}}{6}) - \frac{1}{32\pi^2} (\frac{\bar{R}}{6})^2 (\ln \frac{1}{\Lambda^2} \cdot \frac{\bar{R}}{6} - \frac{1}{2}) = \frac{1+2\ln 3}{4608\pi^2} \bar{R}^2 - \frac{1}{2304\pi^2} \bar{R}^2 \ln \frac{\bar{R}}{\Lambda^2} = \frac{1+2\ln 3}{768\pi^2} \bar{R}_{\mu\nu\alpha\beta}^2 - \frac{1}{384\pi^2} \bar{R}_{\mu\nu\alpha\beta}^2 \ln \frac{\sqrt{6\bar{R}_{\mu\nu\alpha\beta}^2}}{\Lambda^2}$$
(58)

Now by shifting $\bar{R}_{\mu\nu\alpha\beta} \rightarrow \bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle$ we get,

$$\mathcal{L}_{eff}^{(1)} \simeq \frac{1+2\ln 3}{768\pi^2} (\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2 -\frac{1}{384\pi^2} (\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2 \ln \frac{\sqrt{6(\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2}}{\Lambda^2} = -\frac{1}{384\pi^2} (\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2 \left\{ \ln \frac{\sqrt{(\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2}}{\Lambda^2} + \ln \frac{\sqrt{6}}{3} - \frac{1}{2} \right\}$$
(59)

Note that this is not exact but only approximation in the constant curvature background. With normalization condition $\frac{\partial^2 \mathcal{V}_{eff}}{(\partial R_{\mu\nu\alpha\beta})^2}\Big|_{R_{\mu\nu\alpha\beta}=\Lambda} = \frac{1}{2}$ the renormalized effective potential is now

$$\mathcal{V}_{eff} \simeq \frac{1}{4} (\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2 + \frac{1}{384\pi^2} (\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2 \left\{ \ln \frac{\sqrt{(\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2}}{\Lambda^2} - \frac{3}{2} \right\}$$
(60)

This potential has the minimum value \mathcal{V}_{min} when $\bar{R}_{\mu\nu\alpha\beta} = 0$ and $\langle Q_{\mu\nu\alpha\beta} \rangle \neq 0$.

$$\mathcal{V}_{min} = -\frac{1}{768\pi^2} < Q_{\mu\nu\alpha\beta} >^2, \tag{61}$$

$$\langle Q_{\mu\nu\alpha\beta} \rangle = e^{1-96\pi^2} \Lambda^2 \tag{62}$$

Expanding the original classical Lagrangian around the new vacuum we obtain

$$\mathcal{L}_{eff} \simeq -\frac{1}{4} (\bar{R}_{\mu\nu\alpha\beta} + \langle Q_{\mu\nu\alpha\beta} \rangle)^2 = -\frac{1}{4} \bar{R}_{\mu\nu\alpha\beta}^2 - \frac{1}{2} \bar{R} M^2 - \frac{3}{2} M^4.$$
(63)

where we put $\langle Q_{\mu\nu\alpha\beta} \rangle = \frac{1}{2}M^2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$ [3]. So we can get the Einstein-Hilbert type terms in the effective Lagrangian (in units $\hbar = c = 1$)

$$\mathcal{L}_{EHeff} = -\frac{1}{4}\bar{R}^2_{\mu\nu\alpha\beta} - \frac{1}{16\pi G}(\bar{R} + 2\lambda).$$
(64)

Thus we have similar result with the Weyl fields as with the torsion case in [3]. This means that the Weyl fields can be important player of quantum gravity like the torsion fields. And in low energy limit the metric gets the dynamics and Einstein-Hilbert term becomes dominant. This Einstein-Hilbert term contains only metric field without torsion and Weyl vector fields. So this describes the conventional general relativity. However if we introduce the scalar fields with non-minimal coupling term $\xi R \phi^2$ and integrate out with respect to the scalars then we get the full Einstein-Hilbert term which contains not only metric but also torsion and Weyl vector fields. The Lagrangian of scalar field together with all permissible non-minimal coupling is given by [5]

$$\mathcal{L}_{scalar} = -\frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi - \frac{1}{2}m^{2}\phi^{2} - \frac{\lambda}{4!}\phi^{4}$$

$$+\frac{1}{2}(\xi_{1}R + \xi_{2}\bar{\nabla}_{\mu}K^{\mu} + \xi_{3}K_{\mu}K^{\mu} + \xi_{4}S_{\mu}S^{\mu} + \xi_{5}M_{\mu\nu\lambda}M^{\mu\nu\lambda})\phi^{2}$$

$$(65)$$

where $K_{\mu} = K^{\sigma}_{\ \sigma\mu}, S^{\mu} = \tilde{\epsilon}^{\mu\nu\lambda\kappa}K_{\nu\lambda\kappa}, M^{\sigma}_{\ \sigma\mu} = 0$ and $\tilde{\epsilon}^{\mu\nu\lambda\kappa}M_{\nu\lambda\kappa} = 0.$

But for the conformal invariant Lagrangian we set $\xi_1 = \frac{1}{6}$, $m = \xi_i = 0$ (i = 2, 3, 4, 5). So the conformal invariant scalar Lagrangian becomes

$$\mathcal{L}_{scalar} = -\frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + \frac{1}{12}R\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{66}$$

In next section we will investigate the relation between the torsion and Weyl fields with the full Einstein-Hilbert action using Palatini formalism.

IV. GENERAL R^2 -TYPE WEYL-CARTAN GRAVITY WITH THE PALATINI CONNECTIONS

A. The Palatini Connections

Palatini's approach is the first order formalism treating the metric and connection as independent degrees of freedom and varying separately with respect to them. With this method we can naturally derive Weyl gravity from Einstein-Hilbert action.

Now we consider the low energy case, so let us start from Einstein-Hilbert Lagrangian density

$$\mathcal{L} = \sqrt{-g}R\tag{67}$$

and take variation with respect to the connections.

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^{\lambda}} = \partial_{\lambda} g^{\mu\nu} - \delta^{\nu}_{\lambda} \partial_{\rho} g^{\mu\rho} + \frac{1}{2} g^{\mu\nu} \partial_{\lambda} g_{\sigma\kappa} \cdot g^{\sigma\kappa} - \frac{1}{2} \delta^{\nu}_{\lambda} \partial^{\mu} g_{\sigma\kappa} \cdot g^{\sigma\kappa} - g^{\mu\nu} \Gamma_{\sigma\lambda}^{\ \sigma} + g^{\rho\mu} \Gamma_{\rho\lambda}^{\ \nu} - \delta^{\nu}_{\lambda} g^{\rho\sigma} \Gamma_{\sigma\rho}^{\ \mu} + g^{\nu\rho} \Gamma_{\lambda\rho}^{\ \mu},$$
$$= 0$$

and using the following identity

$$\begin{split} \nabla_{\lambda}g^{\mu\nu} &= \partial_{\lambda}g^{\mu\nu} + \Gamma_{\lambda\kappa}^{\ \mu}g^{\kappa\nu} + \Gamma_{\lambda\kappa}^{\ \nu}g^{\mu\kappa} \\ \nabla_{\rho}g^{\mu\rho} &= \partial_{\rho}g^{\mu\rho} + \Gamma_{\rho\kappa}^{\ \mu}g^{\kappa\rho} + \Gamma_{\rho\kappa}^{\ \rho}g^{\mu\kappa} \\ \nabla_{\lambda}g_{\sigma\kappa} &= \partial_{\lambda}g_{\sigma\kappa} - \Gamma_{\lambda\sigma}^{\ \alpha}g_{\alpha\kappa} - \Gamma_{\lambda\kappa}^{\ \alpha}g_{\sigma\alpha} \\ \nabla^{\mu}g_{\sigma\kappa} &= g^{\mu\beta}(\partial_{\beta}g_{\sigma\kappa} - \Gamma_{\beta\sigma}^{\ \alpha}g_{\alpha\kappa} - \Gamma_{\beta\kappa}^{\ \alpha}g_{\sigma\alpha}) \end{split}$$

we can get

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^{\lambda}} = \nabla_{\lambda} g^{\mu\nu} - \Gamma_{\lambda\kappa}^{\ \mu} g^{\kappa\nu} - \Gamma_{\lambda\kappa}^{\ \nu} g^{\mu\kappa} - \delta_{\lambda}^{\nu} (\nabla_{\rho} g^{\mu\rho} - \Gamma_{\rho\kappa}^{\ \mu} g^{\kappa\rho} - \Gamma_{\rho\kappa}^{\ \rho} g^{\mu\kappa})
+ \frac{1}{2} g^{\mu\nu} g^{\sigma\kappa} (\nabla_{\lambda} g_{\sigma\kappa} + \Gamma_{\lambda\sigma}^{\ \alpha} g_{\alpha\kappa} + \Gamma_{\lambda\kappa}^{\ \alpha} g_{\sigma\alpha})
- \frac{1}{2} \delta_{\lambda}^{\nu} (\nabla^{\mu} g_{\sigma\kappa} + \Gamma_{\sigma}^{\mu} g_{\alpha\kappa} + \Gamma_{\kappa}^{\mu} g_{\sigma\alpha}) g^{\sigma\kappa}
- g^{\mu\nu} \Gamma_{\sigma\lambda}^{\ \sigma} + g^{\rho\mu} \Gamma_{\rho\lambda}^{\ \nu} - \delta_{\lambda}^{\nu} g^{\rho\sigma} \Gamma_{\sigma\rho}^{\ \mu} + g^{\nu\rho} \Gamma_{\lambda\rho}^{\ \mu}
= 0$$
(68)

and now let us define the non-metricity as $\nabla_{\mu}g_{\nu\lambda} \equiv -Q_{\mu\nu\lambda}^{\ \ 2}$ and the torsion as $t_{\mu\nu}^{\ \lambda} = \Gamma_{\mu\nu}^{\ \lambda} - \Gamma_{\nu\mu}^{\ \lambda}$. And let $\Gamma_{\mu\nu\lambda} = g_{\lambda\alpha}\Gamma_{\mu\nu}^{\ \ \alpha}, Q_{\alpha} = -\frac{1}{4}Q_{\alpha\kappa}^{\ \kappa}$. Then we have the following equation,

$$t_{\lambda\mu}^{\ \nu} - \delta^{\nu}_{\mu} t_{\lambda\sigma}^{\ \sigma} + \delta^{\nu}_{\lambda} t_{\mu\sigma}^{\ \sigma} - Q_{\lambda\mu}^{\ \nu} + \delta^{\nu}_{\lambda} Q_{\sigma\mu}^{\ \sigma} - 2\delta^{\nu}_{\mu} Q_{\lambda} + 2\delta^{\nu}_{\lambda} Q_{\mu} = 0$$
(69)

If we treat the metric compatible connections, *i.e.* $Q_{\lambda\mu}^{\ \nu} = 0$, then above equation (69) becomes

$$t_{\lambda\mu}^{\ \nu} - \delta^{\nu}_{\mu} t_{\lambda\sigma}^{\ \sigma} + \delta^{\nu}_{\lambda} t_{\mu\sigma}^{\ \sigma} = 0, \tag{70}$$

and contracting this equation with δ^{μ}_{ν} , we get

$$t_{\lambda\sigma}^{\ \sigma} - 4t_{\lambda\sigma}^{\ \sigma} + t_{\lambda\sigma}^{\ \sigma} = -2t_{\lambda\sigma}^{\ \sigma} = 0, \tag{71}$$

again we put this result to eq.(70), then we have $t_{\lambda\mu}^{\nu} = 0$. So if this theory is metric compatible, it also should be torsion free.

Conversely, when $t_{\lambda\mu}^{\nu} = 0$, from the eq.(69) it is easy to get $Q_{\lambda\mu}^{\nu} = 0$. Therefore torsion free connection should be metric compatible.

Now let us find the general relation between the torsion and the non-metricity. First, after contracting eq.(69) with δ^{μ}_{ν} and δ^{λ}_{ν} respectively we can get

$$-2t_{\lambda\sigma}^{\ \sigma} + Q_{\sigma\lambda}^{\ \sigma} - 2Q_{\lambda} = 0 \tag{72}$$

and

$$2t_{\mu\sigma}^{\ \sigma} + 3Q_{\sigma\mu}^{\ \sigma} + 6Q_{\mu} = 0.$$
⁽⁷³⁾

And adding both equations (72) and (73), we get $Q_{\sigma\mu}^{\sigma} = -Q_{\mu}$. And put this into eq.(73), then we also get $t_{\mu\sigma}^{\sigma} = -\frac{3}{2}Q_{\mu}$. And again put these results into eq.(69), then

² Note that $\nabla_{\mu}g^{\nu\lambda} = Q_{\mu}^{\ \nu\lambda}$

$$t_{\lambda\mu}^{\ \nu} - Q_{\lambda\mu}^{\ \nu} - \frac{1}{2} \delta_{\lambda}^{\nu} Q_{\mu} - \frac{1}{2} \delta_{\mu}^{\nu} Q_{\lambda} = 0.$$
(74)

And by symmetrizing and anti-symmetrizing eq.(74) about the lower indices λ and μ , we get $Q_{(\lambda\mu)}^{\nu} = -Q_{(\lambda}\delta_{\mu)}^{\nu}$ and $Q_{[\lambda\mu]}^{\nu} = t_{\lambda\mu}^{\nu}$.

Therefore

$$Q_{\lambda\mu}^{\nu} = Q_{(\lambda\mu)}^{\nu} + Q_{[\lambda\mu]}^{\nu}$$

$$= -\frac{1}{2}(Q_{\lambda}\delta_{\mu}^{\nu} + Q_{\mu}\delta_{\lambda}^{\nu}) + t_{\lambda\mu}^{\nu},$$
(75)

and

$$Q_{\lambda\mu\nu} = -\frac{1}{2}Q_{\lambda}g_{\mu\nu} - \frac{1}{2}Q_{\mu}g_{\lambda\nu} + t_{\lambda\mu\nu}.$$
(76)

Now the left hand side of eq.(76) is $\mu\nu$ -symmetric, so the right hand side also should be. So $-\frac{1}{2}Q_{\mu}g_{\lambda\nu} + t_{\lambda\mu\nu} \equiv A_{\lambda\mu\nu} = A_{\lambda(\mu\nu)}$, and

$$t_{\lambda\mu\nu} = \frac{1}{2}Q_{\mu}g_{\lambda\nu} + A_{\lambda\mu\nu}.$$
(77)

Now eq.(77) should be $\lambda\mu$ -antisymmetric so we can express eq.(77) like following

$$t_{\lambda\mu\nu} = \frac{1}{2}Q_{\mu}g_{\lambda\nu} - \frac{1}{2}Q_{\lambda}g_{\mu\nu} + B_{\lambda\mu\nu}, \qquad (78)$$

where $B_{\lambda\mu\nu}$ is some tensor which is symmetric about the second and third indices and antisymmetric about the first and second indices, but such a tensor cannot be exist. So $B_{\lambda\mu\nu} = 0$. Finally we have got the relation between the torsion and the non-metricity and it is following

$$t_{\lambda\mu\nu} = \frac{1}{2}Q_{\mu}g_{\lambda\nu} - \frac{1}{2}Q_{\lambda}g_{\mu\nu}$$
⁽⁷⁹⁾

and from eq.(76)

$$Q_{\lambda\mu\nu} = -\frac{1}{2}Q_{\lambda}g_{\mu\nu} - \frac{1}{2}Q_{\mu}g_{\lambda\nu} + \frac{1}{2}Q_{\mu}g_{\lambda\nu} - \frac{1}{2}Q_{\lambda}g_{\mu\nu}$$
$$= -Q_{\lambda}g_{\mu\nu}$$
$$\therefore \nabla_{\lambda}g_{\mu\nu} = Q_{\lambda}g_{\mu\nu}$$
(80)

Now let us express the connection with the metric and Weyl vector field using eq.(80), that is,

$$\partial_{\mu}g_{\nu\lambda} - \Gamma_{\mu\nu\lambda} - \Gamma_{\mu\lambda\nu} = Q_{\mu}g_{\nu\lambda}$$

$$\therefore \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} = \partial_{\mu}g_{\nu\lambda} - Q_{\mu}g_{\nu\lambda}.$$
 (81)

By changing the order of indices, we get two expressions like following

$$\Gamma_{\nu\mu\lambda} + \Gamma_{\nu\lambda\mu} = \partial_{\nu}g_{\mu\lambda} - Q_{\nu}g_{\mu\lambda}, \tag{82}$$

$$\Gamma_{\lambda\mu\nu} + \Gamma_{\lambda\nu\mu} = \partial_{\lambda}g_{\mu\nu} - Q_{\lambda}g_{\mu\nu}.$$
(83)

And now adding (81) and (82) then subtracting (83), and with some algebra we have the final expression

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2} (\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}) - \frac{1}{2} (Q_{\mu}g_{\nu\lambda} + Q_{\nu}g_{\mu\lambda} - Q_{\lambda}g_{\mu\nu}) + \frac{1}{2} (t_{\mu\nu\lambda} + t_{\lambda\mu\nu} + t_{\lambda\nu\mu}). = \frac{1}{2} (\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}) - \frac{1}{2} Q_{\mu}g_{\nu\lambda}.$$
(84)

By the same method we have got the same result (84) in a Lagrangian such as

$$\mathcal{L} = \sqrt{-g} (aR + b\tilde{\epsilon}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}) \tag{85}$$

where a and b are some constants and $\tilde{\epsilon}_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$, $\tilde{\epsilon}^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}$, $\epsilon_{0123} = 1$, $\epsilon^{0123} = -1$.

In higher derivative gravity we cannot say that the above results is valid in general. And it is not proper to apply the Palatini approach to higher derivative gravity[6]. So instead of applying this method to the higher derivative gravity, we will just try to take eq.(84) as a constraint of connections and call it the Palatini connection. But at least we can say that in low energy regime with non-zero vacuum expectation value of the scalar field the Weyl vector fields can be the source of the torsion.

B. General Lagrangian under the Palatini connections

Since $Q_{\mu} = -f J W_{\mu}$ where J(=2) is the weight of $g_{\mu\nu}$, we can express the torsion in terms of the Weyl vector fields. That is,

$$t_{\mu\nu}^{\ \lambda} = \frac{1}{2} (\delta_{\mu}^{\lambda} Q_{\nu} - \delta_{\nu}^{\lambda} Q_{\mu}) = -\frac{1}{2} f J (\delta_{\mu}^{\lambda} W_{\nu} - \delta_{\nu}^{\lambda} W_{\mu})$$

$$= f (W_{\mu} \delta_{\nu}^{\lambda} - W_{\nu} \delta_{\mu}^{\lambda}).$$
(86)

This means that the Weyl vector fields generate the torsion fields in low energy regime. And the Palatini connection (84) becomes

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2} (\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}) + \frac{1}{2} f J W_{\mu}g_{\nu\lambda}$$
(87)

$$\equiv \bar{\Gamma}_{\mu\nu\lambda} + fW_{\mu}g_{\nu\lambda}.$$
(88)

Or

$$\Gamma^{\lambda}_{\mu\nu} = \bar{\Gamma}^{\lambda}_{\mu\nu} + fW_{\mu}\delta^{\lambda}_{\nu}.$$
(89)

The curvature tensor is

$$\begin{aligned} R_{\mu\nu\lambda}^{\ \rho} &= 2\left\{\partial_{[\mu}\Gamma_{\nu]\lambda}^{\ \rho} - \Gamma_{[\mu|\lambda|}^{\ \sigma}\Gamma_{\nu]\sigma}^{\ \rho}\right\} \\ &= 2\left\{\partial_{[\mu}\left(\bar{\Gamma}_{\nu]\lambda}^{\ \rho} + fW_{\nu]}\delta_{\lambda}^{\rho}\right) - \left(\bar{\Gamma}_{[\mu|\lambda|}^{\ \sigma} + fW_{[\mu}\delta_{|\lambda|}^{\sigma}\right)\left(\bar{\Gamma}_{\nu]\sigma}^{\ \rho} + fW_{\nu]}\delta_{\sigma}^{\rho}\right)\right\} \\ &= 2\left\{\partial_{[\mu}\bar{\Gamma}_{\nu]\lambda}^{\ \rho} - \bar{\Gamma}_{[\mu|\lambda|}^{\ \sigma}\bar{\Gamma}_{\nu]\sigma}^{\ \rho}\right\} + 2f\left\{\delta_{\lambda}^{\rho}\partial_{[\mu}W_{\nu]} - \delta_{\sigma}^{\rho}W_{[\nu}\bar{\Gamma}_{\mu]\lambda}^{\ \sigma} - \delta_{\lambda}^{\sigma}W_{[\mu}\bar{\Gamma}_{\nu]\sigma}^{\ \rho}\right\} \\ &= 2\left\{\partial_{[\mu}\bar{\Gamma}_{\nu]\lambda}^{\ \rho} - \bar{\Gamma}_{[\mu|\lambda|}\bar{\Gamma}_{\nu]\sigma}^{\ \rho}\right\} + 2f\delta_{\lambda}^{\rho}\partial_{[\mu}W_{\nu]} \\ &= \bar{R}_{\mu\nu\lambda}^{\ \rho} + 2f\delta_{\lambda}^{\rho}\partial_{[\mu}W_{\nu]} = \bar{R}_{\mu\nu\lambda}^{\ \rho} + 2f\delta_{\lambda}^{\rho}\bar{\nabla}_{[\mu}W_{\nu]}. \end{aligned}$$
(90)

And the Ricci tensor and scalar are

$$R_{\mu\nu} = R_{\mu\sigma\nu}^{\ \sigma} = \bar{R}_{\mu\nu} + 2f\bar{\nabla}_{[\mu}W_{\nu]} \tag{91}$$

$$R = R_{\mu}^{\ \mu} = \bar{R}. \tag{92}$$

Since the Weyl-Cartan curvature does not have all the symmetry of the Riemann curvature, we have the another Ricci tensor which is defined by following

$$\dot{R}_{\mu\nu} \equiv R_{\rho\mu\lambda\nu}g^{\rho\lambda} = (\bar{R}_{\rho\mu\lambda\nu} + 2fg_{\lambda\nu}\bar{\nabla}_{[\rho}W_{\mu]})g^{\rho\lambda}
= \bar{R}_{\mu\nu} + 2f\bar{\nabla}_{[\nu}W_{\mu]} = \bar{R}_{\mu\nu} - 2f\bar{\nabla}_{[\mu}W_{\nu]} = R_{\nu\mu}$$
(93)

With these quantities we can compute the square of them. The squares of the curvature tensors are following

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = (\bar{R}_{\mu\nu\lambda\rho} + 2fg_{\lambda\rho}\bar{\nabla}_{[\mu}W_{\nu]})(\bar{R}^{\mu\nu\lambda\rho} + 2fg^{\lambda\rho}\bar{\nabla}^{[\mu}W^{\nu]})$$
$$= \bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\nu\lambda\rho} + 16f^{2}\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]}$$
(94)

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\rho\lambda} = -\bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\nu\lambda\rho} + 16f^2\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]}$$
(95)

$$R_{\mu\nu\lambda\rho}R^{\lambda\rho\mu\nu} = (\bar{R}_{\mu\nu\lambda\rho} + 2fg_{\lambda\rho}\bar{\nabla}_{[\mu}W_{\nu]})(\bar{R}^{\lambda\rho\mu\nu} + 2fg^{\mu\nu}\bar{\nabla}^{[\lambda}W^{\rho]})$$
$$= \bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\lambda\rho\mu\nu} = \bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\nu\lambda\rho} = -R_{\mu\nu\lambda\rho}R^{\lambda\rho\nu\mu}$$
(96)

$$R_{\mu\nu\lambda\rho}R^{\mu\lambda\nu\rho} = (\bar{R}_{\mu\nu\lambda\rho} + 2fg_{\lambda\rho}\bar{\nabla}_{[\mu}W_{\nu]})(\bar{R}^{\mu\lambda\nu\rho} + 2fg^{\nu\rho}\bar{\nabla}^{[\mu}W^{\lambda]})$$

$$= \bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\lambda\nu\rho} + 4f^{2}\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]}$$

$$= -R_{\mu\nu\lambda\rho}R^{\nu\lambda\mu\rho}$$
(97)

$$R_{\mu\nu\lambda\rho}R^{\mu\lambda\rho\nu} = (\bar{R}_{\mu\nu\lambda\rho} + 2fg_{\lambda\rho}\bar{\nabla}_{[\mu}W_{\nu]})(-\bar{R}^{\mu\lambda\nu\rho} + 2fg^{\nu\rho}\bar{\nabla}^{[\mu}W^{\lambda]})$$

$$= -R_{\mu\nu\lambda\rho}R^{\mu\lambda\nu\rho} + 4f^2\nabla_{[\mu}W_{\nu]}\nabla^{[\mu}W^{\nu]}$$
(98)

$$\begin{aligned} R_{\mu\nu\lambda\rho}R^{\nu\rho\lambda\mu} &= (\bar{R}_{\mu\nu\lambda\rho} + 2fg_{\lambda\rho}\bar{\nabla}_{[\mu}W_{\nu]})(-\bar{R}^{\mu\lambda\nu\rho} + 2fg^{\lambda\mu}\bar{\nabla}^{[\nu}W^{\rho]}) \\ &= -\bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\lambda\nu\rho} - 4f^2\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]} \end{aligned}$$

$$= -R_{\mu\nu\lambda\rho}R^{\mu\lambda\nu\rho} = -R_{\mu\nu\lambda\rho}R^{\rho\nu\lambda\mu}$$
⁽⁹⁹⁾

$$R_{\mu\nu\lambda\rho}R^{\rho\lambda\nu\mu} = \bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho}R^{\lambda\rho\mu\nu}$$
(100)

$$R_{\mu\nu\lambda\rho}R^{\rho\nu\mu\lambda} = -\bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\lambda\nu\rho} + 4f^2\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]} = R_{\mu\nu\lambda\rho}R^{\mu\lambda\rho\nu}.$$
 (101)

And the square of the Ricci tensors and scalars are following

$$R_{\mu\nu}R^{\mu\nu} = (\bar{R}_{\mu\nu} + 2f\bar{\nabla}_{[\mu}W_{\nu]})(\bar{R}^{\mu\nu} + 2f\bar{\nabla}^{[\mu}W^{\nu]})$$

$$= \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + 4f^2\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]}$$
(102)

$$R_{\mu\nu}R^{\nu\mu} = \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - 4f^2\bar{\nabla}_{[\mu}W_{\nu]}\bar{\nabla}^{[\mu}W^{\nu]}$$
(103)

$$R^2 = \bar{R}^2. \tag{104}$$

Now let us consider the Gauss-Bonnet like identity.

$$I_{BF} \equiv R^2 - (R_{\mu\nu} + \check{R}_{\mu\nu})(R^{\nu\mu} + \check{R}^{\nu\mu}) + R_{\mu\nu\alpha\beta}R^{\alpha\beta\mu\nu}.$$
 (105)

But in our case it is the same with the Gauss-Bonnet identity of Riemannian. That is,

$$I_{BF} \equiv R^{2} - (R_{\mu\nu} + \check{R}_{\mu\nu})(R^{\nu\mu} + \check{R}^{\nu\mu}) + R_{\mu\nu\alpha\beta}R^{\alpha\beta\mu\nu} = \bar{R}^{2} - (\bar{R}_{\mu\nu} + 2f\bar{\nabla}_{[\mu}W_{\nu]} + \bar{R}_{\mu\nu} - 2f\bar{\nabla}_{[\mu}W_{\nu]})(\bar{R}^{\nu\mu} + 2f\bar{\nabla}^{[\nu}W^{\mu]} + \bar{R}^{\nu\mu} - 2f\bar{\nabla}^{[\nu}W^{\mu]}) + \bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta} = \bar{R}^{2} + 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - \bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta} = I_{GB}(\bar{R})$$
(106)

$$\mathcal{L}_{gen} = a_0 R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} + a_1 R_{\mu\nu\lambda\kappa} R^{\lambda\kappa\mu\nu} + a_2 R_{\mu\nu\lambda\kappa} R^{\mu\lambda\nu\kappa} + a_3 R_{\mu\nu\lambda\kappa} R^{\mu\nu\kappa\lambda} + a_4 R_{\mu\nu\lambda\kappa} R^{\mu\lambda\kappa\nu} + a_5' R_{\mu\nu} R^{\mu\nu} + a_6' R_{\mu\nu} R^{\nu\mu} + a_7' \check{R}_{\mu\nu} \check{R}^{\mu\nu} + a_8' R_{\mu\nu} \check{R}^{\mu\nu} + a_9' \check{R}_{\mu\nu} \check{R}^{\nu\mu} + a_{10}' R_{\mu\nu} \check{R}^{\nu\mu} + a_7 R^2 + a_8 A_{\mu\nu\lambda\kappa} A^{\mu\nu\lambda\kappa} = a_0 R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} + a_1 R_{\mu\nu\lambda\kappa} R^{\lambda\kappa\mu\nu} + a_2 R_{\mu\nu\lambda\kappa} R^{\mu\lambda\nu\kappa} + a_3 R_{\mu\nu\lambda\kappa} R^{\mu\nu\kappa\lambda} + a_4 R_{\mu\nu\lambda\kappa} R^{\mu\lambda\kappa\nu} + (a_5' + a_7' + a_{10}') R_{\mu\nu} R^{\mu\nu} + (a_6' + a_8' + a_9') R_{\mu\nu} R^{\nu\mu} + a_7 R^2 + a_8 A_{\mu\nu\lambda\kappa} A^{\mu\nu\lambda\kappa}$$
(107)

where $A_{\mu\nu\lambda\kappa} = \frac{1}{6}(R_{\mu\nu\lambda\kappa} + R_{\mu\lambda\kappa\nu} + R_{\mu\kappa\nu\lambda} + R_{\lambda\kappa\mu\nu} + R_{\nu\lambda\kappa\mu} + R_{\nu\lambda\mu\kappa})$ which vanishes in Riemann space-time. After redefining the coefficients, we can write the Lagrangian like following

$$\mathcal{L}_{gen} = a_0 R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} + a_1 R_{\mu\nu\lambda\kappa} R^{\lambda\kappa\mu\nu} + a_2 R_{\mu\nu\lambda\kappa} R^{\mu\lambda\nu\kappa} + a_3 R_{\mu\nu\lambda\kappa} R^{\mu\nu\kappa\lambda}
+ a_4 R_{\mu\nu\lambda\kappa} R^{\mu\lambda\kappa\nu} + a_5 R_{\mu\nu} R^{\mu\nu} + a_6 R_{\mu\nu} R^{\nu\mu} + a_7 R^2 + a_8 A_{\mu\nu\lambda\kappa} A^{\mu\nu\lambda\kappa}
= (a_0 + a_1 - a_3 + \frac{1}{3}a_8) \bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\nu\lambda\kappa} + (a_2 - a_4 + \frac{7}{18}a_8) \bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\lambda\nu\kappa}
+ (a_5 + a_6) \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + a_7 \bar{R}^2
+ 4(4a_0 + a_2 + 4a_3 + a_4 + a_5 - a_6 + \frac{8}{9}a_8) f^2 \bar{\nabla}_{[\mu} W_{\nu]} \bar{\nabla}^{[\mu} W^{\nu]}
= (a_0 + a_1 + \frac{1}{2}a_2 - a_3 - \frac{1}{2}a_4 + \frac{19}{36}a_8) \bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\nu\lambda\kappa}
+ (a_5 + a_6) \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + a_7 \bar{R}^2
+ (a_5 + a_6) \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + a_7 \bar{R}^2
+ (a_5 + a_6) \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + a_7 \bar{R}^2$$
(109)
+ 4(4a_0 + a_2 + 4a_3 + a_4 + a_5 - a_6 + \frac{8}{9}a_8) f^2 \bar{\nabla}_{[\mu} W_{\nu]} \bar{\nabla}^{[\mu} W^{\nu]}

where we have used following

$$0 = \bar{R}^{\mu\nu\lambda\kappa}(\bar{R}_{\mu\nu\lambda\kappa} + \bar{R}_{\mu\lambda\kappa\nu} + \bar{R}_{\mu\kappa\nu\lambda})$$

$$= \bar{R}^{\mu\nu\lambda\kappa}(\bar{R}_{\mu\nu\lambda\kappa} - \bar{R}_{\mu\lambda\nu\kappa} - \bar{R}_{\mu\lambda\nu\kappa})$$

$$= \bar{R}^{\mu\nu\lambda\kappa}(\bar{R}_{\mu\nu\lambda\kappa} - 2\bar{R}_{\mu\lambda\nu\kappa}).$$
(110)

Thus if we put some constants into the coefficients, we can get some special Lagrangian such as the Yang-Mills type Lagrangian by setting $a_0 = -\frac{1}{4}$, $a_i = 0$ $(i = 1, 2, \dots, 8)$. But we will keep the general form and for simplicity we want to write the Lagrangian as

$$\mathcal{L}_{gen} = \alpha \bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\nu\lambda\kappa} + \beta \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + a_7 \bar{R}^2 + \gamma \bar{\nabla}_{[\mu} W_{\nu]} \bar{\nabla}^{[\mu} W^{\nu]}$$
(111)

where $\alpha = a_0 + a_1 + \frac{1}{2}a_2 - a_3 - \frac{1}{2}a_4 + \frac{19}{36}a_8$, $\beta = a_5 + a_6$ and $\gamma = 4f^2(4a_0 + a_2 + 4a_3 + a_4 + a_5 - a_6 + \frac{8}{9}a_8)$.

Now $(\bar{\nabla}_{[\mu}W_{\nu]})^2 = \frac{1}{2}(\bar{\nabla}_{\mu}W_{\nu})^2 - \frac{1}{2}\bar{R}_{\mu\nu}W^{\mu}W^{\nu} - \frac{1}{2}\bar{\nabla}_{\mu}(W^{\nu}\bar{\nabla}_{\nu}W^{\mu}) + \frac{1}{2}W^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}W^{\mu}$, so we can express the Lagrangian like following

$$\mathcal{L}_{gen} = \alpha \bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\nu\lambda\kappa} + \beta \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + a_7 \bar{R}^2$$

$$+ \frac{\gamma}{2} \Big\{ (\bar{\nabla}_{\mu} W_{\nu})^2 - \bar{R}_{\mu\nu} W^{\mu} W^{\nu} - \bar{\nabla}_{\mu} (W^{\nu} \bar{\nabla}_{\nu} W^{\mu}) + W^{\nu} \bar{\nabla}_{\nu} \bar{\nabla}_{\mu} W^{\mu} \Big\}.$$
(112)

If we want the Lagrangian which reduces to Gauss-Bonnet invariant in the limit of Riemannian geometry, then we have only to put $\alpha = a_7$ and $\beta = -4a_7$.

Note that the Lagrangian (111) is just the form of the Maxwell theory in the curved space-time. So we can say that in low energy regime the Palatini connection make the Weyl symmetry broken but we have another gauge symmetry of U(1), that is, $\delta W_{\mu} = \partial_{\mu} \Lambda'$ and $\delta g_{\mu\nu} = 0$. The Weyl fields and torsion fields are the geometric fields. And by some symmetry breaking we have got the Maxwell fields from the geometric fields. So this can be another type of the unification. Note that in [1] we showed the contortion field has U(1) symmetry. Maybe there is a relation between the two U(1) symmetry of contortion and Weyl vector fields. If we think this two U(1) is identical and we set $K_{\mu} = 3fW_{\mu}$, then the total connection becomes

$$\Gamma_{\mu\nu\lambda} = \bar{\Gamma}_{\mu\nu\lambda} + K_{\mu\nu\lambda} + f(g_{\mu\lambda}W_{\nu} + g_{\nu\lambda}W_{\mu} - g_{\mu\nu}W_{\lambda}) \\
= \frac{1}{2}(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}) + M_{\mu\nu\lambda} + \frac{1}{3}(g_{\mu\nu}K_{\lambda} - g_{\mu\lambda}K_{\nu}) \\
+ \frac{1}{6}\tilde{\epsilon}_{\mu\nu\lambda\kappa}S^{\kappa} + f(g_{\mu\lambda}W_{\nu} + g_{\nu\lambda}W_{\mu} - g_{\mu\nu}W_{\lambda}) \\
= \frac{1}{2}(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}) + M_{\mu\nu\lambda} + \frac{1}{6}\tilde{\epsilon}_{\mu\nu\lambda\kappa}S^{\kappa} + fW_{\mu}g_{\nu\lambda}$$

where $K_{\mu} = K^{\sigma}_{\sigma\mu}$, $S^{\mu} = \tilde{\epsilon}^{\mu\nu\lambda\kappa}K_{\nu\lambda\kappa}$, $M^{\sigma}_{\sigma\mu} = 0$ and $\tilde{\epsilon}^{\mu\nu\lambda\kappa}M_{\nu\lambda\kappa} = 0$. So with this connection we can construct the theory of torsion and Weyl vector fields which has U(1) symmetry. Note that in this case it is not that the Weyl gauge symmetry changes into U(1) symmetry. The U(1) symmetry comes from the torsion's symmetry. That is, we can say that the Weyl vector fields eat the torsion and become the Maxwell vector fields.

By the way there may be someone who wants to change the scale symmetry to the phase symmetry directly, then he should expand the real Weyl gauge transformation to the complex Weyl gauge transformation. That is, the vielbein transform like

$$e'_{\mu}^{\ a} = e^{\Lambda(x) + i\phi(x)} e_{\mu}^{\ a} \tag{113}$$

$$e^{\prime \mu}_{\ a} = e^{-\Lambda(x) + i\phi(x)} e^{\mu}_{\ a} \tag{114}$$

where the Roman alphabet letters (a, b, c, \cdots) indicate the Lorentz index and the Greek alphabet letters $(\mu, \nu, \lambda, \cdots)$ indicate the coordinate index.

And the metric can be induced from vielbein like following way,

$$g_{\mu\nu} = 2e^{*a}_{\ (\mu}e_{\nu)a} \tag{115}$$

$$\eta_{ab} = 2e^{*\mu}{}_{(a}e_{|\mu|b)} \tag{116}$$

Then the metric transforms like same way of the real Weyl transformation : $g'_{\mu\nu} = e^{2\Lambda(x)}g_{\mu\nu}$. Now we can give weight to the tensor. The weight (J_1, J_2) tensors transform like $T'(x) = e^{J_1\Lambda(x)+iJ_2\phi(x)}T(x)$. So the vielbeins e^a_μ have the weight (1, 1) and e^a_μ have the weight (-1, 1). In this case the covariant derivative becomes

$$D_{\mu} = \partial_{\mu} - \Gamma_{\mu} + f_1 J_1 W_{1\mu} + f_2 J_2 W_{2\mu} \tag{117}$$

where the two types of Weyl vector fields transform like following,

$$W_{1\mu}' = W_{1\mu} - \frac{1}{f_1} \partial_\mu \Lambda$$
 (118)

$$W'_{2\mu} = W_{2\mu} - \frac{i}{f_2} \partial_\mu \phi$$
 (119)

In this way he may obtain the Maxwell fields from the Weyl vector fields. But this is a another story. So we don't go further here, and leave it for another study.

V. CONCLUSIONS

The Weyl vector fields also can play an important role in quantum gravity with the torsion. And in low energy regime with scalar field there is a relation between the Weyl vector fields and the torsion fields. If this condition is

given to Weyl vector fields and torsions, then the Lagrangian becomes like Maxwell type. And this Maxwell symmetry comes from the symmetry of torsion, not from the Weyl gauge symmetry.

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