On the causal properties of symmetric Lorentzian spaces *

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Abstract

The causal properties of Lorentzian symmetric spaces are investigated in the paper. There is proved the global hyperbolicity of the Cahen–Wallach Lorentzian symmetric spaces, corresponding to solvable Lie algebras.

1 Introduction

Symmetric Lorentzian spaces play a significant role in the foundations of modern supergravity theories [1, 2]. In [3], a classification of all Lorentzian symmetric spaces was given. Besides the universal covers of multi-dimensional de Sitter and anti-de Sitter spaces, the only simply-connected symmetric spaces are the Cahen–Wallach spaces $CW_n(A)$, which are constructed in the following way [1]. Consider the Euclidean space $\mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$ and the plane $\mathbb{R}^2$ with coordinates (ξ, η). Let $A = (a_{ij})$ be a symmetric matrix. Then $CW_n(A) = (\mathbb{R}^2 \times \mathbb{R}^n, ds^2_{n,A})$, where

$$ds^2_{n,A} = 2d\eta \left(d\xi + \sum_{i,j=1}^{n} a_{ij} x^i x^j d\eta \right) + (dx^1)^2 + \ldots + (dx^n)^2.$$

One of the main results of the paper is the following

Theorem 1. *The space $CW_n(A)$ is globally hyperbolic.*

Remark 1. The spaces $CW_n(A)$ correspond to solvable Lie algebras in the classification of [3]. It can be shown in the remaining cases that de Sitter space is globally hyperbolic (for instance, [2]), and anti-de Sitter space is not globally hyperbolic [4].

Theorem 1 turns out to be an immediate consequence of the following more general result.

*The author was supported by the Russian Foundation of Basic Research (grants 09-01-00142-a and 09-01-12130-ofi-m), the Leading Scientific Schools grant NSh-7256.2010.1 and the joined project of SB RAS and UrB RAS N 46

Theorem 2. *Let (M, g_1) be a Riemannian C^1-manifold and let there exist a point p_0 with the property that the exponential map $\exp_{p_0} : T_{p_0}M \to M$ is a diffeomorphism. Denote by $\rho(p) = \rho(p_0, p)$ the Riemannian distance function to the point p_0.*

Consider the following Lorentzian metric g_2 on $N = \mathbb{R}^2 \times M$:

$$g_2 = 2d\eta\,(d\xi - Fd\eta) + g_1,$$

where ξ, η are the coordinates on $\mathbb{R}^2$ and $F : M \to \mathbb{R}$ is a continuous function. Assume that

$$|F(p)| \leq c_1^2 + c_2^2 \rho(p)^2$$

for some constants c_1, c_2. Then (N, g_2) is globally hyperbolic.

Remark 2. One can relax the conditions of Theorem 2: instead of the function ρ one can choose the distance function to some subset $S \subset M$; and the exponential map condition can be substituted by the following one: the restriction of the function ρ to every piecewise C^1-smooth curve in M is a piecewise C^1-smooth real function. The last property is closely related to the cut locus structure of M.

2 Proof of the main results

Let (N, g) be a time-oriented Lorentzian manifold. The manifold (N, g) is called *strongly causal* if for each point $p \in N$ and for each neighborhood U of p there exists an open neighborhood $V \subset U$ of $p \in V$ such that each causal curve in N, starting and ending in V, is entirely contained in U. As usual, a *causal future (past)* $J^+(p)$ $(J^-(p))$ of the point p is the set of all such points q that either $p = q$ or there exists a future directed (past directed) causal curve starting at p and ending at q. A strongly causal Lorentzian manifold (N, g) is called *globally hyperbolic* if for each pair $p, q \in N$ the set $J^+(p) \cap J^-(q)$ is compact.

It is proved in [6, 7] that global hyperbolicity of the time-oriented Lorentzian manifold (N, g) is equivalent to the following condition: (N, g) is isometric to the direct product $\mathbb{R} \times S$ with the metric $-\beta dt^2 + g_t$, where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on t, and each "section" $\{t\} \times S$ is a smooth space-like Cauchy hypersurface in M.

We say that a space-time (N_1, g_1) is *asymptotically dominated* by a space-time (N_2, g_2) if there exists a continuous map $f : N_1 \to N_2$, satisfying the following two properties:

1) for each causal curve γ in N_1 the curve $f(\gamma)$ is causal in N_2;

2) for each point $p \in N_2$ the set $f^{-1}(p)$ is compact.

Lemma 1. *Let (N_1, g_1) be asymptotically dominated by (N_2, g_2). Then if (N_2, g_2) is globally hyperbolic, then (N_1, g_1) is also globally hyperbolic.*

Proof. Since (N_2, g_2) is globally hyperbolic, there exists a time function $T' : N_2 \to \mathbb{R}$ [6, 7]. Put $T = T' \circ f$. The property 1) of the map f implies that $T : N_1 \to \mathbb{R}$ is a time function on N_1. Then (N_1, g_1) is stably causal and, consequently, it is strongly causal [5].

Let us prove that for each compact set $K' \subset N_2$ the set $K = f^{-1}(K') \subset N_1$ is compact. Introducing Riemannian metrics on N_1 and N_2, we can assume that N_1 and N_2 are metric spaces with countable topology bases. So, it suffices for us to prove that from each sequence $p_i \in K$ we can extract a convergent subsequence. Therefore, in view of compactness of K', we can assume that $f(p_i) \to q \in K'$. This means that there exits a subsequence p_i converging to the set $K_0 = f^{-1}(q)$, which is compact by the property 2) of the map f. Consider an open ε-neighborhood U of the compact K_0 with the compact closure $K_1 = \bar{U}$. Then we can assume that $p_i \in U$ and, passing to a subsequence, we obtain that $p_i \to p \in K_1$. So far as p_i converges to K_0, we see that $p \in K_0 \subset K$.

Further, if $x \in J^+(p) \cap J^-(q)$, then $f(x) \in J^+(f(p)) \cap J^-(f(q))$. Therefore, $J^+(p) \cap J^-(q)$ is a subset of the compact set $f^{-1}(J^+(f(p)) \cap J^-(f(q)))$. So far as the set $J^+(p) \cap J^-(q)$ is closed in a strongly causal space-time [5], we can conclude that it is compact. This argument completes the proof of the global hyperbolicity of (N_1, g_1). The lemma is proved.

Consider the following metric on $N = \mathbb{R}^3$ with coordinates (η, ξ, τ):

$$g = 2d\eta\,(d\xi - f(\tau)d\eta) + d\tau^2,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let (N, g) be oriented in time by the coordinate vector field $\frac{\partial}{\partial \eta}$.

Theorem 3. *Let there exist constants c_1, c_2 such that $|f(\tau)| \leq c_1^2\tau^2 + c_2^2$. Then the metric g is globally hyperbolic.*

Remark 3. In the paper [8] the global hyperbolicity of the metric g was proved in the case when $f(\tau) \sim \tau$. However, the method that was used there does not work in the general case.

Proof. Let $f_0(\tau) = c_1^2\tau^2 + c_2^2$. Let us consider the metric

$$\tilde{g} = 2d\eta\,(d\xi - f_0(\tau)d\eta) + d\tau^2,$$

which dominates g, and let us show its global hyperbolicity.

First of all, we construct a global time function $T = T(\eta, \xi, \tau)$. We will seek T in the form:

$$T = \eta - \Phi(\xi, \tau), \qquad (1)$$

for some smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}$. It is easy to calculate the gradient of T with respect to the metric $\tilde{g}$:

$$\nabla T = (1 - 2f_0(\tau)\Phi_\xi)\,\partial_\xi - \Phi_\xi\partial_\eta - \Phi_\tau\partial_\tau.$$

We will try to find a time function T, whose gradient is time-like, that is

$$|\nabla T|^2 = 2f_0(\tau)\Phi_\xi^2 + \Phi_\tau^2 - 2\Phi_\xi < 0,$$

or, equivalently,

$$\Phi_\xi^2 + \frac{\Phi_\tau^2}{2f_0(\tau)} < \frac{\Phi_\xi}{f_0(\tau)}. \tag{2}$$

Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be some smooth functions. More precisely, we will try to find the function Φ in the following form:

$$\Phi(\xi, \tau) = \phi\left(\frac{\xi}{\psi(\tau)}\right).$$

Then the condition (2) changes into the following inequality:

$$\phi'\left(\frac{\xi}{\psi(\tau)}\right)\left[1 + \frac{\psi'(\tau)^2}{2f_0(\tau)}\left(\frac{\xi}{\psi(\tau)}\right)^2\right] < \frac{\psi(\tau)}{f_0(\tau)}. \tag{3}$$

Take $\varepsilon > 0$ and put $\psi(\tau) = \varepsilon f_0(\tau) = \varepsilon(c_1^2\tau^2 + c_2^2)$, that is $\psi(\tau)/f_0(\tau) = \varepsilon$. Then

$$\frac{\psi'(\tau)^2}{2f_0(\tau)} = \frac{2\varepsilon^2 c_1^4 \tau^2}{c_1^2\tau^2 + c_2^2} \leq 2\varepsilon^2 c_1^2.$$

Consequently, the inequality (3) holds everywhere if we require ϕ to satisfy the following equation:

$$\phi'(x)\left(1 + 2\varepsilon^2 c_1^2 x^2\right) = \frac{\varepsilon}{2}.$$

An elementary integration yields

$$\phi_\varepsilon(x) = \frac{1}{2\sqrt{2}c_1}\arctan\left(\sqrt{2}\varepsilon c_1 x\right).$$

So we have proved that the everywhere defined smooth function

$$T_\varepsilon(\xi, \eta, \tau) = \eta - \frac{1}{2\sqrt{2}c_1}\arctan\left(\frac{\sqrt{2}\varepsilon c_1 \xi}{c_1^2\tau^2 + c_2^2}\right)$$

is a time function with the time-like gradient. In particular, this implies strong causality of the metric g [5].

Consider two points $p_i = (\xi_i, \eta_i, \tau_i)$, $i = 1, 2$ in N, where $T(p_1) < T(p_2)$ (in the opposite case the intersection of the cones $K = J^+(p_1) \cap J^-(p_2)$ is empty). Note that for all η_0, ξ_0 the transformation

$$(\xi, \eta, \tau) \mapsto (\xi + \xi_0, \eta + \eta_0, \tau)$$

is an isometry of the Lorentzian manifold (N, g). Consequently, we can assume, without loss of generality, that $\eta_1 = 0$.

Lemma 2. *Assuming the conditions described above, if* $\eta_2 < \frac{\pi}{4\sqrt{2}c_1}$, *then* $K = J^+(p_1) \cap J^-(p_2)$ *is compact.*

Proof of Lemma 2. In the further calculations we put

$$x=\frac{\xi}{c_1^2\tau^2+c_2^2}, x_i=\frac{\xi_i}{c_1^2\tau_i^2+c_2^2}, i=1,2.$$

Now choose $\varepsilon>0$ small enough so that

$$|\phi_\varepsilon(x_i)|\leq\frac{\pi}{4\sqrt{2}c_1}-\eta_2, i=1,2.$$

Consider some point $p\in K$ and two causal curves: $\gamma(T)$, starting at p_1 and ending at p, and $\delta(T)$, starting at p and ending at p_2 (it is evident that we can choose the time function T as a regular parameter on these curves). Joining these curves, we can assume that we have one piecewise causal curve $\gamma(T)$, starting at p_1, going through p and ending at p_2. Then, using monotonicity of the functions T (strict) and η (non-strict) along γ, we have

$$\phi_\varepsilon(x(T))\leq\phi_\varepsilon(x(T))-\phi_\varepsilon(x_1)+\frac{\pi}{4\sqrt{2}c_1}-\eta_2<\eta(T)-\eta_1+\frac{\pi}{4\sqrt{2}c_1}-\eta_2\leq$$
$$\leq\frac{\pi}{4\sqrt{2}c_1}.$$

The definition of the function ϕ_ε immediately implies that there exists a constant d such that $x(T)\leq d$ for all T. Since the initial point p was taken to be arbitrary, we have $x(p)\leq d$ for all $p\in K$. Analogously,

$$\phi_\varepsilon(x(T))\geq\phi_\varepsilon(x(T))-\phi_\varepsilon(x_2)-\frac{\pi}{4\sqrt{2}c_1}+\eta_2>\eta(T)-\eta_2-\frac{\pi}{4\sqrt{2}c_1}+\eta_2\geq$$
$$\geq-\frac{\pi}{4\sqrt{2}c_1}.$$

Therefore, we can assume that for all $p\in K$ the following inequality takes place:

$$|x(p)|\leq d. \tag{4}$$

Further, from the relation (1) we have:

$$1=\frac{d\eta}{dT}-\phi'_\varepsilon(x)\frac{dx}{dT},$$

that is

$$\frac{dx}{dT}\geq-\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2x^2\right)\geq-\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right).$$

Consequently,

$$\frac{dx}{dT}=\frac{1}{f_0}\frac{d\xi}{dT}-\frac{\xi}{f_0}\frac{2c_1^2\tau}{f_0}\frac{d\tau}{dT}=\frac{1}{f_0}\frac{d\xi}{dT}-x\frac{2c_1^2\tau}{f_0}\frac{d\tau}{dT}\geq-\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right).$$

This immediately implies

$$\frac{d\xi}{dT}\geq-f_0\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)+2xc_1^2\tau\frac{d\tau}{dT}\geq-f_0\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)-2d_1c_1^2\left|\tau\frac{d\tau}{dT}\right|$$

Further, we use the causality condition for the curve γ:

$$0 \geq 2\frac{d\eta}{dT}\left(\frac{d\xi}{dT} - f_0\frac{d\eta}{dT}\right) + \left(\frac{d\tau}{dT}\right)^2 \geq$$

$$2\frac{d\eta}{dT}\left(-f_0\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right) - 2dc_1^2\left|\tau\frac{d\tau}{dT}\right| - f_0\frac{d\eta}{dT}\right) + \left(\frac{d\tau}{dT}\right)^2 \tag{5}$$

Let us assume, first, that $\frac{d\eta}{dT} \geq 1$. Then, continuing the previous inequality, we obtain

$$-2f_0\left(1+\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)\right)\left(\frac{d\eta}{dT}\right)^2 - 4dc_1^2\frac{d\eta}{dT}\left|\tau\frac{d\tau}{dT}\right| + \left(\frac{d\tau}{dT}\right)^2 \leq 0$$

In view of our assumption about the growth of η, we can change the parameter T on the curve γ, taking the coordinate η as a new parameter. In this case (dividing by f_0), we have:

$$\frac{1}{f_0}\left(\frac{d\tau}{d\eta}\right)^2 - 4dc_1^2\left|\frac{\tau}{f_0}\frac{d\tau}{d\eta}\right| - 2\left(1+\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)\right) \leq 0.$$

So far as

$$\frac{|\tau|}{\sqrt{f_0}} = \frac{1}{\sqrt{c_1^2+\frac{c_2^2}{\tau^2}}} \leq \frac{1}{c_1},$$

we have

$$\left(\frac{1}{\sqrt{f_0}}\frac{d\tau}{d\eta}\right)^2 - 4dc_1\left|\frac{1}{\sqrt{f_0}}\frac{d\tau}{d\eta}\right| - 2\left(1+\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)\right) \leq 0$$

The last estimate implies

$$\left|\frac{1}{\sqrt{f_0}}\frac{d\tau}{d\eta} - 2dc_1\right| \leq \sqrt{4d^2c_1^2 + 2\left(1+\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)\right)}.$$

By virtue of

$$\frac{1}{\sqrt{f_0}}\frac{d\tau}{d\eta} = \frac{d}{d\eta}\left(\frac{1}{c_1}\,\text{arcsinh}\,\left(\frac{c_1\tau}{c_2}\right)\right),$$

we can integrate the last inequality and, using a priori boundedness of η, we obtain that the total increment of the function τ on those subsegments of γ, where $\frac{d\eta}{dT} \geq 1$, is bounded by some constant D depending only on c_1, c_2, η_2 and on the constants $\xi_1, \xi_2, d, \varepsilon$ which are determined by the choice of the points p_1, p_2.

If $\frac{d\eta}{dT} \leq 1$, the inequality (5) can be immediately rewritten as:

$$\left(\frac{d\tau}{dT}\right)^2 - 4dc_1^2\left|\tau\frac{d\tau}{dT}\right| - 2f_0\left(1+\frac{2}{\varepsilon}\left(1+2\varepsilon^2c_1^2d^2\right)\right) \leq 0.$$

As in the previous case, dividing by f_0 and integrating, we obtain that the total increment of τ on those subsegments, where $\frac{d\eta}{dT} \leq 1$, is also universally bounded by the constant D.

Thus we have proved that the coordinate τ is universally bounded along each causal curve. The estimate (4) implies that the coordinate ξ is also universally bounded. Therefore, the set K is contained in a bounded domain in $\mathbb{R}^3$, that is, the closure of K is compact. Strong causality implies the closeness and, consequently, compactness of K [5]. The lemma is proved.

Now let p_1 and p_2 be the same as above but without any restriction on η_2. As above, we consider the intersection of the cones $K = J^+(p_1) \cap J^-(p_2)$. Take a constant $C > 0$ and consider the following Lorentzian metric on the manifold $N' = \mathbb{R}^3(\xi', \eta', \tau')$:

$$\tilde{g}'^2 = 2d\eta' \left(d\xi' - f_0'(\tau)d\eta'\right) + d\tau'^2,$$

where

$$f_0'(\tau') = \frac{c_1^2}{C^2}\tau'^2 + \frac{c_2^2}{C^2} = c_1'^2\tau'^2 + c_2'^2.$$

Now choose C large enough for the inequality

$$\eta_2 < \frac{\pi}{4\sqrt{2}c_1'} \tag{6}$$

to hold. Consider the transformation $\sigma : N \to N'$:

$$\sigma(\xi, \eta, \tau) = \left(\frac{\xi}{C^2}, \eta, \frac{\tau}{C}\right).$$

It is evident that the transformation of the Lorentzian spaces $\sigma : (N, ds^2) \to (N', ds'^2)$ is a conformal diffeomorphism. Consequently, it maps homeomorphically the intersection K of the causal future and past of the points p_1, p_2 to the intersection K' of the corresponding causal future and past of the points $p_1' = (\xi_1/C^2, 0, \tau_1/C)$ and $p_2' = (\xi_2/C^2, \eta_2, \tau_2/C)$. Inequality (6) and Lemma 2 imply that K' is compact and, therefore, K is also compact. Our theorem is proved.

Remark 4. The asymptotical growth of the function f_0 in Theorem 3 is optimal in the class of power functions. Indeed, if, in the conditions of Theorem 3, we consider $f = \tau^{2+\varepsilon}$ for $\varepsilon > 0$, then it is sufficient to take a class of the causal curves of the form $\gamma(s) = (\xi_0, \eta(s), \tau(s))$. Such curves are causal on the plane with coordinates (η, τ) with respect to the Lorentzian metric

$$-2d\eta^2 + \frac{d\tau^2}{f(\tau)}.$$

An elementary integration shows that this metric is not globally hyperbolic: there exist light-like curves, both future- and past-directed, which escape to $+\infty$ with respect to the variable τ for a finite increment of η.

Proof of Theorem 2. It suffices to note that the metric g_2 in the conditions of Theorem 2 is asymptotically dominated by the metric

$$ds^2 = g_2 = 2d\eta\left(d\xi - \left(c_1^2 + c_2^2\rho^2\right)d\eta\right) + d\rho^2.$$

Indeed, let $\gamma(s) = (\xi(s), \eta(s), p(s))$ be a causal curve in N. Let $\delta_s(t)$, $0 \leq t \leq \rho(p(s))$ be the shortest normal geodesic joining the points $p_0 = \delta_s(0)$ and $p(s) = \delta_s(\rho(p(s)))$. Let

$$\frac{dp}{ds} = d\rho\left(\frac{dp}{ds}\right)\frac{\partial}{\partial\rho} + \left(\frac{dp}{ds} - d\rho\left(\frac{dp}{ds}\right)\frac{\partial}{\partial\rho}\right)$$

be the decomposition of the tangent vector into radial and tangential components with respect to the level hypersurface of the function ρ (which is, in fact, the geodesic sphere of radius ρ centered at p_0). Moreover, we can interpret $\frac{\partial}{\partial\rho}$ as a tangent vector field to the normal geodesic δ_s. In this case

$$\left|\frac{dp}{ds}\right| \geq \left|d\rho\left(\frac{dp}{ds}\right)\right| = \frac{d\rho(p(s))}{ds}, \tag{7}$$

By the assumptions of the theorem, each piecewise C^1-smooth curve in N is projected under the map $(\xi, \eta, p) \mapsto (\xi, \eta, \rho(p))$ to a piecewise C^1-smooth curve in $\mathbb{R}^3$. Then, (7) implies that each causal curve in N is projected under the same map to a causal curve in $\mathbb{R}^3$ with the Lorentzian metric

$$g = 2d\eta\left(d\xi - \left(c_1^2 + c_2^2\rho^2\right)d\eta\right) + d\rho^2.$$

Obviously, the level sets of the function ρ (that is, the geodesic spheres in M) are compact. It remains to apply Theorem 3 and Lemma 1.

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