CONTINUOUS OBSERVATIONS AND THE WAVE FUNCTION COLLAPSE

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We propose to modify the collapse axiom of quantum measurement theory by replacing the instantaneous with a continuous collapse of the wave function in finite time τ. We apply it to coordinate measurement of a free quantum particle that is initially confined to a domain $D \subset \mathbb{R}^d$ and is observed continuously by illuminating $\mathbb{R}^d - D$. The continuous collapse axiom (CCA) defines the post-measurement wave function (PMWF)in D after a negative measurement as the solution of Schrödinger's equation at time τ with instantaneously collapsed initial condition and homogeneous Dirichlet condition on the boundary of D. The CCA applies to all cases that exhibit the Zeno effect. It rids quantum mechanics of the unphysical artifacts caused by instantaneous collapse and introduces no new artifacts.

I. INTRODUCTION

Schrödinger's equation does not describe the results of measurements. Rather, a separate wave function collapse axiom [1], [2] is needed to connect between the Schrödinger evolution of the wave function and the possible results of laboratory measurements. According to this axiom, as applied to a quantum particle's coordinate, a measurement collapses the wave function instantaneously to one that vanishes on a subset of positive measure in the

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Euclidean space of the coordinate [19]. Because all the possible collapsed wave functions of a given measurement form a subspace of $L^2(\mathbb{R}^n)$, the collapse is referred to as a projection into this subspace [1]. According to the collapse axiom, the post-measurement wave function (PMWF) evolves from its collapsed form according to Schrödinger's equation.

Consider, for example, an ideal coordinate measurement of a particle by illuminating instantaneously the positive axis $\mathbb{R}^+$ or a finite interval and assume that no particle is observed there [3]. The collapse axiom implies that the PMWF is truncated instantaneously to zero on the positive axis and that it is renormalized on the negative axis. This implies that after this measurement the wave function is discontinuous at the origin and that it evolves from its truncated form according to the Schrödinger equation [2]. It was shown, however [4], that the evolved PMWF has no moments, no average momentum, and has infinite energy, which is unphysical, because only finite energy is expended in this measurement. The same phenomenon occurs if the result of the measurement is that the particle is in the positive axis, without specifying its location there. Infinite measurable physical quantities, such as moments, are not encountered in the physical world, therefore infinities are incompatible with classical, relativistic, or quantum physics. The problem of infinities permeates to the foundations of theoretical physics (the Copenhagen interpretation [5] and weak causality [6]).

Despite these difficulties, the results of the collapse axiom describe faithfully laboratory quantum measurements, at least as a phenomenological and operational theory. So far, unfortunately, Hamiltonian theories have failed to reconcile the collapse axiom with Schrödinger's equation. Roughly speaking, there are two main approaches for alleviating the problem, one is to modify the collapse axiom, as we propose below, and the other is to modify the Schrödinger equation as in e.g. [7], [8], [9].

Our aim in this paper is to reconcile the collapse axiom for coordinate measurement with Schrödinger's equation by giving up the assumption of instantaneous truncation as the description of a coordinate measurement. Rather, we assume that a single coordinate measurement is continuous and has finite duration. Specifically, we propose to modify the collapse axiom for the negative coordinate measurement by postulating that it lasts for a positive time τ, specific to each measurement apparatus, and the PMWF the unmeasured domain is the solution of Schrödinger's equation at time τ with initial condition that is the truncated wave function and a zero Dirichlet (totally reflecting) condition on the boundary of the domain.

The resulting PMWF is called the continuously collapsed wave function. Although, to the best of our knowledge, our continuous collapse axiom, similarly to the instantaneous collapse axiom, has no Hamiltonian realization, we show below that, unlike the collapse axiom, it does not introduce artifacts into single particle quantum theory. We introduce the finite duration of the continuous coordinate measurement to reflect the fact that all physical measurements require finite time and this time is the property of every individual measurement apparatus [9]. The continuously collapsed wave function has a finite first moment. We note that continuous collapse for measurements that exhibit the Zeno effect leaves the state unchanged, thus it introduces no new phenomenology.

II. CONTINUOUS OBSERVATION OF A BROWNIAN PARTICLE

We calculate the probability density function of a Brownian particle by considering its intermittent observations. A free Brownian particle is initially confined to a domain $D \subset \mathbb{R}^d$ and is observed intermittently by instantaneous illuminations of $\Omega = \mathbb{R}^d - D$ at times Δt apart. Between observations the particle diffuses freely. It is our purpose to evaluate the pdf $p(\boldsymbol{x}, t)$ of the particle in D at observation times $\Delta t, 2\Delta t, \ldots, N\Delta t$, given that it was not observed in Ω. We begin with evaluating the pdf in the first interval $[0, \Delta t]$. At time Δt the pdf in $\mathbb{R}^d$ is given by

$$p_1(\boldsymbol{x}_1, \Delta t) = (2\pi\Delta t)^{-d/2} \int_D p_0(\boldsymbol{x}_0) \exp\left\{-\frac{|\boldsymbol{x}_1 - \boldsymbol{x}_0|^2}{2\Delta t}\right\} d\boldsymbol{x}_0. \tag{1}$$

After the instantaneous observation at time Δt the pdf in D is the conditional density $\pi_1(\boldsymbol{x}_1, \Delta t)$, given that the particle is not in Ω. That is,

$$\pi_1(\boldsymbol{x}_1, \Delta t) = \frac{p_1(\boldsymbol{x}_1, \Delta t)}{\int_D p_1(\boldsymbol{x}_1, \Delta t)\, d\boldsymbol{x}_1}, \tag{2}$$

which is normalized in D. At time $2\Delta t$ the propagated density is

$$p_2(\boldsymbol{x}_1, 2\Delta t) = (2\pi\Delta t)^{-d/2} \int_D \pi_1(\boldsymbol{x}_1, \Delta t) \exp\left\{-\frac{|\boldsymbol{x}_2 - \boldsymbol{x}_1|^2}{2\Delta t}\right\} d\boldsymbol{x}_1, \tag{3}$$

the conditional density is

$$\pi_2(\boldsymbol{x}_2, 2\Delta t) = \frac{p_2(\boldsymbol{x}_2, 2\Delta t)}{\int_D p_2(\boldsymbol{x}_2, 2\Delta t)\, d\boldsymbol{x}_2}$$

$$= \frac{(2\pi\Delta t)^{-d/2}\int_D \pi_1(\boldsymbol{x}_1,\Delta t)\exp\left\{-\frac{|\boldsymbol{x}_2-\boldsymbol{x}_1|^2}{2\Delta t}\right\}d\boldsymbol{x}_1}{(2\pi\Delta t)^{-d/2}\int_D\int_D \pi_1(\boldsymbol{x}_1,\Delta t)\exp\left\{-\frac{|\boldsymbol{x}_2-\boldsymbol{x}_1|^2}{2\Delta t}\right\}d\boldsymbol{x}_1\,d\boldsymbol{x}_2}, \tag{4}$$

and so on. The recursion for $\pi(\boldsymbol{x},t)$ on the lattice $t=j\Delta t$ $(j=1,2,\ldots)$ is therefore

$$\pi(\boldsymbol{x},t+\Delta t) = \frac{(2\pi\Delta t)^{-d/2}\int_D \pi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2\Delta t}\right\}d\boldsymbol{y}}{(2\pi\Delta t)^{-d/2}\int_D\int_D \pi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{y}-\boldsymbol{z}|^2}{2\Delta t}\right\}d\boldsymbol{y}\,d\boldsymbol{z}}. \tag{5}$$

The integral in the denominator of (5) can be evaluated by the change of variables $\boldsymbol{y}=\boldsymbol{z}+\boldsymbol{\xi}\sqrt{\Delta t}$ as

$$\begin{aligned}
&(2\pi)^{-d/2}\int\limits_D\int\limits_D \pi(\boldsymbol{z}+\boldsymbol{\xi}\sqrt{\Delta t},t)\exp\left\{-\frac{|\boldsymbol{\xi}|^2}{2}\right\}d\boldsymbol{y}\,d\boldsymbol{z}\\
=\;&(2\pi)^{-d/2}\int\limits_D\int\limits_D\left[\pi(\boldsymbol{z},t)+\sqrt{\Delta t}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}}\pi(\boldsymbol{z},t)+\frac{\Delta t}{2}\sum_{i,j}\xi_i\xi_j\pi_{x_ix_j}(\boldsymbol{z},t)+O\left(|\boldsymbol{\xi}\sqrt{\Delta t}|^3\right)\right]\\
&\times\exp\left\{-\frac{|\boldsymbol{\xi}|^2}{2}\right\}d\boldsymbol{\xi}\,d\boldsymbol{z}=\int\limits_D\pi(\boldsymbol{z},t)\,d\boldsymbol{z}+\frac{\Delta t}{2}\oint\limits_{\partial D}\frac{\partial\pi(\boldsymbol{z},t)}{\partial n}\,dS_{\boldsymbol{z}}+o(\Delta t)\\
=\;&1+\Delta tJ(t)+o(\Delta t),
\end{aligned}\tag{6}$$

where $\boldsymbol{n}$ is the unit outer normal at the boundary and $J(t)$ is the total absorption flux on the boundary, given by

$$J(t)=\frac{1}{2}\oint\limits_{\partial D}\frac{\partial\pi(\boldsymbol{z},t)}{\partial n}\,dS. \tag{7}$$

Expanding the left hand side and the integral in the numerator of equation (5), we obtain from (6) that for small Δt,

$$(1+\Delta tJ(t))[\pi(\boldsymbol{x},t)+\Delta t\pi_t(\boldsymbol{x},t)+o(\Delta t)]=\pi(\boldsymbol{x},t)+\frac{\Delta t}{2}\Delta_{\boldsymbol{x}}\pi(\boldsymbol{x},t)+o(\Delta t). \tag{8}$$

Hence, in the limit $\Delta t\to 0$,

$$\pi_t(\boldsymbol{x},t)=\frac{1}{2}\Delta_{\boldsymbol{x}}\pi(\boldsymbol{x},t)-J(t)\pi(\boldsymbol{x},t)\;\text{ for }\boldsymbol{x}\in D. \tag{9}$$

If $\boldsymbol{x}\in\partial D$ in (5), then in the limit $\Delta t\to 0$ the Gaussian integral in the numerator extends over half the space (see [15], [17]), which leads to the absorbing boundary condition

$$\pi(\boldsymbol{x},t)=0\;\text{ for }\boldsymbol{x}\in\partial D. \tag{10}$$

The initial condition for $\pi(\boldsymbol{x},t)$ is

$$\pi(\boldsymbol{x},0)=p_0(\boldsymbol{x}) \text{ for } \boldsymbol{x}\in D. \tag{11}$$

The solution of the nonlinear initial boundary value problem (9)-(11) can be constructed in the form of the renormalized density

$$\pi(\boldsymbol{x},t)=\frac{p(\boldsymbol{x},t)}{\displaystyle\int_D p(\boldsymbol{y},t)\,d\boldsymbol{y}}, \tag{12}$$

where $p(\boldsymbol{x},t)$ is the solution of the Fokker-Planck equation in D with absorbing boundary conditions on ∂D,

$$p_t(\boldsymbol{x},t) = \frac{1}{2}\Delta_{\boldsymbol{x}} p(\boldsymbol{x},t) \text{ for } \boldsymbol{x}\in D,\ t>0 \tag{13}$$

$$p(\boldsymbol{x},t) = 0 \text{ for } \boldsymbol{x}\in \partial D,\ t>0 \tag{14}$$

$$p(\boldsymbol{x},0) = p_0(\boldsymbol{x}) \text{ for } \boldsymbol{x}\in D. \tag{15}$$

Indeed, differentiating (12) once with respect to t and twice with respect to $\boldsymbol{x}$ gives

$$\begin{aligned}
\pi_t(\boldsymbol{x},t) &= \frac{p_t(\boldsymbol{x},t)}{\displaystyle\int_D p(\boldsymbol{y},t)\,d\boldsymbol{y}} - \frac{p(\boldsymbol{x},t)\displaystyle\int_D p_t(\boldsymbol{y},t)\,d\boldsymbol{y}}{\left(\displaystyle\int_D p(\boldsymbol{y},t)\,d\boldsymbol{y}\right)^2}\\
&= \frac{\frac{1}{2}\Delta_{\boldsymbol{x}} p(\boldsymbol{x},t)}{\displaystyle\int_D p(\boldsymbol{y},t)\,d\boldsymbol{y}} - \frac{p(\boldsymbol{x},t)\frac{1}{2}\displaystyle\int_D \Delta_{\boldsymbol{x}} p(\boldsymbol{y},t)\,d\boldsymbol{y}}{\left(\displaystyle\int_D p(\boldsymbol{y},t)\,d\boldsymbol{y}\right)^2}\\
&= \frac{1}{2}\Delta\pi(\boldsymbol{x},t) - J(t)\pi(\boldsymbol{x},t),
\end{aligned}$$

which is (9). We have used the identity

$$J(t)=\frac{\frac{1}{2}\displaystyle\int_D \Delta_{\boldsymbol{x}} p(\boldsymbol{y},t)\,d\boldsymbol{y}}{\displaystyle\int_D p(\boldsymbol{y},t)\,d\boldsymbol{y}},$$

which is (7).

The boundary value (9)-(11) suggests the following Brownian simulation of the continuous observation process. At each time step Δt of the simulation returns the escaping particles to D and distributes them there according to the existing (empirical) density. This, in effect, amounts to putting sources distributed in D according to the instantaneous density and the strength of each source is the total efflux on the boundary.

III. CONTINUOUS COORDINATE MEASUREMENT OF A QUANTUM PARTICLE

We adopt the above procedure to the coordinate observation of a quantum particle. The support of the freely propagating wave function is collapsed to D at times of negative observations. It follows that at observation times

$$\psi(\boldsymbol{x},t+\Delta t)=(2\pi i\Delta t)^{-d/2}\int\limits_D\psi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2i\Delta t}\right\}\,d\boldsymbol{y}, \tag{16}$$

as shown in [13], [14]. In the limit $\Delta t\to 0,\ N\Delta t\to t$ the solution of equation (16) converges to the solution of Schrödinger's equation in D with the totally reflecting boundary condition $\psi(\boldsymbol{x},t)=0$ for all $\boldsymbol{x}\in\partial D$ [4]. Therefore continuous observations do not allow the particle to exit D. This is the Zeno paradox in the sense that the wave function in the observed domain $\mathbb{R}^+$ is frozen at 0 [18], [11], [12].

However, if the wave function is renormalized after each observation, a procedure analogous to that of Section II gives for the renormalized wave function the recursion

$$\pi(\boldsymbol{x},t+\Delta t)=\frac{(2\pi i\Delta t)^{-d/2}\displaystyle\int_D\pi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2i\Delta t}\right\}\,d\boldsymbol{y}}{\left\{\displaystyle\int_D\left|(2\pi i\Delta t)^{-d/2}\int_D\pi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{y}-\boldsymbol{z}|^2}{2i\Delta t}\right\}\,d\boldsymbol{y}\right|^2\,d\boldsymbol{z}\right\}^{1/2}}. \tag{17}$$

If $\boldsymbol{x}\in\partial D$, then in the limit $\Delta t\to 0$ we obtain, as in Section II, that $\pi(\boldsymbol{x},t)=0$.

To determine the differential equation that $\pi(\boldsymbol{x},t)$ satisfies in D, we expand the denominator as in (6). With the substitution $\boldsymbol{y}=\boldsymbol{z}+\boldsymbol{\xi}\sqrt{\Delta t}$, the inner integral is

$$\begin{aligned}&(2\pi i\Delta t)^{-d/2}\int\limits_D\pi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{y}-\boldsymbol{z}|^2}{2i\Delta t}\right\}\,d\boldsymbol{y}\\
=\ &(2\pi i)^{-d/2}\int\limits_{(D-z)/\sqrt{\Delta t}}\pi(\boldsymbol{z}+\boldsymbol{\xi}\sqrt{\Delta t},t)\exp\left\{-\frac{|\boldsymbol{\xi}|^2}{2i}\right\}\,d\boldsymbol{\xi}\\
=\ &(2\pi i)^{-d/2}\int\limits_{(D-z)/\sqrt{\Delta t}}\left[\pi(\boldsymbol{z},t)+\sqrt{\Delta t}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}}\pi(\boldsymbol{z},t)+\frac{\Delta t}{2}\sum_{i,j}\xi_i\xi_j\pi_{x_ix_j}(\boldsymbol{z},t)+O\left(|\boldsymbol{\xi}\sqrt{\Delta t}|^3\right)\right]\\
&\times\exp\left\{-\frac{|\boldsymbol{\xi}|^2}{2i}\right\}\,d\boldsymbol{\xi}.\end{aligned} \tag{18}$$

The divergent integrals have to be summed by replacing $\exp\{-|\boldsymbol{\xi}|^2/2i\}$ with $\exp\{-|\boldsymbol{\xi}|^2(1+i\varepsilon)/2i\}$ for a positive ε and taking first the limit $\Delta t\to 0$ and then $\varepsilon\to 0$.

In the one-dimensional case $D=[-a,a]$, so the first term gives

$$\begin{aligned}&(2\pi i)^{-1/2}\pi(z,t)\int\limits_{(-a-z)/\sqrt{\Delta t}}^{(a-z)/\sqrt{\Delta t}}\exp\left\{-\frac{\xi^2(1+i\varepsilon)}{2i}\right\}\,d\xi\\ =\;&\pi(z,t)(1+o(\Delta t)),\end{aligned}$$

the second term gives

$$\frac{\sqrt{\Delta t}}{\sqrt{2\pi i}}\pi_x(z,t)\int\limits_{(-a-z)/\sqrt{\Delta t}}^{(a-z)/\sqrt{\Delta t}}\xi\exp\left\{-\frac{\xi^2(1+i\varepsilon)}{2i}\right\}\,d\xi=o(\Delta t),$$

and the third term is

$$\begin{aligned}&\frac{\Delta t}{2\sqrt{2\pi i}}\pi_{xx}(z,t)\int\limits_{(-a-z)/\sqrt{\Delta t}}^{(a-z)/\sqrt{\Delta t}}\xi^2\exp\left\{-\frac{\xi^2(1+i\varepsilon)}{2i}\right\}\,d\xi\\ =\;&\frac{i\Delta t}{2(1+i\varepsilon)}\pi_{xx}(z,t)\,[1+o(1)]\quad\text{for }\Delta t\to 0.\end{aligned}$$

Thus the only term in the denominator of (17) that is $O(\Delta t)$ is

$$\begin{aligned}&\frac{\Delta t}{2}\int\limits_{-a}^{a}\left[\frac{i}{1+i\varepsilon}\bar{\pi}(z,t)\pi_{xx}(z,t)-\frac{i}{1-i\varepsilon}\pi(z,t)\bar{\pi}_{xx}(z,t)\right]\,dz\\ =\;&i\varepsilon\Delta t\int\limits_{-a}^{a}\frac{|\pi_x(z,t)|^2}{1+\varepsilon^2}\,dz.\end{aligned}$$

In higher dimensions this term is

$$i\varepsilon\Delta t\int\limits_{D}\frac{|\nabla\pi(\boldsymbol{z},t)|^2}{1+\varepsilon^2}\,d\boldsymbol{z}(1+o(1)). \tag{19}$$

Now it follows from (17) and (19) that

$$\pi(\boldsymbol{x},t+\Delta t)=\frac{(2\pi i\Delta t)^{-d/2}\int_D\pi(\boldsymbol{y},t)\exp\left\{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2i\Delta t}\right\}\,d\boldsymbol{y}}{\left[1+i\varepsilon\Delta t\int_D\frac{|\nabla\pi(\boldsymbol{z},t)|^2}{1+\varepsilon^2}\,d\boldsymbol{z}(1+o(1))\right]^{1/2}}, \tag{20}$$

which, as in (9), gives the nonlinear Schrödinger equation

$$\pi_t(\boldsymbol{x},t)=\frac{i}{2}\Delta_{\boldsymbol{x}}\pi(\boldsymbol{x},t)-J(t)\pi(\boldsymbol{x},t), \tag{21}$$

where

$$J(t)=\frac{i\varepsilon}{2}\int\limits_{D}\frac{|\nabla\pi(\boldsymbol{z},t)|^2}{1+\varepsilon^2}\,d\boldsymbol{z}\to 0\ \text{ as }\varepsilon\to 0. \tag{22}$$

As far as the question of the need for renormalization is concerned, similarly to the case of diffusion (see Section II), the conditional probability density of the quantum particle in D after a negative observation is $|\pi(\boldsymbol{x}, t + \Delta t)|^2$, as given in (17). Therefore (17) holds up to a pure phase factor, which can be assumed to be 1. It follows from (21) and (22) that the wave function of a quantum particle under continuous observation is the solution of the Schrödinger equation in D with homogeneous Dirichlet boundary conditions, with or without renormalization. Summa summarum, (21) and (22) show that the matter is mute.

Thus the wave function of a quantum particle under continuous negative observations of the spatial coordinate for any period τ is the solution $\pi(\boldsymbol{x}, \tau)$ of the Schrödinger equation in D with truncated initial conditions in D and homogeneous Dirichlet boundary conditions on ∂D. One consequence of this observation is that if the wave function collapses in $\mathbb{R}^+$ at time $t = 0$, it has to stay collapsed throughout the measurement period $[0, \tau]$. Indeed, if it vanishes in the illuminated domain $\mathbb{R}^+$ at $t = 0$, but does not vanish there at positive time $0 < t \leq \tau$, then the wave stays collapsed throughout the interval $[0, t)$, so according to Schrödinger's equation, as described above, it vanishes on $\mathbb{R}^+$ at time t as well and there can be no particle detected at this time. This fact is a manifestation of the Zeno effect, when applied to coordinate measurement [3], [11], [12].

We call the solution of Schrödinger's equation with zero boundary conditions the continuously collapsed wave function. The above discussion shows that unlike the assertion of the collapse axiom, that after a single collapse in the observed region the wave function in the unmeasured region D stays intact, continuous spatial coordinate observations for any time τ change the wave function in the unmeasured domain, as described above, but freezes in the measured region. More specifically, the continuously collapsed wave function in D becomes $\pi(\boldsymbol{x}, \tau)$.

IV. DISCUSSION

An experimental realization of continuous measurement is to place photographic plates parallel to the direction of the light in the measured region $\mathbb{R}^+$ so that photons scattered from a particle that entered $\mathbb{R}^+$ are absorbed by the plates. The laboratory absorption times can be recorded to construct a time histogram of measured particles. If at the end of the exposure period τ the photographic plate detects no particle, we have to conclude that throughout this period there was no particle in the measured region and the wave function vanished there. It follows that the wave function at time τ is the solution of Schrödinger's equation with zero boundary conditions. Pursuing the same logic, as mentioned above, if there is collapse of the wave function at time $t = 0$, the wave function stays collapsed throughout the measurement period $0 \leq t \leq \tau$. Indeed, if the time histogram shows a particle at time $0 < t \leq \tau$, then the wave function stayed collapsed throughout the interval $[0, t)$, so according to Schrödinger's equation, as described in Section III, it vanishes on $\mathbb{R}^+$ at time t as well and there can be no particle detected at this time. This fact is a manifestation of the Zeno effect, when applied to coordinate measurement.

As mentioned in the Introduction, we propose here a modification of the collapse axiom for the negative coordinate measurement by postulating that it lasts for a positive time τ and the PMWF in the unmeasured domain is the solution of Schrödinger's equation at time τ with initial condition that is the truncated wave function and a zero Dirichlet condition on the boundary of the domain. It follows that the PMWF $\pi(x, \tau)$ is continuous on the entire line and has a finite derivative at the boundary. It was shown in [15] that the solution of Schrödinger's equation on the entire line, with a continuous initial condition and finite one-sided derivatives decays at infinity as $|x|^{-2}$, so it has a finite first moment. Thus the continuous collapse axiom alleviates the artifact, mentioned in the Introduction, introduced by the instantaneous collapse. Note that continuous collapse for measurements that exhibit the Zeno effect leaves the state unchanged, thus it introduces no new phenomenology.

There is, however, a difference between the experimental measurement of the wave function at time τ in the instantaneous and continuous cases. Consider the initial pre-measurement plane-wave (in normalized dimensionless variables)

$$\psi(x) = e^{-ix}, \tag{23}$$

which is measured at time $t = 0$ a negative measurement on the positive axis. In case of an instantaneous collapse, at time $t = \tau$ the truncated wave function

$$\varphi^I(x, 0) = \Theta(-x)e^{-ix} \tag{24}$$

will have propagated into

$$\varphi^I(x, \tau) = \frac{1}{\sqrt{i\pi\tau}} \int\limits_{-\infty}^{0} e^{-iy} e^{-i(x-y)^2/2\tau}\, dy. \tag{25}$$

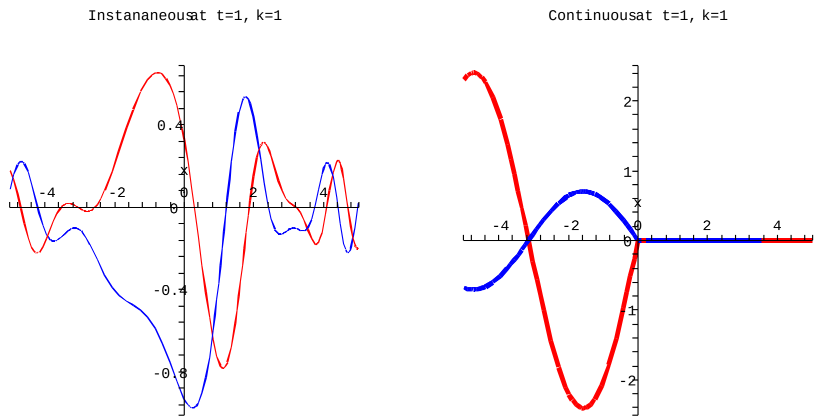

FIG. 1: The real (red) and imaginary (blue) parts of $\varphi^I(x,1)$ (left) and of $\varphi^C(x,1)$ (right).

According to Section III, in the case of a continuous collapse that ends at time $t=\tau$ the continuously collapsed wave function is the solution of Schrödinger's equation in $\mathbb{R}^-$ at time τ with zero boundary condition at $x=0$ and initial condition (24) and it vanishes in $\mathbb{R}^+$. This solution is constructed by reflection in the origin as will have propagated into

$$\varphi^C(x,\tau)=\frac{2i\Theta(-x)}{\sqrt{i\pi\tau}}\int\limits_{-\infty}^{\infty}\sin ky\,e^{-i(x-y)^2/2\tau}\,dy. \tag{26}$$

Figure 1 shows the real (red) and imaginary (blue) parts of $\varphi^I(x,1)$ (left) and $\varphi^C(x,1)$ (right). The two different statistics should be observable. Comparing the PMWF, we see that $\varphi^I(x,0)=\Theta(-x)$ while $\varphi^C(x,\tau)$ is that given in Figure There should be also a difference in the measurement of the average spatial coordinate in a given interval after an instantaneous negative coordinate measurement and after a negative continuous coordinate measurement. Specifically, in the former case, the average coordinate in the interval $[x_1,x_2]$ at time Δt after the collapse is $O(\Delta t^{1/2}\log(x_2-x_1))$, whereas in the latter case it is $O(\Delta t^{3/2}(x_1^{-2}-x_2^{-2}))$ [15], [14].

The implication of the last observation in Section III is that if intermittent absorption of a free quantum particle on the positive axis $\mathbb{R}^+$ is defined as a process of instantaneous truncation of the wave function on $\mathbb{R}^+$ at times Δt apart, then, in contrast to the case of intermittent measurements, the wave function on the negative axis is not renormalized. Thus, similarly to the case of measurement, intermittent instantaneous absorption does not permit the particle to propagate into $\mathbb{R}^+$, resulting in no absorption at all.

The analogous situation in diffusion is different. Assume that intermittent instantaneous absorption in $\mathbb{R}^+$ is defined as turning on intermittently an instantaneous infinite killing measure $k(x,t)$ in $\mathbb{R}^+$, for example,

$$k(x,t) = \sum_{k=1}^{N} \delta(t - k\Delta t)\Theta(x),$$

where $N = t/\Delta t$ and $\Theta(x)$ is the Heaviside step function. Then, according to the Feynman-Kac formula [17], the transition probability density of the killed Brownian motion is the solution of the problem

$$\frac{\partial p_N(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p_N(x,t)}{\partial x^2} - \sum_{k=1}^{N} \delta(t - k\Delta t)\Theta(x) p_N(x,t). \tag{27}$$

Integrating over space and time, we obtain

$$\int_{-\infty}^{\infty} p_N(x,t)\,dx - 1 = \frac{1}{\Delta t}\int_0^{\infty}\sum_{k=1}^{N} p_N(x, t - k\Delta t)\Delta t\,dx. \tag{28}$$

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a sufficiently small interval that is the resolution of the measuring device.