

# Alternative construction of gauge-invariant variables for linear metric perturbation on general background spacetime

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Construction of the gauge-invariant variables for the linear metric perturbation, which was proposed in the paper [K. Nakamura, arXiv:1101.1147], is discussed through an alternative approach. Our starting point of the construction of the gauge-invariant variables is an non-trivial non-local decomposition of the linear metric perturbation. Assuming the existence of some Green functions, we reproduce results in the above paper. This supports the consistency of the result and implies that one can develop the general-relativistic higher-order gauge-invariant perturbation theory on general background spacetime.

## I. INTRODUCTION

General relativity is a theory in which the construction of exact solutions is not so easy. In this situation, perturbation theories are powerful techniques and the developments of perturbation theories lead physically fruitful results and interpretations of natural phenomena. For this reason, general relativistic *linear* perturbation theory has been widely used in many area[1]. Further, the investigation of *higher-order* general-relativistic perturbations is also necessary due to the precise observations[2] in cosmology[3, 4], and to prepare more precise wave form of gravitational wave[5] for the gravitational wave detection.

As well-known, general relativity is based on the concept of general covariance. Due to this general covariance, the “gauge degree of freedom”, which is an unphysical degree of freedom, arises in general-relativistic perturbations. To obtain physical results, we have to

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fix this gauge degrees of freedom or to extract some invariant quantities of perturbations. This situation becomes more complicated in higher-order perturbations. Since there are so-called *gauge-invariant* linear perturbation theories in some background spacetimes, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view to avoid this gauge issues.

According to these motivation, the general framework of higher-order general-relativistic gauge-invariant perturbation theory has been discussed in some papers by the present author[6]. In this general framework, we consider the perturbative expansion of the physical metric  $\bar{g}_{ab} = g_{ab} + \lambda h_{ab} + O(\lambda^2)$ , where  $\lambda$  is an infinitesimal parameter for the perturbation theory.  $h_{ab}$  is the linear-order metric perturbation. Further, in our general framework, we assumed the following conjecture:

**Conjecture I.1** *If there is a tensor field  $h_{ab}$  of the second rank, whose gauge transformation rule is*

$$\mathcal{Y}h_{ab} - \mathcal{X}h_{ab} = \mathcal{L}_\xi g_{ab}, \quad (1)$$

*then there exist a tensor field  $\mathcal{H}_{ab}$  and a vector field  $X^a$  such that  $h_{ab}$  is decomposed as*

$$h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}, \quad (2)$$

*where  $\mathcal{H}_{ab}$  and  $X^a$  are transformed as*

$$\mathcal{Y}\mathcal{H}_{ab} - \mathcal{X}\mathcal{H}_{ab} = 0, \quad \mathcal{Y}X^a - \mathcal{X}X^a = \xi^a. \quad (3)$$

*Here,  $\mathcal{Y}$  and  $\mathcal{X}$  denote different gauge choices.*

In the case of cosmological perturbations, we confirmed that this conjecture is almost true, and then, we developed the second-order gauge-invariant cosmological perturbations[4]. Since our general framework of higher-order perturbations does not depend on details of the background metric  $g_{ab}$ , we will be able to develop general-relativistic higher-order gauge-invariant perturbation theory if Conjecture I.1 is true[6]. Although we recently propose a scenario of the proof of Conjecture I.1, in this article, we give an alternative explanation of this proof. This is the main purpose of this article.

## II. CONSTRUCTION OF GAUGE-INVARIANT VARIABLES

Now, we give a scenario of a proof of Conjecture I.1 on general background spacetimes. Here, we assume that the background spacetimes considered in this paper admit ADM decomposition[8]. Therefore, the background spacetime  $\mathcal{M}_0$  considered here is  $n + 1$ -dimensional spacetime which is foliated by spacelike hypersurfaces  $\Sigma(t)$  ( $\dim \Sigma = n$ ). Each  $\Sigma$  may have its boundary  $\partial\Sigma$ . The background metric  $g_{ab}$  is given by

$$g_{ab} = -\alpha^2(dt)_a(dt)_b + q_{ij}(dx^i + \beta^i dt)_a(dx^j + \beta^j dt)_b, \quad (4)$$

where  $\alpha$  is the lapse function,  $\beta^i$  is the shift vector, and  $q_{ab} = q_{ij}(dx^i)_a(dx^j)_b$  is the metric on  $\Sigma(t)$ .

To consider the decomposition (2) of  $h_{ab}$ , we, first, consider the components of the metric  $h_{ab}$  as

$$h_{ab} = h_{tt}(dt)_a(dt)_b + 2h_{ti}(dt)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b. \quad (5)$$

The gauge transformation rule (1) gives the transformation rules for the components  $\{h_{tt}, h_{ti}, h_{ij}\}$ , which are given by

$$\begin{aligned} y h_{tt} - x h_{tt} &= 2\partial_t \xi_t - \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) \xi_t \\ &\quad - \frac{2}{\alpha} \left( \beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j \right. \\ &\quad \left. + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha \right) \xi_i, \end{aligned} \quad (6)$$

$$\begin{aligned} y h_{ti} - x h_{ti} &= \partial_t \xi_i + D_i \xi_t - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) \xi_t \\ &\quad - \frac{2}{\alpha} (-\alpha^2 K^j_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j) \xi_j, \end{aligned} \quad (7)$$

$$y h_{ij} - x h_{ij} = 2D_{(i} \xi_{j)} + \frac{2}{\alpha} K_{ij} \xi_t - \frac{2}{\alpha} \beta^k K_{ij} \xi_k, \quad (8)$$

where  $K_{ij}$  is the extrinsic curvature:

$$K_{ij} = -\frac{1}{2\alpha} \left[ \frac{\partial}{\partial t} q_{ij} - D_i \beta_j - D_j \beta_i \right], \quad (9)$$

and  $D_i$  is the covariant derivative associated with the spatial metric  $q_{ij}$ . Inspecting these gauge transformation rules and assuming the existence of the Green function of the derivative

operator  $\Delta := D^i D_i$ , we consider the decompositions of the components  $h_{ti}$  and  $h_{ij}$  as follows:

$$\begin{aligned}
 h_{ti} =: & D_i h_{(VL)} + h_{(V)i} - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) h_{(VL)} \\
 & - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) \Delta^{-1} \left[ \partial_t D^k h_{(TV)k} - 2D_l \{ h_{(TV)k} (\alpha K^{kl} - D^{(k} \beta^{l)}) \} \right. \\
 & \quad \left. + h_{(TV)k} D^k (\alpha K - D^l \beta_l) \right] \\
 & - \frac{2}{\alpha} (-\alpha^2 K^j{}_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j) h_{(TV)j}, \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 h_{ij} =: & \frac{1}{n} q_{ij} h_{(L)} + h_{(T)ij} + \frac{2}{\alpha} K_{ij} h_{(VL)} - \frac{2}{\alpha} \beta^k K_{ij} h_{(TV)k} \\
 & - \frac{2}{\alpha} K_{ij} \Delta^{-1} \left[ \partial_t D^k h_{(TV)k} - 2D_l \{ h_{(TV)k} (\alpha K^{kl} - D^{(k} \beta^{l)}) \} \right. \\
 & \quad \left. + h_{(TV)k} D^k (\alpha K - D^l \beta_l) \right], \tag{11}
 \end{aligned}$$

$$h_{(T)ij} =: D_i h_{(TV)j} + D_j h_{(TV)i} - \frac{2}{n} q_{ij} D^l h_{(TV)l} + h_{(TT)ij}, \tag{12}$$

$$D^i h_{(V)i} = 0, \quad q^{ij} h_{(TT)ij} = 0, \quad D^i h_{(TT)ij} = 0. \tag{13}$$

From the gauge-transformation rule (6) and the definitions (10) and (13) of the variable  $h_{(LV)}$  and  $h_{(V)i}$ , we can derive the gauge-transformation rules for these variables by assuming the existence of the Green function  $\mathcal{F}^{-1}$  of the elliptic derivative operator

$$\mathcal{F} := \Delta - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) D^i - 2D^i \left\{ \frac{1}{\alpha} (D_i \alpha - \beta^j K_{ij}) \right\}. \tag{14}$$

However, in these gauge-transformation rules, the variable  $h_{(TV)i}$  defined in Eqs. (11)–(13) is also included. This means that the gauge-transformation rules for variables  $h_{(LV)}$  and  $h_{(V)i}$  are specified if we have the gauge-transformation rule for  $h_{(TV)i}$ . From the trace part of the gauge-transformation rule (8), we can derive the gauge-transformation rule for the variable  $h_{(L)}$ , which is defined by Eqs. (11)–(13). This gauge-transformation rule also includes the variables  $h_{(LV)}$  and  $h_{(TV)i}$ . Since the gauge-transformation rule for the variable  $h_{(LV)}$  is determined if the gauge-transformation rule for the variable  $h_{(TV)i}$  is specified, the gauge-transformation rule for  $h_{(L)}$  is also specified if the gauge-transformation rule for  $h_{(TV)i}$  is given. Thus, the gauge-transformation rules for the variables  $h_{(LV)}$ ,  $h_{(V)i}$ , and  $h_{(L)}$  are specified if the gauge-transformation for the variable  $h_{(TV)i}$  is specified.

Taking the divergence of the traceless part of the gauge-transformation rule (8) and using Eqs. (11)–(13) and the gauge-transformation rule for the variable  $h_{(LV)}$  obtained above, we

reach to a single equation for a unknown single vector field  $A_i := y h_{(TV)i} - x h_{(TV)i} - \xi_i$ :

$$\mathcal{D}^{jl} A_l = D_i \left[ \frac{2}{\alpha} \tilde{K}^{ij} \left\{ \mathcal{F}^{-1} \left[ \partial_t D^k A_k - 2D_l \left\{ A_k (\alpha K^{kl} - D^{(k} \beta^{l)}) + \frac{1}{\alpha} L^{ml} A_m \right\} \right. \right. \right. \\ \left. \left. \left. + A_k D^k (\alpha K - D^l \beta_l) \right] - \beta^k A_k \right\} \right], \quad (15)$$

where the derivative operator  $\mathcal{D}^{ij}$  and the tensors  $\tilde{K}^{ij}$  and  $L^{ij}$  are defined by

$$\mathcal{D}^{ij} := q^{ij} \Delta + \left( 1 - \frac{2}{n} \right) D^i D^j + R^{ij}, \quad \tilde{K}^{ij} := K^{ij} - \frac{1}{n} q^{ij} K, \quad K := q^{ij} K_{ij}, \\ L^{ij} := -\alpha^2 K^{ij} + \beta^i \beta_k K^{kj} - \beta^i D^j \alpha + \alpha D^j \beta^i. \quad (16)$$

As a trivial solution to Eq. (15), we have  $A_i = 0$ , i.e.,

$$y h_{(TV)i} - x h_{(TV)i} = \xi_i \quad (17)$$

Thus, we have obtained the gauge-transformation rule for the variable  $h_{(TV)i}$ .

Using the gauge-transformation rule (17) for the variable  $h_{(TV)i}$ , the gauge-transformation rules for variables  $h_{tt}$ ,  $h_{(VL)}$ ,  $h_{(V)i}$ ,  $h_{(L)}$ , and  $h_{(TT)ij}$  are given by

$$y h_{(VL)} - x h_{(VL)} = \xi_t + \Delta^{-1} \left[ \partial_t D^k \xi_k - 2D_l \left\{ \xi_k (\alpha K^{kl} - D^{(k} \beta^{l)}) \right\} \right. \\ \left. + \xi_k D^k (\alpha K - D^l \beta_l) \right], \quad (18)$$

$$y h_{(V)i} - x h_{(V)i} = \partial_t \xi_i - D_i \Delta^{-1} \left[ \partial_t D^k \xi_k - 2D_l \left\{ \xi_k (\alpha K^{kl} - D^{(k} \beta^{l)}) \right\} \right. \\ \left. + \xi_k D^k (\alpha K - D^l \beta_l) \right], \quad (19)$$

$$y h_{(L)} - x h_{(L)} = 2D^i \xi_i, \quad (20)$$

$$y h_{(TT)ij} - x h_{(TT)ij} = 0. \quad (21)$$

Inspecting the gauge-transformation rules (17) and (18), we first construct the variables  $X_t$  and  $X_i$  which satisfy the properties  $y X_t - x X_t = \xi_t$ ,  $y X_i - x X_i = \xi_i$ , respectively. We can easily find these variables as follows:

$$X_t := h_{(VL)} - \Delta^{-1} \left[ \partial_t D^k h_{(TV)k} - 2D_l \left\{ h_{(TV)k} (\alpha K^{kl} - D^{(k} \beta^{l)}) \right\} \right. \\ \left. + h_{(TV)k} D^k (\alpha K - D^l \beta_l) \right], \quad (22)$$

$$X_i := h_{(TV)i}. \quad (23)$$

These  $X_t$  and  $X_i$  satisfy the desired gauge-transformation rules, respectively.

Through these  $X_t$  and  $X_i$  and the gauge-transformation rules (6), (19)–(21), we can easily define gauge-invariant variables  $\Phi$ ,  $\Psi$ ,  $\nu_i$ , and  $\chi_{ij}$  by

$$\begin{aligned}
 -2\Phi &:= h_{tt} - 2\partial_t X_t + \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) X_t \\
 &\quad + \frac{2}{\alpha} \left( \beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j + \alpha^2 D^i \alpha \right. \\
 &\quad \left. - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha \right) X_i,
 \end{aligned} \tag{24}$$

$$-2n\Psi := h_{(L)} - 2D^i X_i, \tag{25}$$

$$\begin{aligned}
 \nu_i &:= h_{(V)i} - \partial_t X_i \\
 &\quad + D_i \Delta^{-1} \left[ \partial_t D^k X_k - 2D_l \{ X_k (\alpha K^{kl} - D^{(k} \beta^{l)}) \} \right. \\
 &\quad \left. + X_k D^k (\alpha K - D^l \beta_l) \right],
 \end{aligned} \tag{26}$$

$$\chi_{ij} := h_{(TT)ij}. \tag{27}$$

Actually, we can easily confirm that these variables are gauge invariant. We also note that the variable  $\nu_i$  and  $\chi_{ij}$  satisfy the properties  $D^i \nu_i = 0$ ,  $q^{ij} \chi_{ij} = 0$ ,  $\chi_{[ij]} = 0$ , and  $D^i \chi_{ij} = 0$ .

Through the original components  $h_{tt}$ ,  $h_{ti}$ , and  $h_{ij}$  in terms of the variables  $\Phi$ ,  $\Psi$ ,  $\nu_i$ ,  $\chi_{ij}$ ,  $X_t$ , and  $X_i$ , the first-order metric perturbation  $h_{ab}$  are given in the form as Eq. (2) by choosing

$$\mathcal{H}_{ab} = -2\Phi(dt)_a(dt)_b + 2\nu_i(dt)_{(a}(dx^i)_{b)} + (-2q_{ij}\Psi + \chi_{ij})(dx^i)_a(dx^j)_b, \tag{28}$$

$$X_a = X_t(dt)_a + X_i(dx^i)_a. \tag{29}$$

Thus, we have confirmed Conjecture I.1.

### III. SUMMARY

We have shown an alternative scenario of a proof of Conjecture I.1. In our previous paper[7], we first assumed the existence of the gauge-variant variables  $X_t$  and  $X_i$  and confirmed this existence through the explicit construction of these variables. Logically speaking, this is a no-trivial logic and we have explicitly constructed gauge-invariant variables  $\Phi$ ,  $\Psi$ ,  $\nu_i$ , and  $\chi_{ij}$ . In this previous derivation, we assume the existence of the Green functions of the Laplacian  $\Delta$  and the elliptic derivative operator  $\mathcal{D}_{ij}$  in Eqs. (16). Therefore, special modes which belong to the kernels of these two derivative operators are excluded in our consideration. To include these modes, different treatments will be necessary. We called

this problem as *zero-mode problem*. In this paper, we gave an alternative construction of the variable  $X_t$  and  $X_i$  to support our previous result[7]. In this sense, we may say that the results in our previous paper[7] are correct.

However, in the approach in this article, we assumed the existence of the Green function of the elliptic derivative operator  $\mathcal{F}$ , which defined by Eq. (14), instead of  $\mathcal{D}_{ij}$ . Further, the role of nontrivial solutions to Eq. (15) is not clear. In this sense, the set up of the above “zero-mode problem” is still ambiguous in general case. To clarify this “zero-mode problem” itself, it will be necessary to discuss this problem through the concrete background metric  $g_{ab}$  and appropriate boundary conditions at  $\partial\Sigma$  at first. We will leave these issues as future works.

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