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On the capabilities of grammars, automata, and transducers controlled by monoids

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During the last decades, classical models in language theory have been extended by control mechanisms defined by monoids. We study which monoids cause the extensions of context-free grammars, finite automata, or finite state transducers to exceed the capacity of the original model. Furthermore, we investigate when, in the extended automata model, the nondeterministic variant differs from the deterministic one in capacity. We show that all these conditions are in fact equivalent and present an algebraic characterization. In particular, the open question of whether every language generated by a valence grammar over a finite monoid is context-free is provided with a positive answer.

1 Introduction

The idea to equip classical models of theoretical computer science with a monoid (or a group) as a control mechanism has been pursued by several authors in the last decades [FS02, IMVM01, Kam09, MS01, Pău80, RK09]. This interest is justified by the fact that these extensions allow for a uniform treatment of a wide range of automata and grammar models: Suppose a storage mechanism can be regarded as a set of states on which a set of partial transformations operates and a computation is considered valid if the composition of the executed transformations is the identity. Then, this storage constitutes a certain monoid control.

For example, in a pushdown storage, the operations *push* and *pop* (for each participating stack symbol) and compositions thereof are partial transformations on the set of words over some alphabet. In this case, a computation is considered valid if, in the end, the stack is brought back to the initial state, i.e., the identity transformation has been applied. As further examples, blind and partially blind multicounter automata (see [Gre78]) can be regarded as finite automata controlled by a power of the integers and of the bicyclic monoid (see [RK09]), respectively.

Another reason for studying monoid controlled automata, especially in the case of groups, is that the word problems of a group G are contained in a full trio (such as the context-free or the indexed languages) if and only if the languages accepted by valence automata over G are contained in this full trio (see, for example, [Kam09, Proposition 2]). Thus, valence automata offer an automata theoretic interpretation of word problems for groups.

A similar situation holds for context-free grammars where each production is assigned a monoid element such that a derivation is valid as soon as the product of the monoid elements (in the order of the application of the rules) is the identity. Here, the integers, the multiplicative group of $\mathbb{Q}$, and powers of the bicyclic monoid lead to additive and multiplicative valence grammars and Petri net controlled grammars, respectively. The latter are in turn equivalent to matrix grammars (with erasing and without appearance checking, see [DT09] for details). Therefore, the investigation of monoid control mechanisms promises very general insights into a variety of models.

One of the most basic problems for these models is the characterization of those monoids whose use as control mechanism actually increases the power of the respective model. For monoid controlled automata, such a characterization has been achieved by Mitrana and Stiebe [MS01] for the case of groups, but has not been established for monoids. For valence grammars, that is, context-free grammars with monoid control, very little was known in this respect up to date. It was an open problem whether valence grammars over finite monoids are capable of generating languages that are not context-free (see [FS02, p. 387]).

Another important question is for which monoids the extended automata can be determinized, that is, for which monoids the deterministic variant is as powerful as the nondeterministic one. Mitrana and Stiebe [MS01] have shown that automata controlled by a group cannot be determinized if the group contains at least one element of infinite order. However, the exact class of monoids for which automata can be determinized was not known to date.

The contribution of this work is twofold. On the one hand, the open question of whether all languages generated by valence grammars over finite monoids are context-free is settled affirmatively. On the other hand, we present an algebraic dichotomy of monoids that turns out to provide a characterization for all the conditions above. Specifically, we show that the following assertions are equivalent:

- Valence grammars over M generate only context-free languages.
- Valence automata over M accept only regular languages.
- Valence automata over M can be determinized.
- Valence transducers over M perform only rational transductions.
- In each finitely generated submonoid of M, only finitely many elements possess a right inverse.

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2 Basic notions

A *monoid* is a set M together with an associative operation and a neutral element. Unless defined otherwise, we will denote the neutral element of a monoid by 1 and its operation by juxtaposition. That is, for a monoid M and $a, b \in M$, $ab \in M$ is their product. The *opposite monoid* M^{op} of M has the same set of elements as M, but has the operation $\circ$ with $a \circ b := ba$, $a, b \in M$. For $a, b \in M$, we write $a \sqsubseteq b$ iff there are $c, d \in M$ such that $b = ac = da$. Let $a \in M$. An element $b \in M$ with $ab = 1$ is called a *right inverse* of a. If $b \in M$ obeys $ba = 1$, it is a *left inverse* of a. An element that is both a left and a right inverse is said to be a *two-sided inverse*. By $\mathbf{1}$, we denote the trivial monoid that consists of just one element. M is said to be *left-cancellative* if $ab = ac$ implies $b = c$ for $a, b, c \in M$. Whenever M^{op} is left-cancellative, we say that M is *right-cancellative.*

A subset $N \subseteq M$ is said to be a *submonoid of* M iff $1 \in N$ and $a, b \in N$ implies $ab \in N$. For a subset $N \subseteq M$, let $\langle N \rangle$ be the intersection of all submonoids N' of M that contain N. That is, $\langle N \rangle$ is the smallest submonoid of M that contains N. $\langle N \rangle$ is also called the *submonoid generated by* N. We call a monoid *finitely generated* if it is generated by a finite subset. In each monoid M, we have the following submonoids:

$$\begin{aligned} \mathfrak{R}(M) &:= \{a \in M \mid \exists b \in M : ab = 1\}, \\ \mathfrak{L}(M) &:= \{a \in M \mid \exists b \in M : ba = 1\}. \end{aligned}$$

The elements of $\mathfrak{R}(M)$ and $\mathfrak{L}(M)$ are called *right invertible* and *left invertible*, respectively. In addition, for every element $a \in M$, we define the sets

$$\begin{aligned} \overrightarrow{\mathfrak{I}}(a) &:= \{b \in M \mid ab = 1\}, \\ \overleftarrow{\mathfrak{I}}(a) &:= \{b \in M \mid ba = 1\}. \end{aligned}$$

When using a monoid M as part of a control mechanism, the subset

$$\mathfrak{C}(M) := \{a \in M \mid \exists b, c \in M : bac = 1\}$$

will play an important role. If in M every element has a two-sided inverse, we call M a *group.*

Let Σ be a fixed countable set of abstract symbols, the finite subsets of which are called *alphabets.* For an alphabet X, we will write X^* for the set of words over X. The empty word is denoted by $\lambda \in X^*$. In particular, $\emptyset^* = \{\lambda\}$. Together with the concatenation as its operation, X^* is a monoid. We will regard every $x \in X$ as an element of X^*, namely the word consisting only of one occurence of x. For a symbol $x \in X$ and a word $w \in X^*$, let $|w|_x$ be the number of occurrences of x in w. For a subset $Y \subseteq X$, let $|w|_Y := \sum_{x \in Y} |w|_x$. By $|w|$, we will refer to the length of w. By

$X^{\leq n} \subseteq X^*$, for $n \in \mathbb{N}$, we denote the set of all words over X of length $\leq n$. Given alphabets X, Y, subsets of X^* and $X^* \times Y^*$ are called *languages* and *transductions*, respectively. We define the *shuffle* $L_1 \sqcup\!\sqcup L_2$ of two languages $L_1, L_2 \subseteq X^*$ to be the set of all words $w \in X^*$ such that $w = u_1 v_1 \cdots u_n v_n$ for some $u_i, v_i \in X^*$, $1 \leq i \leq n$, with $u_1 \cdots u_n \in L_1$, $v_1 \cdots v_n \in L_2$. When $\{w\}$ is used as an operand for $\sqcup\!\sqcup$, we also just write w instead of $\{w\}$. For $x_1, \ldots, x_n \in X$, let $(x_1 \cdots x_n)^{\mathrm{rev}} := x_n \cdots x_1$.

Let M be a monoid. An *automaton over M* is a tuple $A = (M, Q, E, q_0, F)$, in which Q is a finite set of *states*, E is a finite subset of $Q \times M \times Q$, called the set of *edges*, $q_0 \in Q$ is the *initial state*, and $F \subseteq Q$ is the set of *final states*. The *step relation* $\Rightarrow_A$ of A is a binary relation on $Q \times M$, for which $(p, a) \Rightarrow_A (q, b)$ iff there is an edge (p, c, q) such that $b = ac$. The set generated by A is then

$$S(A) := \{a \in M \mid \exists q \in F : (q_0, 1) \Rightarrow_A^* (q, a)\}.$$

A *valence automaton over M* is an automaton A over $X^* \times M$, where X is an alphabet. A is said to be *deterministic* if all its edges are in $Q \times (X \times M) \times Q$ and, for each pair $(q, x) \in Q \times X$, there is at most one edge $(q, (x, m), p)$ for $m \in M, p \in Q$. The *language accepted by A* is defined as

$$L(A) := \{w \in X^* \mid (w, 1) \in S(A)\}.$$

A *finite automaton* is a valence automaton over $\mathbf{1}$. For a finite automaton $A = (X^* \times \mathbf{1}, Q, E, q_0, F)$, we also write $A = (X, Q, E, q_0, F)$. Languages accepted by finite automata are called *regular languages*. A *valence transducer over M* is an automaton A over $X^* \times Y^* \times M$, where X and Y are alphabets. The *transduction performed by A* is

$$T(A) := \{(x, y) \in X^* \times Y^* \mid (x, y, 1) \in S(A)\}.$$

A *finite state transducer* is a valence transducer over $\mathbf{1}$. For a finite state transducer $A = (X^* \times Y^* \times \mathbf{1}, Q, E, q_0, F)$, we also write $A = (X, Y, Q, E, q_0, F)$. Transductions performed by finite state transducers are called *rational transductions*.

A *valence grammar over M* is a tuple $G = (N, T, M, P, S)$, where N, T are disjoint alphabets, called the *nonterminal* and *terminal alphabet*, respectively, $P \subseteq N \times (N \cup T)^* \times M$ is a finite set of *productions*, and $S \in N$ is the *start symbol*. For a production $(A, w, m) \in P$ we also write $(A \to w; m)$. The *derivation relation* $\Rightarrow_G$ of G is a binary relation on $(N \cup T)^* \times M$, for which $(u, a) \Rightarrow_G (v, b)$ iff there is a $(A \to w; c) \in P$ and words $r, s \in (N \cup T)^*$ such that $u = rAs$, $v = rws$, and $b = ac$. The *language generated by G* is defined as

$$L(G) := \{w \in T^* \mid (S, 1) \Rightarrow_G^* (w, 1)\}.$$

Valence grammars were introduced by Păun in [Pău80]. A thorough treatment, including normal form results and a classification of the resulting language classes for commutative monoids, has been carried out by Fernau and Stiebe [FS02]. Valence grammars over $\mathbf{1}$ are called *context-free grammars*. For a context-free grammar $G = (N, T, \mathbf{1}, P, S)$, we also write $G = (N, T, P, S)$. Furthermore, a production $(A \to w; 1) \in P$ in a context-free grammar is also written $A \to w$. Languages generated by context-free grammars are called *context-free*.

3 A dichotomy of monoids

An *infinite ascending chain* in M is an infinite sequence $x_1, x_2, \dots$ of pairwise distinct elements of M such that $x_i \sqsubseteq x_{i+1}$ for all $i \in \mathbb{N}$.

Lemma 1. *Let M be left- or right-cancellative. Then exactly one of the following holds:*

1. *M is a finite group.*
2. *M contains an infinite ascending chain.*

Proof. Suppose M does not contain an infinite ascending chain.

First, we prove that M is a group. Let $r \in M$ and consider the elements $r^i \in M$, $i \in \mathbb{N}$. Since $r^i \sqsubseteq r^j$ for $i \leq j$, our assumption implies that there are $i, j \in \mathbb{N}$, $i < j$, with $r^i = r^j$. Since M is left- or right-cancellative, this implies $r^{j-i} = 1$, meaning that r has in r^{j-i-1} a two-sided inverse. Thus, M is a group.

This implies that $r \sqsubseteq s$ for any $r, s \in M$. Therefore, if M were infinite, it would contain an infinite ascending chain. □

Lemma 2. *Let $s, t \in M$, $s \neq t$, and $s \sqsubseteq t$. Then, $\overrightarrow{\mathfrak{I}}(s) \cap \overrightarrow{\mathfrak{I}}(t) = \emptyset$ and $\overleftarrow{\mathfrak{I}}(s) \cap \overleftarrow{\mathfrak{I}}(t) = \emptyset$.*

Proof. We only show $\overrightarrow{\mathfrak{I}}(s) \cap \overrightarrow{\mathfrak{I}}(t) = \emptyset$ since then $\overleftarrow{\mathfrak{I}}(s) \cap \overleftarrow{\mathfrak{I}}(t) = \emptyset$ follows by applying the former to the opposite monoid. Write $t = us$, $u \in M$, and suppose there were a $z \in \overrightarrow{\mathfrak{I}}(s) \cap \overrightarrow{\mathfrak{I}}(t)$. Then, $1 = tz = usz = u$ and thus $t = s$. □

Theorem 3. *For every monoid M, exactly one of the following holds:*

1. *The subsets $\mathfrak{R}(M)$, $\mathfrak{L}(M)$, and $\mathfrak{E}(M)$ coincide and constitute a finite group.*
2. *$\mathfrak{R}(M)$ and $\mathfrak{L}(M)$ each contain an infinite ascending chain. In particular, there exist infinite sets $S \subseteq \mathfrak{R}(M)$ and $S' \subseteq \mathfrak{L}(M)$ such that $\overrightarrow{\mathfrak{I}}(s) \cap \overrightarrow{\mathfrak{I}}(t) = \emptyset$ for $s, t \in S$, $s \neq t$, and $\overleftarrow{\mathfrak{I}}(s') \cap \overleftarrow{\mathfrak{I}}(t') = \emptyset$ for $s', t' \in S'$, $s' \neq t'$.*

Proof. First, we claim that $\mathfrak{R}(M)$ is infinite if and only if $\mathfrak{L}(M)$ is infinite. Here, it suffices that $\mathfrak{R}(M)$ being infinite implies the infinity of $\mathfrak{L}(M)$, since the other direction follows by considering the opposite monoid. If $\mathfrak{R}(M)$ is infinite, it contains an infinite ascending chain according to Lemma 1. By Lemma 2, the elements of the chain have pairwise disjoint sets of right inverses, which are non-empty. Since right inverses are left invertible, $\mathfrak{L}(M)$ is infinite.

Suppose $\mathfrak{R}(M)$ and $\mathfrak{L}(M)$ are both finite. Since $\mathfrak{R}(M)$ is right-cancellative, it is a group by Lemma 1. Thus, we have $\mathfrak{R}(M) \subseteq \mathfrak{L}(M)$ and analogously $\mathfrak{L}(M) \subseteq \mathfrak{R}(M)$. In order to prove $\mathfrak{E}(M) = \mathfrak{R}(M)$, we observe that $\mathfrak{R}(M) \subseteq \mathfrak{E}(M)$ by definition. Now, suppose $a \in \mathfrak{E}(M)$ to be witnessed by $bac = 1$, $b, c \in M$. By this equation, we have $b \in \mathfrak{R}(M)$ and can multiply b^{-1} on the left and then b on the right. We obtain $acb = 1$ and thus $a \in \mathfrak{R}(M)$. This proves that $\mathfrak{R}(M) = \mathfrak{L}(M) = \mathfrak{E}(M)$ and that this is a finite group.

In case $\mathfrak{R}(M)$ and $\mathfrak{L}(M)$ are both infinite, the infinite ascending chains are provided by Lemma 1. By Lemma 2, their elements form sets $S \subseteq \mathfrak{R}(M)$ and $S' \subseteq \mathfrak{L}(M)$ with the desired properties. □

4 Capabilities of valence automata and transducers

In this section, we show that the following conditions are equivalent:

- Valence automata over M accept only regular languages.
- Valence automata over M can be determinized.
- Valence transducers over M perform only rational transductions.
- $\mathfrak{R}(N)$ is finite for every finitely generated submonoid N of M.

Lemma 4. *Let $\mathfrak{R}(N)$ be finite for every finitely generated submonoid N of M. Then, valence automata over M accept only regular languages and valence transducers over M perform only rational transductions. In particular, valence automata over M can be determinized.*

Proof. Let $A = (X^* \times M, Q, E, q_0, F)$ be a valence automaton over M. Since E is finite, the set of $m \in M$ such that there is some edge $(p, (w, m), q)$ in E is finite. If N is the submonoid of M generated by these $m \in M$, we can regard A as a valence automaton over N. Thus, let $A = (X^* \times N, Q, E, q_0, F)$. Furthermore, removing edges of the form $(p, (w, m), q)$ such that $m \notin \mathfrak{E}(N)$ will not alter the accepted language, since such edges cannot be used in a successful run. By Theorem 3, $\mathfrak{E}(N)$ is a finite group and we can assume $A = (X^* \times \mathfrak{E}(N), Q, E, q_0, F)$. Since $\mathfrak{E}(N)$ is finite, a finite automaton accepting $L(A)$ can now easily be constructed by incorporating the monoid elements into the states. The proof for the valence transducers over M works completely analogously.

Since finite automata can be determinized and we have seen that valence automata over M accept only regular languages, it follows that valence automata over M can be determinized. □

In [MS01], Mitrana and Stiebe proved that valence automata over groups with at least one element of infinite order cannot be determinized. We can now use a similar idea and our dichotomy theorem to provide a characterization of those monoids over which valence automata can be determinized.

Lemma 5. *Let M be a finitely generated monoid such that $\mathfrak{R}(M)$ is infinite. Then, there is a valence automaton over M whose accepted language cannot be accepted by a deterministic valence automaton over M. In particular, valence automata over M can accept non-regular languages and valence transducers over M can perform non-rational transductions.*

Proof. Let M be generated by the finite set $\{a_1, \dots, a_n\}$ and let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$ be disjoint alphabets. Let $\varphi : (X \cup Y)^* \to M$ be the epimorphism defined by $\varphi(x_i) := \varphi(y_i) := a_i$ and $K := X^* \cup \{w \in X^*Y^* \mid \varphi(w) = 1\}$. Then, K is clearly accepted by a (nondeterministic) valence automaton over M. Suppose K were accepted by a deterministic valence automaton A over M. Let $S \subseteq \mathfrak{R}(M)$ be the infinite set provided by Theorem 3. The infinity of S implies that we can find an infinite set $S' \subseteq X^*$ such that $\varphi(S') = S$ and $\varphi(u) \neq \varphi(v)$ for $u, v \in S'$, $u \neq v$. Since A is deterministic and $S' \subseteq L(A)$, each word $w \in S'$ causes A to enter a configuration $(q(w), 1)$, where $q(w)$ is a final state. Choose $u, v \in S'$ such that $u \neq v$ and $q(u) = q(v)$. Let $u' \in Y^*$ be a word such that $\varphi(u)\varphi(u') = 1$, this is possible since $\varphi(u) \in \mathfrak{R}(M)$ and φ is surjective. The word u' causes A to go from $(q(u), 1) = (q(v), 1)$ to $(q, 1)$ for some final state q, since $uu' \in K$. Thus, vu' is also contained in K and hence $\varphi(v)\varphi(u') = 1$, but $\overrightarrow{\mathfrak{I}}(\varphi(u)) \cap \overrightarrow{\mathfrak{I}}(\varphi(v)) = \emptyset$, a contradiction.

Thus, K is not accepted by a deterministic valence automaton over M. In particular, K is not regular. Furthermore, from the valence automaton accepting K, a valence transducer can be constructed that maps $\{\lambda\}$ to K. Since K is not regular, the transduction performed by the transducer is not rational. □

5 Capabilities of valence grammars

In this section, it is shown that the following conditions are equivalent:

1. Valence grammars over M generate only context-free languages.

2. $\mathfrak{R}(N)$ is finite for every finitely generated submonoid N of M.

In one of the directions, we have to construct a context-free grammar for valence grammars over monoids that fulfill the second condition. Because of the limited means available in the context-free case, the constructed grammar can simulate only a certain fragment of the derivations in the valence grammar. Thus, we will have to make sure that every word generated by the valence grammar has a derivation in the aforementioned fragment. These derivations are obtained by considering the derivation tree of a given derivation and then choosing a suitable linear extension of the tree order. The construction of these linear extensions can already be described for a simpler kind of partial order, *valence trees*.

Let X be an alphabet and $U \subseteq X$ a subset. Then, each word $w \in X^*$ has a unique decomposition $w = y_0x_1y_1 \cdots x_ny_n$ such that $y_0, y_n \in (X \setminus U)^*$, $y_i \in (X \setminus U)^+$ for $1 \leq i \leq n-1$ and $x_i \in U^+$ for $1 \leq i \leq n$. This decomposition is called *U-decomposition* of w and we define $\rho(w, U) := n$.

A *tree* is a finite partially ordered set $(\mathcal{T}, \leq)$ that has a least element and where, for each $t \in \mathcal{T}$, the set $\{t' \in \mathcal{T} \mid t' \leq t\}$ is totally ordered by $\leq$. The least element is also called the *root* and the maximal elements are called *leaves*. A *valence tree* $\mathcal{T}$ over M is a tuple $(\mathcal{T}, \leq, \varphi)$, where $(\mathcal{T}, \leq)$ is a tree and $\varphi : \mathcal{T}^* \to M$ is a homomorphism[1] assigning a *valence* to each node. An *evaluation* defines an order in which the nodes in a valence tree can be traversed that is compatible with the tree order. Thus, an *evaluation* of $\mathcal{T}$ is a linear extension $\preceq$ of $(\mathcal{T}, \leq)$. Let $w \in \mathcal{T}^*$ correspond to $\preceq$, i.e., let $\mathcal{T} = \{t_1, \ldots, t_n\}$ such that $t_1 \preceq \cdots \preceq t_n$ and $w = t_1 \cdots t_n$. Then the *value* of $\preceq$ is defined to be $\varphi(w)$. An element $v \in M$ is called a *value of $\mathcal{T}$* if there exists an evaluation of $(\mathcal{T}, \leq)$ with value v. Given a node $t \in \mathcal{T}$, let $U_t := \{t' \in \mathcal{T} \mid t \leq t'\}$. If $w = y_0x_1y_1 \cdots x_ny_n$ is the

[1]We will often assume, without loss of generality, that $\mathcal{T}$ is an alphabet.

U_t-decomposition of w, then $\varphi(x_1), \dots, \varphi(x_n)$ is called the *valence sequence* of t in w and n its *length.* By the *excursiveness* of an evaluation, we refer to the maximal length of a valence sequence. Hence, the excursiveness of an evaluation is the maximal number of times one has to enter any given subtree when traversing the nodes in the order given by the evaluation. We are interested in finding evaluations of valence trees with small excursiveness. Of course, for every valence tree, there are evaluations with excursiveness one (take, for example, the order induced by a preorder traversal), but these might not be able to cover all possible values. However, we will see in Lemma 8 that, in the case of a finite group, there exists a bound m such that every value can be attained by an evaluation of excursiveness at most m.

Lemma 6. *For each finite group G, there is a constant $m \in \mathbb{N}$ with the following property: For elements $g_i, h_i \in G$, $i = 1, \dots, n$, $n \geq m$, there are indices $k, \ell \in \mathbb{N}$, $1 \leq k < \ell \leq n$, such that $g_k h_k \cdots g_\ell h_\ell = g_k \cdots g_\ell h_k \cdots h_\ell$.*

Proof. Let $m = 2(|G|^3 + 1)$ and $D \subseteq \{1, \dots, n\}$ be the set of odd indices. Define the map $\alpha : D \to G^3$ by $\alpha(i) := (g_1 \cdots g_i,\ h_1 \cdots h_i,\ g_1 h_1 \cdots g_i h_i)$ for $i \in D$. Since $|D| \geq |G|^3 + 1$, there are indices $i, j \in D$, $i < j$, such that $\alpha(i) = \alpha(j)$. This means that $g_{i+1} \cdots g_j = 1$, $h_{i+1} \cdots h_j = 1$, and $g_{i+1} h_{i+1} \cdots g_j h_j = 1$. Since i, j are both odd, letting $k = i + 1$ and $\ell = j$ implies $k < \ell$ and yields the desired equality. $\square$

Lemma 7. *Let X be an alphabet and $U, V \subseteq X$ subsets such that either $U \subseteq V$, $V \subseteq U$, or $U \cap V = \emptyset$. Furthermore, let $r \in X^*U$, $x \in U^+$, $y \in (X \setminus U)^+$, and $s \in X^* \setminus UX^*$. Then, we have $\rho(rxys, V) \leq \rho(ryxs, V)$.*

Proof. Suppose $V \subseteq U$. Since y does not contain any symbols in V, we have

$$\begin{aligned}\rho(ryxs, V) &= \rho(r, V) + \rho(x, V) + \rho(s, V),\\ \rho(rxys, V) &= \rho(rx, V) + \rho(s, V).\end{aligned}$$

Thus,

$$\begin{aligned}\rho(rxys, V) &= \rho(rx, V) + \rho(s, V)\\ &\leq \rho(r, V) + \rho(x, V) + \rho(s, V)\\ &= \rho(ryxs, V).\end{aligned}$$

In the case $U \cap V = \emptyset$, x does not contain any symbol in V. Hence,

$$\begin{aligned}\rho(ryxs, V) &= \rho(r, V) + \rho(y, V) + \rho(s, V),\\ \rho(rxys, V) &= \rho(r, V) + \rho(ys, V),\end{aligned}$$

which implies

$$\begin{aligned}\rho(rxys, V) &= \rho(r, V) + \rho(ys, V)\\ &\leq \rho(r, V) + \rho(y, V) + \rho(s, V)\\ &= \rho(ryxs, V).\end{aligned}$$

Now suppose $U \subseteq V$. Since the rightmost letter of r is in V and x lies in V^+, we have $\rho(rxys, V) = \rho(rys, V)$. Thus, $\rho(rxys, V) = \rho(rys, V) \leq \rho(ryxs, V)$. $\square$

Lemma 8. *For each finite group G, there is a constant m such that each value of a valence tree over G has an evaluation of excursiveness at most m.*

Proof. For an alphabet X, we denote the set of *multisets* over X, i.e., maps $X \to \mathbb{N}$, by $X^{\oplus}$. $X^{\oplus}$ carries a (commutative) monoid structure by way of $(\alpha+\beta)(x) := \alpha(x)+\beta(x)$ for $x \in X$. To every evaluation w of $(\mathcal{T}, \leq)$, we assign the multiset $\mu_w \in \mathcal{T}^{\oplus}$, which is defined by $\mu_w(t) := \rho(w, U_t)$ for every $t \in \mathcal{T}$. That is, $\mu_w(t)$ is the length of the valence sequence of t in w.

Let m be the constant provided by Lemma 6 and let $w \in \mathcal{T}^*$ be an evaluation of $(\mathcal{T}, \leq)$ such that μ_w is minimal with respect to $\sqsubseteq$ among all evaluations with value v. If we can prove that $\mu_w(t) \leq m$ for all $t \in \mathcal{T}$, the lemma follows. Therefore, suppose that there is a $t \in \mathcal{T}$ with $n := \mu_w(t) > m$. Specifically, let $w = y_0x_1y_1 \cdots x_ny_n$ be the U_t-decomposition of w. Use Lemma 6 to find indices $1 \leq k < \ell \leq n$ with

$$\varphi(x_k)\varphi(y_k)\cdots\varphi(x_\ell)\varphi(y_\ell) = \varphi(x_k)\cdots\varphi(x_\ell)\varphi(y_k)\cdots\varphi(y_\ell). \tag{1}$$

Furthermore, let

$$w' := (y_0x_1y_1\cdots x_{k-1}y_{k-1})(x_k\cdots x_\ell y_k \cdots y_\ell)(x_{\ell+1}y_{\ell+1}\cdots x_ny_n). \tag{2}$$

That is, we obtain w' from w by replacing $x_ky_k\cdots x_\ell y_\ell$ with $x_k\cdots x_\ell y_k \dots y_\ell$. Then, (1) means that $\varphi(w') = \varphi(w)$. We shall prove that w' is an evaluation of $(\mathcal{T}, \leq)$ and obeys $\mu_{w'} \sqsubset \mu_w$, which contradicts the choice of w.

First, we prove that w' is an evaluation. Let $u_1, u_2 \in \mathcal{T}$ be nodes with $u_1 \leq u_2$. If $u_1 < t$, then u_1 appears in y_0, and thus u_2 is on the right side of u_1 in w'. If $u_1 \geq t$, then each of the nodes u_1, u_2 appears in some x_i and therefore do not change their relative positions. If u_1 and t are incomparable, then u_2 and t are also incomparable and each of u_1, u_2 appears in some y_i. Again, u_1 and u_2 do not change their relative positions. Thus, w' corresponds to a linear extension of $\leq$.

We want to show that $\mu_{w'} \sqsubseteq \mu_w$. To this end, we consider the words

$$w_i := (y_0x_1y_1\cdots x_{k-1}y_{k-1})(x_k\cdots x_{k+i}y_k\cdots y_{k+i})(x_{k+i+1}y_{k+i+1}\cdots x_ny_n)$$

for $0 \leq i \leq \ell - k$. With these, we have $w = w_0$ and $w' = w_{\ell-k}$. Since $(\mathcal{T}, \leq)$ is a tree, we have $U_u \subseteq U_t$, $U_t \subseteq U_u$, or $U_u \cap U_t = \emptyset$ for every $u \in \mathcal{T}$. Therefore, we can apply Lemma 7 to $U := U_t$, $V := U_u$, and

$$\begin{aligned} r &:= (y_0x_1y_1\cdots x_{k-1}y_{k-1})(x_k\cdots x_{k+i}), & x &:= x_{k+i+1}, \\ y &:= y_k\cdots y_{k+i}, & s &:= y_{k+i+1}(x_{k+i+2}y_{k+i+2}\cdots x_ny_n), \end{aligned}$$

which yields $\rho(w_{i+1}, U_u) \leq \rho(w_i, U_u)$ for $0 \leq i < \ell - k$. This implies $\mu_{w'}(u) \leq \mu_w(u)$ and therefore $\mu_{w'} \sqsubseteq \mu_w$.

It remains to be shown that $\mu_{w'}$ is strictly smaller than μ_w. In w', the node t has the valence sequence

$$\varphi(x_1), \dots, \varphi(x_{k-1}), \varphi(x_k\cdots x_\ell), \varphi(x_{\ell+1}), \cdots \varphi(x_n),$$

which has length $\mu_{w'}(t) = n - (\ell - k) < n = \mu_w(t)$. □

We define a *derivation tree* for a valence grammar $G = (N, T, M, P, S)$ to be a tuple $(\mathcal{T}, \leq, \varphi, (\leq_t)_{t\in\mathcal{T}}, \Lambda)$, where

- $(\mathcal{T}, \leq, \varphi)$ is a valence tree,
- for each $t \in \mathcal{T}$, $\leq_t$ is a total order on the set of successors of t,
- $\Lambda : \mathcal{T} \to N \cup T \cup \{\lambda\}$ defines a *label* for each node,
- if $t \in \mathcal{T}$ is a node with the successors $s_1, \ldots, s_n$ such that $s_1 \leq_t \ldots \leq_t s_n$, then we either have $\Lambda(t) \in T \cup \{\lambda\}$, $n = 0$, and $\varphi(t) = 1$ or we have $\Lambda(t) \in N$ and there is a production $(\Lambda(t) \to \Lambda(s_1)\cdots\Lambda(s_n); \varphi(t))$ in P.

The total orders $\leq_t$, $t \in \mathcal{T}$, induce a total order on the set of leaves (see [HU79, Section 4.3] for details), which in turn defines a word $w \in T^*$. This word is called the *yield* of the derivation tree.

Each derivation tree can be regarded as a valence tree. An evaluation then defines a derivation $(A, 1) \Rightarrow_G^* (w, v)$, where $A \in N$ is the label of the root, w is the yield, and $v \in M$ is the value of the evaluation. Conversely, every derivation induces a derivation tree and an evaluation. Thus, a word $w \in T^*$ is in $L(G)$ iff there exists a derivation tree for G with yield w, a root labeled S, and an evaluation with value 1. See [FS02, Section 4.2] for details.

Lemma 9. *Let $\mathfrak{R}(N)$ be finite for every finitely generated submonoid N of M. Furthermore, let $G = (N, T, M, P, S)$ be a valence grammar over M. Then, $L(G)$ is context-free.*

Proof. As in the proof of Lemma 4, we can assume that M is finitely generated and thus has a finite $\mathfrak{R}(M)$. Since productions $(A \to w; m)$ with $m \notin \mathfrak{E}(M)$ cannot be part of a successful derivation, their removal does not change the generated language. Furthermore, by Theorem 3, $\mathfrak{E}(M)$ is a finite group. Thus, we can assume that $G = (N, T, H, P, S)$, where $H = \mathfrak{E}(M)$ is a finite group. By a simple construction, we can further assume that in G, every production is of the form $(A \to w; h)$ with $w \in N^*$ or $(A \to w; 1)$ with $w \in T \cup \{\lambda\}$.

We shall construct a context-free grammar $G' = (N', T, P', S')$ for $L(G)$. The basic idea is that G' will simulate derivations of bounded excursiveness. This is done by letting the nonterminals in G' consist of a nonterminal $A \in N$ and a finite sequence σ of elements from H. G' then simulates the generation of a nonterminal A by generating a pair (A, σ) and thereby guesses that the corresponding node in the derivation tree of G will have σ as its valence sequence. Lemma 8 will then guarantee that this allows G' to derive all words in $L(G)$ when the sequences σ are of bounded length.

Formally, we will regard H as an alphabet and a sequence will be a word over H. In order to be able to distinguish between the concatenation of words in H^* and the group operation in H, we will denote the concatenation in H^* by $\Box$. Thus, let $N' = N \times H^{\leq m}$, in which $m \in \mathbb{N}$ is the constant provided by Lemma 8 for the group H. The set of sequences that can be obtained from another sequence σ by "joining" subsequences is denoted by $J(\sigma)$:

$$J(h_1 \Box h_2 \Box \sigma) := J((h_1 h_2)\Box\sigma) \cup \{h_1 \Box \sigma' \mid \sigma' \in J(h_2 \Box \sigma)\}$$

for $h_1, h_2 \in H$ and $\sigma \in H^*$ and $J(\sigma) := \{\sigma\}$ if $|\sigma| \leq 1$. J is defined for subsets $S \subseteq H^*$ by $J(S) := \bigcup_{\sigma \in S} J(\sigma)$.

For each production $(A \to w; h) \in P$, $w = B_1 \cdots B_n$, $B_i \in N$ for $1 \leq i \leq n$, we include the production

$$(A, \sigma) \to (B_1, \sigma_1) \cdots (B_n, \sigma_n),$$

for each $\sigma \in H^{\leq m} \setminus \{\lambda\}$ and $\sigma_1, \ldots, \sigma_n \in H^{\leq m}$ such that for $\sigma = h_1 \Box \sigma'$, $h_1 \in H$, $\sigma' \in H^{\leq m-1}$, one of the following holds:

- $(h^{-1}h_1)\Box\sigma' \in J(\sigma_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \sigma_n)$.
- $h_1 = h$ and $\sigma' \in J(\sigma_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \sigma_n)$.

Furthermore, for every production $(A \to w, 1)$, $w \in T \cup \{\lambda\}$, we include $(A, \lambda) \to w$. Finally, the start symbol of G' is $(S, 1)$.

It remains to be shown that $L(G') = L(G)$. In order to prove $L(G') \subseteq L(G)$, one can show by induction on n that for $w \in T^*$, $(A, \sigma) \Rightarrow^n_{G'} w$ implies that there is a derivation $(A, 1) \Rightarrow^*_G (w, h)$ for some $h \in H$ using productions $(A_1 \to w_1; h_1), \ldots, (A_k \to w_k; h_k)$ such that $\sigma \in J(h_1 \Box \cdots \Box h_k)$. This implies that for $(S, 1) \Rightarrow^*_{G'} w$, $w \in T^*$, we have $w \in L(G)$. Thus, $L(G') \subseteq L(G)$.

Let $w \in L(G)$ with derivation tree $(\mathcal{T}, \leq, \varphi, (\leq_t)_{t \in \mathcal{T}}, \Lambda)$. By Lemma 8, there is an evaluation $\preceq$ of the tree of excursiveness $\leq m$. From the tree and the evaluation, we construct a derivation tree $(\mathcal{T}, \leq, \varphi', (\leq_t)_{t \in \mathcal{T}}, \Lambda')$ for w in G' as follows. The components $\mathcal{T}$, $\leq$, and $\leq_t$, $t \in \mathcal{T}$, stay unaltered, but φ' will assign 1 to each node and Λ' is defined by $\Lambda'(t) := \Lambda(t)$ if $\Lambda(t) \in T \cup \{\lambda\}$ and $\Lambda'(t) := (\Lambda(t), h_1 \Box \cdots \Box h_k)$ if $\Lambda(t) \in N$, where $h_1, \ldots, h_k$ is the valence sequence of t in $\preceq$. Now, one can see that the new tree is a derivation tree for G' that generates w with any evaluation. Hence, $L(G) \subseteq L(G')$. $\Box$

In order to prove the main result of this section, we need to exhibit a valence grammar over M that generates a non-context-free language when given a finitely generated monoid M with infinite $\mathfrak{R}(M)$. In the proof that the generated language is not context-free, we will use the following well-known Iteration Lemma by Ogden [Ogd68].

Lemma 10 (Ogden). *For each context-free language L, there is an integer m such that for any word $z \in L$ and any choice of at least m distinct marked positions in z, there is a decomposition $z = uvwxy$ such that:*

1. *w contains at least one marked position.*
2. *Either u and v both contain marked positions, or x and y both contain marked positions.*
3. *vwx contains at most m marked positions.*
4. *$uv^iwx^iy \in L$ for every $i \geq 0$.*

Lemma 11. *Let $\mathfrak{R}(M)$ be infinite for some finitely generated monoid M. Then, there is a valence grammar over M that generates a language that is not context-free.*

Proof. Let M be generated by $a_1, \dots, a_n$ and let $X = \{x_1, \dots, x_n\}$ be an alphabet. Furthermore, let $\varphi : X^* \to M$ be the surjective homomorphism defined by $\varphi(x_i) = a_i$. The valence grammar $G = (N, T, M, P, S_0)$ is defined as follows. Let $N = \{S_0, S_1\}$, $T = X \cup \{c\}$, and let P consist of the productions

$$(S_0 \to x_i S_0 x_i, a_i), \quad (S_0 \to c S_1 c, 1), \quad (S_1 \to x_i S_1, a_i), \quad (S_1 \to \lambda, 1)$$

for $1 \leq i \leq n$. Then, clearly $L(G) = K := \{rcscr^{\mathrm{rev}} \mid r, s \in X^*,\ \varphi(rs) = 1\}$. It remains to be shown that K is not context-free. Suppose K is context-free and let m be the constant provided by Lemma 10. By Theorem 3, we can find an infinite subset $S \subseteq \mathfrak{L}(M)$ such that $\overleftarrow{\mathfrak{I}}(a) \cap \overleftarrow{\mathfrak{I}}(b) = \emptyset$ for $a, b \in S$, $a \neq b$. Since φ is surjective, we can define $\ell(a)$ for every $a \in S$ to be the minimal length of a word $w \in X^*$ such that $\varphi(w)a = 1$. If $\ell(a) < m$ for all $a \in S$, the finite set $\{\varphi(w) \mid w \in X^*, |w| < m\}$ contains a left inverse for every $a \in S$. This, however, contradicts the fact that the infinitely many elements of S have disjoint sets of left inverses. Thus, there exists an $a \in S$ with $\ell(a) \geq m$. We choose words $r, s \in X^*$ such that $\varphi(s) = a$ and r is of minimal length among those words satisfying $\varphi(rs) = 1$. Then, by the choice of a, we have $|r| \geq m$.

We apply the Iteration Lemma to the word $z = rcscr^{\mathrm{rev}} \in K$, where we choose the first $|r|$ symbols to be marked. Let $z = uvwxy$ be the decomposition from the lemma. Condition 1 implies $|uv| < |r|$. Because of 4, x cannot contain a c. Furthermore, x cannot be a subword of r, since then pumping would lead to words with mismatching first and third segment. In particular, from condition 2, the first part holds and v is not empty. Thus, if x were a subword of s, pumping would again lead to a mismatching first and third segment. Hence, x is a subword of r^{rev}. If we now pump with $i = 0$, we obtain a word $r'cscr'' \in K$, where $|r'| < |r|$. In particular, we have $\varphi(r's) = 1$, in contradiction to the choice of r. □

Theorem 12. *Let M be a monoid. The following conditions are equivalent:*

1. *Valence grammars over M generate only context-free languages.*
2. *Valence automata over M accept only regular languages.*
3. *Valence automata over M can be determinized.*
4. *Valence transducers over M perform only rational transductions.*
5. *$\mathfrak{R}(N)$ is finite for every finitely generated submonoid N of M.*
6. *$\mathfrak{L}(N)$ is finite for every finitely generated submonoid N of M.*
7. *$\mathfrak{E}(N)$ is finite for every finitely generated submonoid N of M.*

Proof. Theorem 3 immediately implies that 5, 6 and 7 are equivalent. 1 is equivalent to 5 by Lemma 11 and Lemma 9. Lemmas 5 and 4 prove that 2, 3, and 4 are each equivalent to 5. □

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