

(In)finite extensions of algebras from their İnönü-Wigner contractions

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The way to obtain massive non-relativistic states from the Poincaré algebra is twofold. First, following İnönü and Wigner the Poincaré algebra has to be contracted to the Galilean one. Second, the Galilean algebra is to be extended to include the central mass operator. We show that the central extension might be properly encoded in the non-relativistic contraction. In fact, any İnönü-Wigner contraction of one algebra to another, corresponds to an infinite tower of abelian extensions of the latter. The proposed method is straightforward and holds for both central and non-central extensions. Apart from the Bargmann (non-zero mass) extension of the Galilean algebra, our list of examples includes the Weyl algebra obtained from an extension of the contracted $SO(3)$ algebra, the Carrollian (ultra-relativistic) contraction of the Poincaré algebra, the exotic Newton-Hooke algebra and some others.

The paper is dedicated to the memory of Laurent Houart (1967-2011).

1. INTRODUCTION

The *İnönü-Wigner contraction* (IW) [1] (see [2] for the generalizations) plays a very important rôle in physics. To see the most basic example of the contraction, start with the $SO(3)$ algebra of $3d$ rotations. The algebra consists of three operators J_i satisfying the commutator relations:

$$[J_2, J_3] = iJ_1, [J_3, J_1] = iJ_2 \quad \text{and} \quad [J_1, J_2] = iJ_3. \quad (1)$$

Rescaling the operators J_1 and J_2 :

$$J_{i=1,2} = \tilde{J}_{i=1,2}/\sigma \quad J_3 = \tilde{J}_3 \quad (2)$$

and taking the $\sigma \rightarrow 0$ limit one ends up with a new algebra:

$$[\tilde{J}_2, \tilde{J}_3] = i\tilde{J}_1, [\tilde{J}_3, \tilde{J}_1] = i\tilde{J}_2 \quad \text{and} \quad [\tilde{J}_1, \tilde{J}_2] = 0, \quad (3)$$

which is the algebra of $ISO(2)$, the group of the \mathbb{R}^2 isometries, where $\tilde{J}_{1,2}$ and \tilde{J}_3 are the translations and the rotation respectively. One can think of this contraction as of taking to infinity the radius of the 2-sphere, while zooming near a point on it.

Probably the most famous physical application of the IW contraction is the *non-relativistic* limit of the Poincaré algebra. As we will briefly review in the paper, the contraction parameter is the speed of light c and the rescaled operators are the time translation (the Hamiltonian) and the boosts. By the construction, however, the $c \rightarrow \infty$ limit leads to the Galilean algebra, which has exactly the same number of generators as the original Poincaré algebra. For physical applications, though, it

is necessarily to *centrally extend* the non-relativistic algebra by including the mass parameter M . The $M \neq 0$ version of the Galilean algebra is commonly known as the Bargmann algebra [3].

In fact, the Poincaré algebra is itself a contraction of the de Sitter algebra $SO(4,1)$, where the contraction parameter now is the cosmological constant Λ . As was shown [4] by Bacry and Lévy-Leblond in 1967, under some physical assumptions (like parity, time-reversal, etc.) the full list of possible kinematic groups consists of the (anti) de Sitter groups and their IW contractions. For example, one of the possibilities is the $(c, \Lambda) \rightarrow (\infty, 0)$ contraction of the de Sitter algebra with the parameter $\omega = c\sqrt{\Lambda}$ kept fixed. The contraction produces the so-called Newton-Hooke (NH) algebra, which further reduces to the Galilean algebra for $\omega = 0$. One can also take the *ultra-relativistic* ($c \rightarrow 0$) limit of the Poincaré algebra arriving at the Carrollian algebra [5, 6], the much less studied counterpart of the Galilean algebra.

In quantum mechanics we are interested in general in *projective* representations of groups. On the other hand, a projective representation of a group G is essentially equivalent to a regular representation of the central extension of G . For this reason central extensions are of special importance in quantum physics. The canonical (and the simplest) example is the Weyl algebra:

$$[q, p] = i\hbar. \quad (4)$$

Unlike the Galilean group, both the Poincaré algebra has no central extensions except for the two dimensional

case.¹ This can be seen from the group cohomology arguments (see [8] for a detailed review). These groups, although, still have plenty of *non-central* extensions, some of which are interesting from the physical point of view, e.g. the Maxwell algebra [9, 10], a specific non-central extension of the Poincaré algebra.

The main goal of this paper is to show that starting from an IW contraction one can straightforwardly find an extension of the contracted algebra. Although the proposed procedure explicitly yields an infinite extension, it can be easily truncated at any level. It works both for central and non-central extensions, and we find a criterion for an extension to be central.

Our first example is a IW contraction of $SO(3)$ given by the rescaling:

$$J_{1,2} = \tilde{J}_{1,2}/\sigma^2 \quad J_3 = \tilde{J}_3/\sigma^3. \quad (5)$$

For $\sigma \rightarrow 0$ we find an abelian algebra (all the commutators vanish), but as we demonstrate in the paper the first level extension dictated by this contraction reproduces the central extension of the Weyl algebra. The higher level extensions, however, are not anymore central.

The organization of the paper is as follows. In the next section we present our method for constructing algebra extensions from its İnönü-Wigner contraction. We then describe various applications, like the non-central extension of the Poincaré algebra from the de Sitter algebra contraction, the Galilean and the Carrollian contractions of the Poincaré algebra and the exotic extension of the Galilean and the NH algebra. We end up with a short list of open questions.

2. THE METHOD

2.1. Lie algebra bundles

For our construction we will need the notion of bundles of Lie algebras [11] (see also [12] for a pedagogical introduction). A *Lie algebra bundle* (Lie bundle for short) is a vector bundle for which each fiber has a smoothly varying Lie algebra structure.² More explicitly, the vector bundle $(\mathcal{E}, \pi, \mathcal{S})$ should be equipped with a morphism $\theta : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}$, which induces a Lie algebra structure on each fiber \mathcal{E}_σ . Here \mathcal{E} is the total space, π is the projection map and σ is a point on the base \mathcal{S} . This definition is sometimes called in the literature a *weak* Lie algebra

bundle, in contrast to the *strong* one which requires also local triviality of the Lie structure. It means that for any $\sigma \in \mathcal{S}$ there exists a neighborhood U_σ of σ , a Lie algebra g and a morphism $\phi : g \times U_\sigma \rightarrow \pi^{-1}(U_\sigma)$ such that $\phi|_{\sigma'} : g \rightarrow \pi^{-1}(\sigma')$ is a Lie algebra isomorphism for each $\sigma' \in U_\sigma$. We will refer to ϕ as a Lie bundle trivialization. If ϕ extends to the entire base, the Lie algebra bundle will be called *trivial*.

Next, let us restrict our attention to weak Lie bundles, where the base \mathcal{S} is just an affine line³ \mathbb{R}^1 . In this case any Lie algebra bundle will be trivial as a vector bundle, but not necessarily so from the Lie algebra point of view. The Lie algebra g at any fiber is the same as a vector space, but its brackets are in general σ -dependent. If a^i are the generators of g and $a_i^\sigma = (a_i, \sigma)$ denotes a point on the Lie bundle, then the most general form of the commutators is:

$$[a_i^\sigma, a_j^\sigma] = f_{ij}^k(\sigma) a_k^\sigma, \quad (6)$$

where $f_{ij}^k(\sigma)$ are some smooth functions.

Let us consider the following example. Assume that i runs from 1 to 3 and:

$$f_{ij}^k(\sigma) = \frac{1}{2} \epsilon_{ij}^k \cdot f(\sigma). \quad (7)$$

If $f(\sigma) \neq 0$ for any $\sigma \in \mathbb{R}^1$, then the Lie bundle is trivial (namely strong) with the trivialization ϕ given by:

$$\phi(a_i, \sigma) = \frac{a_i^\sigma}{f(\sigma)} \quad (8)$$

and a_i 's identified with the $SO(3)$ generators J_i 's. On the other hand, the Lie bundle will be non-trivial⁴ (weak but not strong) if $f(\sigma_0) = 0$ for some $\sigma_0 \in \mathbb{R}^1$, since in this case ϕ is ill-defined for $\sigma = \sigma_0$. We still, however, can claim that the Lie bundle is trivial everywhere *except* at $\sigma = \sigma_0$.

Similarly we can see the rescaling (2) as a trivialization map for the Lie algebra bundle given by:

$$[\tilde{J}_2, \tilde{J}_3] = i\tilde{J}_1, [\tilde{J}_3, \tilde{J}_1] = i\tilde{J}_2 \quad \text{and} \quad [\tilde{J}_1, \tilde{J}_2] = \sigma^2 \tilde{J}_3. \quad (9)$$

At $\sigma = 0$ the trivialization map is singular and the Lie algebra structure reduces to (3), but at any other point the algebra is isomorphic to $SO(3)$.

2.2. The İnönü-Wigner bundle

We are now in a position to define the main ingredient of our construction.

Definition. *The İnönü-Wigner (IW) bundle of a Lie algebra g is a (weak) Lie algebra bundle over an affine*

¹ The Galilean and the NH algebra have an additional, the so called *exotic*, central extension in 2 + 1 dimensions [7] as we will describe in the paper.

² We will deal exclusively with Lie algebras and not with Lie groups. Let us only notice here that every bundle of Lie groups defines a bundle of Lie algebras, and every bundle of algebras can be integrated to a bundle of groups. The group bundle, however, is not necessarily Hausdorff (see [12]).

³ Our discussion can be easily generalized to complex numbers.

⁴ If the base space is \mathbb{R}^1 local triviality implies also global triviality.

line \mathbb{R}^1 , such that its restriction to $\mathbb{R}^1 \setminus \{0\}$ is a trivial bundle $g \times \mathbb{R}^1 \setminus \{0\}$.

From the above discussion the Lie bundle defined by (9) and (7) are examples of IW bundles. The latter is non-trivial if and only if $f(0) = 0$.

As yet another example consider the following IW Lie algebra bundle⁵:

$$[J_2^\sigma, J_3^\sigma] = i\sigma^3 J_1^\sigma, [J_3^\sigma, J_1^\sigma] = i\sigma^3 J_2^\sigma, [J_1^\sigma, J_2^\sigma] = i\sigma J_3^\sigma. \quad (10)$$

Upon using the trivialization map:

$$\phi(J_{1,2}, \sigma) = \frac{J_{1,2}^\sigma}{\sigma^2} \quad \text{and} \quad \phi(J_3, \sigma) = \frac{J_3^\sigma}{\sigma^3}. \quad (11)$$

we find that for $\sigma \neq 0$ the algebra is isomorphic to $SO(3)$. We see that the trivialization morphism ϕ is actually the rescaling of the generators in (5). The fact that ϕ fails to be an isomorphism at $\sigma = 0$ indicates that we find a new algebra at this point.

2.3. The extensions

Our goal in this subsection is to explore the $\sigma \rightarrow 0$ limit.

Let $\tilde{\mathfrak{g}}$ stand for a vector space of *smooth* sections of an IW bundle of an algebra g . Again, as a vector space g is the same all over the base (and so we drop here the σ -index), but its commutators are not. The (infinite) vector space of smooth sections $\tilde{\mathfrak{g}}$ is spanned by the vectors $\sigma^n a_i$, where $a_i \in g$ and $n \geq 0$. Moreover, this vector space is a Lie algebra by its own right, since we can multiply the sections pointwise. Sections vanishing at $\sigma = 0$ is an ideal of this algebra. This ideal is $\sigma\tilde{\mathfrak{g}}$.

The *classical IW algebra* is the quotient:

$$\mathfrak{g}_0 \equiv \tilde{\mathfrak{g}}/\sigma\tilde{\mathfrak{g}}. \quad (12)$$

To arrive at this algebra it is enough to directly take the $\sigma \rightarrow 0$ limit. Notice that \mathfrak{g}_0 and g are identical as vector spaces, but not as Lie algebras. For (10) the algebra \mathfrak{g}_0 is an abelian algebra of J_1, J_2 and J_3 . Similarly, the scaling (3) leads to the $ISO(2)$ algebra.

Up to this point, the $\sigma \rightarrow 0$ contraction is identical to the original Inönü and Wigner prescription. We, however, don't want to stop here. Notice that $\sigma^n \tilde{\mathfrak{g}}$ is an ideal of $\tilde{\mathfrak{g}}$ for any $n \geq 1$. The *semi-classical IW algebra* is defined by:

$$\mathfrak{g}_1 \equiv \tilde{\mathfrak{g}}/\sigma^2\tilde{\mathfrak{g}}. \quad (13)$$

The algebra \mathfrak{g}_1 has *twice* more generators than \mathfrak{g}_0 , but it might happen that some of these new generators form an ideal and can be quotiented out. For the IW bundle defined by (10) let us introduce $J_i^{(0)} \equiv J_i$ and $J_i^{(1)} \equiv \sigma J_i$. We find that the semi-classical algebra \mathfrak{g}_1 has only one non-trivial commutator:

$$[J_1^{(0)}, J_2^{(0)}] = iJ_3^{(1)}. \quad (14)$$

As was announced in Introduction this is the central extension of the Weyl algebra. To be more precise, in order to identify \mathfrak{g}_1 with the Weyl algebra we have to quotient the algebra by its abelian ideal $\{J_3^{(0)}, J_1^{(1)}, J_2^{(1)}\}$.

We can easily repeat this procedure for any n . Clearly, the new algebra will have n times more generators than the original algebra. For the IW bundle in (10) the *level* algebra $\mathfrak{g}_n \equiv \tilde{\mathfrak{g}}/\sigma^{n+1}\tilde{\mathfrak{g}}$ is (here $J_i^{(n)}$ stands for $\sigma^n J_i$ and $J_i^{(n)} = 0$ for negative n):

$$\begin{aligned} [J_2^{(n)}, J_3^{(m)}] &= iJ_1^{(n+m-3)}, & [J_3^{(n)}, J_1^{(m)}] &= iJ_2^{(n+m-3)}, \\ [J_1^{(n)}, J_2^{(m)}] &= iJ_3^{(n+m-1)} \end{aligned} \quad (15)$$

with all the other commutators vanishing. One may argue that the output algebra is not particularly interesting. The situation changes drastically, however, for a more sophisticated trivialization (rescaling) ϕ than the one in (11).

Before concluding this section, let us discuss how our semi-classical extension \mathfrak{g}_1 may lead to a *central* extension of the \mathfrak{g}_0 algebra. To this end it is worth recalling the rigorous definition of a (not necessarily central) extension. An algebra \mathfrak{a}' is called an extension of \mathfrak{a} by an ideal \mathfrak{n} , if \mathfrak{a} is isomorphic to the quotient $\mathfrak{a}'/\mathfrak{n}$. This definition can be shortly summarized with the following *short exact sequence*:

$$0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{a}' \rightarrow \mathfrak{a} \rightarrow 0. \quad (16)$$

If \mathfrak{n} is also inside the center of \mathfrak{a}' , then the extension is called *central*.

Consider the quotient $\mathfrak{n} \equiv \sigma\tilde{\mathfrak{g}}/\sigma^2\tilde{\mathfrak{g}} = \sigma\mathfrak{g}_1$. Obviously, \mathfrak{n} is an ideal of \mathfrak{g}_1 . Moreover, \mathfrak{n} is an abelian ideal and is isomorphic⁶ to $(\mathfrak{g}_0)_{\text{abelian}}$. Our first level extension is actually given by the following exact sequence:

$$0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g}_1 \xrightarrow{\pi} \mathfrak{g}_0 \rightarrow 0. \quad (17)$$

Since \mathfrak{n} is by construction abelian, our \mathfrak{g}_1 extension is *always* abelian. The ideal \mathfrak{n} is, however, not necessarily inside the center of \mathfrak{g}_1 . In fact, \mathfrak{n} consists of elements of the form σa_i and becomes central if and only if \mathfrak{g}_0 is

⁵ The form of the commutators in (10) suggests an analogy with *loop algebras*. The similarity is, however, illusive, since the parameter σ takes values in \mathbb{R}^1 and not in \mathbb{S}^1 , and, even more importantly, for the loop algebra construction, the Lie algebra necessarily remains the same all over the loop, while in our case, a new algebra emerges at $\sigma = 0$.

⁶ A Lie algebra g_{abelian} is isomorphic to the algebra g as a vector space but all its commutators are vanishing.

abelian. This was exactly the case for the IW bundle defined by (5), which led to the Weyl algebra (14).

It seems, therefore, that our construction produces central extensions only upon very restricting conditions. In particular, it rules out the Bargmann (non-zero mass) extension of the Galilean algebra. Luckily, we can slightly modify the extension described by (17).

Assume that \mathfrak{m}_0 is an ideal of \mathfrak{g}_0 . It automatically implies that $\pi^{-1}(\mathfrak{m}_0)$ is an ideal of \mathfrak{g}_1 , where π is the projection map from (17). Furthermore, $\mathfrak{m}_1 \equiv \sigma\pi^{-1}(\mathfrak{m}_0)$ is an abelian ideal of \mathfrak{g}_1 and the quotient $\mathfrak{n}/\mathfrak{m}_1 = \sigma\mathfrak{g}_1/\mathfrak{m}_1$ is isomorphic to $(\mathfrak{g}_0/\mathfrak{m}_0)_{\text{abelian}}$. We get the following *abelian* extension of \mathfrak{g}_0 :

$$0 \rightarrow \mathfrak{n}/\mathfrak{m}_1 \rightarrow \mathfrak{g}_1/\mathfrak{m}_1 \xrightarrow{\pi} \mathfrak{g}_0 \rightarrow 0. \quad (18)$$

The above extension is central if and only if $\mathfrak{g}_0/\mathfrak{m}_0$ is abelian. Indeed, the latter requirement is equivalent to the statement that $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{m}_0$. But that means that $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \pi^{-1}(\mathfrak{m}_0)$ and so $[\sigma\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{m}_1$. The latter immediately implies that $\mathfrak{n}/\mathfrak{m}_1$ is central.

This way the number of the new generators in the extended algebra will be smaller than total number of generators in g . We will see in the next section that (18) naturally leads to the central (mass) extension of the Galilean algebra.

3. EXAMPLES

We have already seen how the central extension of the Weyl algebra emerges from the $SO(3)$ contraction. In the rest of the paper we will see other applications of the proposed extension method.

3.1. The non-central extensions of the Poincaré algebra from the de Sitter algebra contraction

In d space-time dimensions the commutators of the de Sitter and the anti de Sitter algebras ($SO(d, 1)$ and $SO(d - 1, 2)$) are:

$$\begin{aligned} [M_{\mu\nu}, M_{\lambda\rho}] &= \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - (\lambda \leftrightarrow \rho), \\ [M_{\lambda\rho}, P_\mu] &= \eta_{\mu\lambda}P_\rho + \eta_{\mu\rho}P_\lambda, \\ [P_\mu, P_\nu] &= \varepsilon M_{\mu\nu}. \end{aligned} \quad (19)$$

Here $\eta = \text{diag}(-1, 1, \dots, 1)$ and $\varepsilon = 1, -1$ for the de Sitter and the anti de Sitter algebra respectively.

We are interested in the zero cosmological constant limit, $\Lambda \rightarrow 0$, of the (anti) de Sitter algebra. As such the algebra (19) is Λ -independent, so we have to properly rescale the operators introducing Λ into the commutators. The new Λ -dependent algebra will be isomorphic to the original one for any Λ except for $\Lambda = 0$, where it should reduce to the Poincaré algebra. The right rescaling of the generators is, in fact, well known in the literature. The metric on the (anti) de Sitter space can be

written as:

$$ds_{(a)dS}^2 = R^2 \cdot \frac{\eta_{\mu\nu}dy^\mu dy^\nu}{1 + \varepsilon y^2/4}, \quad (20)$$

where $y^2 \equiv \eta_{\mu\nu}y^\mu y^\nu$ and $R = \Lambda^{-1/2}$ is the (anti) de Sitter radius. In terms of these coordinates the isometry generators are:

$$\begin{aligned} M_{\mu\nu} &= \eta_{\nu\lambda}y^\lambda \frac{\partial}{\partial y^\mu} - \eta_{\mu\lambda}y^\lambda \frac{\partial}{\partial y^\nu}, \\ P_\mu &= \left(\varepsilon - \frac{y^2}{4} \right) \frac{\partial}{\partial y^\mu} + \frac{1}{2} \eta_{\mu\lambda}y^\lambda y^\rho \frac{\partial}{\partial y^\rho}. \end{aligned} \quad (21)$$

In the limit $(R, y^\mu) \rightarrow (\infty, 0)$ with $x^\mu = R \cdot y^\mu$ kept fixed, the metric (20) reduces to the Minkowski metric, while the leading scaling behavior of the generators (32) is:

$$M_{\mu\nu} \rightarrow M_{\mu\nu}, \quad P_\mu \rightarrow R \cdot P_\mu. \quad (22)$$

It may be also interesting to see the limit in terms of the (ξ^μ, ξ^d) coordinates used for the (anti) de Sitter space embedding in $\mathbb{R}^{d,1}$:

$$\xi^\mu \xi_\mu + \varepsilon (\xi^d)^2 = \varepsilon R^2. \quad (23)$$

These coordinates are related to y^μ 's by:

$$\frac{\xi^\mu}{R} = \frac{y^\mu}{1 + \varepsilon y^2/4}, \quad \frac{\xi^d}{R} = \frac{1 - \varepsilon y^2/4}{1 + \varepsilon y^2/4}. \quad (24)$$

In the Poincaré limit this reduces to $(\xi^\mu, \xi^d) = (x^\mu, R)$ and so, analogously to the $SO(3)$ example in the first paragraph of Introduction, the limit corresponds to the zooming around the $\xi^d = R$ point of the (anti) de Sitter space.

With this rescaling we find the following IW bundle:

$$[P_\mu^\sigma, P_\nu^\sigma] = \sigma \cdot \varepsilon M_{\mu\nu}^\sigma, \quad (25)$$

where the affine parameter σ is just the cosmological constant $\sigma = R^{-2} = \Lambda$ and we left out the commutators from the first two lines in (19) since they remain completely σ -independent.

As expected, the ‘‘classical’’ \mathfrak{g}_0 algebra is precisely the Poincaré algebra:

$$\begin{aligned} [M_{\mu\nu}^{(0)}, M_{\lambda\rho}^{(0)}] &= \eta_{\mu\lambda}M_{\nu\rho}^{(0)} + \text{permutations}, \\ [M_{\lambda\rho}^{(0)}, P_\mu^{(0)}] &= \eta_{\mu\lambda}P_\rho^{(0)} + \eta_{\mu\rho}P_\lambda^{(0)}, \\ [P_\mu^{(0)}, P_\nu^{(0)}] &= 0. \end{aligned} \quad (26)$$

The next level ‘‘semi-classical’’ algebra \mathfrak{g}_1 includes also the operators $M_{\mu\nu}^{(1)} = \sigma M_{\mu\nu}$ and $P_\mu^{(1)} = \sigma P_\mu$. Notice, however, that the translations $P_\mu^{(0)}$ form an ideal of the Poincaré algebra \mathfrak{g}_0 . We therefore can use the modified exact sequence (18) with \mathfrak{m}_0 being the subalgebra of translations $P_\mu^{(0)}$, so that the extension will include only

the $M_{\mu\nu}^{(1)}$ generators and not $P_\mu^{(1)}$'s. Defining $Z_{\mu\nu} = \varepsilon M_{\mu\nu}^{(1)}$ we arrive at the Maxwell algebra:

$$\begin{aligned} [M_{\mu\nu}^{(0)}, Z_{\lambda\rho}] &= \eta_{\mu\lambda} Z_{\nu\rho} + \eta_{\nu\rho} Z_{\mu\lambda} - \eta_{\mu\rho} Z_{\nu\lambda} - \eta_{\nu\lambda} Z_{\mu\rho}, \\ [P_\mu^{(0)}, P_\nu^{(0)}] &= Z_{\mu\nu}, \end{aligned} \quad (27)$$

with the first two commutators in (26) remaining intact. The extension is non-central since $Z_{\mu\nu}$ does not commute with the Lorentz generators. It becomes a scalar only for $d = 2$. Following the recipe from the previous section we can extend the algebra to higher levels. The commutators between $Z_{\mu\nu}$'s become non-trivial already at the \mathfrak{g}_2 algebra:

$$[Z_{\mu\nu}, Z_{\lambda\rho}] = \eta_{\mu\lambda} Y_{\nu\rho} + \eta_{\nu\rho} Y_{\mu\lambda} - \eta_{\mu\rho} Y_{\nu\lambda} - \eta_{\nu\lambda} Y_{\mu\rho}, \quad (28)$$

where $Y_{\mu\nu} \equiv M_{\mu\nu}^{(2)} = \sigma^2 M_{\mu\nu}$ and:

$$[P_\mu^{(0)}, P_\nu^{(1)}] = Y_{\mu\nu}. \quad (29)$$

For similar extensions of the Maxwell algebra recently studied in the literature see [13], [14], [15] and [16].

Before ending this section let us remark that the de Sitter algebra (19) permits an additional, infinite cosmological constant limit [17]. For fixed $x^\mu = R \cdot y^\mu$ but with $(R, y^\mu) \rightarrow (0, \infty)$ the momenta scale like $P_\mu \rightarrow P_\mu/R$. The contracted algebra is equivalent to the Poincaré but its physical meaning is different, since the $P_\mu^{(0)}$'s now are special conformal transformations and not translations.

3.2. The Galilean contraction of the Poincaré algebra

Our next example is the non-relativistic contraction of the Poincaré algebra.

In d space-time dimensions the Poincaré algebra consists of⁷ $(d-1)(d-2)/2$ space-space rotations \mathbf{J} , $d-1$ boosts \mathbf{B} , $d-1$ space translations \mathbf{P} and the time translation H . In all of the remaining examples but the last one we will explicitly consider the $d = 4$ case, with a straightforward generalization for other dimensions. The commutators are⁸:

$$[\mathbf{J}, H] = 0, \quad [\mathbf{J}, \mathbf{J}] = \mathbf{J}, \quad [\mathbf{J}, \mathbf{B}] = \mathbf{B}, \quad [\mathbf{J}, \mathbf{P}] = \mathbf{P} \quad (30)$$

and:

$$[H, \mathbf{B}] = \mathbf{P}, \quad [\mathbf{B}, \mathbf{B}] = -\mathbf{J}, \quad [\mathbf{P}, \mathbf{B}] = H, \quad (31)$$

with all other commutators vanishing. The first set of the commutators (30) simply implies that \mathbf{B} , \mathbf{P} and \mathbf{J} transform as vectors under space rotations, while H is a

scalar. The differential representation of the operators in terms of the Minkowskian coordinates is:

$$\begin{aligned} J_i &= -\epsilon_{ij}^k x^j \frac{\partial}{\partial x^k}, & B_i &= \delta_{ij} x^j \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^i}, \\ P_i &= \frac{\partial}{\partial x^i}, & H &= \frac{\partial}{\partial x^0}. \end{aligned} \quad (32)$$

The non-relativistic limit corresponds to $c, x^0 \rightarrow \infty$ with $t = x^0/c$ held fixed. The Poincaré algebra generators scale in this limit as:

$$\mathbf{J} \rightarrow \mathbf{J}, \quad \mathbf{B} \rightarrow c\mathbf{B}, \quad \mathbf{P} \rightarrow \mathbf{P}, \quad H \rightarrow \frac{H}{c}. \quad (33)$$

This rescaling leads to the following IW bundle:

$$[H, \mathbf{B}] = \mathbf{P}, \quad [\mathbf{B}, \mathbf{B}] = -\sigma\mathbf{J}, \quad [\mathbf{P}, \mathbf{B}] = \sigma H, \quad (34)$$

where $\sigma = c^{-2}$ and for simplicity we dropped the σ -indices. We also left out the σ -independent part of the algebra (30). Putting $\sigma = 0$ in (34) we of course find the Galilean algebra \mathfrak{g}_0 :

$$[H^{(0)}, \mathbf{B}^{(0)}] = \mathbf{P}^{(0)}, \quad [\mathbf{B}^{(0)}, \mathbf{B}^{(0)}] = 0, \quad [\mathbf{P}^{(0)}, \mathbf{B}^{(0)}] = 0 \quad (35)$$

together with the regular transformations of $H^{(0)}$, $\mathbf{B}^{(0)}$ and $\mathbf{P}^{(0)}$ under the $\mathbf{J}^{(0)}$ rotations.

As we have already emphasized before the \mathfrak{g}_1 extension has by construction twice more operators than \mathfrak{g}_0 . The new operators are $\mathbf{J}^{(1)}$, $\mathbf{B}^{(1)}$, $\mathbf{P}^{(1)}$ and $H^{(1)}$. In order to get the central non-zero mass extension of (35) we have to use (18), where the ideal \mathfrak{m}_0 is the subalgebra of all Galilean rotations, boosts and space translations, namely $\mathbf{J}^{(0)}$, $\mathbf{B}^{(0)}$ and $\mathbf{P}^{(0)}$. This allows us to mode out $\mathbf{J}^{(1)}$, $\mathbf{B}^{(1)}$ and $\mathbf{P}^{(1)}$ from the extended algebra. Denoting $M = H^{(1)}$ we find that $\mathfrak{g}_1/\mathfrak{m}_1$ is exactly the Bargmann algebra:

$$[H^{(0)}, \mathbf{B}^{(0)}] = \mathbf{P}^{(0)}, \quad [\mathbf{B}^{(0)}, \mathbf{B}^{(0)}] = 0, \quad [\mathbf{P}^{(0)}, \mathbf{B}^{(0)}] = M. \quad (36)$$

It is noteworthy that one can formally derive the Bargmann algebra by plugging $H = M/\sigma + H^{(0)}$ into (34) and matching the σ -expansion from the both sides. This approach is, however, not rigorously defined and unlike our method cannot be used for the next order extension.

3.3. The Carrollian contraction of the Poincaré algebra

The Carrollian contraction of the Poincaré algebra (30, 31) is given by:

$$\mathbf{J} \rightarrow \mathbf{J}, \quad \mathbf{B} \rightarrow \frac{\mathbf{B}}{c}, \quad \mathbf{P} \rightarrow \mathbf{P}, \quad H \rightarrow \frac{H}{c}. \quad (37)$$

It naturally follows from the $c \rightarrow 0$ limit of (32). The IW bundle is:

$$[H, \mathbf{B}] = \sigma\mathbf{P}, \quad [\mathbf{B}, \mathbf{B}] = -\sigma\mathbf{J}, \quad [\mathbf{P}, \mathbf{B}] = H, \quad (38)$$

⁷ We will use bold font to indicate 3-dimensional space vectors.
⁸ Here $[\mathbf{A}, \mathbf{B}] = \mathbf{C}$ is a shorthand notation for $[A_i, B_j] = \epsilon_{ijk} C_k$ and $[\mathbf{A}, \mathbf{B}] = C$ stands for $[A_i, B_j] = \delta_{ij} C_k$.

where this time $\sigma = c^2$. For $\sigma = 0$ we find the Carrollian algebra:

$$[H^{(0)}, \mathbf{B}^{(0)}] = 0, \quad [\mathbf{B}^{(0)}, \mathbf{B}^{(0)}] = 0, \quad [\mathbf{P}^{(0)}, \mathbf{B}^{(0)}] = H^{(0)}. \quad (39)$$

By analogy with the Galilean case we can mode out some operators from the next level \mathfrak{g}_1 extension. Notice that $\mathbf{J}^{(0)}$, $\mathbf{B}^{(0)}$ and $H^{(0)}$ form an ideal \mathfrak{m}_0 and so we can exclude $\mathbf{J}^{(1)}$, $\mathbf{B}^{(1)}$ and $H^{(1)}$ from the \mathfrak{g}_1 extension. With $\mathbf{Z} = \mathbf{P}^{(1)}$ the $\mathfrak{g}_1/\mathfrak{m}_1$ algebra takes the following form:

$$[H^{(0)}, \mathbf{B}^{(0)}] = \mathbf{Z}, \quad [\mathbf{B}^{(0)}, \mathbf{B}^{(0)}] = 0, \quad [\mathbf{P}^{(0)}, \mathbf{B}^{(0)}] = H^{(0)}. \quad (40)$$

This extension is central only for $d = 2$ when \mathbf{Z} becomes a scalar.

3.4. The exotic Newton-Hooke algebra

In our discussion of the Galilean algebra extension we excluded the operator $\mathbf{J}^{(1)}$ from the ‘‘semi-classical’’ algebra. We did so because this operator is not central in the extended $4d$ Galilean algebra, for it does not commute with the $\mathbf{J}^{(0)}$ rotations. The situation changes for $d = 3$, since in this case \mathbf{J} is a scalar. The $d = 3$ Galilean algebra with the J central extension is known in the literature as the exotic Galilean algebra.⁹

Instead of focusing our attention on this algebra let us make the discussion a bit more general by starting with the Newton-Hooke (NH) contraction. The NH algebra is the non-relativistic limit of the de Sitter algebra. To be more precisely, one sends both c and R to infinity with the parameter $\omega = c/R$ held fixed.

For $d = 3$ the de Sitter algebra is:

$$[J, H] = 0, \quad [J, B_i] = \epsilon_{ij} B_j, \quad [J, P_i] = \epsilon_{ij} P_j \quad (41)$$

and:

$$\begin{aligned} [H, B_i] &= P_i, & [B_i, B_j] &= -\epsilon_{ij} J, & [P_i, B_j] &= \delta_{ij} H, \\ [P_i, H] &= B_i, & [P_i, P_j] &= \epsilon_{ij} J, \end{aligned} \quad (42)$$

where $(H, P_i) = P_\mu$, $B_i = M_{0i}$ and $J = M_{12}$. Combining the rescalings (22) and (33):

$$J \rightarrow J, \quad B_i \rightarrow cB_i, \quad P_i \rightarrow \frac{P_i}{R}, \quad H \rightarrow \frac{H}{cR} \quad (43)$$

we obtain a new IW bundle with the $\sigma = c^{-2}$ affine parameter:

$$\begin{aligned} [H, B_i] &= P_i, & [B_i, B_j] &= -\sigma \cdot \epsilon_{ij} J, & [P_i, B_j] &= \sigma \cdot \delta_{ij} H, \\ [P_i, H] &= \omega^2 B_i, & [P_i, P_j] &= \sigma \cdot \epsilon_{ij} \omega^2 J, \end{aligned} \quad (44)$$

where we omitted again all the commutators with J , because they are σ -independent.

Obviously the boosts $B_i^{(0)}$ and the momenta $P_i^{(0)}$ will form an ideal of \mathfrak{g}_0 . Using this ideal for (18) we arrive at

the *exotic 3d Newton-Hooke algebra*:

$$\begin{aligned} [H^{(0)}, B_i^{(0)}] &= P_i, & [B_i^{(0)}, B_j^{(0)}] &= -\epsilon_{ij} \tilde{Z}, \\ [P_i^{(0)}, B_j^{(0)}] &= \delta_{ij} M, \\ [P_i^{(0)}, H^{(0)}] &= \omega^2 B_i^{(0)}, & [P_i^{(0)}, P_j^{(0)}] &= \epsilon_{ij} \omega^2 \tilde{Z}, \end{aligned} \quad (45)$$

where $\tilde{Z} \equiv J^{(1)}$ and $M \equiv H^{(1)}$. For $\omega = 0$ (45) transforms into the exotic Galilean algebra.

4. OPEN QUESTIONS

In our paper we have shown that given an İnönü-Wigner contraction of one algebra to another, one can easily find an infinite extension of the latter. The method works for both central and non-central extensions and the extension may be truncated at any level.

There are plenty of open problems to be explored. Let us list some of them:

- It will be interesting to establish a connection between our extension approach and the expansion method presented in [18]. This method is somewhat similar in spirit to ours. It is based on the Maurer-Cartan one-forms expansion in powers of a real parameter which is related to the rescaling of the Lie group coordinates. Our results should also be compared to those of [15].
- It is not clear what is the physical meaning of the new (in)finite algebras. In some cases the answer is already known. For example, the Maxwell algebra (27) corresponds to a charged relativistic particle moving in a constant electromagnetic field. The meaning of the second level extension (28,29) is yet to be understood.
- Finding irreducible representations of the infinitely extended algebras and their truncated versions might be a challenging problem.

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⁹ One may also consider a similar exotic central extension of the 3d Carrollian algebra.

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