

Electromagnetic Excitations of Hall Systems on Four Dimensional Space

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Abstract

The noncommutativity of a four-dimensional phase space is introduced from a purely symplectic point of view. We show that there is always a coordinate map to locally eliminate the gauge fluctuations inducing the deformation of the symplectic structure. This uses the Moser's lemma; a refined version of the celebrated Darboux theorem. We discuss the relation between the coordinates change arising from Moser's lemma and the Seiberg–Witten map. As illustration, we consider the quantum Hall systems on \mathbf{CP}^2 . We derive the action describing the electromagnetic interaction of Hall droplets. In particular, we show that the velocities of the edge field, along the droplet boundary, are noncommutativity parameters-dependents.

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1 Introduction

Recently, there has been considerable interest in the noncommutative geometry as framework for physical theories and as tool for study certain mathematical structures, which appears in some physical models. This is mainly motivated by the new development in string theory [1]. Subsequently, the idea of non commutative space time at small length scales [2] has been drawn much attention in various fields and found interesting implications, see for instance [3-4].

Since the noncommutative space resembles a quantum phase space (with noncommutativity parameter θ playing the role of \hbar), many papers have been devoted to study various aspects of quantum mechanics [5-9] on the noncommutative space where space-space is non commuting and/or momentum-momentum is non commuting. The usual way of investigating the noncommutative quantum mechanics is to map the noncommutative space to a commutative one. At classical level, this map turns out to be similar to the celebrated Darboux transformation. In this respect, the noncommutative quantum mechanics can be viewed as quantization of a phase space equipped with modified symplectic structure. To eliminate the fluctuation, one has to define a diffeomorphism, which maps the modified symplectic form to its counter part in the commutative case. Hence, one of the main aims of the present work is to give a general prescription to perform this "dressing" transformation for arbitrary modified closed two-form on a curved phase space. This prescription uses the Moser's lemma [10] which is a refined version of Darboux theorem. We will discuss many facets and consequences of this transformation. We also compare this method with the transformation, which arises from the Hilbert–Schmidt orthonormalization method in four-dimensional phase space.

On the other hand, the prototypical topic at the interface between the noncommutative geometry and condensed matter physics was in the last decade, the quantum Hall effect. Indeed, according to the Laughlin [11], a large collection of fermions in a strong magnetic field behaves like a rigid droplet of liquid. This incompressible quantum fluid picture constitutes the basis of the main advances in this field of research, especially its connection with the noncommutative structures. Indeed, it was shown that Laughlin states at filling factor $1/k$ can be provided by an appropriate noncommutative finite Chern–Simons matrix model at level k and hence reproduces the basic features of quantum Hall states [12-13]. In connection with quantum Hall systems in higher dimensions [14-25], the ideas of the noncommutative geometry were useful to show that the effective action for the edge excitations of a quantum hall droplet is generically given by a chiral boson action [21-25]. In relation with these issues, the second main task of this paper concerns the electromagnetic excitations of Hall droplets in four-dimensional complex projective space. The electromagnetic field is introduced as a variation of the \mathbf{CP}^2 symplectic two-form.

The outline of the paper is as follows. In section 2, we first review the basic structure of quantum systems whose elementary transitions (excitations) operators close the Lie algebra $su(d + 1)$. We define the Bargmann phase space and the corresponding symplectic structure ω_0 of such system. This is realized by making use of the coherent states formalism, which offers a very nice way in the study of the quantum classical correspondence. We introduce the noncommutative Bargmann space by shifting the symplectic two-form $\omega_0 \rightarrow \omega_0 + F$ where F is the perturbation induced by a external gauge field. Consequently, the position as well as momenta coordinates cease to Poisson commute. Thus, to study the dynamics of a given system whose phase space is noncommutative, it is more appropriate to find

out a dressing transformation that converts the modified symplectic form to ω_0 . This issue is presented in section 3. We give a general procedure based on the Moser's lemma to eliminate the fluctuations of the symplectic structure. This generalizes the maps based on the Darboux transformations to include also curved phase spaces. The effects of the modification become then encoded in the Hamiltonian of the system. We discuss the relation between the obtained transformation and the famous Seiberg–Witten map, which was initially introduced in the context of the noncommutative gauge theory [1], see also [26–28]. In section 4, we treat the case where the matrix elements of the fluctuation form F are constants. We show that, in this particular case, one can obtain an exact dressing transformation contrarily to Moser's procedure (which is in some sense perturbative). This exact transformation is similar to Hilbert–Schmidt orthonormalization procedure. As illustration of our results, we consider, in Section 5, the problem of the electromagnetic excitations of a quantum Hall droplet in the complex projective space \mathbf{CP}^2 . The coupling of the quantum Hall droplet with electromagnetic field is done from a purely symplectic point of view. We give the Wess–Zumino–Witten action describing the edge excitations on the boundary of the quantum Hall droplet. We show that the electromagnetic field modify the velocities of the propagation of the chiral field along the angular directions. Concluding remarks close the present paper.

2 Symplectic deformation and noncommutative Bargmann space

2.1 General considerations

It is well established that for an exact solvable quantum system, there is always a well-defined group structure. We denote by \mathcal{G} the corresponding operator algebra. The dynamical properties of this system are described within a Hilbert space \mathcal{F} and the dynamical observables are represented by operators acting on it. This space is completely specified by determining the subset of \mathcal{G} generated by the elementary transition or excitation operators of the system, i.e. annihilation t_i^- and creation t_i^+ , with $i = 1, 2, \dots, d$. The Hamiltonian system and various transition operators can be expressed in terms of the scale operators.

On the other hand, for a classical system, the dynamical observables are differential analytic functions defined on a phase space endowed with a symplectic structure. The classical limit can occur only if such structure can emerge from the quantum system in question. In other words, one must construct a geometry originated from the Hilbert space, which must possess the necessary symplectic structure. Indeed, for a quantum system, namely an algebraic structure $(\mathcal{G}, \mathcal{F})$, there exist $2d$ -dimensional symplectic manifold \mathcal{M} , which is isomorphic to the so-called coset space G/H , where G is the covering group of \mathcal{G} and H is the maximal stability subgroup of G with respect to the fixed state $|\psi_0\rangle$, i.e. the highest weight vector.

In the present analysis, we mainly focus on the $su(d+1)$ quantum systems. For the Lie algebra $su(d+1)$, there are $2d$ generators, which are not in its subalgebra $u(d)$. These can be separated into the lowering t_{-i} and raising t_{+i} types. It is interesting to note that $su(d+1)$ can be introduced through the Weyl generators $t_{\pm i}$ and the triple commutation relations, such as

$$[[t_{+i}, t_{-j}], t_{+k}] = \delta_{jk}t_{+i} + \delta_{ij}t_{+k} \quad (1)$$

$$[[t_{+i}, t_{-j}], t_{-k}] = -\delta_{ik}t_{-j} - \delta_{ij}t_{-k} \quad (2)$$

implemented by the mutual commutators

$$[t_{+i}, t_{+j}] = 0, \quad [t_{-i}, t_{-j}] = 0. \quad (3)$$

Recall that, the mentioned description was introduced for the first time by Jacobson [29] in the context of Lie triple systems. This provides a minimal alternative to the Chevally description. The corresponding Hilbert space [30], see also [31-33], is

$$\mathcal{F} = \{|n_1, n_2, \dots, n_d\rangle; \quad n_i \in \mathbb{N}\}. \quad (4)$$

The elementary excitations operators act on \mathcal{F} as

$$t_{\pm i}|n_1, \dots, n_i, \dots, n_d\rangle = \sqrt{F_i(n_1, \dots, n_i \pm 1, \dots, n_d)}|n_1, \dots, n_i \pm 1, \dots, n_d\rangle \quad (5)$$

where the structure function $F(n_1, \dots, n_i, \dots, n_d)$ is given by

$$F_i(n_1, \dots, n_i, \dots, n_d) = n_i [k + 1 - (n_1 + n_2 + \dots + n_d)] \quad (6)$$

and k is a real number labeling the representation. The Hilbert space has a finite dimension if the quantum numbers n_i fulfilled the condition $(n_1 + n_2 + \dots + n_d) \leq k$. This dimension is

$$\dim \mathcal{F} = \frac{(k + d)!}{k!d!}$$

which is nothing but the dimension of the symmetric representations of the Lie algebra $su(d + 1)$.

To obtain the manifold \mathcal{M} , one can use an unitary exponential mapping. This is

$$\sum_{i=1}^d (\eta_i t_{+i} - \bar{\eta}_i t_{-i}) \longrightarrow \Omega = \exp \sum_{i=1}^d (\eta_i t_{+i} - \bar{\eta}_i t_{-i}) \quad (7)$$

where η_i are complex parameters and Ω is an unitary coset representative of the coset space $G/H \equiv SU(d+1)/U(d)$. This gives the complex projective space \mathbf{CP}^d as geometrical realization corresponding to \mathcal{F} . This correspondence can be better visualized using the formalism of generalized coherent states of G , such as

$$\Omega \longrightarrow |\Omega\rangle \equiv \Omega|\psi_0\rangle = \Omega|0, 0, \dots, 0\rangle. \quad (8)$$

This gives (see for instance in [33] where the notations are more or less similar)

$$|\Omega\rangle = \sum_{\{n_i\}} \left[\frac{k!}{n_1 n_2! \dots n_d! (k - n)!} \right]^{\frac{1}{2}} \frac{z_1^{n_1} z_2^{n_2} \dots z_d^{n_d}}{(1 + \bar{z} \cdot z)^{k/2}} |n_1, n_2, \dots, n_d\rangle \quad (9)$$

where $n = n_1 + n_2 + \dots + n_d$ and the complex variables are $z_i = \frac{\eta_i}{\sqrt{\eta \cdot \eta}} \tan \sqrt{\eta \cdot \eta}$. Obviously, these states constitute an complete set with respect to the measure

$$d\mu(\bar{z}, z) = \frac{(k + d)!}{\pi^d k!} \frac{d^2 z_1 d^2 z_2 \dots d^2 z_d}{(1 + \bar{z} \cdot z)^{d+1}}. \quad (10)$$

The space of analytical functions (Bargmann space) defined by the above coherent states is equipped with a symplectic (Kähler) two-form. This makes it into classical phase space and hence it connects the quantum model to its semiclassical limit. It can be realized by introducing the Kähler potential

$$K_0(\bar{z}, z) = \ln |\langle \psi_0 | \Omega \rangle|^{-2} = k \ln(1 + \bar{z} \cdot z) \quad (11)$$

which allows us to define a closed symplectic two-form

$$\omega_0 = ig_{i\bar{j}} dz^i \wedge d\bar{z}^j. \quad (12)$$

The corresponding Poisson bracket is given by

$$\{f, g\} = -ig^{i\bar{j}} \left(\frac{\partial f}{\partial z^i} \frac{\partial g}{\partial \bar{z}^j} - \frac{\partial g}{\partial z^i} \frac{\partial f}{\partial \bar{z}^j} \right). \quad (13)$$

The components of the metric tensor take the form

$$g_{i\bar{j}} = \frac{\partial^2 K_0(\bar{z}, z)}{\partial z_i \partial \bar{z}_j} = k(1 + \bar{z} \cdot z)^{-2} [(1 + \bar{z} \cdot z)\delta_{ij} - \bar{z}_i z_j]$$

and therefore the matrix elements of its inverse are

$$g^{i\bar{j}} = \frac{1}{k}(1 + \bar{z} \cdot z)(\delta_{ij} + z_i \bar{z}_j).$$

By introducing the canonical coordinates (q, p) of $G/H = SU(d+1)/U(d)$

$$\frac{1}{\sqrt{2k}}(q_i + ip_i) = \frac{z_i}{\sqrt{1 + \bar{z} \cdot z}} \quad (14)$$

it is easily seen that the Poisson two-form can be transformed into the canonical one. This is

$$\omega_0 = \sum_i dq_i \wedge dp_i. \quad (15)$$

Now the Poisson bracket becomes

$$\{f, g\} = \sum_{i=1,2} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial p^i} \frac{\partial f}{\partial q^i} \right) \quad (16)$$

This re-parametrization offers a familiar phase space structure with $\sum_i (p_i^2 + q_i^2) \leq 2k$, which shows that the phase space of the system is compact. As mentioned in the introduction, we will essentially be interested by the four-dimensional phase space, namely $d = 2$ in the above analysis.

2.2 Deformed symplectic structure

We now assume that the symplectic structure of the phase space is modified due to the presence of an external electromagnetic background. This can be formulated by replacing the canonical two-form ω_0 by a closed new one, such as

$$\omega = \omega_0 + F = \omega_0 - \frac{1}{2} \mathcal{B}_{ij}(q) dq^i \wedge dq^j + \frac{1}{2} \mathcal{E}_{ij}(p) dp_i \wedge dp_j \quad (17)$$

where the deformation is encoded in the antisymmetric tensors \mathcal{E}_{ij} and \mathcal{B}_{ij} . This modification requires a condition on the space dimension, namely $d > 1$. Note that, ω can be mapped, in a compact form, as

$$\omega = \frac{1}{2} \omega_{IJ}(\xi) d\xi^I \wedge d\xi^J \quad (18)$$

where $I, J = 1, 2, 3, 4$, with $\xi^i = q^i$ and $\xi^{i+2} = p^i$ for $i = 1, 2$. The nonvanishing elements of the antisymmetric matrix ω are

$$\omega_{12} = -\mathcal{B}_{12}, \quad \omega_{34} = \mathcal{E}_{12}, \quad \omega_{13} = \omega_{24} = 1. \quad (19)$$

It is nondegenerate i.e. $\det \omega \neq 0$, when the antisymmetric tensors \mathcal{E}_{ij} and \mathcal{B}_{ij} satisfy the condition $\det(1_{2 \times 2} - \mathcal{E}\mathcal{B}) \neq 0$. This conclusion can easily be reached by writing ω in terms of matrix. Here we assume that such a condition is satisfied. To find the classical equations of motion and establish the connection between the classical and quantum theory, it is necessary to define the Poisson brackets associated with the new phase space geometry in a consistent way. Indeed, since the Poisson brackets for the coordinates on the phase space are the inverse of the symplectic form as matrix, we have

$$\{\mathcal{F}, \mathcal{G}\} = (\omega^{-1})^{IJ} \frac{\partial \mathcal{F}}{\partial \xi^I} \frac{\partial \mathcal{G}}{\partial \xi^J} \quad (20)$$

where $(\omega^{-1})^{IJ}$ is the inverse matrix of ω_{IJ} (17) and $(\mathcal{F}, \mathcal{G})$ are two functions defined on the phase space. After a straightforward calculation, one can show

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{ik} (\Theta_1^{-1})_{ik} \frac{\partial \mathcal{F}}{\partial q^i} \left[\frac{\partial \mathcal{G}}{\partial p^k} - \sum_j \mathcal{E}_{kj} \frac{\partial \mathcal{G}}{\partial q^j} \right] - (\Theta_2^{-1})_{ik} \frac{\partial \mathcal{F}}{\partial p^i} \left[\frac{\partial \mathcal{G}}{\partial q^k} - \sum_j \mathcal{B}_{kj} \frac{\partial \mathcal{G}}{\partial p^j} \right] \quad (21)$$

where the matrix elements of Θ_1 and Θ_2 are defined by

$$(\Theta_1)_{ij} = \delta_{ij} - \mathcal{E}_{ik} \mathcal{B}_{kj} \quad (22)$$

$$(\Theta_2)_{ij} = \delta_{ij} - \mathcal{B}_{ik} \mathcal{E}_{kj}. \quad (23)$$

They can also be read in matrices form as $\Theta_1 = 1 - \mathcal{E}\mathcal{B}$ and $\Theta_2 = 1 - \mathcal{B}\mathcal{E}$, respectively. It follows that, the modified canonical Poisson brackets are

$$\{q^i, q^j\} = - \sum_k (\Theta_1^{-1})_{ik} \mathcal{E}_{kj} \quad (24)$$

$$\{p^i, p^j\} = \sum_k (\Theta_2^{-1})_{ik} \mathcal{B}_{kj} \quad (25)$$

$$\{q^i, p^j\} = (\Theta_1^{-1})_{ij} = (\Theta_2^{-1})_{ji}. \quad (26)$$

These relations traduce the noncommutativity of the phase space generated by the symplectic modification. Clearly, in the limiting case $\mathcal{E} = 0$ and $\mathcal{B} = 0$, the noncommutative relations (24-26) reduce to the canonical Poisson brackets. According to the modified symplectic structure of the phase space, we introduce the vector fields $X_{\mathcal{F}}$ associated to a given function $\mathcal{F}(q^i, p^j)$. This is

$$X_{\mathcal{F}} = \sum_i X^i \frac{\partial}{\partial q^i} + Y^i \frac{\partial}{\partial p^i} \quad (27)$$

such that the interior contraction of ω with $X_{\mathcal{F}}$ gives

$$\iota_{X_{\mathcal{F}}} \omega = d\mathcal{F}. \quad (28)$$

A simple calculation leads

$$X^i = \sum_j (\Theta_1^{-1})_{ij} \left(\frac{\partial \mathcal{F}}{\partial p^j} - \sum_k \mathcal{E}_{jk} \frac{\partial \mathcal{F}}{\partial q^k} \right) \quad (29)$$

$$Y^i = - \sum_j (\Theta_2^{-1})_{ij} \left(\frac{\partial \mathcal{F}}{\partial q^j} - \sum_k \mathcal{B}_{jk} \frac{\partial \mathcal{F}}{\partial p^k} \right). \quad (30)$$

One can check

$$\iota_{X_{\mathcal{F}}} \iota_{X_{\mathcal{G}}} \omega = \{\mathcal{F}, \mathcal{G}\}. \quad (31)$$

3 Noncommutative dynamics in Bargmann space

The celebrated Darboux theorem guarantees the existence of local coordinates (Q_i, P_i) such that ω takes a canonical form. Such Darboux coordinates transformation are easily obtained once of the tensors \mathcal{B} and \mathcal{E} vanishes. This can be done by using one-form potential $A_i(q) dq_i$ and $\bar{A}_i(p) dp_i$ that defines a $U(1)$ abelian potential A . It is

$$F = dA, \quad A = A_I d\xi^I = A_i(q) dq^i + \bar{A}_i(p) dp^i \quad (32)$$

where bar is just a notation and has nothing to do with the usual complex conjugate. Consequently, for $\mathcal{E} = 0$, the Darboux coordinates are given by

$$Q_i = q_i, \quad P_i = p_i - A_i(q). \quad (33)$$

However, for $\mathcal{B} = 0$, one obtains

$$Q_i = q_i + \bar{A}_i(p) \quad P_i = p_i. \quad (34)$$

In the case where both of forms \mathcal{B} and \mathcal{E} are constant, ω can be re-written in canonical form. This can be achieved by making use of a linear symplectic orthonormalization procedure à la Hilbert Schmidt, which will be treated in section 4. However, for nonconstant \mathcal{B} and \mathcal{E} , the Darboux procedure fails in converting the symplectic two-form $\omega_0 + F$ in Darboux canonical form. As alternative method, one has to employ is based on the Moser's lemma, which constitutes a refined version of Darboux theorem. This will be detailed in what follows.

3.1 Symplectic dressing through Moser's lemma

Let us start by revisiting the derivation of Moser's lemma which behind a nice procedure to locally eliminate the fluctuation $\mathcal{E} + \mathcal{B}$ of the initial symplectic two form ω_0 . To give a general algorithm to realize a dressing transformation through Moser's lemma, we will consider the general case where the matrix elements of ω_0 are phase space dependents.

According to Moser's lemma, there always exists a diffeomorphism on the phase space ϕ whose pullback maps ω to ω_0 . This is

$$\phi^*(\omega_0 + F) = \omega_0 \quad (35)$$

namely, we have

$$\phi : \xi^I \mapsto \phi(\xi^I), \quad \frac{\partial \phi(\xi^K)}{\partial \xi^I} \frac{\partial \phi(\xi^L)}{\partial \xi^J} \omega_{KL}(\phi(\xi)) = \omega_{0IJ}(\xi). \quad (36)$$

To find out this change of coordinates, one can start by defining a family of one parameter of symplectic forms

$$\omega(t) = \omega_0 + tF \quad (37)$$

interpolating ω_0 and $\omega_0 + F$ for $t = 0$ and $t = 1$, respectively, with $0 \leq t \leq 1$. Note that, t is just an affine parameter labeling the flow generated by a smooth t -dependent vector field $X(t)$. Accordingly, one also define a family of diffeomorphisms

$$\phi^*(t)\omega(t) = \omega_0 \quad (38)$$

satisfying $\phi^*(t = 0) = id$ and $\phi^*(t = 1)$ will be the required solution of our problem, i.e. (35). Differentiating (38), one check that $X(t)$ must satisfy the identity

$$0 = \frac{d}{dt} [\phi^*(t)\omega(t)] = \phi^*(t) \left[L_{X(t)}\omega(t) + \frac{d\omega(t)}{dt} \right]. \quad (39)$$

where $L_{X(t)}$ denotes the Lie derivative of the field $X(t)$. Using the Cartan identity $L_X = \iota_X \circ d + d \circ \iota_X$ and the fact that $d\omega(t) = 0$, we obtain

$$\phi^*(t) \{ d [\iota_{X(t)}\omega(t)] + F \} = 0 \quad (40)$$

where ι_X stands for interior contraction as above. It follows that $X(t)$ is verifying the linear equation

$$\iota_{X(t)}\omega(t) + A = 0 \quad (41)$$

which solves (39). Therefore, the components of $X(t)$ are given by

$$X^I(t) = -A_J \omega^{-1JI}(t). \quad (42)$$

For small fluctuations of the symplectic structure, i.e. $F \ll \omega_0$, one can write the inverse of ω as

$$\omega^{-1}(t) = \omega_0^{-1} - t\omega_0^{-1}F\omega_0^{-1} + t^2\omega_0^{-1}F\omega_0^{-1}F\omega_0^{-1} + \dots \quad (43)$$

This determines the components of $X(t)$ in terms of the $U(1)$ connection A and its derivatives and allows us to write down the explicit form of the transformation ϕ . Indeed, since the t evolution of $\omega(t)$ is governed by the first order differential equation

$$[\partial_t + X(t)]\omega(t) = 0 \quad (44)$$

it is easy to show that

$$[\exp(\partial_t + X(t)) \exp(-\partial_t)]\omega(t+1) = \omega(t). \quad (45)$$

This leads to the relation

$$[\exp(\partial_t + X(t)) \exp(-\partial_t)]|_{(t=0)}(\omega_0 + F) = \phi^*(\omega_0 + F) = \omega_0 \quad (46)$$

where ϕ^* is given by

$$\phi^* = id + X(0) + \frac{1}{2}(\partial_t X)(0) + \frac{1}{2}X^2(0) + \dots \quad (47)$$

More explicitly, using (42), the contribution arising from the second term in (47) read as

$$X(0) = \omega_0^{-1IJ} A_J \partial_I. \quad (48)$$

The contribution of the third term in (47) is

$$\frac{1}{2}(\partial_t X)(0) = -\frac{1}{2}(\omega_0^{-1} F \omega_0^{-1})^{IJ} A_J \partial_I. \quad (49)$$

The last term in (47) gives

$$\frac{1}{2}X^2(0) = \frac{1}{2}(\omega_0^{-1IJ} A_J \partial_I)(\omega_0^{-1I'J'} A_{J'} \partial_{I'}) \quad (50)$$

Finally, in terms of local coordinates, the coordinate transformation ϕ whose pullback maps $\omega_0 + F \rightarrow \omega_0$ is given by

$$\phi(\xi^L) = \xi^L + \xi_1^L + \xi_2^L + \dots \quad (51)$$

where ξ_1^L is

$$\xi_1^L = \omega_0^{-1LJ} A_J \quad (52)$$

and ξ_2^L takes the form

$$\xi_2^L = -\frac{1}{2}\omega_0^{-1LK} F_{KL'} \omega_0^{-1L'J} A_J + \frac{1}{2}\omega_0^{-1IJ} A_J (\partial_I \omega_0^{-1LJ'}) A_{J'} + \frac{1}{2}\omega_0^{-1IJ} A_J \omega_0^{-1LJ'} (\partial_I A_{J'}). \quad (53)$$

Using the relations

$$\partial_{J'} A_{I'} = (\partial_{J'} \omega_{0I'I}) \xi_1^{I'} + \omega_{0I'I} (\partial_{J'} \xi_1^{I'}) \quad (54)$$

$$\partial_I \omega_0^{-1LJ'} = -\omega_0^{-1LJ''} (\partial_I \omega_{0J''K}) \omega_0^{-1KJ'} \quad (55)$$

and the antisymmetry property of the symplectic form, keep in mind that ω_0 is assumed closed and nonconstant, one can check

$$\begin{aligned} \xi_2^L &= -\omega_0^{-1LK} F_{KL'} \xi_1^{L'} + \frac{1}{2}\omega_0^{-1LK} \omega_0^{-1MJ} A_J \omega_0^{-1N'J'} A_{J'} \partial_M \omega_{0NK} \\ &\quad + \frac{1}{2}\omega_0^{-1LK} \omega_0^{-1MS} A_S \omega_{0MN} \partial_K (\omega_0^{-1NS'} A_{S'}). \end{aligned} \quad (56)$$

It is remarkable that this dressing transformation coincides with the Susskind map derived in connection with the quantum Hall systems and noncommutative Chern–Simons theory [12]. It leads also to the very familiar Seiberg–Witten map [1] used in the context of the string and noncommutative gauge theories. This will be clarified in the next subsection.

3.2 Seiberg–Witten map in four-dimensional phase space

In fact, one can see from (52) and (56) that the dressing transformation can be written as

$$\phi(\xi^L) = \xi^L + \hat{A}^L \quad (57)$$

where we have set

$$\begin{aligned} \hat{A}^L = & \omega_0^{-1LK} \left[A_K - F_{KL'} \omega_0^{-1L'M} A_M + \frac{1}{2} \omega_0^{-1MJ} A_J \omega_0^{-1NJ'} A_{J'} \partial_M \omega_{0NK} \right. \\ & \left. + \frac{1}{2} \omega_0^{-1MS} A_S \omega_{0MN} \partial_K (\omega_0^{-1NS'} A_{S'}) \right]. \end{aligned} \quad (58)$$

The transformation (57) is similar to the so-called Susskind map. It encodes the geometrical fluctuations induced by the external magnetic field F . Also, it coincides with the Seiberg–Witten map in a curved manifold for the noncommutative abelian gauge theory [30]. Indeed, under the gauge transformation

$$A \longrightarrow A + d\Lambda \quad (59)$$

the components (58) transform as

$$\hat{A}^L \longrightarrow \hat{A}^L + \omega_0^{-1LJ} \partial_J \hat{\Lambda} + \{ \hat{A}^L, \hat{\Lambda} \} + \dots \quad (60)$$

where the noncommutative gauge parameter $\hat{\Lambda}$

$$\hat{\Lambda} = \Lambda + \frac{1}{2} \omega_0^{-1IJ} A_J \partial_I \Lambda + \dots \quad (61)$$

is written as function of Λ and the abelian connection A . The equations (58), (60) and (61) are the semiclassical versions of the Seiberg–Witten map. The connection \hat{A} is the induced noncommutative gauge potential given in terms of its commutative counter part A . This establish a correspondence between symplectic deformations and non commutative gauge theories.

Now we return to the situation of our purpose where the phase space is four-dimensional and equipped with the canonical Darboux form ω_0 given in (15). In this particular case, one can verify, by using (32), (51), (52) and (56), that the deformed two-form $\omega_0 + F$ (17) takes the canonical form

$$\omega_0 + F = dQ^i \wedge dP^i \quad (62)$$

where the new phase space variables Q^i and P^i are given by

$$Q^i = \phi^{-1}(q^i) = q^i + \bar{A}_i(p) - \sum_{j=1,2} A_j(q) \left[\mathcal{E}_{ij}(p) - \frac{1}{2} \frac{\partial \bar{A}_j(p)}{\partial p_i} \right] + \dots \quad (63)$$

$$P^i = \phi^{-1}(p^i) = p^i - A_i(q) + \sum_{j=1,2} \bar{A}_j(p) \left[\mathcal{B}_{ij}(q) + \frac{1}{2} \frac{\partial A_j(q)}{\partial q_i} \right] + \dots \quad (64)$$

It is interesting to note that for $\bar{A}_i(p) = 0$ (respectively $A_i(q) = 0$) we obtain (33) (respectively (34)) and recover the Darboux transformations discussed above when one of the tensors \mathcal{B} and \mathcal{E} vanishes. On the other hand, when the gauge potential (32) is defined as

$$A = -\frac{1}{2} (\bar{\theta} \epsilon_{ij} q_i dq_j - \theta \epsilon_{ij} p_i dp_j) \quad (65)$$

corresponding to a constant electromagnetic fields F (θ and $\bar{\theta}$ real constants), the dressing transformation (63-64) gives

$$Q^i = \left(1 + \frac{3}{8}\theta\bar{\theta}\right) q^i + \frac{\theta}{2} \sum_k \epsilon_{ki} p^k \quad (66)$$

$$P^i = \left(1 + \frac{3}{8}\theta\bar{\theta}\right) p^i + \frac{\bar{\theta}}{2} \sum_k \epsilon_{ki} q^k. \quad (67)$$

ϵ_{ij} , appearing in (65), is the usual antisymmetric tensor, namely $\epsilon_{12} = -\epsilon_{21} = 1$.

3.3 Hamiltonian system

Let $\mathcal{H} \equiv \mathcal{H}(p, q)$ to be the original classical Hamiltonian. In modifying the symplectic structure, the dynamics becomes described by two-form $\omega_0 + F$. The dressing transformation converts the dynamical system of $(\omega_0 + F, \mathcal{H})|_{qp}$ to $(\omega_0, \mathcal{H}_A)|_{QP}$ where we use the old symplectic form but a different Hamiltonian, which can be obtained by simply replacing the old phase space variables in terms of the new ones. In this respect, using (57) (or inverting (63) and (64)), one obtains

$$q^i = \phi(Q^i) = Q^i - \bar{A}_i(P) + \sum_{j=1,2} A_j(Q) \left[\mathcal{E}_{ij}(P) - \frac{1}{2} \frac{\partial \bar{A}_j(P)}{\partial P_i} \right] + \dots \quad (68)$$

$$p^i = \phi(P^i) = P^i + A_i(Q) - \sum_{j=1,2} \bar{A}_j(P) \left[\mathcal{B}_{ij}(Q) + \frac{1}{2} \frac{\partial A_j(Q)}{\partial Q_i} \right] + \dots \quad (69)$$

This result can be used to write down the required Hamiltonian system to the second order in terms of A 's. This is

$$\begin{aligned} \mathcal{H}_A = & \mathcal{H} - \sum_i (\bar{A}_i \bar{u}_i - A_i u_i) + \frac{1}{2} \sum_{ij} \left[\bar{A}_i \bar{A}_j \frac{\partial \bar{u}_i}{\partial Q_j} + A_i A_j \frac{\partial u_i}{\partial P_j} - 2 \bar{A}_i A_j \frac{\partial u_j}{\partial Q_i} \right] \\ & + \sum_{ij} A_j \left[\mathcal{E}_{ij} - \frac{1}{2} \frac{\partial \bar{A}_j}{\partial P_i} \right] \bar{u}_i - \sum_{ij} \bar{A}_j \left[\mathcal{B}_{ij} + \frac{1}{2} \frac{\partial A_j}{\partial Q_i} \right] u_i + \dots \end{aligned} \quad (70)$$

where we the quantities u_i and \bar{u}_i are defined by

$$u_i = \frac{\partial \mathcal{H}}{\partial P_i}, \quad \bar{u}_i = \frac{\partial \mathcal{H}}{\partial Q_i}. \quad (71)$$

Here again bar is just a notation. It is clear that the dressing transformation eliminates the fluctuations of the symplectic form, which become incorporated in the Hamiltonian.

4 Constant symplectic fluctuation

4.1 Poisson structure

As mentioned above the dressing transformation in the special case of a constant symplectic fluctuation can be achieved by making use of the Hilbert–Schmidt procedure. This can be seen as an exact alternative to one described in the former section. From (65), one can verify that the matrix element of the fluctuating tensors are

$$\mathcal{E}_{ij} = \theta \epsilon_{ij}, \quad \mathcal{B}_{ij} = \bar{\theta} \epsilon_{ij}. \quad (72)$$

The nondegeneracy of ω is provided by the condition $1 + \theta\bar{\theta} \neq 0$. In addition, hereafter we assume that $1 + \theta\bar{\theta} > 0$ is fulfilled. With the above particular modification of the symplectic structure, the Poisson brackets (24-26) simply read as

$$\{q^i, q^j\} = -\frac{\theta}{1 + \theta\bar{\theta}}\epsilon_{ij} \quad (73)$$

$$\{p^i, p^j\} = \frac{\bar{\theta}}{1 + \theta\bar{\theta}}\epsilon_{ij} \quad (74)$$

$$\{q^i, p^j\} = \frac{1}{1 + \theta\bar{\theta}}\delta_{ij} \quad (75)$$

reflecting a deviation from the canonical brackets.

In this section, we specify the form of the classical Hamiltonian. More precisely, we consider a bidimensional harmonic oscillator Hamiltonian of the type

$$\mathcal{V}(p, q) = \frac{1}{2} \sum_i (p_i^2 + q_i^2). \quad (76)$$

This will be studied in subsection (4.3).

4.2 Dressing transformation and Quantization

We start by noting that under the transformation

$$Q^i = aq^i + \frac{1}{2}b\theta \sum_k \epsilon_{ki}p^k \quad (77)$$

$$P^i = cp^i + \frac{1}{2}d\bar{\theta} \sum_k \epsilon_{ki}q^k \quad (78)$$

the Poisson brackets (73-75) give the canonical ones

$$\begin{aligned} \{Q^i, Q^j\} &= 0 \\ \{P^i, P^j\} &= 0 \\ \{Q^i, P^j\} &= \delta_{ij} \end{aligned} \quad (79)$$

once the real scalars a , b , c and d satisfy the following set of constraints

$$\begin{aligned} 4a^2 - 4ab - \theta\bar{\theta}b^2 &= 0 \\ 4c^2 - 4cd - \theta\bar{\theta}d^2 &= 0 \\ 4ac + 2\theta\bar{\theta}(ad + bc) - \theta\bar{\theta}bd &= 4(1 + \theta\bar{\theta}). \end{aligned}$$

A simple solution of such set is

$$a = c = \frac{1}{b} = \frac{1}{d} = \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 + \theta\bar{\theta}}}. \quad (80)$$

On the other hand, in terms of the above new dynamical variables, ω can be written as

$$\omega = \sum_i dQ^i \wedge dP^i. \quad (81)$$

Inverting the transformation (77-78), we obtain

$$q^i = \frac{a}{\sqrt{1+\theta\bar{\theta}}} \left[Q^i + \frac{\theta}{2a^2} \sum_k \epsilon_{ik} P^k \right] \quad (82)$$

$$p^i = \frac{a}{\sqrt{1+\theta\bar{\theta}}} \left[P^i + \frac{\bar{\theta}}{2a^2} \sum_k \epsilon_{ik} Q^k \right]. \quad (83)$$

For small values of θ and $\bar{\theta}$, we can see that (82) and (83) give

$$Q^i = \left(1 + \frac{1}{8}\theta\bar{\theta} \right) q^i + \frac{\theta}{2} \sum_k \epsilon_{ki} p^k \quad (84)$$

$$P^i = \left(1 + \frac{1}{8}\theta\bar{\theta} \right) p^i + \frac{\bar{\theta}}{2} \sum_k \epsilon_{ki} q^k \quad (85)$$

which are sensitively comparable to the expressions (66) and (67).

4.3 New induced dynamics

The Hamiltonian \mathcal{V} (76) becomes

$$\mathcal{V} = \frac{a^2}{2(1+\theta\bar{\theta})} \left[\sum_i \left(1 + \frac{\theta^2}{4a^4} \right) P^i P^i + \left(1 + \frac{\bar{\theta}^2}{4a^4} \right) Q^i Q^i + \left(\frac{\theta}{a^2} - \frac{\bar{\theta}}{a^2} \right) \sum_j \epsilon_{ij} Q^i P^j \right]. \quad (86)$$

Evidently the $(\theta, \bar{\theta})$ -dependent terms in (86) arise from the deformation of the symplectic structure. It follows that the deformation of the symplectic structure can be thought as a perturbation reflecting the action of some external potential on the system. This feature is very similar to the Landau problem in quantum mechanics. For the purpose of the next section, we shall convert the Hamiltonian (86) in complex notation. This can be achieved by introducing the variables

$$Z^i = \sqrt{\frac{\Delta}{2}} \left(Q^i + i \frac{P^i}{\Delta} \right), \quad \bar{Z}^i = \sqrt{\frac{\Delta}{2}} \left(Q^i - i \frac{P^i}{\Delta} \right) \quad (87)$$

where the involved parameter is

$$\Delta = \sqrt{\frac{4a^4 + \bar{\theta}^2}{4a^4 + \theta^2}}. \quad (88)$$

They satisfy the usual Poisson relations

$$\begin{aligned} \{Z^i, Z^j\} &= 0 \\ \{Z^i, \bar{Z}^j\} &= -i\delta_{ij} \\ \{\bar{Z}^i, \bar{Z}^j\} &= 0. \end{aligned}$$

The Hamiltonian \mathcal{V} can be written as the sum of two contributions, such as

$$\mathcal{V} - \mathcal{V}_0 = \frac{1}{4a^2} \frac{1}{1+\theta\bar{\theta}} \sqrt{(4a^4 + \theta^2)(4a^4 + \bar{\theta}^2)} (Z^1 \bar{Z}^1 + Z^2 \bar{Z}^2) \quad (89)$$

where \mathcal{V}_0 is given by

$$\mathcal{V}_0 = -\frac{i}{2} \frac{\theta - \bar{\theta}}{1 + \theta\bar{\theta}} \sum_{ij} \epsilon_{ij} \bar{Z}^i Z^j. \quad (90)$$

It can be also written in a form that is more appropriate for our purpose. Indeed, by considering new variables

$$Z_+ = \frac{1}{\sqrt{2}} (Z^1 + iZ^2), \quad Z_- = \frac{1}{\sqrt{2}} (Z^1 - iZ^2) \quad (91)$$

and substituting (91) in (89-90), we end up with

$$\mathcal{V} = (\Omega - \delta) Z_+ \bar{Z}_+ + (\Omega + \delta) Z_- \bar{Z}_- \quad (92)$$

where Ω is

$$\Omega = \frac{\sqrt{(4a^4 + \theta^2)(4a^4 + \bar{\theta}^2)}}{4a^2(1 + \theta\bar{\theta})} \quad (93)$$

and δ takes the form

$$\delta = \frac{\theta - \bar{\theta}}{2(1 + \theta\bar{\theta})}. \quad (94)$$

Note that, two-form (81) can be rewritten as

$$\omega = i (dZ_+ \wedge d\bar{Z}_+ + dZ_- \wedge d\bar{Z}_-). \quad (95)$$

Upon quantization, all canonical variables become the Heisenberg operators satisfying commutation rules according to the canonical prescription, i.e. Poisson bracket \longrightarrow -i commutator. It follows that the nonvanishing commutators are

$$[Z_+, \bar{Z}_+] = 1, \quad [Z_-, \bar{Z}_-] = 1. \quad (96)$$

Note that, the Hamiltonian (92) is a superposition of two one dimensional harmonic oscillators. Thus, the symplectic modification induces a splitting of energy levels (degeneracy lifting). This effect is very important and will have interesting consequences on the electromagnetic excitations of quantum Hall effect in four-dimensional space. This is the main task of the next section.

5 Four-dimensional quantum Hall droplet

5.1 Brief review

To illustrate the results of the previous sections, we consider a large number of particles, evolving in four-dimensional complex projective manifold \mathbf{CP}^2 , under the action of a magnetic field generated by two-form ω_0 (12). In this situation the spectrum is highly degenerate, splitting in Landau levels, and it was shown [21] that there is one-to-one correspondence between the lowest Landau levels (LLL) or ground state wavefunctions and the coherent states given by (9), with $d = 2$ ($\mathcal{F} \equiv$ LLL). For a

strong magnetic field ($k \rightarrow \infty$), the gap between Landau levels becomes large and the particles are constrained to be accommodated in the LLL forming a quantum Hall droplet.

The dynamics of the droplet is characterized as follows. Since the LLL are highly degenerated, one can fill states with $M = M_1 + M_2$ particles where M_i stands for the particle number in a given mode i . The corresponding density operator is then

$$\rho_0 = \sum_{n_1, n_2} |n_1, n_2\rangle \langle n_1, n_2|. \quad (97)$$

The fluctuations, preserving the number of states, are described by an unitary transformation

$$\rho_0 \longrightarrow \rho = U \rho_0 U^\dagger \quad (98)$$

and the equation of motion is the quantum Liouville equation

$$i \frac{\partial \rho}{\partial t} = [V, \rho] \quad (99)$$

where V is the confining potential ensuring the degeneracy lifting of the LLL, see [21-22, 24] for more details. Furthermore, since the LLL wavefunctions coincide with $SU(3)$ coherent states in the symmetric representation, this offers a simple way to perform the semiclassical analysis. This can be done by associating to every operator A a symbol, such as

$$\mathcal{A}(\bar{z}, z) = \langle z | A | z \rangle = \langle 0 | \Omega^\dagger A \Omega | 0 \rangle. \quad (100)$$

An associative star product of two functions $\mathcal{A}(\bar{z}, z)$ and $\mathcal{B}(\bar{z}, z)$ is then defined by

$$\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \langle z | AB | z \rangle \quad (101)$$

which rewrites, for large k , as

$$\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \mathcal{A}(\bar{z}, z) \mathcal{B}(\bar{z}, z) - g^{j\bar{m}} \partial_j \mathcal{A}(\bar{z}, z) \partial_{\bar{m}} \mathcal{B}(\bar{z}, z). \quad (102)$$

Then, the symbol or function associated with the commutator of two operators A and B is given by

$$\langle z | [A, B] | z \rangle = -g^{j\bar{m}} \{ \partial_j \mathcal{A}(\bar{z}, z) \partial_{\bar{m}} \mathcal{B}(\bar{z}, z) - \partial_j \mathcal{B}(\bar{z}, z) \partial_{\bar{m}} \mathcal{A}(\bar{z}, z) \} \quad (103)$$

which leads to the result

$$\langle z | [A, B] | z \rangle = i \{ \mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z) \} \equiv \{ \mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z) \}_\star \quad (104)$$

where $\{, \}$ stands for the Poisson bracket defined by (13) and the notation $\{, \}_\star$ stands for Moyal brackets.

With the above semiclassical correspondence, we can give the symbol of the density matrix (97) in the limit of large number of states, i.e. large magnetic field, and large number of fermions M ($M < \dim \mathcal{F}$). This is [21]

$$\rho_0(\bar{z}, z) \simeq \exp(-k\bar{z} \cdot z) \sum_{n=0}^M \frac{(k\bar{z} \cdot z)^n}{n!} \simeq \Theta(M - k\bar{z} \cdot z). \quad (105)$$

where Θ is the usual step function. It corresponds to an abelian droplet configuration with boundary defined by $k\bar{z} \cdot z = M$ and its radius is proportional to \sqrt{M} .

The confining potential can be defined in terms of the Fock number operators $N_i|n_1, n_2\rangle = n_i|n_1, n_2\rangle$, with $i = 1, 2$. This is

$$V = N_1 + N_2. \quad (106)$$

The associated symbol is given by

$$\mathcal{V}(\bar{z}, z) = \langle z|V|z\rangle = k \frac{\bar{z} \cdot z}{1 - \bar{z} \cdot z}. \quad (107)$$

which is exactly the potential given by (76).

This brief review gives the necessary tools needed to examine the electromagnetic excitations of a quantum Hall droplet in four-dimensional manifold by using the results obtained in the previous sections. We will mainly focus on the situation where the matrix \mathcal{B} and \mathcal{E} are constants.

5.2 Electromagnetic excitations of quantum Hall droplets

It is clear that we may think the Hilbert \mathcal{F} as the quantization of the phase space \mathbf{CP}^2 where the symplectic form ω_0 is proportional to the Kahler form on \mathbf{CP}^2 . The modification of the symplectic structure of the phase space induces electromagnetic interactions of the quantum Hall droplets. The symplectic dressing methods, discussed previously, give a prescription to eliminate the gauge fluctuations by encoding their effects in the expression of the Hamiltonian of the system. Hence, in the case of constants \mathcal{B} and \mathcal{E} , as shown above, the symplectic two form is mapped, via the relations (82-83), (87) and (91), to its canonical form (95) in terms of the new variables Z_+ and Z_- . The Poisson brackets become the canonical ones. Also, it is easily seen that the confining potential (107) can be mapped as

$$\mathcal{V}(\bar{Z}, Z) = \Omega_+ Z_+ \bar{Z}_+ + \Omega_- Z_- \bar{Z}_- \quad (108)$$

where $\Omega_{\pm} = \Omega \mp \delta$ and the density function is given by

$$\rho_0(\bar{Z}, Z) = \Theta [M - k (\Omega_+ Z_+ \bar{Z}_+ + \Omega_- Z_- \bar{Z}_-)]. \quad (109)$$

These are the main ingredients to evaluate the effective action describing the quantum Hall droplets interacting with an external magnetic field F . This action is given by [34]

$$S = \int dt \text{Tr} [\rho_0 U^\dagger (i\partial_t - V) U]. \quad (110)$$

For a strong magnetic field or k large, the quantities appearing in this action can be evaluated as classical functions.

Along similar lines as in [34, 21,24], we start by computing the kinetic term. In this order, we set $U = e^{+i\Phi}$ ($\Phi^\dagger = \Phi$) to get

$$i \int dt \text{Tr} (\rho_0 U^\dagger \partial_t U) \simeq \frac{1}{2k} \int d\mu \{\Phi, \rho_0\} \partial_t \Phi \quad (111)$$

where the symbol $\{, \}$ is the Poisson bracket. This gives

$$\{\Phi, \rho_0\} = (\Omega_+ \mathcal{L}_+ \Phi + \Omega_- \mathcal{L}_- \Phi) \frac{\partial \rho_0}{\partial r^2} \quad (112)$$

where $r^2 = \Omega_+ Z_+ \bar{Z}_+ + \Omega_- Z_- \bar{Z}_-$ and the first order differential operators are defined by

$$\mathcal{L}_\alpha = i \left(Z_\alpha \frac{\partial}{\partial Z_\alpha} - \bar{Z}_\alpha \frac{\partial}{\partial \bar{Z}_\alpha} \right), \quad \alpha = +, -. \quad (113)$$

In (112), the derivative of the density function gives a δ function with support on the boundary $\partial\mathcal{D}$ of the droplet \mathcal{D} defined by $kr^2 = M$. Then, we have

$$i \int dt \text{Tr} \left(\rho_0 U^\dagger \partial_t U \right) \approx -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt (\Omega_+ \mathcal{L}_+ \Phi + \Omega_- \mathcal{L}_- \Phi) \partial_t \Phi. \quad (114)$$

We come now to the evaluation of the potential term in (110), which can be written as

$$\text{Tr}(\rho_0 U^\dagger V U) = \text{Tr}(\rho_0 V) + i \text{Tr}([\rho_0, V] \Phi) + \frac{1}{2} \text{Tr}([\rho_0, \Phi] [V, \Phi]) + \dots \quad (115)$$

It can be easily verified that the first term in the second line in (115) gives a bulk contribution that can be ignored since we are interested to the edge dynamics. Further, remark that it is Φ -independent and contains no information about the dynamics of the edge excitations. From (97) and (106), we have $[\rho_0, V] = 0$, thus the second term in (115) vanishes. The last term in (115) is evaluated similarly to (114). Finally, we have

$$\int dt \text{Tr} \left(\rho_0 U^\dagger \mathcal{H} U \right) \approx \frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt (\Omega_+ \mathcal{L}_+ \Phi + \Omega_- \mathcal{L}_- \Phi)^2. \quad (116)$$

Combining (114) and (116), we get

$$S \approx -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt [\Omega_+ (\mathcal{L}_+ \Phi) + \Omega_- (\mathcal{L}_- \Phi)] [(\partial_t \Phi) + \Omega_+ (\mathcal{L}_+ \Phi) + \Omega_- (\mathcal{L}_- \Phi)]. \quad (117)$$

This action involves only the time derivative of Φ and the tangential derivatives $\mathcal{L}_\alpha \Phi$. It is a generalization of a chiral abelian Wess–Zumino–Witten (WZW) theory. For $\theta = 0$ and $\bar{\theta} = 0$, we recover the WZW usual action for the edge states associated with un-gauged Hall droplets in four-dimensional space [21]. This is given by

$$S \approx -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt [(\partial_t \Phi)(\mathcal{L}\Phi) + \omega(\mathcal{L}\Phi)^2]. \quad (118)$$

where $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$.

5.3 Edge fields

The action (117) is minimized by the fields Φ , which are satisfying the equation of motion

$$\sum_{\alpha=\pm} (\Omega_\alpha \mathcal{L}_\alpha) [\partial_t \Phi + \Omega_\alpha \mathcal{L}_\alpha \Phi] = 0. \quad (119)$$

The edge field Φ can be expanded in powers of the phase space variables Z_α . Note that, since the excitations are moving on the real 3-sphere $\mathbf{S}^3 \sim SU(2)$, it is convenient to introduce the $SU(2)$ parametrization. This is

$$\Omega_+ Z_+ = \sqrt{\frac{M}{k}} \frac{\sqrt{\zeta \bar{\zeta}}}{\sqrt{1 + \zeta \bar{\zeta}}} e^{i\phi_+}, \quad \Omega_- Z_- = \sqrt{\frac{M}{k}} \frac{1}{\sqrt{1 + \bar{\zeta} \zeta}} e^{i\phi_-} \quad (120)$$

where ζ and $\bar{\zeta}$ are the local complex coordinates for $SU(2)$. The operators \mathcal{L}_\pm reduce to partial derivatives ∂_{ϕ_\pm} with respect to ϕ_\pm . Thus, the field Φ is given as

$$\Phi = \sum_{n_+, n_-} c_{n_+, n_-}(t) e^{i\phi_+ n_+} e^{i\phi_- n_-} \quad (121)$$

where the coefficients c_{n_+, n_-} are (ϕ_+, ϕ_-) -independents for $(n_+ \neq 0, n_- \neq 0)$. It follows that the solution of the equation of motion (119) takes the form

$$\Phi = (\phi_+ - \Omega_+ t) + (\phi_- - \Omega_- t) + \sum_{n_+, n_-} c_{n_+, n_-}(0) e^{i(\phi_+ - \Omega_+ t)n_+} e^{i(\phi_- - \Omega_- t)n_-}. \quad (122)$$

It is clear, from the last equation, that the noncommutativity arising from the symplectic modification changes the propagation velocities of the edge field along the angular directions. It is also important to stress that the velocities Ω_+ and Ω_- are different (respectively equal) for $\theta \neq \bar{\theta}$ (respectively $\theta = \bar{\theta}$).

6 Concluding remarks

We close the present analysis by summarizing the main points and results. We first introduced the Bargman phase space of a quantum system whose elementary excitations close the $su(3)$ Lie algebra. This space is interesting in three respects. First, it equipped with a symplectic structure that one can vary in order to describe the electromagnetic excitations of the system. Second, the points of this space are in correspondence with the $SU(3)$ coherent states, which respect the over completion property. This provides us with an elegant tool to perform the semiclassical analysis (definition of star product and Moyal brackets). Third, this phase space is four-dimensional manifold and one can consider a symplectic modification (17) such both positions q and momentum p cease to Poisson commute. This can not be realized in two dimensional case.

In connection with this phase space, the present work addresses three major issues: First, the variation (or perturbation) of the symplectic two-form $\omega_0 \rightarrow \omega_0 + F$, which induces the noncommutative structures, can be eliminated through the Moser's lemma that is a refined version of Darboux theorem. This leads to a dressing transformation (51), see also (68-69), which converts the modified two-form in its undeformed form. The effects of the fluctuations become encoded in the Hamiltonian of the system (70). The dynamics remains unchanged. We showed the dressing transformation is equivalent to the Seiberg–Witten map (57-58). This means that a symplectic modification and a noncommutative abelian gauge transformation are equivalents.

The second issue concerns the particular case where the matrix elements of the components \mathcal{E} and \mathcal{B} of electromagnetic fluctuation F are constants (72). We used the Hilbert–Schmidt orthonormalization procedure to write down an exact dressing transformation (82-83). Here again the effect of the non commutativity becomes encoded in the Hamiltonian (86). This induces the anisotropy of the harmonic oscillator potential (92) and upon quantization generates a degeneracy lifting analogously to the well known Zeeman effect.

Finally, as application of the tools developed in this paper, we considered the problem of quantum Hall effect in the complex projective space $\mathbf{CP}^2 = SU(3)/U(2)$. We derived the Wess–Zumino–Witten action (117) governing the electromagnetic excitations of a large collection of fermions in the lowest

Landau levels. We obtained explicitly the edge field excitations (122) traveling with modified velocities as consequence of the noncommutativity effects.

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References

- [1] N. Seiberg and E. Witten, *JHEP* **9909** (1999) 032, [hep-th/9908142].
- [2] S. Doplicher, K. Fredenhagen and J.E. Roberts, *Commun. Math. Phys.* **172** (1995) 187, [hep-th/0303037].
- [3] M.R. Douglas and N.A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977, [hep-th/0106048].
- [4] R.J. Szabo, *Phys. Rep.* **378** (2003) 207, [hep-th/0109162].
- [5] V.P. Nair, *Phys. Lett.* **B505** (2001) 249, [hep-th/0008027].
- [6] V.P. Nair and A.P. Polychronakos, *Phys. Lett.* **B505** (2001) 267, [hep-th/0011172]; A. Jellal, *J. Phys. A: Math. Gen* **34** (2001) 10159, [hep-th/0502040].
- [7] O.F. Dayi and A. Jellal, *Phys. Lett.* **A287** (2001) 349, [cond-mat/0103562]; *J. Math. Phys.* **43** (2002) 4592; (Erratum-ibid. **45** (2004) 827), [hep-th/0111267]; O.F. Dayi and L.T. Kelleyane, *Mod. Phys. Lett.* **A17** (2002) 1937, [hep-th/0202062].
- [8] C. Duval and P.A. Horváthy, *Phys. Lett.* **B479** (2000) 284, [hep-th/0002233]; *J. Phys. A: Math. Gen.* **34** (2001) 10097, [hep-th/0106089]; *Phys. Lett.* **B547** (2002) 306, (Erratum-ibid. **588** (2004) 228), [hep-th/0209166]; P.A. Horváthy, *Ann. Phys.* **299** (2002) 128, [hep-th/0201007]; *SIGMA* **2** (2006) 090, [cond-mat/0609571]; *Phys. Lett.* **A359** (2006) 705, [cond-mat/0606472]; P.A. Horváthy and M.S. Plyushchay, *JHEP* **0206** (2002) 033, [hep-th/0201228]; *Nucl. Phys.* **B714** (2005) 269, [hep-th/0502040]; *Phys. Lett.* **B595** (2004) 547, [hep-th/0404137].
- [9] F. Delduc, Q. Duret, F. Gieres and M. Lefrancois, *J. Phys. Conf. Ser.* **103** (2008) 012020, [arXiv:0710.2239].
- [10] J. Moser, *Trans. Amer. Math. Soc* **120** (1965) 286.
- [11] R.B. Laughlin, *Phys. Rev.* **B23**, (1981) 5632; *Phys. Rev. Lett.* **50** (1983) 1395.
- [12] L. Susskind, *The Quantum Hall Fluid and NonCommutative Chern-Simons Theory*, [hep-th/0101029].
- [13] A.P. Polychronakos, *JHEP* **0104** (2001) 011, [hep-th/0103013].

- [14] S.C. Zhang and J.P. Hu, *Science* **294** (2001) 823, [cond-mat/0110572].
- [15] Y-X. Chen, B-Y. Hou and B-Y. Hou, *Nucl. Phys.* **B638** (2002) 220, [hep-th/0203095].
- [16] M. Fabinger, *JHEP* **0205** (2002) 037, [hep-th/0201016].
- [17] H. Elvang and J. Polchinski, *The Quantum Hall Effect on \mathbb{R}^4* , [hep-th/0209104].
- [18] B.A. Bernevig, J.P. Hu, N. Toumbas and S.C. Zhang, *Phys. Rev. Lett.* **91** (2003) 236803, [cond-mat/0306045].
- [19] Brian P. Dolan, *JHEP* **0305** (2003) 018, [hep-th/0304037].
- [20] G. Meng, *J. Phys. A: Math. Gen.* **36** (2003) 9415, [cond-mat/0306351].
- [21] D. Karabali and V.P. Nair, *Nucl. Phys.* **B641** (2002) 533, [hep-th/0203264]; *Nucl. Phys.* **B679** (2004) 427, [hep-th/0307281]; *Nucl. Phys.* **B697** (2004) 513, [hep-th/0403111]; *J. Phys. A: Math. Gen.* **39** (2006) 12735, [hep-th/0606161].
- [22] V.P. Nair and S. Randjbar-Daemi, *Nucl. Phys.* **B679** (2004) 447, [hep-th/0309212].
- [23] A.P. Polychronakos, *Nucl. Phys.* **B711** (2005) 505, [hep-th/0411065]; *Nucl. Phys.* **B705** (2005) 457, [hep-th/0408194].
- [24] M. Daoud and A. Jellal, *Nucl. Phys.* **B764** (2007) 109, [hep-th/0605289]; *Inter. J. Geom. Meth. Mod. Phys* **4** (2007) 1187, [hep-th/0605290]; *Int. J. Mod. Phys.* **A23** (2008) 3129, [hep-th/0610157].
- [25] D. Karabali, *Nucl. Phys.* **B726** (2005) 407, [hep-th/0507027]; *Nucl. Phys.* **B750** (2006) 265, [hep-th/0605006].
- [26] B. Jurco, P. Schupp and J. Wess, *Nucl. Phys* **B584** (2000) 784, [hep-th/0005005].
- [27] B. Jurco, L. Möller, S. Schraml, P. Schupp and J. Wess, *Eur. Phys. J.* **C21** (2001) 383, [hep-th/0104153].
- [28] W. Behr and A. Sykora, *Nucl. Phys.* **B698** (2004) 473, [hep-th/0309145].
- [29] N. Jacobson, *Amer. J. Math.* **71** (1949) 149.
- [30] T.D. Palev, *Lie Algebraical Aspects of the Quantum Statistics. Unitary Quantization (A-quantization)*, [hep-th/9705032].
- [31] T.D. Palev and J. Van der Jeugt, *J. Math. Phys.* **43** (2002) 3850, [hep-th/0010107].
- [32] N.I. Stoilova and J. Van der Jeugt, *J. Math. Phys.* **46** (2005) 033501, [math-ph/0409002]; *J. Math. Phys.* **48** (2007) 043504, [math-ph/0611085].
- [33] M. Daoud, *J. Phys. A: Math. Gen.* **39** (2006) 889, [math-ph/0606050].
- [34] B. Sakita, *Phys. Lett.* **B387** (1996) 118, [hep-th/9607047].