

# BANACH SPACES OF UNIVERSAL DISPOSITION

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ABSTRACT. In this paper we present a method to obtain Banach spaces of universal and almost-universal disposition with respect to a given class  $\mathfrak{M}$  of normed spaces. The method produces, among other, the Gurariĭ space  $\mathcal{G}$  (the only separable Banach space of almost-universal disposition with respect to the class  $\mathfrak{F}$  of finite dimensional spaces), or the Kubis space  $\mathcal{K}$  (under CH, the only Banach space with the density character the continuum which is of universal disposition with respect to the class  $\mathfrak{S}$  of separable spaces). We moreover show that  $\mathcal{K}$  is not isomorphic to a subspace of any  $C(K)$ -space – which provides a partial answer to the injective space problem– and that –under CH– it is isomorphic to an ultrapower of the Gurariĭ space.

We study further properties of spaces of universal disposition: separable injectivity, partially automorphic character and uniqueness properties.

## 1. SPACES OF UNIVERSAL AND ALMOST-UNIVERSAL DISPOSITION

In [9] Gurariĭ introduces the notions of spaces of universal and almost-universal disposition for a given class  $\mathfrak{M}$  as follows.

**Definition 1.1.** Let  $\mathfrak{M}$  be a class of Banach spaces.

- (1) A Banach space  $U$  is said to be of almost universal disposition for the class  $\mathfrak{M}$  if, given  $A, B \in \mathfrak{M}$ , isometric embeddings  $u : A \rightarrow U$  and  $v : A \rightarrow B$ , and  $\varepsilon > 0$ , there is a  $(1 + \varepsilon)$ -isometric embedding  $u' : B \rightarrow U$  such that  $u = u'v$ .
- (2) A Banach space  $U$  is of universal disposition for the class  $\mathfrak{M}$  if, given  $A, B \in \mathfrak{M}$  and isometric embeddings  $u : A \rightarrow U$  and  $v : A \rightarrow B$ , there is an isometric embedding  $u' : B \rightarrow U$  such that  $u = u'v$ .

Gurariĭ shows that there exists a separable Banach space of almost-universal disposition for the class  $\mathfrak{F}$  of finite dimensional spaces [9, Theorem 2]. We recall now the main properties of Gurariĭ's creature. First, it is clear that two separable Banach spaces of almost-universal disposition for finite dimensional spaces are almost isometric —this is shown by an obvious back-and-forth argument in [9, Theorem 4]. A different and simpler description of Gurariĭ space(s) by means of triangular matrices was provided by Lazar and Lindenstrauss in [15, Theorem 5.6]. On the other hand, Pełczyński and Wojtaszczyk show in [20] that the family of separable Lindenstrauss spaces has a maximal member: there is a separable Lindenstrauss space  $\mathcal{PW}$  having the following property: for every separable Lindenstrauss space  $X$  and each  $\varepsilon > 0$ , there is an operator  $u : X \rightarrow \mathcal{PW}$  such that  $\|x\| \leq \|u(x)\| \leq (1 + \varepsilon)\|x\|$  and a contractive projection of  $\mathcal{PW}$  onto the range of  $u$ . One year later Wojtaszczyk [23] himself shows that  $\mathcal{PW}$  can be constructed as a space of almost universal disposition for finite dimensional spaces. Finally, Lusky shows in [18] that two separable spaces of almost-universal disposition for

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finite-dimensional Banach spaces are isometric. Therefore, there exists a unique separable space of almost-universal disposition for finite dimensional Banach spaces, that we will call the Gurarii space and denote by  $\mathcal{G}$ .

Gurarii conjectured the existence of spaces of universal disposition for the classes  $\mathfrak{F}$  of finite dimensional spaces and  $\mathfrak{S}$  of separable spaces: see the footnote to Theorem 5 in [9]. We will present a method able to effectively generate such examples, as well as other spaces of universal or almost-universal disposition, such as the Gurarii space [9] or the Fraïssé limit constructed by Kubis [14].

## 2. BACKGROUND

Our notation is fairly standard, as in [17]. A Banach space  $X$  is said to be an  $\mathcal{L}_{\infty, \lambda}$ -space with  $\lambda \geq 1$ ) if every finite dimensional subspace  $F$  of  $X$  is contained in another finite dimensional subspace of  $X$  whose Banach-Mazur distance to the corresponding  $\ell_{\infty}^n$  is at most  $\lambda$ . A space  $X$  is said to be a  $\mathcal{L}_{\infty}$ -space if it is a  $\mathcal{L}_{\infty, \lambda}$ -space for some  $\lambda \geq 1$ ; we will say that it is a Lindenstrauss space if it is a  $\mathcal{L}_{\infty, 1+\varepsilon}$ -space for all  $\varepsilon > 0$ . Throughout the paper, ZFC denotes the usual setting of set theory with the Axiom of Choice, while CH denotes the continuum hypothesis ( $\mathfrak{c} = \aleph_1$ ).

**2.1. The push-out construction.** The push-out construction appears naturally when one considers a couple of operators defined on the same space, in particular in any extension problem. Let us explain why. Given operators  $\alpha : Y \rightarrow A$  and  $\beta : Y \rightarrow B$ , the associated push-out diagram is

$$(1) \quad \begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \beta' \\ B & \xrightarrow{\alpha'} & \text{PO} \end{array}$$

Here, the push-out space  $\text{PO} = \text{PO}(\alpha, \beta)$  is quotient of the direct sum  $A \oplus_1 B$ , the product space endowed with the sum norm, by the closure of the subspace  $\Delta = \{(\alpha y, -\beta y) : y \in Y\}$ . The map  $\alpha'$  is given by the inclusion of  $B$  into  $A \oplus_1 B$  followed by the natural quotient map  $A \oplus_1 B \rightarrow (A \oplus_1 B)/\overline{\Delta}$ , so that  $\alpha'(b) = (0, b) + \overline{\Delta}$  and, analogously,  $\beta'(a) = (a, 0) + \overline{\Delta}$ .

The diagram (1) is commutative:  $\beta'\alpha = \alpha'\beta$ . Moreover, it is ‘minimal’ in the sense of having the following universal property: if  $\beta'' : A \rightarrow C$  and  $\alpha'' : B \rightarrow C$  are operators such that  $\beta''\alpha = \alpha''\beta$ , then there is a unique operator  $\gamma : \text{PO} \rightarrow C$  such that  $\alpha'' = \gamma\alpha'$  and  $\beta'' = \gamma\beta'$ . Clearly,  $\gamma((a, b) + \overline{\Delta}) = \beta''(a) + \alpha''(b)$  and one has  $\|\gamma\| \leq \max\{\|\alpha''\|, \|\beta''\|\}$ . Regarding the behaviour of the maps in diagram (1), apart from the obvious fact that both  $\alpha'$  and  $\beta'$  are contractive, we have:

### Lemma 2.1.

- (a) *If  $\alpha$  is an isomorphic embedding, then  $\Delta$  is closed.*
- (b) *If  $\alpha$  is an isometric embedding and  $\|\beta\| \leq 1$  then  $\alpha'$  is an isometric embedding.*
- (c) *If  $\alpha$  is an isomorphic embedding then  $\alpha'$  is an isomorphic embedding.*
- (d) *If  $\|\beta\| \leq 1$  and  $\alpha$  is an isomorphism then  $\alpha'$  is an isomorphism and*

$$\|(\alpha')^{-1}\| \leq \max\{1, \|\alpha\|\}.$$

*Proof.* (a) is clear. (b) If  $\|\beta\| \leq 1$ ,

$$\|\alpha'(b)\| = \|(0, b) + \Delta\| = \inf_{y \in Y} \|\alpha y\| + \|b - \beta y\| \geq \inf_y \|\beta y\| + \|b - \beta y\| \geq \|b\|,$$

as required. (c) is clear after (b). (d) To prove the assertion about  $(\alpha')^{-1}$ , notice that for all  $a \in A$  and  $b \in B$  one has  $(a, b) + \delta = (0, b + \beta y) + \delta$  for  $y \in Y$  such that  $\alpha y = a$ . Therefore, for all  $y' \in Y$

one has

$$\begin{aligned} \|b + \beta y\| &\leq \|b + \beta y + \beta y'\| + \|\beta y'\| \\ &\leq \|b + \beta y + \beta y'\| + \|y'\| \\ &\leq \|b + \beta y + \beta y'\| + \|\alpha^{-1}\| \|\alpha y'\| \end{aligned}$$

from where the assertion follows.  $\square$

A Banach space  $E$  is said to be (separably) injective if for every (separable) Banach space  $X$  and each subspace  $Y \subset X$ , every operator  $t : Y \rightarrow E$  extends to an operator  $T : X \rightarrow E$ . If some extension  $T$  exists with  $\|T\| \leq \lambda \|t\|$  we say that  $E$  is  $\lambda$ -separably injective. Following [2], a Banach space  $E$  is said to be universally separably injective if for every Banach space  $X$  and each separable subspace  $Y \subset X$ , every operator  $t : Y \rightarrow E$  extends to an operator  $T : Y \rightarrow X$ . If some extension  $T$  exists with  $\|T\| \leq \lambda \|t\|$  we say that  $E$  is universally  $\lambda$ -separably injective.

### 3. THE BASIC CONSTRUCTION

Let us consider an isometric embedding  $u : A \rightarrow B$  and an operator  $t : A \rightarrow E$ . We want to extend  $t$  through  $u$ , probably at the cost of replacing  $E$  by a larger space. The push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ t \downarrow & & \downarrow t' \\ E & \xrightarrow{u'} & \text{PO} \end{array}$$

does exactly what we ask:  $t'u = u't$ . It is important to realize that  $u'$  is again an isometric embedding and that  $t'$  is a contraction (resp. an isometric embedding) if  $t$  is; see Lemma 2.1. What we need is to be able to do the same with a previously established family of embeddings. The input data for the construction are:

- A Banach space  $E$ .
- A family  $\mathfrak{J}$  of isometric embeddings between certain Banach spaces.
- A family  $\mathfrak{L}$  of norm one operators from certain Banach spaces to  $E$ .

Each member of  $\mathfrak{J}$  is, by definition, an isometric embedding  $u : A \rightarrow B$ , where  $A$  and  $B$  are Banach spaces. Then  $A = \text{dom } u$  is the domain of  $u$  and  $B = \text{cod } u$  is the codomain. Let us remark that  $\text{cod } u$  is usually larger than the image of  $u$ . This implicitly yields three families of Banach spaces:

- $\text{dom } \mathfrak{J} = \{\text{dom } u : u \in \mathfrak{J}\}$ ,
- $\text{cod } \mathfrak{J} = \{\text{cod } u : u \in \mathfrak{J}\}$ ,
- $\text{dom } \mathfrak{L} = \{\text{dom } t : t \in \mathfrak{L}\}$ .

To avoid complications we will assume that  $\mathfrak{J}$  and  $\mathfrak{L}$  are sets and also that  $\text{dom } \mathfrak{J} = \text{dom } \mathfrak{L}$ . Notice that the only element of  $\text{cod } \mathfrak{L}$  is  $E$ .

Set  $\Gamma = \{(u, t) \in \mathfrak{J} \times \mathfrak{L} : \text{dom } u = \text{dom } t\}$  and consider the Banach spaces of summable families  $\ell_1(\Gamma, \text{dom } u)$  and  $\ell_1(\Gamma, \text{cod } u)$ . We have an obvious isometric embedding

$$\oplus \mathfrak{J} : \ell_1(\Gamma, \text{dom } u) \longrightarrow \ell_1(\Gamma, \text{cod } u)$$

defined by  $(x_{(u,t)})_{(u,t) \in \Gamma} \mapsto (u(x_{(u,t)}))_{(u,t) \in \Gamma}$ ; and a contraction

$$\Sigma \mathfrak{L} : \ell_1(\Gamma, \text{dom } u) \longrightarrow E,$$

given by  $(x_{(u,t)})_{(u,t) \in \Gamma} \mapsto \sum_{(u,t) \in \Gamma} t(x_{(u,t)})$ . (Observe that the notation is slightly imprecise since both  $\oplus \mathfrak{J}$  and  $\Sigma \mathfrak{L}$  depend on  $\Gamma$ ). We can form their push-out diagram

$$\begin{array}{ccc} \ell_1(\Gamma, \text{dom } u) & \xrightarrow{\oplus \mathfrak{J}} & \ell_1(\Gamma, \text{cod } u) \\ \Sigma \mathfrak{L} \downarrow & & \downarrow \\ E & \longrightarrow & \text{PO}. \end{array}$$

In this way we obtain an isometric enlargement  $\iota : E \rightarrow \text{PO}$  such that every operator  $t : A \rightarrow E$  in  $\mathfrak{L}$  can be extended to an operator  $t' : B \rightarrow \text{PO}$  through any embedding  $u : A \rightarrow B$  in  $\mathfrak{J}$  provided  $\text{dom } u = \text{dom } t = A$ . In the step  $\alpha$  we leave the family  $\mathfrak{J}$  fixed, replace  $E$  by  $\text{PO}$  and  $\mathfrak{L}$  by another family  $\mathfrak{L}_\alpha$  and proceed again.

In this way, one can iterate this construction until any countable or uncountable ordinal. Moreover, a careful choice of the families  $\mathfrak{L}_\alpha$  in the successive steps allows one to produce spaces of universal disposition, as we will see now.

We pass to present some specific constructions in detail. We fix a Banach space  $X$ . We would take  $\mathfrak{J}$  as the family of all isometric embeddings between separable Banach spaces. Avoiding the details required to fix the inconvenient that the class of separable Banach spaces is not a set, let  $\mathfrak{S}$  be the family of all separable Banach spaces up to isometries. The initial step of the construction is performed with the space  $X$ , the set  $\mathfrak{J}$  of all isometric embeddings acting between the elements of  $\mathfrak{S}$  and the set  $\mathfrak{L}$  of all norm one operators  $t : S \rightarrow X$ , where  $S \in \mathfrak{S}$ .

We are going to define Banach spaces  $\mathfrak{S}^\alpha = \mathfrak{S}^\alpha(X)$  for all ordinals  $\alpha$  starting with  $\mathfrak{S}^0 = X$  and in such a way that, for  $\alpha < \beta$ , there is an isometric embedding  $\iota_{(\alpha,\beta)} : \mathfrak{S}^\alpha \rightarrow \mathfrak{S}^\beta$ . The embeddings must satisfy the obvious compatibility condition that  $\iota_{(\beta,\gamma)} \iota_{(\alpha,\beta)} = \iota_{(\alpha,\gamma)}$  if  $\alpha < \beta < \gamma$ . In particular, for each ordinal  $\alpha$ , we have an embedding  $\iota_{(0,\alpha)} : X \rightarrow \mathfrak{S}^\alpha$ .

We use transfinite induction as follows. Suppose  $\mathfrak{S}^\beta$  and the corresponding embeddings defined for each  $\beta < \alpha$ . If  $\alpha = \beta + 1$  is a successor ordinal, we consider the following data:

- The Banach space  $\mathfrak{S}^\beta$ ,
- the set  $\mathfrak{J}$  of all isometric embeddings acting between the elements of  $\mathfrak{S}$ , and
- the set  $\mathfrak{L}_\beta$  of all norm one operators  $t : S \rightarrow \mathfrak{S}^\beta$ , where  $S \in \mathfrak{S}$ .

Then we set  $\Gamma_\beta = \{(u, t) \in \mathfrak{J} \times \mathfrak{L}_\beta : \text{dom } u = \text{dom } t\}$  and we form the push-out diagram

$$(2) \quad \begin{array}{ccc} \ell_1(\Gamma_\beta, \text{dom } u) & \xrightarrow{\oplus \mathfrak{J}_\beta} & \ell_1(\Gamma_\beta, \text{cod } u) \\ \Sigma \mathfrak{L}_\beta \downarrow & & \downarrow \\ \mathfrak{S}^\beta & \longrightarrow & \text{PO} \end{array}$$

thus obtaining  $\mathfrak{S}^\alpha = \mathfrak{S}^{\beta+1} = \text{PO}$ . The embedding  $\iota_{(\beta,\alpha)}$  is the lower arrow in the above diagram and the other embeddings are given by composition with  $\iota_{(\beta,\alpha)}$ . If  $\alpha$  is a limit ordinal we take  $\mathfrak{S}^\alpha$  as the direct limit  $\lim_{\beta < \alpha} \mathfrak{S}^\beta$ , with the obvious embeddings. Two variations of this construction will be considered:

- Replace  $\mathfrak{S}$  by the smaller family  $\mathfrak{F}$  generated by the finite dimensional spaces in  $\mathfrak{S}$ . We will call the final space  $\mathfrak{F}^\alpha(X)$ .
- We leave the initial family of Banach spaces  $\mathfrak{S}$ , but replace  $\mathfrak{J}$  by the set  $\mathfrak{J}^\infty$  of all isometric embeddings of the spaces of  $\mathfrak{S}$  into  $\ell_\infty$ , so that  $\text{cod } u = \ell_\infty$  for every  $u \in \mathfrak{J}^\infty$ . The choice of  $\mathfrak{L}$  is the same as before: all contractive operators from the spaces in  $\mathfrak{S}$  to  $X$ . Now, we proceed as before to construct a family of Banach spaces  $\mathfrak{U}^\alpha = \mathfrak{U}^\alpha(X)$  together with the corresponding compatible linking embeddings. The passage from  $\mathfrak{U}^\alpha$  to  $\mathfrak{U}^{\alpha+1}$  is as follows. We set  $\mathfrak{L}_\alpha$  as

the set of all norm one operators from the spaces of  $\mathfrak{S}$  to  $\mathfrak{U}^\alpha$  and  $\Gamma_\alpha = \{(u, t) \in \mathfrak{J}^\infty \times \mathfrak{L}_\alpha : \text{dom } u = \text{dom } t\}$  and we form the push-out diagram

$$(3) \quad \begin{array}{ccc} \ell_1(\Gamma_\alpha, \text{dom } u) & \xrightarrow{\oplus \mathfrak{J}^\infty} & \ell_1(\Gamma_\alpha, \ell_\infty) \\ \Sigma \mathfrak{L}_\alpha \downarrow & & \downarrow \\ \mathfrak{U}^\alpha & \longrightarrow & \text{PO} \end{array}$$

thus obtaining  $\mathfrak{U}^{\alpha+1} = \text{PO}$ . The embedding  $\iota_{(\alpha, \alpha+1)}$  is the lower arrow in the above diagram and the other embeddings are given by composition with  $\iota_{(\alpha, \alpha+1)}$ .

**Proposition 3.1.** *Let  $X$  be a Banach space.*

- (a) *The spaces  $\mathfrak{S}^{\omega_1}(X)$  and  $\mathfrak{U}^{\omega_1}(X)$  are of universal disposition for separable Banach spaces.*
- (b) *The space  $\mathfrak{F}^{\omega_1}(X)$  is of universal disposition for finite-dimensional Banach spaces.*

*Proof.* We write the proof for  $\mathfrak{S}^{\omega_1} = \mathfrak{S}^{\omega_1}(X)$ . The case  $\mathfrak{U}^{\omega_1}(X)$  is analogous and we leave it to the reader.

We must show that is  $v : A \rightarrow B$  and  $\ell : A \rightarrow \mathfrak{S}^{\omega_1}$  are isometric embeddings and  $B$  is separable, then there is an isometric embedding  $L : B \rightarrow \mathfrak{S}^{\omega_1}$  such that  $Lv = \ell$ . We may and do assume  $A, B \in \mathfrak{S}$  so that  $v$  is in  $\mathfrak{J}$ . On the other hand there is  $\alpha < \omega_1$  such that  $\ell(A) \subset \mathfrak{S}^\alpha$  and we may consider that  $\ell$  is one of the operators in  $\mathfrak{L}_\alpha$ . Therefore  $\ell$  has an extension  $\ell'$  making the following square commutative:

$$\begin{array}{ccc} A & \xrightarrow{v} & B \\ \ell \downarrow & & \downarrow \ell' \\ \mathfrak{S}^\alpha & \xrightarrow{\iota_{(\alpha, \alpha+1)}} & \mathfrak{S}^{\alpha+1}. \end{array}$$

Actually  $\ell'$  is the composition of the inclusion  $J_{(v, \ell)}$  of  $B = \text{cod } v$  into the  $(v, \ell)$ -th coordinate of  $\ell_1(\Gamma_\alpha, \text{cod } u)$  with the right descending arrow in the diagram

$$(4) \quad \begin{array}{ccc} \ell_1(\Gamma_\alpha, \text{dom } u) & \xrightarrow{\oplus \mathfrak{J}} & \ell_1(\Gamma_\alpha, \text{cod } u) \\ \Sigma \mathfrak{L}_\alpha \downarrow & & \downarrow \\ \mathfrak{S}^\alpha & \longrightarrow & \text{PO} = \mathfrak{S}^{\alpha+1}. \end{array}$$

We know that  $\ell'$  is a contraction and we must prove it is isometric. We have

$$\text{PO} = (\mathfrak{S}^\alpha \oplus_1 \ell_1(\Gamma_\alpha, \text{cod } u)) / \Delta \quad \text{with} \quad \Delta = \left\{ \left( \sum_{(u, t) \in \Gamma_\alpha} tx_{(u, t)}, - \sum_{(u, t) \in \Gamma_\alpha} ux_{(u, t)} \right) \right\}.$$

Thus, for  $b \in B$  we have  $\ell'(b) = (0, J_{(v, \ell)}b) + \Delta$  and

$$\|\ell'(b)\|_{\text{PO}} = \text{dist}((0, J_{(v, \ell)}b), \Delta) = \inf_{a \in A} \{\|\ell(a)\|_{\mathfrak{S}^\alpha} + \|b - v(a)\|_B\} = \|b\|_B$$

since both  $\ell$  and  $v$  preserve the norm.

The proof of (b) is left to the reader. □

This is the space Gurariĭ conjectured. We will later show that –under CH– such space is unique and coincides with the Fraïssé limit in the category of separable Banach spaces and into isometries constructed by Kubis [14]; and also with an ultrapower of Gurariĭ space. For our purposes it is enough to stop the constructions at  $\omega_1$ , however it is not hard to believe that a careful choice of the cardinal  $\alpha$  can produce spaces  $\mathfrak{S}^\alpha$  with special properties. Observe that, say,  $\mathfrak{S}^{\omega_1+1}(X)$  is not of universal disposition for  $\mathfrak{S}$ .

## 4. PROPERTIES OF SPACES OF UNIVERSAL DISPOSITION

Gurarii shows in [9] that a space of universal disposition for all finite-dimensional spaces cannot be separable since no separable Banach space can be of universal disposition for the couple  $\{\mathbb{R}, \mathbb{R} \times \mathbb{R}\}$ : indeed, a Banach space of universal disposition for  $\mathbb{R}$  has also been called transitive –and the still open Mazur’s rotation problem asks whether a separable transitive Banach space must be Hilbert–; and it is well known that the norm of a transitive space must be differentiable at every point. This prevents the space from being of universal disposition for  $\mathbb{R}^2$ . On the other hand it is clear that, if the starting space  $X$  is separable, then the Banach spaces appearing in Propositions 3.1 have density character  $\mathfrak{c}$  since each of them is the union of an  $\omega_1$  sequence formed by Banach spaces of density  $\mathfrak{c}$ .

**Proposition 4.1.** *A Banach space of universal disposition for separable spaces must have density at least  $\mathfrak{c}$  and contains an isometric copy of each Banach space of density  $\aleph_1$  or less.*

*Proof.* The first part is a juxtaposition of forthcoming Lemma 4.2 –which asserts that a Banach space of universal disposition for separable spaces must be 1-separably injective– and the result in [2] asserting that a  $\lambda$ -separably injective space with  $\lambda < 2$  is either finite-dimensional or has density character at least  $\mathfrak{c}$ . To prove the second part, assume that  $U$  is a space of universal disposition for separable Banach spaces and let  $X$  have density  $\aleph_1$ . Write  $X$  as an  $\omega_1$ -sequence of separable Banach spaces, beginning with  $X_0 = 0$  and use the argument given in the proof of (i)  $\Rightarrow$  (v) in [16, p. 221], using norm preserving operators in every step.  $\square$

**Problem 1.** Must a space of universal disposition for finite-dimensional spaces contain an isometric copy of each Banach space of density  $\aleph_1$  or less? Our guess is no.

**Problem 2.** Does there exist consistently a Banach space of universal disposition for finite dimensional spaces having density character strictly smaller than  $\mathfrak{c}$ ?

**Lemma 4.2.** *Let  $E$  be a Banach space. Suppose there is a constant  $\lambda$  such that for each (separable) Banach space  $X$  and every pair of into isometries  $u : Y \rightarrow X$  and  $v : Y \rightarrow E$  there exists an operator  $V : X \rightarrow E$  such that  $Vu = v$  with  $\|V\| \leq \lambda$ . Then  $E$  is  $\lambda$ -(separably) injective*

*Proof.* Let  $t : Y \rightarrow E$  have norm one. Denote by  $Y'$  the closure of the range of  $t$  and make the push-out of  $(u, t)$ :

$$(5) \quad \begin{array}{ccc} Y & \xrightarrow{u} & X \\ t \downarrow & & \downarrow t' \\ Y' & \xrightarrow{u'} & \text{PO} . \end{array}$$

By Lemma 2.1,  $u'$  is an into isometry, and the hypothesis yields and operator  $t'' : \text{PO} \rightarrow E$  such that  $t''u'$  is the inclusion of  $Y'$  into  $E$ , with  $\|t''\| \leq \lambda$ . Taking  $T = t''t'$  we end the proof.  $\square$

**Proposition 4.3.** *Let  $E$  be a space of universal disposition for separable spaces.*

- (a) *Given a separable Banach space  $X$  and a subspace  $Y \subset X$ , every isomorphic embedding  $t : Y \rightarrow E$  extends to an isomorphic embedding  $T : X \rightarrow E$  with  $\|T\| = \|t\|$  and  $\|T^{-1}\| = \|t^{-1}\|$ .*
- (b) *Consequently, if  $\text{dens } E \leq \aleph_1$ , then  $E$  is separably automorphic; namely, any isomorphism between two subspaces of  $E$  can be extended to an automorphism of  $E$ .*

*Proof.* (a) Let  $u$  denote the inclusion of  $Y$  into  $X$  and assume, without loss of generality, that  $\|t\| = 1$ . We follow the same notation as in Lemma 4.2. Looking at Diagram 5 we have  $\|t'\| = 1$  and  $u'$  is isometric, so there is an isometric embedding  $t'' : \text{PO} \rightarrow E$  such that  $t''u'$  is the inclusion of  $Y' = \text{ran } t$  into  $E$ . Now  $T = t''t'$  is the extension of  $t$  we wanted. Clearly,  $\|T\| = \|t\| = 1$ . On the other hand, by [7, lemma 1.3.b],  $\|(t')^{-1}\| \leq \max\{1, \|t^{-1}\|\}$  hence  $\|T^{-1}\| = \|t^{-1}\|$ .

(b) For the second part, it suffices to show that if  $Y$  is a separable subspace of  $E$ , every isomorphic embedding  $\varphi_0 : Y \rightarrow E$  extends to an automorphism of  $E$ . This is proved through the obvious back-and-forth argument: write  $E = \bigcup_{\alpha < \omega_1} E_\alpha$  as an  $\omega_1$ -sequence of separable subspaces starting with  $E_0 = Y$ . Consider the embedding  $\varphi_0 : E_0 \rightarrow E$ . Let  $\psi_1 : \varphi(E_0) + E_1 \rightarrow E$  be an extension of  $\varphi_0^{-1} : \varphi(E_0) \rightarrow E$ , with  $\|\psi_1\| = \|\varphi_0^{-1}\|$  and  $\|\psi_1^{-1}\| = \|\varphi_0\|$ . Notice that  $\text{ran } \psi_1 = E_0 + \psi_1(E_1)$ . Let  $\varphi_2$  be the extension of  $\psi_1^{-1}$  to  $E_0 + \psi_1(E_1) + E_2$  provided by Part (a) and so on. Proceeding by transfinite induction one gets a couple of endomorphisms  $\varphi$  and  $\psi$  such that  $\psi\varphi = \varphi\psi = \mathbf{1}_E$ , with  $\|\varphi\| = \|\varphi_0\|$  and  $\|\psi\| = \|\varphi_0^{-1}\|$  and  $\varphi = \varphi_0$  on  $Y$ .  $\square$

Our next result shows that under CH there is no dependence on the initial separable space  $X$  in the constructions appearing in Proposition 3.1

**Proposition 4.4.** *Under CH there is a unique space of universal disposition for separable spaces with density character  $\aleph_1$ , up to isometries.*

*Proof.* Let  $X$  and  $Y$  be spaces of universal disposition for separable spaces and with density character  $\aleph_1$ . It is obvious that they contain isometric copies of all separable spaces. Let us write  $X = \bigcup_{\alpha < \omega_1} X_\alpha$  and  $Y = \bigcup_{\beta < \omega_1} Y_\beta$  as increasing  $\omega_1$ -sequences of separable subspaces. Pick  $\beta_1$  such that there is an isometric embedding  $\varphi_0 : X_0 \rightarrow Y_{\beta_1}$ . Let  $\psi_1 : Y_{\beta_1} \rightarrow X$  be an isometric extension of  $\varphi_0^{-1}$ . As  $\psi_1$  has separable range there is  $\alpha_2 < \omega_1$  such that  $\text{ran } \psi_1 \subset X_{\alpha_2}$ . Let  $\varphi_2 : X_{\alpha_2} \rightarrow Y$  be an isometric extension of  $\psi_1^{-1}$ . A transfinite iteration of the process produces an isometry  $X \rightarrow Y$ .  $\square$

Let us show that there are spaces of universal disposition for all finite-dimensional Banach spaces which are not of universal disposition for all separable Banach spaces.

**Lemma 4.5.** *A  $c_0$ -valued operator defined on a finite-dimensional Banach space admits a compact extension with the same norm to any superspace*

*Proof.* Let  $F \subset X$  be a finite dimensional subspace of a Banach space  $X$ , and let  $\tau : F \rightarrow c_0$  be a norm one operator. Assume that  $\tau = (\tau_n)$  comes defined by a pointwise null sequence of functionals. Since  $F$  is finite dimensional, the sequence  $(\tau_n)$  is actually norm null. Thus, any sequence of Hahn-Banach extensions will also be norm null, and the operator they define is a compact extension of  $T$ .  $\square$

**Proposition 4.6.** *The space  $\mathfrak{F}^{\omega_1}(c_0)$  is of universal disposition for finite dimensional spaces and not of universal disposition for separable spaces.*

*Proof.* It follows from Lemma 4.5 that the embedding  $X \rightarrow \mathfrak{F}^{\omega_1}(X)$  has the property that every operator  $X \rightarrow c_0$  can be extended to  $\mathfrak{F}^{\omega_1}(X)$ . Therefore  $\mathfrak{F}^{\omega_1}(c_0)$  contains  $c_0$  complemented, and thus it cannot be 1-separably injective.  $\square$

This suggests that quite plausibly there is –even under CH– a continuum of mutually non-isomorphic spaces of universal disposition for finite-dimensional spaces. Let us show that such is the case –outside CH, of course– for separable spaces.

**Proposition 4.7.** *Assume that no Banach space of density character  $\mathfrak{c}$  is universal for all Banach spaces with density character  $\mathfrak{c}$ . Then there is at least a continuum of non-isomorphic spaces of universal disposition for  $\mathfrak{S}$  with density  $\mathfrak{c}$ .*

*Proof.* We proceed by transfinite induction. To make the induction start, form the space  $\mathfrak{S}(1) = \mathfrak{S}^{\omega_1}(\mathbb{R})$ . Take, by hypothesis, a Banach space  $X(1)$  with density character  $\mathfrak{c}$  not contained in  $\mathfrak{S}(\mathbb{R})$  and form then  $\mathfrak{S}(2) = \mathfrak{S}^{\omega_1}(X(1) \oplus \mathfrak{S}(1))$ . Take a new Banach space  $X(2)$  with density character  $\mathfrak{c}$  not contained  $\mathfrak{S}(2)$  and continue in this way.

Let  $\beta < \mathfrak{c}$ , and assume that for each  $\alpha < \beta$  a Banach space  $\mathfrak{S}(\alpha)$  of universal disposition for  $\mathfrak{S}$  has already been constructed verifying:

- (1) For all  $\alpha$ , the space  $\mathfrak{S}(\alpha)$  has density character  $\mathfrak{c}$ .
- (2) For  $\gamma \leq \alpha$  the space  $\mathfrak{S}(\gamma)$  is isometric to a subspace of  $\mathfrak{S}(\alpha)$ .
- (3) For  $\alpha \neq \gamma$  the spaces  $\mathfrak{S}(\alpha)$  and  $\mathfrak{S}(\gamma)$  are not isomorphic.

If  $\beta = \beta' + 1$  is not a limit ordinal, then get a Banach space  $X(\beta')$  with density character  $\mathfrak{c}$  not contained in  $\mathfrak{S}(\beta')$  and form  $\mathfrak{S}(\beta) = \mathfrak{S}^{\omega_1}(\mathfrak{S}(\beta') \oplus X(\beta'))$ .

If  $\beta$  is a limit ordinal, then  $\mathfrak{S}(\beta) = \mathfrak{S}^{\omega_1}(\overline{\bigcup_{\alpha < \beta} \mathfrak{S}(\alpha)})$ .

All this yields a continuum  $\mathfrak{S}(\alpha)$ ,  $\alpha < \mathfrak{c}$ , of mutually non-isomorphic spaces of universal disposition for separable spaces.  $\square$

*Remark 4.8.* The hypothesis is consistent by a result of Brech and Koszmider [6]. The paper [1] contains further results on the existence of spaces of universal disposition for  $\mathfrak{S}$  under different cardinality assumptions.

## 5. GURARIĬ'S SPACE AND ITS ULTRAPOWERS

We can construct the Gurariĭ space as follows. We fix a countable system of isometric embeddings  $\mathfrak{J}_0$  having the following density property: given an isometric embedding  $w : A \rightarrow B$  between finite dimensional spaces, and  $\varepsilon > 0$ , there is  $u \in \mathfrak{J}_0$ , and surjective  $(1 + \varepsilon)$ -isometries  $\alpha : A \rightarrow \text{dom } u$  and  $\beta : B \rightarrow \text{cod } u$  making the square

$$(6) \quad \begin{array}{ccc} A & \xrightarrow{w} & B \\ \alpha \downarrow & & \downarrow \beta \\ \text{dom } u & \xrightarrow{u} & \text{cod } u \end{array}$$

commutative. Set  $\mathfrak{F}_0 = \text{dom } \mathfrak{J}_0$ .

Let now  $X$  be a separable Banach space. We define an increasing sequence of Banach spaces  $G^n = G^n(X)$  as follows. We start with  $G^0 = X$ . Assuming  $G^n$  has been defined we get  $G^{n+1}$  from the basic construction explained in Section 3 just taking as  $\mathfrak{L}_n$  a countable set of  $G^n$ -valued contractions with domain in  $\mathfrak{F}_0$  such that, for every  $\varepsilon > 0$ , and every  $(1 + \varepsilon)$ -isometric embedding  $s : F \rightarrow G^n$ , with  $F \in \mathfrak{F}_0$ , there is  $t \in \mathfrak{L}_n$  such that  $\|s - t\| < \varepsilon$ . We consider the index set  $\Gamma_n = \{(u, t) \in \mathfrak{J}_0 \times \mathfrak{L}_n : \text{dom } u = \text{dom } t\}$  and the push-out diagram

$$(7) \quad \begin{array}{ccc} \ell_1(\Gamma_n, \text{dom } u) & \xrightarrow{\oplus \mathfrak{J}_0} & \ell_1(\Gamma_n, \text{cod } u) \\ \Sigma \mathfrak{L}_n \downarrow & & \downarrow \\ G^n & \longrightarrow & \text{PO} \end{array}$$

Then we set  $G^{n+1} = \text{PO}$ . The linking map  $G^n \rightarrow G^{n+1}$  is given by the lower arrow in the push-out diagram.

**Proposition 5.1.** *Let  $X$  be a separable Banach space. The space*

$$G^\omega(X) = \lim_n G^n = \overline{\bigcup_n G^n}$$

*is a separable Banach space of almost-universal disposition for finite dimensional spaces.*

*Proof.* Let  $w : A \rightarrow B$  and  $s : A \rightarrow G^\omega$  be isometric embeddings, with  $B$  a finite dimensional space and fix  $\varepsilon > 0$ . Choose  $u \in \mathfrak{J}_0$ , as in (6). Clearly, for  $m$  large enough there is a contractive  $(1 + \varepsilon)$ -isometry  $t : \text{dom } u \rightarrow G^m$  satisfying  $\|s - t\alpha\| < \varepsilon$ . Let  $t' : \text{cod } u \rightarrow G^{m+1}$  be the extension provided by Diagram 7, so that  $t'$  is a contractive  $(1 + \varepsilon)$ -isometry such that  $t'u = t$ . Therefore  $t'\beta$  is a contractive  $(1 + \varepsilon)^2$ -isometry satisfying  $\|s - t'\beta w\| \leq \varepsilon$ . The following perturbation result ends the proof.  $\square$

**Lemma 5.2.** *A Banach space  $U$  is of almost universal disposition for finite-dimensional spaces if and only if, given isometric embeddings  $u : A \rightarrow U$  and  $v : A \rightarrow B$  with  $B$  finite-dimensional, and  $\varepsilon > 0$ , there is an  $(1 + \varepsilon)$ -isometric embedding  $u' : B \rightarrow U$  such that  $\|u - u'v\| \leq \varepsilon$ .*

Since Gurariĭ space is unique, for all separable spaces  $X$ , one has  $G^\omega(X) = \mathcal{G}$ . Moreover, the embedding of  $X$  into  $G^\omega(X)$  enjoys the following universal property:

**Proposition 5.3.** *Every norm one operator from  $X$  into a Lindenstrauss space admits, for every  $\varepsilon > 0$ , an extension to  $G^\omega(X)$  of norm at most  $1 + \varepsilon$ .*

*Proof.* Given  $\varepsilon > 0$  we fix a sequence  $(\varepsilon_n)$  such that  $\prod(1 + \varepsilon_n) \leq 1 + \varepsilon$ . Now, let  $\mathcal{L}$  be a Lindenstrauss space and  $\tau : X \rightarrow \mathcal{L}$  be a norm one operator. Look at the diagram

$$\begin{array}{ccc} \ell_1(\Gamma_0, \text{dom } u) & \xrightarrow{\oplus \mathcal{J}_0} & \ell_1(\Gamma_0, \text{cod } u) \\ \Sigma \mathcal{L}_0 \downarrow & & \downarrow \\ X & \longrightarrow & \text{PO} = G^1 \\ \tau \downarrow & & \\ \mathcal{L} & & \end{array}$$

and consider the composition  $\tau \circ \Sigma \mathcal{L}_0$ . Since  $\mathcal{L}$  is a Lindenstrauss space, for each fixed  $(u, t) \in \Gamma_0$ , the restriction of  $\tau \circ \Sigma \mathcal{L}_0$  to the corresponding ‘coordinate’ maps  $\text{dom } u = \text{dom } t$  into a finite-dimensional subspace of  $\mathcal{L}$  and so it is contained in a  $(1 + \varepsilon_1)$ -isomorph of some finite-dimensional  $\ell_\infty^n$ . Therefore, it can be extended to  $\text{cod } u$  through  $u$  with norm at most  $(1 + \varepsilon_1)$ . The  $\ell_1$ -sum of all these extensions yields thus an extension  $T : \ell_1(\Gamma_0, \text{cod } u) \rightarrow \mathcal{L}$  of  $\tau \circ \Sigma \mathcal{L}_0$  with norm at most  $(1 + \varepsilon_1)$ . The push-out property of  $\text{PO} = G^1$  yields therefore an operator  $\tau_1 : G^1 \rightarrow \mathcal{L}$  that extends  $\tau$  with norm at most  $(1 + \varepsilon_1)$ . Iterating the process  $\omega$  times, working with  $(1 + \varepsilon_n)$  at step  $n$ , one gets an extension  $\tau_\omega : G^\omega \rightarrow \mathcal{L}$  of  $\tau$  with norm at most  $\prod(1 + \varepsilon_n) \leq 1 + \varepsilon$ .  $\square$

Therefore every separable Banach space is isometric to a subspace of  $\mathcal{G}$ . Taking as  $X$  a separable Lindenstrauss space, the universal property of the embedding yields:

**Corollary 5.4.** *Every separable Lindenstrauss space is isometric to a  $(1 + \varepsilon)$ -complemented subspace of  $\mathcal{G}$ . Hence  $\mathcal{G}$  is not isomorphic to a complemented subspace of any  $C(K)$ -space (or, in general, any  $\mathcal{M}$ -space).*

Recall that an  $\mathcal{M}$ -space is a Banach lattice where  $\|x + y\| = \max\{\|x\|, \|y\|\}$  provided  $x$  and  $y$  are disjoint, that is,  $|x| \wedge |y| = 0$ . Each (abstract)  $\mathcal{M}$ -space is representable as a (concrete) sublattice in some  $C(K)$ . The second part follows from the fact, proved in [5], that there exist separable Lindenstrauss spaces that are not complemented subspaces of any  $\mathcal{M}$ -space. Proposition 8 in Lusky paper [19] shows that the above Corollary is true even with  $\varepsilon = 0$ .

Everyone acquainted with ultraproducts will realize the obvious fact that ultrapowers of Gurariĭ space  $\mathcal{G}$  are of universal disposition for finite dimensional spaces. Less obvious is that they also are of universal disposition for separable spaces. To show that, let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. For a detailed study of this construction at the elementary level needed here we refer the reader to Heinrich’s survey paper [12] or Sims’ notes [21]. Let  $I$  be a set,  $\mathcal{U}$  be an ultrafilter on  $I$ , and  $(X_i)_{i \in I}$  a family of Banach spaces. Then  $\ell_\infty(X_i)$  endowed with the supremum norm, is a Banach space, and  $c_0^{\mathcal{U}}(X_i) = \{(x_i) \in \ell_\infty(X_i) : \lim_{\mathcal{U}(i)} \|x_i\| = 0\}$  is a closed subspace of  $\ell_\infty(X_i)$ . The ultraproduct of the spaces  $(X_i)_{i \in I}$  following  $\mathcal{U}$  is defined as the quotient

$$[X_i]_{\mathcal{U}} = \ell_\infty(X_i) / c_0^{\mathcal{U}}(X_i).$$

We denote by  $[(x_i)]$  the element of  $[X_i]_{\mathcal{U}}$  which has the family  $(x_i)$  as a representative. It is not difficult to show that  $\|[(x_i)]\| = \lim_{\mathcal{U}(i)} \|x_i\|$ . In the case  $X_i = X$  for all  $i$ , we denote the ultrapower by  $X_{\mathcal{U}}$ , and call it the ultrapower of  $X$  following  $\mathcal{U}$ . If  $T_i : X_i \rightarrow Y_i$  is a uniformly bounded family of operators, the ultrapower operator  $[T_i]_{\mathcal{U}} : [X_i]_{\mathcal{U}} \rightarrow [Y_i]_{\mathcal{U}}$  is given by  $[T_i]_{\mathcal{U}}[(x_i)] = [T_i(x_i)]$ . Quite clearly,  $\|[T_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}(i)} \|T_i\|$ .

**Definition 5.5.** An ultrafilter  $\mathcal{U}$  on a set  $I$  is countably incomplete if there is a decreasing sequence  $(I_n)$  of subsets of  $I$  such that  $I_n \in \mathcal{U}$  for all  $n$ , and  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

Notice that  $\mathcal{U}$  is countably incomplete if and only if there is a function  $n : I \rightarrow \mathbb{N}$  such that  $n(i) \rightarrow \infty$  along  $\mathcal{U}$  (equivalently, there is a family  $\varepsilon(i)$  of strictly positive numbers converging to zero along  $\mathcal{U}$ ). It is obvious that any countably incomplete ultrafilter is non-principal and also that every non-principal (or free) ultrafilter on  $\mathbb{N}$  is countably incomplete. Assuming all free ultrafilters countably incomplete is consistent with ZFC, since the cardinal of a set supporting a free countably complete ultrafilter should be measurable, hence strongly inaccessible.

We will need the following result (see [11, II, Thm. 2.1])

**Theorem 5.6.** *Let  $J$  be an  $M$ -ideal in the Banach space  $E$  and  $\pi : E \rightarrow E/J$  the natural quotient map. Let  $Y$  be a separable Banach space and  $t : Y \rightarrow E/J$  be an operator. Assume further that one of the following conditions is satisfied:*

- (1)  $Y$  has the  $\lambda$ -AP.
- (2)  $J$  is a Lindenstrauss space.

*Then  $t$  can be lifted to  $E$ , that is, there is an operator  $T : Y \rightarrow E$  such that  $\pi T = t$ . Moreover one can get  $\|T\| \leq \lambda \|t\|$  under the assumption (1) and  $\|T\| = \|t\|$  under (2).*

One has.

**Proposition 5.7.** *Ultrapowers of the Gurarii space (or more generally, of any Banach space almost universal disposition for finite dimensional spaces) with respect to countably incomplete ultrafilters are of universal disposition for separable Banach spaces.*

*Proof.* Let  $\mathcal{U}$  be a countably incomplete ultrafilter on the index set  $I$ . It suffices to see the following: if  $S'$  is a separable Banach space containing a subspace  $S$  and we are given an (into) isometry  $u : S \rightarrow \mathcal{G}_{\mathcal{U}}$ , then there is an isometry  $u' : S' \rightarrow \mathcal{G}_{\mathcal{U}}$  extending  $u$ . We can assume and do that  $S$  has codimension 1 in  $S'$ . It is easy to check that  $c_0^{\mathcal{U}}(I, X_i)$  is an  $M$ -ideal in  $\ell_{\infty}(I, M_i)$ ; hence, as every Lindenstrauss space,  $\mathcal{G}$  has the 1-AP, using Theorem 5.6, the operator  $u$  can be lifted to an isometry  $\tilde{u} : S \rightarrow \ell^{\infty}(I, \mathcal{G})$  which we will write as  $\tilde{u}(x) = (u_i(x))$  for certain operators  $u_i : S \rightarrow \mathcal{G}$  with norm at most 1. Write

$$S = \overline{\bigcup_{k=1}^{\infty} S_k},$$

where  $(S_k)$  is an increasing sequence of finite dimensional subspaces of  $S$ . Pick  $s' \in S' \setminus S$  and let  $S'_k$  denote the subspace spanned by  $S_k$  and  $s'$  in  $S'$ . Notice that for each  $x \in S$  one has  $\|u_i(x)\|_{\mathcal{G}} \rightarrow \|x\|_S$  following  $\mathcal{U}$ . This implies that, given a finite dimensional  $E \subset S$  and  $\varepsilon > 0$ , the set

$$\{i \in I : u_i \text{ is an } (1 + \varepsilon)\text{-isometry on } E\}$$

belongs to  $\mathcal{U}$ . Let  $(I_n)$  be a decreasing sequence of elements of  $\mathcal{U}$  with intersection not in  $\mathcal{U}$ , and consider the sets

$$J_n = \{i \in I : u_i \text{ is an } (1 + 1/n)\text{-isometry on } S_n\} \cap I_n.$$

Then  $J_n \in \mathcal{U}$  for all  $n$ , the sequence  $(J_n)$  is decreasing and  $\bigcap_{n=1}^{\infty} J_n = \emptyset$ . For  $i \in I$ , we put  $n(i) = \max\{n : i \in J_n\}$ . Then, of course  $n(i) \rightarrow \infty$  with respect to  $\mathcal{U}$ . Next notice that, for each

$i \in I$ , the operator  $u_i : S_{n(i)} \rightarrow \mathcal{G}$  is an  $(1 + 1/n(i))$ -isometry and so it can be extended to an  $(1 + 2/n(i))$ -isometry  $u'_i : S'_{n(i)} \rightarrow \mathcal{G}$ . Let us consider the ultraproduct operator

$$w = [u'_i]_{\mathcal{U}} : [S'_{n(i)}]_{\mathcal{U}} \rightarrow \mathcal{G}_{\mathcal{U}}$$

and the operator  $j : S' \rightarrow [S'_{n(i)}]_{\mathcal{U}}$  given by  $j(y) = [(x_i)]$ , where  $x_i$  is any point in  $S'_{n(i)}$  minimizing  $\text{dist}(y, S'_{n(i)})$ . The map  $j$  is a linear isometric embedding, and thus the sought-after extension is  $u' = w \circ j$ .  $\square$

**Corollary 5.8.** *Under CH the following Banach spaces are all isometrically isomorphic:*

- Any space of universal disposition for separable spaces of density  $\aleph_1$ .
- The space  $\mathfrak{S}^{\omega_1}(\mathbb{R})$ .
- The Kubis space.
- Any ultrapower of Gurarii's space built over a nontrivial ultrafilter on the integers.

## 6. REMARKS ON THE INJECTIVE SPACE PROBLEM

One of the main open problems about injective spaces is to know if every injective space must be isomorphic to a  $C(K)$ -space. The existence of separably injective spaces which are not isomorphic to  $C(K)$ -spaces has been shown in [8]. We exhibit now a 1-universally separably injective space not isomorphic to a  $C(K)$ -space. Our results are in fact much stronger: we will actually show that spaces of universal disposition for separable spaces cannot be isomorphic to complemented factors of any  $C(K)$  or  $\mathcal{M}$ -space. Since spaces of universal disposition for separable spaces are 1-universally separably injective, the previous assertion follows, in striking contrast with the facts that that injective spaces are complemented in some  $C(K)$ -spaces and 1-injective spaces are moreover isometric to  $C(K)$ -spaces.

**Theorem 6.1.** *Banach spaces of universal disposition for separable spaces are not isomorphic to complemented subspaces of  $C(K)$ -spaces. In particular they are not injective.*

*Proof.* Suppose  $U$  is (nonzero and) of universal disposition for separable spaces and there is an (isomorphic) embedding  $e : U \rightarrow C(K)$  and an operator  $\pi : C(K) \rightarrow U$  such that  $\pi e = \mathbf{1}_U$ . We know that  $U$  contains isometric copies of all separable Banach spaces. Let  $G_0$  be a subspace of  $U$  isometric to  $\mathcal{G}$ ,  $A_0$  the (closed) subalgebra spanned by  $e(G_0)$  in  $C(K)$  and  $B_0$  the closure of  $\pi(A_0)$  in  $U$ . Notice that  $B_0$  is a separable subspace of  $U$  containing  $G_0$ . As  $B_0$  embeds in  $\mathcal{G}$  we can find another copy  $G_1$  of  $\mathcal{G}$  inside  $U$  containing  $B_0$ . Now, replace  $G_0$  by  $G_1$  and continue inductively. This yields a diagram (unlabelled arrows are just inclusions)

$$\begin{array}{ccccccccc} G_0 & \longrightarrow & B_0 & \longrightarrow & G_1 & \longrightarrow & B_1 & \longrightarrow & G_2 & \longrightarrow & \dots \\ e \downarrow & & \pi \uparrow & & e \downarrow & & \pi \uparrow & & e \downarrow & & \\ e(G_0) & \longrightarrow & A_0 & \longrightarrow & e(G_1) & \longrightarrow & A_1 & \longrightarrow & e(G_2) & \longrightarrow & \dots \end{array}$$

The space

$$(8) \quad V = \overline{\bigcup_n G_n} = \overline{\bigcup_n B_n}.$$

is of almost universal disposition for finite dimensional spaces, therefore isometric to  $\mathcal{G}$ . On the other hand  $e$  embeds  $V = \overline{\bigcup_n B_n}$  in  $A = \overline{\bigcup_n A_{n+1}}$  while the restriction of  $\pi$  to  $A$  is left inverse to  $e$  and so  $V$  is (isomorphic to a subspace,) complemented in  $A$ . Finally,  $A$  is a (separable) unital subalgebra of  $C(K)$  hence it is isometrically isomorphic to  $C(M)$  for some compact (metrizable)  $M$ . A contradiction with the fact that  $\mathcal{G}$  is not a subspace of any  $\mathcal{M}$ -space (see remark after Coro. 5.4).  $\square$

The proof is valid replacing  $C(K)$ -spaces by  $\mathcal{M}$ -spaces (replace ‘algebra’ by ‘lattice’ everywhere in the proof and take into account Benyamini’s result in [4] that separable  $\mathcal{M}$ -spaces are isomorphic to  $C(K)$  spaces).

**Corollary 6.2.** *Ultrapowers of the Gurarii space (or more generally, of any Banach space of almost universal disposition for finite dimensional spaces) with respect to countably incomplete ultrafilters are not direct factors in any  $\mathcal{M}$ -space.*

*Remark 6.3.* This statement appears as Theorem 6.8 in [13]. Unfortunately, the argument provided by Henson and Moore needs Stern’s Lemma [22, Theorem 4.5(ii)], which is wrong (see [3]).

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