

NEW LIMIT THEOREMS RELATED TO FREE MULTIPLICATIVE CONVOLUTION

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ABSTRACT. Let \boxplus , \boxtimes , and \boxplus be the free additive, free multiplicative, and boolean additive convolutions, respectively. For a probability measure μ on $[0, \infty)$ with finite second moment, we find a scaling limit of $(\mu^{\boxtimes N})^{\boxplus N}$ as N goes to infinity. The \mathcal{R} -transform of its limit distribution can be represented by the Lambert's W function. From this, we prove that the limiting distribution is freely infinitely divisible as well as the lognormal distribution in classical sense. We also show a similar limit theorem by replacing the free additive convolution with the boolean convolution.

1. INTRODUCTION

In probability theory, limit theorems and infinite divisibility are considered in various situations. The classical references are the books by Gnedenko and Kolmogorov [11] and Petrov [17]. One of the most famous limit theorems is the Central Limit Theorem (for short CLT) that is the scaling limit of the sum of independent, identically distributed (i.i.d.) random variables. Suppose that a random variable Z has the standard normal distribution. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with finite second moment. Then a scaling

$$(1.1) \quad \frac{X_1 + \cdots + X_N - N\mathbb{E}[X_1]}{\sqrt{N\mathbb{V}[X_1]}}$$

converges to Z in distribution as N goes to infinity.

When we consider the product of i.i.d. random variables, we have also a CLT type limit theorem. The simplest case is as follows: for a sequence of i.i.d. random variables $\{X_k\}_{k=1}^{\infty}$ with finite second moment, we consider a scaling

$$(1.2) \quad \prod_{k=1}^N \exp\left(\frac{X_k - \mathbb{E}[X_k]}{\sqrt{N\mathbb{V}[X_k]}}\right).$$

By the CLT, this scaling converges to e^Z in distribution as N goes to infinity. The distribution of e^Z is called the lognormal distribution. It was proved by Thorin [20] that the lognormal distribution is infinitely divisible. The

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product limit theorems are also interested from applications to statistics. For details, see [18] and the book by Galambos and Simonelli [10].

In free probability theory, some limit theorems are known as in classical probability theory. The most famous limit theorem is the free CLT, which was found by Voiculescu. If $\{X_k\}_{k \in \mathbb{N}}$ is a sequence of freely independent identically distributed (for short freely i.i.d.) random variables with finite second moment, then the normalized sum (1.1) converges to the standard Wigner's semi-circle law in distribution as N goes to infinity. In addition, we know the Poisson limit theorem, the stable limit theorem and so on, for details, see [12], [6], and [4]. Recently other new limit theorems with respect to the free convolutions [7], [23], and [21] have been studied.

In this paper, we shall prove a limit theorem involving not only free additive but also free multiplicative convolutions. We introduce a new normalized sum of multiplications of freely independent random variables. For double sequence of freely i.i.d. random variables $\{\{X_i^{(j)}\}_{i \in \mathbb{N}}\}_{j \in \mathbb{N}}$ having a distribution μ on $[0, \infty)$ with finite second moment, we consider a new normalization Y_N ,

$$(1.3) \quad Y_N = \sum_{j=1}^N \frac{\sqrt{X_N^{(j)}} \cdots \sqrt{X_2^{(j)}} X_1^{(j)} \sqrt{X_2^{(j)}} \cdots \sqrt{X_N^{(j)}}}{m_1^N N},$$

where m_1 is the mean of the distribution μ . We shall see that its limit distribution depends only on the first and second moments. In its proof, we shall investigate the Taylor type expansion of the S -transform. In addition, a formula by Belinschi and Nica [2] suggests that the distribution of (1.3) is equal to the one of

$$\widetilde{Y}_N = \frac{\sqrt{\sum_{i=1}^N X_i^{(N)}} \cdots \sqrt{\sum_{i=1}^N X_i^{(1)}} \sqrt{\sum_{i=1}^N X_i^{(1)}} \cdots \sqrt{\sum_{i=1}^N X_i^{(N)}}}{m_1^N N^N},$$

which is corresponding to the scaling (1.2). In this meaning, we may call it free lognormal distribution. Compare to free additive CLT case, it is not exactly the same scaling. The difference may occur because of non-commutativity of random variables. Furthermore a similar limit theorem can be found under boolean independence.

In order to investigate properties of this limit distribution, we show that it is freely infinitely divisible as in classical case lognormal distribution is infinitely divisible. In its proof, the properties of Lambert's W -function play an important role and we obtain its Lévy measure.

This paper is organized as follows. In Section 2, we shall gather the tools for free and boolean probability. Especially, we recall \mathcal{R} , S , and Σ -transforms and infinite divisibility in free probability theory. In Section 3, we shall give the Taylor type expansions for S and Σ -transforms and prove our limit theorems. Finally, in Section 4, we shall discuss the limit distributions with focusing on infinite divisibility and moments.

2. PRELIMINARIES

Let \mathbb{R}_+ be the half line $[0, +\infty)$ and \mathbb{C}^+ be the upper half plane $\{z = x + iy \in \mathbb{C}; y > 0\}$. We fix notation that \mathcal{P} and \mathcal{P}_+ mean the set of all Borel probability measures on \mathbb{R} and \mathbb{R}_+ , respectively. We denote the free additive, free multiplicative, and boolean additive convolutions by \boxplus , \boxtimes , and \uplus , respectively, see for details of convolutions, [19], [22], and [15]. Hereafter, δ_0 stands the Dirac probability measure concentrated on 0.

2.1. Analytic tools for free and boolean convolutions. Here, we shall gather the analytic tools for free and boolean probability and mention some of their important facts.

We denote the Cauchy transform of a probability measure μ on \mathbb{R} by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \mu(dx), \quad z \in \mathbb{C}^+,$$

and

$$\Psi_\rho(z) = \int_{\mathbb{R}} \frac{xz}{1-xz} \rho(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}$$

denotes the moment generating function of ρ on \mathbb{R}_+ . Then the Speicher's R and Voiculescu's \mathcal{R} -transforms of μ are defined as follow: for any given $\alpha > 0$, one can find $\beta > 0$ so that

$$R_\mu(z) = z\mathcal{R}_\mu(z) = zG_\mu^{-1}(z) - 1, \quad 1/z \in \Gamma_{\alpha,\beta},$$

where $G_\mu^{-1}(z)$ is the right inverse of $G_\mu(z)$ with respect to composition and $\Gamma_{\alpha,\beta} = \{z = x+iy \in \mathbb{C}^+; y > \beta, |y| > \alpha x\}$. Note that we will use both R and \mathcal{R} -transforms for convenience. The S and Σ -transforms of ρ are defined by

$$S_\rho(z) = \frac{(z+1)\Psi_\rho^{-1}(z)}{z}, \quad z \in \Psi_\rho(i\mathbb{C}^+)$$

and

$$\Sigma_\rho(z) = S_\rho\left(\frac{z}{1-z}\right), \quad \frac{z}{1-z} \in \Psi_\rho(i\mathbb{C}^+),$$

respectively, where $\Psi_\rho^{-1}(z)$ is the right inverse of $\Psi_\rho(z)$ with respect to composition. Now, we summarize the relations between the transforms and convolutions, see for proofs [6] and [2].

Proposition 2.1. *For $\mu_1 \in \mathcal{P}$, $\mu_2 \in \mathcal{P}$, $\rho_1 \in \mathcal{P}_+$ and $\rho_2 \in \mathcal{P}_+$, which are not δ_0 , there exist $\alpha > 0$ and $\beta > 0$ such that*

$$\begin{aligned} R_{\mu_1 \boxplus \mu_2}(z) &= R_{\mu_1}(z) + R_{\mu_2}(z), \quad 1/z \in \Gamma_{\alpha, \beta}, \\ S_{\rho_1 \boxtimes \rho_2}(z) &= S_{\rho_1}(z)S_{\rho_2}(z), \quad z \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+), \\ S_{\rho_1^{\boxplus t}}(z) &= \frac{1}{t} S_{\rho_1}\left(\frac{z}{t}\right), \\ \Sigma_{\rho_1 \boxtimes \rho_2}(z) &= \Sigma_{\rho_1}(z)\Sigma_{\rho_2}(z), \quad z/(1-z) \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+), \\ \Sigma_{\rho_1^{\boxplus t}}(z) &= \frac{1}{t} \Sigma_{\rho_1}\left(\frac{z}{t}\right). \end{aligned}$$

For $c > 0$, the dilation operator D_c on \mathcal{P} is defined by

$$D_c(\mu)(B) = \mu\left(\frac{1}{c}B\right)$$

for any Borel set B on \mathbb{R}_+ , where $\frac{1}{c}B = \{x \in \mathbb{R}; \frac{1}{c}x \in B\}$. If a random variable X has a distribution μ , then cX is distributed as $D_c(\mu)$. In the paper [2], the authors showed that

$$S_{D_c(\mu)}(z) = \frac{1}{c} S_{\mu}(z)$$

and

$$\Sigma_{D_c(\mu)}(z) = \frac{1}{c} \Sigma_{\mu}(z).$$

2.2. Infinite divisibility for free additive convolution. A probability measure μ is freely infinitely divisible (or \boxplus -infinitely divisible) if for any $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{P}$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}.$$

We denote the class of all \boxplus -infinitely divisible distributions by I^{\boxplus} .

Remark 2.2. We can define other infinite divisibility replacing \boxplus by \boxtimes or \uplus . But for boolean convolution, all probability measures are \uplus -infinitely divisible. So we shall not discuss \uplus -divisibility any longer.

The next proposition characterizes the \boxplus -infinitely divisible laws [22, Theorem3.7.2].

Proposition 2.3. *The followings are equivalent:*

- (1) $\mu \in I^{\boxplus}$.
- (2) \mathcal{R}_{μ} has an analytic extension defined on \mathbb{C}^- with value $\mathbb{C}^- \cup \mathbb{R}$.

(3) *There exist unique $b_\mu \in \mathbb{R}$ and finite measure ν_μ such that*

$$\mathcal{R}_\mu(z) = b_\mu + \int_{\mathbb{R}} \frac{z}{1-tz} \nu_\mu(dt), \quad z \in \mathbb{C}^-.$$

The above expression is called \boxplus -Lévy–Khintchine representation, or simply Lévy–Khintchine representation.

Example 2.4. The typical examples of \boxplus -infinitely divisible distribution are the Wigner’s semi-circle law, the Dirac’s delta distribution, and the free Poisson distribution π_t with parameter $t \geq 0$ having density

$$(2.1) \quad \pi_t(dx) = \max(0, (1-t))\delta_0(dx) + \frac{1}{2\pi x} \sqrt{4t - (x-1-t)^2} \mathbb{1}_{[(1-\sqrt{t})^2, (1+\sqrt{t})^2]}(x) dx.$$

The Lévy measure ν_μ and b_μ of the semi-circle law are δ_0 and 0, and the free Poisson law π_t has $b_\mu = t$ and $\nu_\mu = t\delta_1$. We put π_1 by π .

The following functional equation of the R and S -transforms can be found in, for instance, [15] or [16, Lemma 2]:

Proposition 2.5. *Assume that $\mu \in \mathcal{P}_+$. For some sufficiently small $\varepsilon > 0$, we have a region D_ε which includes $\{-it; 0 < t < \varepsilon\}$ such that*

$$(2.2) \quad z = R_\mu(zS_\mu(z)),$$

for $z \in D_\varepsilon$.

3. NEW LIMIT THEOREMS

In this section, we prove a new limit theorem related to both free additive and multiplicative convolutions. We also discuss a similar result with replacing \boxplus by \uplus . It was proved in [14] by Młotkowski that for the free Poisson law π , we have

$$D_n \left(\left(\pi^{\boxtimes(n-1)} \right)^{\uplus n} \right) \xrightarrow{n \rightarrow \infty} \nu_0 \quad \text{in distribution,}$$

where the p^{th} moment of ν_0 is given by $\frac{p^p}{p!}$. We find that a theorem of this type holds more generally if we replace π by any probability distribution with finite second moment.

3.1. Expansion of the S -transform and Σ -transform. We prove the expansion for the S -transform and Σ -transform under the second moment condition. For the \mathcal{R} -transform, the Taylor type expansion was proved by Benaych-Georges in [3]. For each region A in \mathbb{C} , we denote $z \rightarrow 0$ with $z \in A$ by $z \xrightarrow{z \in A} 0$.

Lemma 3.1. *Let $\rho \in \mathcal{P}_+$ have the moment of order p , that is, for $k = 0, 1, 2, \dots, p$,*

$$m_k(\rho) := \int_{\mathbb{R}_+} x^k \rho(dx) < \infty.$$

Then its moment generating function $\Psi_\rho(z)$ has a Taylor expansion

$$\Psi_\rho(z) = \sum_{k=1}^{p-1} m_k(\rho) z^k + O(z^p), \quad z \xrightarrow{z \in i\mathbb{C}^+} 0.$$

Proof. See [1]. □

Lemma 3.2. *Let $\rho \in \mathcal{P}_+$ have the moment of order $p \geq 2$ and $\rho \neq \delta_0$. Then we have the followings:*

- (1) $\Psi_\rho(z)$ is univalent in $i\mathbb{C}^+$.
- (2) The inverse function $\Psi_\rho^{-1} : \Psi_\rho(i\mathbb{C}^+) \rightarrow i\mathbb{C}^+$ of Ψ_ρ admits Taylor type expansion of order 2

$$\Psi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z - \frac{m_2(\rho)}{(m_1(\rho))^3} z^2 + o(z^2), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0.$$

- (3) $\mathfrak{D}_\rho := \Psi_\rho(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(\rho(\{0\}) - 1, 0)$. In addition, $\Psi_\rho(i\mathbb{C}^+) \cap \mathbb{R} = (\rho(\{0\}) - 1, 0)$,

$$\lim_{t \uparrow 0} \Psi_\rho^{-1}(t) = 0$$

and

$$\lim_{t \downarrow \rho(\{0\}) - 1} \Psi_\rho^{-1}(t) = \infty.$$

Proof. (1) and (3) are proved in Bercovici and Voiculescu [6, Proposition 6.2].

(2) **Step 1** We shall first prove that

$$\Psi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z + o(z), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0.$$

Take any continuous path $\{z(t)\}_{t \in (0,1]}$ in \mathfrak{D}_ρ such that $\lim_{t \downarrow 0} z(t) = 0$. By (1), we can choose a unique continuous path $\{\omega(t)\}_{t \in (0,1]}$ on $t \in (0, 1]$ such that $\lim_{t \downarrow 0} \omega(t) = 0$ and $\Psi_\rho(\omega(t)) = z(t)$.

$$\lim_{t \downarrow 0} \frac{\Psi_\rho^{-1}(z(t))}{z(t)} = \lim_{t \downarrow 0} \frac{\omega(t)}{\Psi_\rho(\omega(t))} = \lim_{t \downarrow 0} \frac{1}{\Psi_\rho(\omega(t))/\omega(t)} = \frac{1}{m_1}.$$

By arbitrary of the paths, it follows that

$$\Psi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z + o(z), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0.$$

Step 2 Using step 1, we shall show the Taylor type expansion of order 2 as $z \xrightarrow{z \in \mathfrak{D}_\rho} 0$.

$$\begin{aligned} \frac{\Psi_\rho^{-1}(\Psi_\rho(z)) - \frac{1}{m_1(\rho)}\Psi_\rho(z)}{\Psi_\rho(z)^2} &= \frac{z - \frac{1}{m_1(\rho)}(m_1(\rho)z + m_2(\rho)z^2 + O(z^3))}{(m_1(\rho)z + m_2(\rho)z^2 + O(z^3))^2} \\ &= \frac{\left(\frac{m_2(\rho)}{m_1(\rho)}z^2 + O(z^3)\right)}{(m_1(\rho))^2 z^2 + O(z^3)} \rightarrow \frac{m_2(\rho)}{(m_1(\rho))^3}, \quad \text{as } z \xrightarrow{z \in \mathfrak{D}_\rho} 0. \end{aligned}$$

As a results, we obtain as follow:

$$\Psi_\rho^{-1}(z) = \frac{1}{m_1(\rho)}z + \frac{m_2(\rho)}{(m_1(\rho))^3}z^2 + o(z^2), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0.$$

□

3.2. Limit theorems. Here we shall state the main theorem.

Theorem 3.3. *We assume that $\rho \in \mathcal{P}_+$ has the second moment and put $\gamma = \frac{\text{Var}(\rho)}{(m_1(\rho))^2}$.*

- (1) *There exist $s_0 > 0$ and $s_1 < 0$ such that the S -transform of ρ is given by*

$$S_\rho(z) = s_0 + s_1 z + o(z), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0,$$

and there exists a probability measure $\mathfrak{h}_\gamma \in \mathcal{P}_+$ such that

$$D_{s_0^n/n} \left((\rho^{\boxtimes n})^{\boxplus n} \right) \rightarrow \mathfrak{h}_\gamma \quad \text{in distribution.}$$

In addition, the S -transform of the limit distribution \mathfrak{h}_γ is

$$S_{\mathfrak{h}_\gamma}(z) = \exp(-\gamma z).$$

- (2) *There exist $\sigma_0 > 0$ and $\sigma_1 < 0$ such that the Σ -transform of ρ is given by*

$$\Sigma_\rho(z) = \sigma_0 + \sigma_1 z + o(z), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0,$$

and there exists a probability measure $\mathfrak{s}_\gamma \in \mathcal{P}_+$ such that

$$D_{s_0^n/n} \left((\rho^{\boxtimes n-1})^{\uplus n} \right) \rightarrow \mathfrak{s}_\gamma \quad \text{in distribution.}$$

In addition, the Σ -transform of the limit distribution \mathfrak{s}_γ is

$$\Sigma_{\mathfrak{s}_\gamma}(z) = \exp(-\gamma z).$$

Proof. Using Lemma 3.2, we have

$$S_\rho(z) = \frac{z+1}{z} \Psi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} - \frac{\text{Var}(\rho)}{(m_1(\rho))^3} z + o(z), \quad z \xrightarrow{z \in \mathfrak{D}_\rho} 0.$$

Let $s_0 = \frac{1}{m_1(\rho)}$ and $s_1 = -\frac{\text{Var}(\rho)}{(m_1(\rho))^3}$. By Proposition 2.1, we obtain

$$\begin{aligned} S_{D_{s_0^n/n}((\rho^{\boxtimes n})^{\boxplus n})}(z) &= \frac{n}{s_0^n} S_{(\rho^{\boxtimes n})^{\boxplus n}}(z) = \frac{1}{s_0^n} S_{\rho^{\boxtimes n}}\left(\frac{z}{n}\right) \\ &= \frac{1}{s_0^n} \left(S_{\rho}\left(\frac{z}{n}\right) \right)^n = \frac{1}{s_0^n} \left(s_0 + s_1 \frac{z}{n} + o\left(\frac{1}{n}\right) \right)^n \\ &= \left(1 + \frac{s_1 z}{s_0 n} + o\left(\frac{1}{n}\right) \right)^n \\ &\rightarrow \exp\left(\frac{s_1}{s_0} z\right) = \exp(-\gamma z) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

From [5, Lemma 7.1] and [6, Theorem 6.13 (ii)], there exists a free multiplicative infinitely divisible measure \mathfrak{h}_γ such that $S_{\mathfrak{h}_\gamma}(z) = \exp(-\gamma z)$. The proof for (2) is the same as for the free additive case. \square

We can exchange the order of free multiplicative and freely additive (or boolean additive) convolutions. The difference is in the scaling speed.

Corollary 3.4. *Under the same setting as in Theorem 3.3, we have*

$$\begin{aligned} D_{s_0^n/n^n} \left((\rho^{\boxplus n})^{\boxtimes n} \right) &\rightarrow \mathfrak{h}_\gamma, \\ D_{s_0^n/n^n} \left((\rho^{\boxplus n-1})^{\boxtimes n} \right) &\rightarrow \mathfrak{s}_\gamma, \end{aligned}$$

as $n \rightarrow \infty$.

Proof. As we have done in the proof of Theorem 3.3, it can be proved by using Proposition 2.1 and Lemma 3.2. \square

4. LAMBERT W FUNCTION AND INFINITE DIVISIBILITY OF THE LIMIT DISTRIBUTION

4.1. On the limit distribution of free case. When we calculate the R -transform or the moment generating function, the Lambert's W -function plays an important role, which satisfies the functional equation

$$z = W(z) \exp(W(z)).$$

This function have been studied for a long period and we have known several good properties of this function as real and complex function. For more details of the Lambert W function, see, for instance, [8]. Let $W_0(z)$ be the principal branch of the Lambert W -function.

By Proposition 2.5 and the S -transform of \mathfrak{h}_γ , we have

$$R_{\mathfrak{h}_\gamma}(ze^{-\gamma z}) = z, \quad 1/z \in \Gamma_{\alpha, \beta}.$$

This functional equation suggests that the R -transform is given by using the Lambert's W -function.

Theorem 4.1. (1) *The \mathcal{R} and R -transforms of probability measure η_γ are given as follows:*

$$\mathcal{R}_{\eta_\gamma}(z) = \frac{-W_0(-\gamma z)}{\gamma z},$$

$$R_{\eta_\gamma}(z) = -\frac{1}{\gamma}W_0(-\gamma z).$$

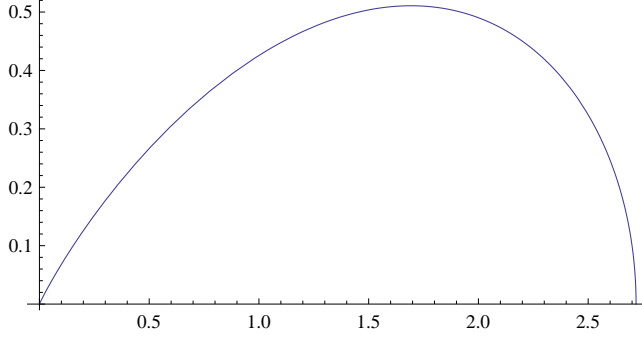
(2) η_γ is both \boxplus -infinitely divisible and \boxtimes -infinitely divisible.

(3) The free cumulant sequence of η_γ is $\left\{ \frac{(\gamma n)^{n-1}}{n!} \right\}_{n \in \mathbb{N}}$.

(4) The Lévy measure ν_{η_γ} of η_γ is given by

$$\nu_{\eta_\gamma}(ds) = \frac{1}{\gamma\pi} s f^{-1}(\gamma/s) 1_{[0, \gamma e]}(s) ds,$$

where $f(u) = u \csc u \exp(-u \cot u)$. In case of $\gamma = 1$, for the shape of the density of ν_{η_1} , see the graph below.



(5) *It holds the following formulas:*

$$\eta_\gamma^{\boxplus t} = D_t(\eta_\gamma^{\boxtimes 1/t}),$$

$$\eta_\gamma^{\boxtimes t} = D_t(\eta_\gamma^{\boxplus 1/t}).$$

The proof of this theorem is helped by the following property and the well-known integral representation of the Lambert's W -function (for instance, see [8, Section 4] and [13, Theorem 3.1]):

Proposition 4.2. (1) *The principal branch of $W_0(z)$ has an analytic extension on $\mathbb{C} \setminus (-\infty, -1/e]$ and it takes $\mathbb{C}^- \cup \mathbb{R}$ on \mathbb{C}^- .*

(2) *For any $z \in \mathbb{C}^+$, we have an integral representation:*

$$\frac{W_0(z)}{z} = \frac{1}{\pi} \int_0^\pi \frac{(1 - u \cot u)^2 + u^2}{z + u \csc u \exp(-u \cot u)} du.$$

Proof of Theorem 11. (1) The \boxtimes -infinitely divisibility is trivial from the form of the S -transform and the facts in [5]. By Proposition 2.5, we have

$$R_{\eta_\gamma}(ze^{-\gamma z}) = z.$$

Then the R -transform is given by using the Lambert's W -function as follows:

$$R_{\eta_\gamma}(z) = -\frac{1}{\gamma}W_0(-\gamma z),$$

and hence,

$$\mathcal{R}_{\eta_\gamma}(z) = \frac{W_0(-\gamma z)}{-\gamma z}.$$

- (2) By Proposition 2.3 and Proposition 4.2 (1), $\mathcal{R}_{\eta_\gamma}$ has an analytic extension defined on \mathbb{C}^- with value $\mathbb{C}^- \cup \mathbb{R}$, which means that η_γ is \boxplus -infinitely divisible.
- (3) The Taylor type expansion of $-W_0(-z)$ at the origin is obtained from Equation (3.1) of [8, pp. 339].
- (4) We put $g(u) = (1 - u \cot(u))^2 + u^2$. Noting that

$$(4.1) \quad g(u) = \frac{uf'(u)}{f(u)},$$

we obtain

$$\begin{aligned} \mathcal{R}_{\eta_\gamma}(z) &= \frac{1}{\pi} \int_0^\pi \frac{g(u)}{-\gamma z + f(u)} du \\ &= \frac{1}{\pi} \int_0^\pi \frac{g(u)/f(u)}{1 - \gamma z/f(u)} du = \frac{1}{\gamma\pi} \int_0^{\gamma e} \frac{f^{-1}(\gamma/s)}{1 - sz} ds, \end{aligned}$$

where we have changed the variables as $s = \gamma/f(u)$.

$$\begin{aligned} \mathcal{R}_{\eta_\gamma}(z) &= \frac{1}{\gamma\pi} \int_0^{\gamma e} \frac{f^{-1}(\gamma/s)}{1 - sz} ds \\ &= \frac{1}{\gamma\pi} \int_0^{\gamma e} \left(\frac{sz}{1 - sz} + 1 \right) f^{-1}(\gamma/s) ds \\ &= \frac{1}{\gamma} + \frac{1}{\gamma\pi} \int_0^{\gamma e} \frac{z}{1 - sz} s f^{-1}(\gamma/s) ds. \end{aligned}$$

Therefore we obtain the Lévy measure $\nu_{\eta_\gamma}(ds) = \frac{s f^{-1}(\gamma/s)}{\gamma\pi} ds$ of η_γ .

- (5) It is direct consequence of Proposition 2.1. □

Remark 4.3. Here we consider the limit distribution with parameter $\gamma = 1$. For example, if ρ is the free Poisson distribution with parameter 1, this is the case. Simply we write η instead of η_1 . There exists a probability measure ρ such that

$$(4.2) \quad \mathcal{R}_\rho(z) = \frac{\mathcal{R}_\eta(z) - 1}{z}.$$

Indeed, if we consider the shifted free cumulant sequence $\{k_n(\rho)\}_{n \in \mathbb{N}} = \left\{ \frac{(n+1)^n}{(n+1)!} \right\}_{n \in \mathbb{N}}$, which is a sequence of coefficients of Taylor expansion of R_ρ

at 0, then it becomes a moment sequence of a probability measure. This means that the measure ρ is a free compound Poisson distribution with the compound measure σ , the moments of which are $m_n(\sigma) = \frac{(n+1)^n}{(n+1)!}$. From (4.2), we have

$$(4.3) \quad z\mathcal{R}_\eta(zM_\rho(z)) = zM_\rho(z).$$

By putting $P(z) = zM_\rho(z)$ and using the Lagrange inversion formula, (4.3) implies that

$$n^{\text{th}} \text{ coefficient of } \{P(z)\} = \frac{1}{n} \times ((n-1)^{\text{st}} \text{ coefficient of } \mathcal{R}_\rho(z)).$$

Hence we obtain the moments of ρ as

$$m_n(\rho) = \frac{(2n+1)^{n-1}}{n!}.$$

4.2. On the limit distribution in boolean case. Let $\mathfrak{s} := \mathfrak{s}_1$ denote a probability measure with the moment sequence $\left\{\frac{n^n}{n!}\right\}_{n \geq 0}$, the positivity of which is ensured by [14]. Then its moment generating function $M_{\mathfrak{s}}(z)$ can be given by

$$M_{\mathfrak{s}}(z) = \sum_{n=0}^{\infty} \frac{n^n}{n!} z^n = \frac{1}{1 - \eta(z)} \quad (1)$$

where the function $\eta(z)$ is defined by

$$\eta(z) = -W_0(-z), \quad z \in \mathbb{C} \setminus \left[\frac{1}{e}, \infty\right).$$

Remark 4.4. The following useful facts on the function η can be found in [9, Sect.2]: The map

$$\theta \mapsto \frac{\sin \theta}{\theta} \exp(\theta \cot \theta)$$

is a bijection of $(0, \pi)$ onto $(0, e)$, and if we define the functions $\eta^+, \eta^- : \left[\frac{1}{e}, \infty\right) \rightarrow \mathbb{C}$ by

$$\eta^\pm \left(\frac{\theta}{\sin \theta} \exp(-\theta \cot \theta) \right) = \theta \cot \theta \pm i\theta, \quad 0 \leq \theta < \pi,$$

then

$$\eta^\pm(x) = \lim_{y \downarrow 0} \eta(x + iy), \quad x \in \left[\frac{1}{e}, \infty\right).$$

From (1), the Cauchy transform of the measure \mathfrak{s} is given by

$$G_{\mathfrak{s}}(\zeta) = \frac{1}{\zeta} \frac{1}{1 - \eta\left(\frac{1}{\zeta}\right)}, \quad \text{for } \zeta \in \mathbb{C} \setminus [0, e].$$

Now we apply the Stieltjes inversion formula to obtain the density function $\varphi_{\mathfrak{s}}(t)$ of the measure \mathfrak{s} , that is, for $t \in [0, e]$,

$$\begin{aligned}
\varphi_{\mathfrak{s}}(t) &= -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \left(G_{\mathfrak{s}}(t + i\varepsilon) \right) \\
&= -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \left(\frac{1}{t + i\varepsilon} \frac{1}{1 - \eta \left(\frac{1}{t + i\varepsilon} \right)} \right) \\
&= -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \left(\frac{t - i\varepsilon}{t^2 + \varepsilon^2} \frac{1}{1 - \eta \left(\frac{t - i\varepsilon}{t^2 + \varepsilon^2} \right)} \right) \\
&= -\frac{1}{\pi} \operatorname{Im} \left(\frac{1}{t} \frac{1}{1 - \eta^{-} \left(\frac{1}{t} \right)} \right),
\end{aligned}$$

where the function η^{-} is defined as in Remark above. Here we change the variables

$$\frac{1}{t} = \frac{\theta}{\sin \theta} \exp(-\theta \cot \theta),$$

then it follows that

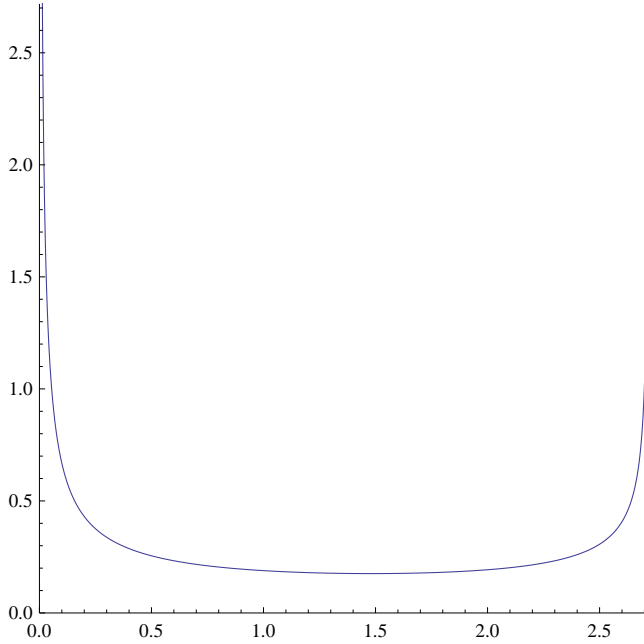
$$\begin{aligned}
\varphi_{\mathfrak{s}}(t) &= -\frac{1}{\pi} \operatorname{Im} \left(\frac{1}{t} \frac{1}{1 - (\theta \cot \theta - i\theta)} \right) \\
&= \frac{1}{\pi} \frac{1}{t} \frac{\theta}{(1 - \theta \cot \theta)^2 + \theta^2} \\
&= \frac{1}{\pi} \left(\frac{\theta}{\sin \theta} \exp(-\theta \cot \theta) \right) \left(\frac{\theta}{(1 - \theta \cot \theta)^2 + \theta^2} \right) \\
&= \frac{1}{\pi} \frac{\theta^2 \exp(-\theta \cot \theta)}{\sin \theta \left((1 - \theta \cot \theta)^2 + \theta^2 \right)}.
\end{aligned}$$

Thus we obtain the following proposition:

Proposition 4.5. *The probability density function $\varphi_{\mathfrak{s}}$ of the measure \mathfrak{s} can be given by the implicit (parametric) form as*

$$\varphi_{\mathfrak{s}} \left(\frac{\sin v}{v} \exp(v \cot v) \right) = \frac{1}{\pi} \frac{v^2 \exp(-v \cot v)}{\sin v \left((1 - v \cot v)^2 + v^2 \right)}, \quad 0 < v < \pi.$$

Remark 4.6. (1) The shape of the density function of φ_s is as the graph below, especially non-unimodal.



(2) The function $(1 - v \cot v)^2 + v^2$ also appears in the integral representation of $W_0(z)/z$ as we mentioned Proposition 4.2:

$$\frac{W_0(z)}{z} = \frac{1}{\pi} \int_0^\pi \frac{(1 - v \cot v)^2 + v^2}{z + \frac{v}{\sin v} \exp(-v \cot v)} dv.$$

Thus using $f(v) = v \csc v \exp(-v \cot v)$ and (4.1) again, the parametric form of the density function can be rewritten as

$$\varphi_s\left(\frac{1}{f(v)}\right) = \frac{1}{\pi} \frac{(f(v))^2}{f'(v)}.$$

4.3. Concluding remark. We also know that there exists the similar moment sequence. In the paper by Dykema and Haagerup [9], they find a limit distribution of DT-operator $DT(1, \delta_0)$. The moment of their one is $m_n = \frac{n^n}{(n+1)!}$. A natural question arises: how do we realize this limit theorem via random matrix model?

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