

# SOME REMARKS ON THE KÄHLER GEOMETRY OF LEBRUN'S RICCI FLAT METRICS ON $\mathbb{C}^2$

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ABSTRACT. In this paper we investigate the balanced condition (in the sense of Donaldson) and the existence of an Engliš expansion for the LeBrun's metrics on  $\mathbb{C}^2$ . Our first result shows that a LeBrun's metric on  $\mathbb{C}^2$  is never balanced unless it is the flat metric. The second one shows that an Engliš expansion of the Rawnsley's function associated to a LeBrun's metric always exists, while the coefficient  $a_3$  of the expansion vanishes if and only if the LeBrun's metric is indeed the flat one.

## 1. INTRODUCTION

In [23] Claude LeBrun constructs, for all positive real numbers  $m \geq 0$ , a family of Kähler metrics  $g_m$  on  $\mathbb{C}^2$ , whose associated Kähler form is given by  $\omega_m = \frac{i}{2} \partial \bar{\partial} \Phi_m$ , where

$$\Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4), \quad (1)$$

$u$  and  $v$  are implicitly defined by

$$x_1 = |z_1| = e^{m(u^2 - v^2)} u, \quad x_2 = |z_2| = e^{m(v^2 - u^2)} v, \quad (2)$$

and  $(z_1, z_2)$  are the standard complex coordinates on  $\mathbb{C}^2$ . For  $m = 0$  one gets the flat metric  $g_0$ , i.e.  $\omega_0 = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ , while for  $m > 0$  each of the  $g_m$ 's represents the first example of complete Ricci flat and non-flat metric on  $\mathbb{C}^2$  having the same volume form of the flat metric  $g_0$ , namely  $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0$ . Moreover, for  $m > 0$ ,  $g_m$  is isometric (up to dilation and rescaling) to the Taub-NUT metric. In this paper we refer to  $g_m$ , for a fixed  $m \geq 0$ , as the *LeBrun metric*. The Kähler form  $\omega_m$  has been

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studied from the symplectic point of view by the first and third author of the present paper in [30] (see also [8]). There we prove that for each  $m$ ,  $(\mathbb{C}^2, \omega_m)$  admits global Darboux coordinates. More precisely, we construct an explicit family  $\Phi_m$  of symplectomorphisms from  $(\mathbb{C}^2, \omega_m)$  into  $(\mathbb{R}^4, \omega_0)$  such that  $\Phi_m^* \omega_0 = \omega_m$  and  $\Phi_0$  equals the identity of  $\mathbb{C}^2 \cong \mathbb{R}^4$ .

In this paper we study the balanced condition, in the sense of Donaldson, and Engliš expansion for the LeBrun metric. The main results are Theorem 4 and Theorem 8 below. The first one states that the LeBrun metric on  $\mathbb{C}^2$  is never balanced unless it is the flat metric, while the second one proves that the coefficient  $a_3$  in Engliš expansion of Rawnsley's function  $\epsilon_{\alpha g}$  vanishes if and only if  $m = 0$ .

The paper is organized as follows. In the next section, after recalling what we need about balanced metrics, we state and prove Theorem 4. Section 3 is devoted to Engliš expansion and to the proof of Theorem 8. Finally, in the Appendix we verify that the LeBrun's metrics are complete and Ricci flat (well-known facts added for completeness' sake) and we compute the norm of the curvature tensor and its Laplacian, which are needed in Section 3.

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## 2. ON THE BALANCED CONDITION FOR THE LEBRUN METRIC

Let  $M$  be an  $n$ -dimensional complex manifold endowed with a Kähler metric  $g$  and let  $\omega$  be the Kähler form associated to  $g$ , i.e.  $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ . Assume that the metric  $g$  can be described by a strictly plurisubharmonic real-valued function  $\Phi : M \rightarrow \mathbb{R}$ , called a *Kähler potential* for  $g$ , i.e.  $\omega = \frac{i}{2} \partial \bar{\partial} \Phi$ .

For a real number  $\alpha > 0$  consider the weighted Hilbert space  $\mathcal{H}_{\alpha\Phi}$  consisting of square integrable holomorphic functions on  $(M, g)$ , with weight  $e^{-\alpha\Phi}$ , namely

$$\mathcal{H}_{\alpha\Phi} = \left\{ f \in \text{Hol}(M) \mid \int_M e^{-\alpha\Phi} |f|^2 \frac{\omega^n}{n!} < \infty \right\}, \quad (3)$$

where  $\frac{\omega^n}{n!} = \det\left(\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}\right) \frac{\omega_0^n}{n!}$  is the volume form associated to  $\omega$  and  $\omega_0 = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$  is the standard Kähler form on  $\mathbb{C}^n$ . If  $\mathcal{H}_{\alpha\Phi} \neq \{0\}$  we can pick an orthonormal basis  $\{f_j^\alpha\}$  and define its reproducing kernel

$$K_{\alpha\Phi}(x, y) = \sum_{j=0}^N f_j^\alpha(x) \overline{f_j^\alpha(y)}, \quad x, y \in M,$$

where  $N + 1$  ( $N \leq \infty$ ) denotes the complex dimension of  $\mathcal{H}_{\alpha\Phi}$ . Consider the function

$$\epsilon_{\alpha g}(x) = e^{-\alpha\Phi(x)} K_{\alpha\Phi}(x, x). \quad (4)$$

As suggested by the notation it is not difficult to verify that this function depends only on the metric  $\alpha g$  and not on the choice of the Kähler potential  $\Phi$  (which is defined up to the sum with the real part of a holomorphic function on  $M$ ) or on the orthonormal basis chosen.

**Definition.** The metric  $\alpha g$  is *balanced* if the function  $\epsilon_{\alpha g}$  is a positive constant.

A balanced metric  $g$  on  $M$  can be viewed as a particular projectively induced Kähler metric. Recall that a Kähler metric  $g$  on a complex manifold  $M$  is *projectively induced* if there exists a Kähler (i.e. a holomorphic and isometric) immersion  $F: M \rightarrow \mathbb{C}P^N$ ,  $N \leq \infty$ , such that  $F^*(g_{FS}) = g$ , where  $g_{FS}$  denotes the Fubini–Study metric on  $\mathbb{C}P^N$ . Projectively induced Kähler metrics enjoy important geometrical properties and were extensively studied in [7] to whom we refer the reader for details). In the case of a balanced metric, the Kähler immersion

$$F_\alpha: M \rightarrow \mathbb{C}P^N, \quad x \mapsto [f_0^\alpha(x), \dots, f_N^\alpha(x)], \quad N \leq \infty,$$

is given by the orthonormal basis  $\{f_j^\alpha\}$  of the Hilbert space  $\mathcal{H}_{\alpha\Phi}$ . Indeed the map  $F_\alpha$  is well-defined since  $\epsilon_{\alpha g}$  is a positive constant and hence for all  $x \in M$  there exists  $\varphi \in \mathcal{H}_{\alpha\Phi}$  such that  $\varphi(x) \neq 0$ . Moreover, if  $\omega_{FS}$  denotes the Fubini–Study Kähler form on  $\mathbb{C}P^N$ , then

$$\begin{aligned} F_\alpha^* \omega_{FS} &= \frac{i}{2} \partial \bar{\partial} \log \sum_{j=0}^N |f_j(z)|^2 = \frac{i}{2} \partial \bar{\partial} \log K_{\alpha\Phi}(z, z) = \\ &= \frac{i}{2} \partial \bar{\partial} \log \epsilon_{\alpha g} + \frac{i}{2} \partial \bar{\partial} \log e^{\alpha\Phi} = \frac{i}{2} \partial \bar{\partial} \log \epsilon_{\alpha g} + \alpha \omega. \end{aligned} \quad (5)$$

Hence if  $\alpha g$  is balanced then it is projectively induced via the holomorphic map  $F_\alpha$ .

In the literature the function  $\epsilon_{\alpha g}$  was first introduced under the name of  *$\eta$ -function* by J. Rawnsley in [36], later renamed as  *$\theta$ -function* in [6]. If  $\epsilon_{\alpha g}$  is balanced for all sufficiently large  $\alpha$  then the geometric quantization (namely the holomorphic line bundle such that  $c_1(L) = [\omega]$ ) is called regular. Regular quantizations plays a prominent role in the theory of quantization by deformation of Kähler manifolds developed in [6] (there the map  $F_\alpha$  is called the *coherent states map*).

**Remark 1.** The definition of balanced metrics was originally given by S. Donaldson [12] (see also [2], [9] and [26]) in the case of a compact polarized Kähler manifold  $(M, g)$  and generalized in [2] (see also [8], [18], [22], [28] [29]) to the noncompact case. Here we give only the definition for those Kähler metrics which admit a globally defined potential such as the family of metrics  $g_m$  on  $\mathbb{C}^2$  in this paper.

We describe now a well-known example which is the prototype of our analysis.

**Example 1.** Let  $(\mathbb{C}^n, g_0)$  be the complex Euclidean space endowed with the flat metric  $g_0$ , whose associated Kähler form is given by

$$\omega_0 = \frac{i}{2} \partial \bar{\partial} \|z\|^2,$$

where  $\Phi_0 = \|z\|^2 = \sum_{j=1}^n |z_j|^2$ . Let  $\mathcal{H}_{\Phi_0}$  be the weighted Hilbert space of squared integrable holomorphic functions on  $(\mathbb{C}^n, g_0)$ , with weight  $e^{-\|z\|^2}$ , namely

$$\mathcal{H}_{\Phi_0} = \left\{ \varphi \in \text{Hol}(\mathbb{C}^n) \mid \int_{\mathbb{C}^n} e^{-\|z\|^2} |\varphi|^2 \frac{\omega_0^n}{n!} < \infty \right\}.$$

Since the potential depends on  $r_j = |z_j|^2$ ,  $j = 1, \dots, n$ , an orthogonal basis for  $\mathcal{H}_{\Phi_0}$  is given by the monomials  $z_1^{j_1} \dots z_n^{j_n}$ . Furthermore, by

$$\begin{aligned} & \int_{\mathbb{C}} |z_1|^{2j_1} \dots |z_n|^{2j_n} e^{-\|z\|^2} \frac{i^n}{2^n} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \\ &= \pi^n \int_0^\infty \dots \int_0^\infty r_1^{j_1} \dots r_n^{j_n} e^{-(r_1 + \dots + r_n)} dr_1 \dots dr_n \\ &= \pi^n j_1! \dots j_n!, \end{aligned} \tag{6}$$

an orthonormal basis for  $\mathcal{H}_{\Phi_0}$  is given by  $\left\{ \frac{z_1^{j_1} \dots z_n^{j_n}}{\sqrt{\pi^n j_1! \dots j_n!}} \right\}$ . The reproducing kernel of  $\mathcal{H}_{\Phi_0}$  reads

$$K_{\Phi_0}(z, z) = \frac{1}{\pi^n} e^{\|z\|^2},$$

hence  $\epsilon_{g_0} = \frac{1}{\pi^n}$ , and so  $g_0$  is a balanced metric. Moreover, since for all  $\alpha > 0$ , the metric  $\alpha g_0$  is holomorphically isometric to  $g_0$  it follows that also  $\alpha g_0$  is a balanced metric.

**Remark 2.** If  $g$  is a balanced (resp. projectively induced) metric on a complex manifold  $M$  and  $\alpha$  is a positive constant then there is not any reason for the metric  $\alpha g$  to be balanced (resp. projectively induced), as it happens in the previous example. Consider for instance a Cartan domain endowed with its Bergman metric  $g_B$ . In accordance with the choice of

$\alpha > 0$ ,  $\alpha g_B$  is either balanced or projectively induced but not balanced or also not projectively induced. In [27] and [29] the first and second author of this paper study the balanced and projectively induced condition for multiples of the Bergman metric  $g_B$  on each Cartan domain in terms of its genus  $\gamma$ .

Here we study the balanced condition for the LeBrun metric. Consider the weighted Hilbert space  $\mathcal{H}_{\alpha\Phi_m}$  for the metric  $\alpha g_m$ ,  $\alpha > 0$  (where  $\Phi_m$  is given by (1)), namely

$$\mathcal{H}_{\alpha\Phi_m} = \left\{ f \in \text{Hol}(\mathbb{C}^2) \mid \int_{\mathbb{C}^2} e^{-\alpha\Phi_m} |f|^2 \frac{\omega_0^2}{2} \right\},$$

where we have taken into account that  $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0 = \omega_0^2$ . The following lemma is needed in the proof of our main results.

**Lemma 3.** *The monomials  $\{z_1^j z_2^k\}$ ,  $j, k = 1, \dots$  are a complete orthogonal system for the Hilbert space  $\mathcal{H}_{\alpha\Phi_m}$ . Moreover Rawnsley's function is given by:*

$$\begin{aligned} \epsilon_{\alpha g} &= e^{-\alpha\Phi_m} K_{\alpha\Phi_m} = e^{-\alpha\Phi_m} \sum_{j,k=0}^{+\infty} \frac{|z_1|^{2j} |z_2|^{2k}}{\|z_1^j z_2^k\|_{m,\alpha}^2} \\ &= e^{-\alpha\Phi_m} \sum_{j,k=0}^{+\infty} \frac{e^{2m(j-k)(U-V)} U^j V^k}{\|z_1^j z_2^k\|_{m,\alpha}^2}. \end{aligned} \quad (7)$$

where  $U = u^2$ ,  $V = v^2$ ,

$$\begin{aligned} \|z_1^j z_2^k\|_{m,\alpha}^2 &= \int_{\mathbb{C}^2} e^{-\alpha(u^2+v^2+m(u^4+v^4))} |z_1|^{2j} |z_2|^{2k} \frac{\omega_0^2}{2} \\ &= \pi^2 [I_{m,\alpha}(j, j, k) I_{m,\alpha}(k, k, j) + 2m I_{m,\alpha}(j+1, j, k) I(k, k, j) + \\ &\quad + 2m I_{m,\alpha}(j, j, k) I_{m,\alpha}(k+1, k, j)], \end{aligned} \quad (8)$$

and

$$I_{m,\alpha}(i, j, k) = \int_0^\infty e^{-\alpha[U+mU^2]+2mU(j-k)} U^i dU. \quad (9)$$

*Proof.* Since the metric depends only on the squared module of the variables it is easy to see that the monomials  $\{z_1^j z_2^k\}$ ,  $j, k = 1, \dots$  are a complete orthogonal system for  $\mathcal{H}_{\alpha\Phi_m}$  (cfr. [14, Lemma 3.11]). By passing to polar coordinates we have

$$\|z_1^j z_2^k\|_{m,\alpha}^2 = \pi^2 \int_0^\infty \int_0^\infty e^{-\alpha[u^2+v^2+m(u^4+v^4)]} r_1^j r_2^k dr_1 dr_2$$

where  $u = u(r_1, r_2), v = v(r_1, r_2)$ . Set  $u^2 = U, v^2 = V$  and consider the map

$$G : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (U, V) \mapsto (r_1 = e^{2m(U-V)}U, r_2 = e^{2m(V-U)}V)$$

and its Jacobian matrix

$$J_G = \begin{pmatrix} (1 + 2mU) e^{2m(U-V)} & -2mU e^{2m(U-V)} \\ -2mV e^{2m(V-U)} & (1 + 2mV) e^{2m(V-U)} \end{pmatrix}.$$

Since  $\det(J_G) = 1 + 2m(U + V) \neq 0$ , by a change of coordinates, the above integral becomes

$$\begin{aligned} & \|z_1^j z_2^k\|_{m, \alpha}^2 = \\ &= \pi^2 \int_0^\infty \int_0^\infty e^{-\alpha[U+V+m(U^2+V^2)]} e^{2mj(U-V)} e^{2mk(V-U)} U^j V^k [1 + 2m(U+V)] dU dV \\ &= \pi^2 \int_0^\infty \int_0^\infty e^{-\alpha[U+V+m(U^2+V^2)]} e^{2mj(U-V)} e^{2mk(V-U)} U^j V^k dU dV + \\ & \quad + 2m\pi^2 \int_0^\infty \int_0^\infty e^{-\alpha[U+V+m(U^2+V^2)]} e^{2mj(U-V)} e^{2mk(V-U)} U^{j+1} V^k dU dV + \\ & \quad + 2m\pi^2 \int_0^\infty \int_0^\infty e^{-\alpha[U+V+m(U^2+V^2)]} e^{2mj(U-V)} e^{2mk(V-U)} U^j V^{k+1} dU dV \\ &= \pi^2 \int_0^\infty e^{-\alpha[U+mU^2]+2mU(j-k)} U^j dU \int_0^\infty e^{-\alpha[V+mV^2]+2mV(k-j)} V^k dV + \\ & \quad + 2m\pi^2 \int_0^\infty e^{-\alpha[U+mU^2]+2mU(j-k)} U^{j+1} dU \int_0^\infty e^{-\alpha[V+mV^2]+2mV(k-j)} V^k dV + \\ & \quad + 2m\pi^2 \int_0^\infty e^{-\alpha[U+mU^2]+2mU(j-k)} U^j dU \int_0^\infty e^{-\alpha[V+mV^2]+2mV(k-j)} V^{k+1} dV. \end{aligned}$$

Formula (8) and hence (7) follows by comparing last equality with (9).  $\square$

We are now in the position to state the first main result of this paper.

**Theorem 4.** *Let  $m \geq 0$ ,  $g_m$  be the LeBrun metric on  $\mathbb{C}^2$  and  $\alpha$  be a positive real number. Then the metric  $\alpha g_m$  is balanced if and only if  $m = 0$ , i.e. it is the flat metric on  $\mathbb{C}^2$ .*

In order to prove it we need the following lemma, interesting on its own sake.

**Lemma 5.** *Let  $m \geq 0$ ,  $g_m$  be the LeBrun metric on  $\mathbb{C}^2$  and  $\alpha$  be a positive real number. Then  $\alpha g_m$  is not projectively induced for  $m > \frac{\alpha}{2}$ .*

*Proof.* Assume by contradiction that  $\alpha g_m$  is projectively induced, namely that there exists  $N \leq \infty$  and a Kähler immersion of  $(\mathbb{C}^2, \alpha g_m)$  into  $\mathbb{C}P^N$ .

Then, it does exist also a Kähler immersion into  $\mathbb{C}P^N$  of the Kähler submanifold of  $(\mathbb{C}^2, \omega_m)$  defined by  $z_2 = 0$ ,  $z_1 = z$ , endowed with the induced metric, having potential  $\tilde{\Phi}_m = u^2 + mu^4$ , where  $u$  is defined implicitly by  $z\bar{z} = e^{2mu^2}u^2$ . Observe that  $\tilde{\Phi}_m$  is the Calabi's diastasis function for this metric, since it is a rotation invariant potential centered at the origin (see [7] or also [27, Th. 3, p. 3]).

Consider the power expansion around the origin of the function  $e^{\alpha\tilde{\Phi}_m} - 1$ , that, by (1) and (2), reads

$$e^{\alpha\tilde{\Phi}_m} - 1 = \alpha|z|^2 + \frac{\alpha}{2}(\alpha - 2m)|z|^4 + \dots$$

Since  $\alpha - 2m \geq 0$  if and only if  $m \leq \frac{\alpha}{2}$ , it follows by Calabi's criterion (see [27, Th. 6, p. 4]) that  $\alpha g_m$  can not admit a Kähler immersion into  $\mathbb{C}P^N$  for any  $m > \frac{\alpha}{2}$ .  $\square$

**Remark 6.** We conjecture that  $\alpha g_m$  is not projectively induced also for  $0 < m \leq \frac{\alpha}{2}$  (for  $m = 0$  it is projectively induced and also balanced by Example 1). We have computer evidences of this fact but we still do not have a proof. Nevertheless we do not need this fact in the following construction. It is also worth pointing out that using Calabi's techniques [7] it is a simple matter to verify that  $(\mathbb{C}^2, \alpha g_m)$  cannot be Kähler immersed into the complex hyperbolic space  $\mathbb{C}H^N$ ,  $N \leq \infty$ , equipped with the hyperbolic metric and, moreover, if  $(\mathbb{C}^2, g_m)$  admits a Kähler immersion into  $(\mathbb{C}^N, g_0)$ ,  $N \leq \infty$ , then  $m = 0$ .

*Proof of Theorem 4.* If  $m = 0$ ,  $\alpha g_0$  is the flat metric on  $\mathbb{C}^2$  which is balanced for all positive values of  $\alpha$  (cfr. Example 1). Furthermore, for  $m > \frac{\alpha}{2}$ , by Lemma 5,  $\alpha g_m$  is not projectively induced and hence it cannot be balanced. Thus the proof of the theorem will be achieved if we show that  $\alpha g_m$  is not balanced for  $0 < m \leq \frac{\alpha}{2}$ . Notice that if  $\alpha g_m$  is balanced then there exists a basis  $\{f_j^\alpha\}$  for  $\mathcal{H}_{\alpha\Phi_m}$  such that

$$\int_{\mathbb{C}^2} e^{-\alpha(u^2+v^2+m(u^4+v^4))} f_j^\alpha \bar{f}_k^\alpha \frac{\omega_0^2}{2} = \lambda \delta_{jk}, \quad (10)$$

for some  $\lambda$  not depending on  $j, k$ . Notice also that the power expansion around the origin of the function  $e^{\alpha\Phi_m}$  reads

$$e^{\alpha\Phi_m} = 1 + \alpha(|z_1|^2 + |z_2|^2) + \alpha \left( \frac{\alpha}{2} - m \right) (|z_1|^4 + |z_2|^4) + \alpha^2 |z_1|^2 |z_2|^2 + \dots$$

as it follows by (1) and (2). Therefore, since the metric  $g_m$  depends on the module of the variables, we can assume without loss of generality that

$f_0^\alpha = 1$  and  $f_1^\alpha = \alpha z_1 z_2$ . We are going to show that  $\|f_0^\alpha\|_{m,\alpha}^2 = \|f_1^\alpha\|_{m,\alpha}^2$  does not hold for  $0 < m \leq \frac{\alpha}{2}$  and hence, for these values of  $m$ ,  $\alpha g_m$  is not balanced. By (8) we have

$$\|f_0^\alpha\|_{m,\alpha}^2 = \int_{\mathbb{C}^2} e^{-\alpha\Phi_m} \frac{\omega_0^2}{2} = \pi^2 [I_{m,\alpha}(0,0,0)^2 + 4mI_{m,\alpha}(1,0,0)I_{m,\alpha}(0,0,0)], \quad (11)$$

$$\begin{aligned} \|f_1^\alpha\|_{m,\alpha}^2 &= \alpha^2 \int_{\mathbb{C}^2} e^{-\alpha\Phi_m} |z_1|^2 |z_2|^2 \frac{\omega_0^2}{2} \\ &= \alpha^2 \pi^2 [I_{m,\alpha}(1,1,1)^2 + 4mI_{m,\alpha}(2,1,1)I_{m,\alpha}(1,1,1)]. \end{aligned} \quad (12)$$

For all  $\alpha > 0$  define  $h_\alpha: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$  as

$$\begin{aligned} h_\alpha(m) &= I_{m,\alpha}(0,0,0)^2 + 4mI_{m,\alpha}(1,0,0)I_{m,\alpha}(0,0,0) + \\ &\quad - \alpha^2 I_{m,\alpha}(1,1,1)^2 - 4m\alpha^2 I_{m,\alpha}(2,1,1)I_{m,\alpha}(1,1,1). \end{aligned}$$

We need to prove that  $h_\alpha(m) \neq 0$  for  $0 < m \leq \frac{\alpha}{2}$ . Observe that  $h_\alpha(0) = 0$ , in accordance with the fact that the flat metric  $g_0$  on  $\mathbb{C}^2$  is balanced for all  $\alpha > 0$ . We have:

$$I_{m,\alpha}(0,0,0) = \int_0^\infty e^{-\alpha(U+mU^2)} dU, \quad (13)$$

$$\begin{aligned} I_{m,\alpha}(1,0,0) &= I_{m,\alpha}(1,1,1) = \int_0^\infty e^{-\alpha(U+mU^2)} U dU \\ &= -\frac{1}{2m} I_{m,\alpha}(0,0,0) + \frac{1}{2\alpha m}, \end{aligned} \quad (14)$$

$$\begin{aligned} I_{m,\alpha}(2,1,1) &= \int_0^\infty e^{-\alpha(U+mU^2)} U^2 dU \\ &= \frac{2m+\alpha}{4m^2\alpha} I_{m,\alpha}(0,0,0) - \frac{1}{4m^2\alpha}. \end{aligned} \quad (15)$$

Hence

$$\begin{aligned} h_\alpha(m) &= \frac{1}{4m^2\alpha} (-4m^2\alpha I_{m,\alpha}(0,0,0)^2 + 8m^2 I_{m,\alpha}(0,0,0) + \alpha^3 I_{m,\alpha}(0,0,0)^2 + \\ &\quad + 4\alpha^2 m I_{m,\alpha}(0,0,0)^2 + \alpha - 4m\alpha I_{m,\alpha}(0,0,0) - 2\alpha^2 I_{m,\alpha}(0,0,0)). \end{aligned}$$

It follows that  $h_\alpha(m) = 0$  if and only if  $I_{m,\alpha}(0,0,0)$  satisfies the following second order equation

$$y^2(-4m^2\alpha + \alpha^3 + 4m\alpha^2) + 2y(4m^2 - 2m\alpha - \alpha^2) + \alpha = 0.$$

Its discriminant

$$\Delta = (4m^2 - 2m\alpha - \alpha^2)^2 - \alpha^2(-4m^2 + \alpha^2 + 4m\alpha) = 16m^3(m - \alpha)$$

is negative for all  $0 < m < \alpha$ , so that  $h_\alpha(m) \neq 0$  for all  $0 < m < \alpha$  and, a fortiori, for  $0 < m \leq \frac{\alpha}{2}$ , which is exactly what we wanted to prove.  $\square$



## 3. ENGLIŠ EXPANSION FOR THE LEBRUN METRIC

Let  $(M, g, \omega = \frac{i}{2}\partial\bar{\partial}\Phi)$  be a Kähler manifold as in the previous section (i.e. admitting a globally defined Kähler potential) and consider Rawnsley's function (4) for a fixed  $\alpha > 0$ . Even if this function is not constant (i.e.  $\alpha g$  is not balanced) it is interesting to understand when  $\epsilon_{\alpha g}$  admits an asymptotic expansion for  $\alpha \rightarrow +\infty$ . For example this turns out to be true when  $M$  is a strongly pseudoconvex bounded domain in  $\mathbb{C}^n$  with real analytic boundary or when  $M$  is an Hermitian symmetric space of noncompact type equipped with its Bergman metric (cfr. Remark 2 above).

Indeed M. Engliš [16] shows that for these domains  $\epsilon_{\alpha g}$  admits the following asymptotic expansion (which has been christened as *Engliš expansion* in [31]) with respect to  $\alpha$

$$\epsilon_{\alpha}(x) \sim \sum_{j=0}^{+\infty} a_j(x) \alpha^{n-j} \quad (16)$$

where  $a_0 = 1$  and  $a_j$  for  $j = 0, 1, 2, \dots$ , are smooth coefficients, in the sense that, for every integers  $l, r$  and every compact  $H \subseteq M$ ,

$$\|\epsilon_{\alpha}(x) - \sum_{j=0}^l a_j(x) \alpha^{n-j}\|_{C^r} \leq \frac{C(l, r, H)}{\alpha^{l+1}} \quad (17)$$

for some constant  $C(l, r, H) > 0$ . Moreover, in [17] M. Engliš also computes the coefficients  $a_j$ ,  $j \leq 3$ , namely:

$$\left\{ \begin{array}{l} a_0 = 1 \\ a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\text{Ric}|^2 + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\text{div div}(R, \text{Ric}) - \frac{1}{6}\text{div div}(\rho\text{Ric}) + \\ \quad + \frac{1}{48}\Delta(|R|^2 - 4|\text{Ric}|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|\text{Ric}|^2 + |R|^2) + \\ \quad + \frac{1}{24}(\sigma_3(\text{Ric}) - \text{Ric}(R, R) - R(\text{Ric}, \text{Ric})), \end{array} \right. \quad (18)$$

where (see also [25] and [33])  $\rho$ ,  $R$ ,  $\text{Ric}$  denote respectively the scalar curvature, the curvature tensor and the Ricci tensor of  $(M, g)$ , and we are using

the following notations (in local coordinates  $z_1, \dots, z_n$ ):

$$\begin{aligned}
|D' \rho|^2 &= \sum_{i=1}^n \left| \frac{\partial \rho}{\partial z_i} \right|^2, \\
|D' \text{Ric}|^2 &= \sum_{i,j,k=1}^n |\text{Ric}_{i\bar{j},k}|^2, \\
|D' R|^2 &= \sum_{i,j,k,l,p=1}^n |R_{i\bar{j}k\bar{l},p}|^2, \\
\text{div div}(\rho \text{Ric}) &= 2|D' \rho|^2 + \sum_{i,j=1}^n \text{Ric}_{i\bar{j}} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_i} + \rho \Delta \rho, \\
\text{div div}(R, \text{Ric}) &= - \sum_{i,j=1}^n \text{Ric}_{i\bar{j}} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_i} - 2|D' \text{Ric}|^2 + \\
&\quad + \sum_{i,j,k,l=1}^n R_{j\bar{i}l\bar{k}} R_{i\bar{j},k\bar{l}} - R(\text{Ric}, \text{Ric}) - \sigma_3(\text{Ric}), \\
R(\text{Ric}, \text{Ric}) &= \sum_{i,j,k,l=1}^n R_{i\bar{j}k\bar{l}} \text{Ric}_{j\bar{i}} \text{Ric}_{l\bar{k}}, \\
\text{Ric}(R, R) &= \sum_{i,j,k,l,p,q=1}^n \text{Ric}_{i\bar{j}} R_{j\bar{k}p\bar{q}} R_{k\bar{i}q\bar{p}}, \\
\sigma_3(\text{Ric}) &= \sum_{i,j,k=1}^n \text{Ric}_{i\bar{j}} \text{Ric}_{j\bar{k}} \text{Ric}_{k\bar{i}}.
\end{aligned} \tag{19}$$

Notice that the term  $a_1$  was already computed by Berezin in his seminal paper [3] on quantization by deformation. Actually Engliš expansion is strongly related to Berezin transform and to asymptotic expansion of Laplace integrals (see [16] and [25]). It is important to point out that for a general noncompact manifold there is not a general theorem which assures the existence of Engliš expansion (see [21] for the case of the Kepler manifold). A partial result in this direction is Theorem 6.1.1 due to X. Ma and G. Marinescu [34], which translated in our situation and with our notations reads as:

**Theorem 7.** *Let  $(M, g, \omega = \frac{i}{2} \partial \bar{\partial} \Phi)$  be a complete Kähler manifold and, for  $\alpha > 0$ ,  $K_{\alpha\Phi}$  be the reproducing kernel of the space*

$$\mathcal{H}_{\alpha\Phi} = \left\{ f \in \text{Hol}(M) \mid \int_M e^{-\alpha\Phi} |f|^2 \frac{\omega^n}{n!} < \infty \right\}. \tag{20}$$

*Then  $\epsilon_{\alpha\Phi} = e^{-\alpha\Phi} K_{\alpha\Phi}$  admits an asymptotic expansion in  $\alpha$  with coefficients given by (18) provided there exists  $c > 0$  such that*

$$iR^{\det} > -c\Theta$$

*where  $R^{\det}$  denotes the curvature of the connection on  $\det(T^{(1,0)}M)$  induced by  $g$ .*

Engliš expansion is the counterpart of the celebrated TYZ (Tian-Yau-Zelditch) expansion of Kempf's distortion function

$$T_{\alpha g}(x) \sim \sum_{j=0}^{\infty} b_j(x) m^{n-j}, \quad \alpha = 0, 1, 2, \dots,$$

for polarized compact Kähler manifolds  $M$  (see [38] and also [1]), where  $b_j$ ,  $j = 0, 1, \dots$ , are smooth coefficients with  $b_0(x) = 1$ . Z. Lu [32], by means

of Tian's peak section method, proved that each of the coefficients  $b_j(x)$  is a polynomial of the curvature of the metric  $g$  and its covariant derivatives at  $x$ . Such a polynomial can be found by finitely many steps of algebraic operations. Furthermore  $b_j(x)$ ,  $j = 1, 2, 3$  have the same values of those given by (18), namely  $b_j(x) = a_j(x)$ ,  $j = 1, 2, 3$ . (see also [24] and [25] for the computations of the coefficients  $b_j$ 's through Calabi's diastasis function).

Due to Donaldson's work (cfr. [12], [13]) in the compact case and respectively to the theory of quantization in the noncompact case (see, e.g. [3] and [6]), it is natural to study metrics with the coefficients  $b_k$ 's of the TYZ expansion (resp.  $a_k$ 's of Engliš expansion) being prescribed. In particular, the vanishing of these coefficients for large enough  $k$  turns out to be related to some important problems in the theory of pseudoconvex manifolds. For instance, Lu and Tian ([33]) showed that the  $b_j$ 's vanish for  $j > n$  provided that the Bergman kernel  $K$  of the pseudoconvex domain  $D$  given by the disc bundle of a polarization of  $M$ , has vanishing log term. Let us recall that this means that  $b \equiv 0$  in the decomposition

$$K(x, x) = \frac{a(x)}{\psi(x)^{n+1}} + b(x) \log \psi(x), \quad (21)$$

proved by Fefferman in his celebrated paper [20] for the Bergman kernel  $K$  of every bounded strictly pseudoconvex domain  $\Omega = \{\psi > 0\} \subseteq \mathbb{C}^n$  given by a defining function  $\psi$  (for analogous results in the case of Szegő kernels or weighted Bergman kernels, see respectively [33] and [19]). The vanishing of the log term is a condition of considerable interest both from an analytic and a geometric point of view. For example, Ramadanov ([35]) conjectured that if the log term of the Bergman kernel vanishes, then the domain is biholomorphic equivalent to the unit ball (this was proved to be true for some special cases, such as domains in  $\mathbb{C}^2$  or domains with rotational symmetries, see for example [4], [5]); Lu and Tian conjectured that if the log term of the Szegő kernel of the unit circle bundle of a polarisation of  $(\mathbb{C}\mathbb{P}^n, \omega)$  vanishes, for some  $\omega$  cohomologous to the Fubini-Study form  $\omega_{FS}$ , then  $\omega$  is holomorphically isometric to  $\omega_{FS}$ . Observe that for  $n = 1$  (see [33, Theorem 1.1]) the proof of this conjecture follows by the above-mentioned result of Lu and Tian and by the expression of  $a_2 = b_2$  in (18) which imply that the scalar curvature  $\rho$  is constant.

The following theorem is the second main result of this paper.

**Theorem 8.** *Let  $m \geq 0$ ,  $g_m$  be the LeBrun metric on  $\mathbb{C}^2$  and  $\alpha$  be a positive real number. Then Engliš expansion of Rawnsley's function  $\epsilon_{\alpha g}$  exists and*

the coefficients  $a_j$  are given by (18). Moreover, if  $a_j = 0$  for  $j > 2$  then  $m = 0$ , i.e.  $g_m = g_0$  is the flat metric on  $\mathbb{C}^2$ .

*Proof.* The first part of the statement follows from Theorem 7 applied to  $M = \mathbb{C}^2$  and  $g = g_m$ , since in this case  $iR^{det}$  equals the Ricci form of  $g$  which vanishes by the Ricci-flatness of LeBrun's metrics.

Assume now that  $a_j = 0$  vanishes for  $j > 2$ . In particular,  $a_3 = 0$ . By (18), (19) and by the fact that  $g_m$  is Ricci flat it follows that the Laplacian of the norm of its curvature tensor vanishes, i.e.  $\Delta|R|^2 = 0$ . On the other hand formula (29) in Lemma A2 yields  $m = 0$  and we are done.  $\square$

Combining the previous theorem with formula (5) we get the following corollary which should be compared with Theorem 4.1 in [37], where it is proven the analogous result for Kähler–Einstein metrics with Einstein constant  $-1$ .

**Corollary 9.** *Let  $g_m$  be the LeBrun metric on  $\mathbb{C}^2$ ,  $m \geq 0$ . Then  $g_m$  can be approximated by suitable normalized projectively induced Kähler metrics on  $\mathbb{C}^2$ .*

*Proof.* Let  $F_\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}P^\infty$  be the coherent states map, namely the holomorphic map constructed with the orthonormal basis of monomials given by Lemma 3. Then, by formula (5), since  $a_0 = 1$ , one gets  $\lim_{\alpha \rightarrow +\infty} \frac{1}{\alpha} F_\alpha^* g_{FS} = g$ .  $\square$

#### APPENDIX A. SOME COMPUTATION

**Proposition A1.** *Fix  $m \geq 0$  and let  $g_m$  be the LeBrun metric on  $\mathbb{C}^2$ . Then  $g_m$  is complete and  $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0$ .*

*Proof.* Consider the Kähler potential

$$\Phi = \Phi_m(U, V) = U + V + m(U^2 + V^2)$$

for the LeBrun's metric  $g_m$ , where  $U = u^2$  and  $V = v^2$ . Here  $U$  and  $V$  are implicitly defined by

$$x_1 = |z_1|^2 = e^{2m(U-V)}U, \quad (22)$$

$$x_2 = |z_2|^2 = e^{2m(V-U)}V. \quad (23)$$

(cfr. (1) and (2)). Since  $\Phi = \Phi(x_1, x_2)$  (i.e. depends on the squared module of the variables), we can write the matrix  $g_m = (g_{\alpha\bar{\beta}})$  as

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \Phi_{11} x_1 + \Phi_1 & \Phi_{12} \bar{z}_1 z_2 \\ \Phi_{12} z_1 \bar{z}_2 & \Phi_{22} x_2 + \Phi_2 \end{pmatrix}, \quad (24)$$

where we write  $\Phi_i = \partial\Phi/\partial x_i$  and  $\Phi_{ij} = \partial^2\Phi/\partial x_i\partial x_j$ . Now observe that

$$\frac{\partial\Phi}{\partial x_i} = \frac{\partial\Phi}{\partial U} \frac{\partial U}{\partial x_i} + \frac{\partial\Phi}{\partial V} \frac{\partial V}{\partial x_i}, \quad i = 1, 2,$$

and

$$\begin{pmatrix} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial U} & \frac{\partial x_1}{\partial V} \\ \frac{\partial x_2}{\partial U} & \frac{\partial x_2}{\partial V} \end{pmatrix}^{-1}.$$

By

$$\begin{aligned} \frac{\partial x_1}{\partial U} &= (1 + 2mU)e^{2m(U-V)}, & \frac{\partial x_1}{\partial V} &= -2mUe^{2m(U-V)}, \\ \frac{\partial x_2}{\partial V} &= (1 + 2mV)e^{2m(V-U)}, & \frac{\partial x_2}{\partial U} &= -2mVe^{2m(V-U)}, \end{aligned}$$

since  $1 + 2m(U + V) \neq 0$ , we get

$$\begin{pmatrix} \frac{\partial x_1}{\partial U} & \frac{\partial x_1}{\partial V} \\ \frac{\partial x_2}{\partial U} & \frac{\partial x_2}{\partial V} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{(1+2mV)e^{2m(V-U)}}{1+2m(U+V)} & \frac{2me^{2m(U-V)}U}{1+2m(U+V)} \\ \frac{2me^{2m(V-U)}V}{1+2m(U+V)} & \frac{(1+2mU)e^{2m(U-V)}}{1+2m(U+V)} \end{pmatrix}.$$

Thus, since  $\partial\Phi/\partial U = 1 + 2mU$  and  $\partial\Phi/\partial V = 1 + 2mV$ , a direct computation gives

$$\begin{aligned} \Phi_1 &= \frac{\partial\Phi}{\partial x_1} = (1 + 2mV)e^{2m(V-U)}, \\ \Phi_2 &= \frac{\partial\Phi}{\partial x_2} = (1 + 2mU)e^{2m(U-V)}, \\ \Phi_{11} &= \frac{\partial\Phi_1}{\partial U} \frac{\partial U}{\partial x_1} + \frac{\partial\Phi_1}{\partial V} \frac{\partial V}{\partial x_1} = -\frac{2me^{4m(V-U)}}{1 + 2m(U + V)}, \\ \Phi_{12} &= \frac{\partial\Phi_1}{\partial U} \frac{\partial U}{\partial x_2} + \frac{\partial\Phi_1}{\partial V} \frac{\partial V}{\partial x_2} = \frac{4m(1 + m(U + V))}{1 + 2m(U + V)}, \\ \Phi_{22} &= \frac{\partial\Phi_2}{\partial U} \frac{\partial U}{\partial x_2} + \frac{\partial\Phi_2}{\partial V} \frac{\partial V}{\partial x_2} = -\frac{2me^{4m(U-V)}}{1 + 2m(U + V)}. \end{aligned}$$

Combining these formulas with (22), (23) and (24) above, we get

$$g_m = \begin{pmatrix} \frac{1+4mV(1+mU+mV)}{1+2m(U+V)} e^{2m(V-U)} & \frac{4m(1+mU+mV)}{1+2m(U+V)} z_2 \bar{z}_1 \\ \frac{4m(1+mU+mV)}{1+2m(U+V)} z_1 \bar{z}_2 & \frac{1+4mU(1+mU+mV)}{1+2m(U+V)} e^{2m(U-V)} \end{pmatrix}, \quad (25)$$

from which follows easily, substituting again (22) and (23), that  $\det(g_m) = 1$ , i.e.  $\omega_m \wedge \omega_m = \det(g_m)\omega_0 \wedge \omega_0 = \omega_0 \wedge \omega_0$ .

In order to prove the completeness of  $g_m$ , consider the metric  $\tilde{g}_m$  on  $\mathbb{C}^2$  defined by

$$\tilde{g}_m = \frac{1}{1 + 2m(U + V)} \left( e^{2m(V-U)} dz_1 d\bar{z}_1 + e^{2m(U-V)} dz_2 d\bar{z}_2 \right).$$

We claim that  $g_m \geq \tilde{g}_m$ . In fact, by (25) given  $(w_1, w_2) \in T_{(z_1, z_2)}\mathbb{C}^2$  we get

$$\begin{aligned} \|(w_1, w_2)\|_{g_m}^2 &= \frac{1 + 4mV(1 + mU + mV)}{1 + 2m(U + V)} e^{2m(V-U)} |w_1|^2 + \\ &\quad + \frac{4m(1 + mU + mV)}{1 + 2m(U + V)} (z_2 \bar{z}_1 w_1 \bar{w}_2 + z_1 \bar{z}_2 w_2 \bar{w}_1) + \\ &\quad + \frac{1 + 4mU(1 + mU + mV)}{1 + 2m(U + V)} e^{2m(U-V)} |w_2|^2 = \\ &= \frac{e^{2m(U-V)}}{1 + 2m(U + V)} |w_1|^2 + \frac{e^{2m(U-V)}}{1 + 2m(U + V)} |w_2|^2 \\ &\quad + \frac{4m(1 + mU + mV)}{1 + 2m(U + V)} |z_2 w_1 + z_1 w_2|^2, \end{aligned}$$

where we used the identity  $z_2 \bar{z}_1 w_1 \bar{w}_2 + z_1 \bar{z}_2 w_2 \bar{w}_1 = |z_2 w_1 + z_1 w_2|^2 - |z_2|^2 |w_1|^2 - |z_1|^2 |w_2|^2$ . Our problem reduces then to show that  $(\mathbb{C}^2, \tilde{g}_m)$  is complete (see e.g. [11, Ex. 7 p. 153]). Actually, it is enough to show that  $(S, \tilde{g}_m|_S)$  is complete, where  $S$  is the surface of real dimension 2 defined by

$$S = \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_1) = 0, \text{Im}(z_2) = 0\}.$$

In fact, since  $\tilde{g}_m$  depends only on the modules of the variables, fixed  $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ , the maps  $\varphi_\theta, \psi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by  $\varphi_\theta(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$  and  $\psi(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$ , are isometries. Thus,  $S$  is totally geodesic in  $(\mathbb{C}^2, \tilde{g}_m)$  being the fixed points' locus of  $\psi$ , and every geodesic through the origin of  $(\mathbb{C}^2, \tilde{g}_m)$  can be viewed as a geodesic of  $S$ , since any vector tangent at the origin of  $\mathbb{C}^2$  is isometric through a map  $\varphi_\theta$ , for some  $\theta$ , to a vector of  $T_{(0,0)}S$ .

Let  $\mu_1 = \text{Re}(z_1)$ ,  $\mu_2 = \text{Re}(z_2)$ . For  $(z_1, z_2) \in S$  we get  $z_1 = \mu_1$ ,  $z_2 = \mu_2$  and by  $|\mu_1| = e^{m(u^2 - v^2)} u$ ,  $|\mu_2| = e^{m(v^2 - u^2)} v$ , it follows that

$$\begin{aligned} d\mu_1 &= \pm [(2mu^2 + 1)du - 2muvdv] e^{m(u^2 - v^2)}, \\ d\mu_2 &= \pm [(2mv^2 + 1)dv - 2muvdu] e^{m(v^2 - u^2)}, \end{aligned}$$

which implies

$$\begin{aligned} \tilde{g}_m|_S &= \frac{[(2mu^2 + 1)du - 2muvdv]^2 + [(2mv^2 + 1)dv - 2muvdu]^2}{1 + 2m(u^2 + v^2)} \\ &= \frac{(1 + 2mu^2)^2 + 4m^2 u^2 v^2}{1 + 2m(u^2 + v^2)} du^2 + \frac{(1 + 2mv^2)^2 + 4m^2 u^2 v^2}{1 + 2m(u^2 + v^2)} dv^2 + \\ &\quad - \frac{8muv(1 + mu^2 + mv^2)}{1 + 2m(u^2 + v^2)} dudv. \end{aligned}$$

We claim that

$$\tilde{g}_m|_S \geq \frac{du^2 + dv^2}{1 + 2m(u^2 + v^2)}. \quad (26)$$

Observe that if (26) holds we are done, since in polar coordinates the right-hand side reads  $(d\rho^2 + \rho^2 d\theta^2)/(1 + 2m\rho^2)$ , and

$$\int_0^{+\infty} \|\alpha'(t)\|_{\tilde{g}_m|_S} dt \geq \int_{\rho_0}^{+\infty} \frac{d\rho}{\sqrt{1 + 2m\rho^2}} = +\infty,$$

for any divergent curve  $\alpha : [0, +\infty) \rightarrow S$  (see e.g. [11, Ex. 5 p. 153]). In order to verify (26), observe that, for any vector  $(\alpha, \beta)$  tangent to  $S \equiv \mathbb{R}^2$  at  $(u, v)$ , it is equivalent to

$$\begin{aligned} & ((1 + 2mu^2)^2 + 4m^2u^2v^2 - 1)\alpha^2 + ((1 + 2mv^2)^2 + 4m^2u^2v^2 - 1)\beta^2 + \\ & - 8muv(1 + mu^2 + mv^2)\alpha\beta \geq 0. \end{aligned} \quad (27)$$

If  $\beta = 0$ , it holds true. Assume  $\beta \neq 0$ . Setting  $\gamma = \alpha/\beta$  we get a second order equation in  $\gamma$  whose discriminant is 0. The conclusion follows by noticing that the coefficient of the leading term, namely  $(1 + 2mu^2)^2 + 4m^2u^2v^2 - 1$ , is nonnegative and it vanishes iff  $u = 0$ , case for which (27) reads  $(1 + 2mv^2)^2 - 1 \geq 0$ .  $\square$

**Lemma A2.** *Fix  $m \geq 0$  and let  $g = g_m$  be the corresponding LeBrun metric on  $\mathbb{C}^2$ . Then the squared norm of its curvature tensor and its Laplacian are given respectively by:*

$$|R|^2 = \frac{96m^2}{(1 + 2m(U + V))^8}, \quad (28)$$

and

$$\Delta|R|^2 = \frac{3072m^3(7m(U + V) - 1)}{(1 + 2m(U + V))^{11}}. \quad (29)$$

*Proof.* Recall that the curvature tensor for a given Kähler metric  $g = (g_{i\bar{j}})$  on a  $n$ -dimensional complex manifold is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{l}}}{\partial z_k \partial \bar{z}_j} + \sum_{pq=1}^n g^{p\bar{q}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{l}}}{\partial \bar{z}_j}, \quad (30)$$

and, in accordance with the general definition of norm of a complex tensor, (see e.g. [39, p. 127]) we have:

$$|R|^2 = \sum_{i,j,k,l,p,q,r,s=1}^n g^{p\bar{i}} g^{j\bar{q}} g^{r\bar{k}} g^{l\bar{s}} R_{i\bar{j}k\bar{l}} \overline{R_{p\bar{q}r\bar{s}}}. \quad (31)$$

Moreover the Laplacian  $\Delta|R|^2$  reads

$$\Delta|R|^2 = \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial^2 |R|^2}{\partial z_i \partial \bar{z}_j}. \quad (32)$$

Here  $g^{i\bar{j}}$ ,  $i, j = 1, \dots, n$  are the entries of the inverse of  $(g_{i\bar{j}})$ , namely  $\sum_{j=1}^n g^{i\bar{j}} g_{j\bar{k}} = \delta_{ik}$ .

Observe that by (24), since  $\det(g) = 1$ , we have

$$\begin{pmatrix} g^{1\bar{1}} & g^{1\bar{2}} \\ g^{2\bar{1}} & g^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \Phi_{22} x_2 + \Phi_2 & -\Phi_{12} \bar{z}_1 z_2 \\ -\Phi_{12} z_1 \bar{z}_2 & \Phi_{11} x_1 + \Phi_1 \end{pmatrix}. \quad (33)$$

Further, by (24) we obtain

$$\begin{aligned} \frac{\partial g_{k\bar{k}}}{\partial z_j} &= \Phi_{kkj} \bar{z}_j x_k + \Phi_{kk} \bar{z}_j \delta_{jk} + \Phi_{kj} \bar{z}_j, \\ \frac{\partial g_{k\bar{j}}}{\partial z_j} &= \Phi_{kj} \bar{z}_k + \Phi_{kjj} \bar{z}_k x_j, \quad (\text{for } k \neq j), \\ \frac{\partial g_{j\bar{k}}}{\partial z_j} &= \Phi_{jkj} \bar{z}_j^2 z_k, \quad (\text{for } k \neq j). \end{aligned} \quad (34)$$

where  $\Phi_{jkl} = \frac{\partial^3 \Phi}{\partial z_j \partial \bar{z}_k \partial z_l}$ ,  $j, k, l = 1, 2$ . It follows easily by the previous computations that

$$\begin{aligned} \Phi_{111} &= \frac{4m^2 e^{6m(V-U)}}{(1+2m(U+V))^3} (3+4m(U+2V)), \\ \Phi_{222} &= \frac{4m^2 e^{6m(U-V)}}{(1+2m(U+V))^3} (3+4m(2U+V)), \\ \Phi_{112} &= -\frac{4m^2 e^{2m(V-U)}}{(1+2m(U+V))^3} (1+4mV), \\ \Phi_{221} &= -\frac{4m^2 e^{2m(U-V)}}{(1+2m(U+V))^3} (1+4mU). \end{aligned}$$

Finally, by (34) we get, for  $j, k = 1, 2$ ,

$$\begin{aligned} \frac{\partial^2 g_{k\bar{k}}}{\partial z_k \partial \bar{z}_k} &= \Phi_{kkkk} x_k^2 + 4\Phi_{kkk} x_k + 2\Phi_{kk}, \\ \frac{\partial^2 g_{k\bar{j}}}{\partial z_k \partial \bar{z}_j} &= \frac{\partial^2 g_{k\bar{k}}}{\partial z_j \partial \bar{z}_j} = \Phi_{kkjj} x_k x_j + \Phi_{kkj} x_k + \Phi_{kjj} x_j + \Phi_{kj}, \quad (\text{for } k \neq j) \\ \frac{\partial^2 g_{j\bar{k}}}{\partial z_k \partial \bar{z}_k} &= \frac{\partial^2 g_{k\bar{k}}}{\partial z_k \partial \bar{z}_j} = \Phi_{kkkj} z_k^2 \bar{z}_k \bar{z}_j + 2\Phi_{kkj} z_k \bar{z}_j, \quad (\text{for } k \neq j), \\ \frac{\partial^2 g_{j\bar{k}}}{\partial z_j \partial \bar{z}_k} &= \Phi_{kkjj} z_k^2 \bar{z}_j^2, \quad (\text{for } k \neq j), \end{aligned}$$

where



$$\begin{aligned}\frac{\partial^4 \Phi}{\partial x_1^4} &= \Phi_{1111} = -\frac{16m^3 e^{8m(V-U)}(8+43mV+19mU+12m^2(4UV+5V^2+U^2))}{(1+2m(U+V))^5}, \\ \frac{\partial^4 \Phi}{\partial x_2^4} &= \Phi_{2222} = -\frac{16m^3 e^{8m(U-V)}(8+43mU+19mV+12m^2(4UV+5U^2+V^2))}{(1+2m(U+V))^5}, \\ \frac{\partial^4 \Phi}{\partial x_1^3 \partial x_2} &= \Phi_{1112} = \frac{16m^3 e^{4m(V-U)}(2+mU+13mV+24m^2V^2)}{(1+2m(U+V))^5}, \\ \frac{\partial^4 \Phi}{\partial x_1 \partial x_2^3} &= \Phi_{2221} = \frac{16m^3 e^{4m(U-V)}(2+mV+13mU+24m^2U^2)}{(1+2m(U+V))^5}, \\ \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} &= \Phi_{1122} = \Phi_{2211} = -\frac{8m^3(1+2m(U+V)+8m^2(U-V)^2)}{(1+2m(U+V))^4}.\end{aligned}$$

Thus, it is not hard to see using (A2) that

$$\begin{aligned}R_{1\bar{1}1\bar{1}} &= \frac{4me^{4m(V-U)}}{(1+2m(U+V))^5}(16m^4V(V^3-4UV^2+U^2V)+ \\ &\quad + 32m^3V^2(V-2U)+8m^2V(3V-2U)+8mV+1), \\ R_{1\bar{1}1\bar{2}} = R_{1\bar{2}1\bar{1}} = \overline{R_{2\bar{1}1\bar{1}}} = \overline{R_{1\bar{1}2\bar{1}}} &= \frac{8m^2z_1\bar{z}_2e^{2m(V-U)}}{(1+2m(U+V))^5}(8m^3V(V^2-4UV+U^2)+ \\ &\quad + 4m^2V(V-5U)-2m(V+2U)-1), \\ R_{1\bar{1}2\bar{2}} = R_{2\bar{2}1\bar{1}} = R_{1\bar{2}2\bar{1}} = R_{2\bar{1}2\bar{1}} &= \frac{4m}{(1+2m(u+v))^5}(16m^4uv(u^2+v^2-4uv)+ \\ &\quad - 16m^3uv(u+v)-4m^2(u^2+v^2+4uv)-4m(u+v)-1), \\ R_{1\bar{2}1\bar{2}} = \overline{R_{2\bar{1}2\bar{1}}} &= \frac{32(2m^2(V^2+U^2-4UV)-2m(U+V)-1)\bar{z}_1^2z_2m^3}{(1+2m(U+V))^5}, \\ R_{2\bar{2}2\bar{1}} = R_{2\bar{1}2\bar{2}} = \overline{R_{1\bar{2}2\bar{2}}} = \overline{R_{2\bar{2}1\bar{2}}} &= \frac{8m^2z_2\bar{z}_1e^{2m(U-V)}}{(1+2m(U+V))^5}(8m^3U(U^2-4VU+V^2)+ \\ &\quad + 4m^2U(U-5V)-2m(U+2V)-1), \\ R_{2\bar{2}2\bar{2}} &= \frac{4me^{4m(U-V)}}{(1+2m(U+V))^5}(16m^4U(U^3-4VU^2+V^2U)+ \\ &\quad + 32m^3U^2(U-2V)+8m^2U(3U-2V)+8mU+1).\end{aligned}$$

Formulas (28) and (29) can be obtained, after a long but straightforward computation, by substituting the previous expressions into (31) and (32).  $\square$

## REFERENCES

- [1] C. Arezzo, A. Loi, *Quantization of Kähler manifolds and the asymptotic expansion of Tian–Yau–Zelditch*, J. Geom. Phys. 47 (2003), 87–99.
- [2] C. Arezzo, A. Loi, *Moment maps, scalar curvature and quantization of Kähler manifolds*, Comm. Math. Phys. 243 (2004), 543–559.
- [3] F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116–1175 (Russian).
- [4] D. Boichu, G. Coeuré, *Sur le noyau de Bergman des domaines de Reinhardt*, Invent. Math. 72 (1983), 131–152.

- [5] L. Boutet de Monvel, *Le noyau de Bergman en dimension 2*, Séminaire sur les Equations aux Dérivées partielles 1987-1988, Exp. no. XXII, 13pp., Ecole Polytech., Palaiseau 1988.
- [6] M. Cahen, S. Gutt, J. Rawnsley, *Quantization of Kähler manifolds. I: Geometric interpretation of Berezin's quantization*, J. Geom. Physics 7 (1990), 45–62.
- [7] E. Calabi, *Isometric Imbeddings of Complex Manifolds*, Ann. Math. 58 (1953), 1-23.
- [8] F. Cuccu, A. Loi, *Global symplectic coordinates on complex domains*, J. Geom. Phys. 56 (2006), 247-259.
- [9] F. Cuccu, A. Loi, *Balanced metrics on  $\mathbb{C}^n$* , J. Geom. Phys. 57 (2007), 1115-1123.
- [10] A. J. Di Scala, A. Loi *Symplectic duality of symmetric spaces*, Adv. Math. 217 (2008), 2336-2352.
- [11] M. P. Do Carmo, *Riemannian Geometry*, Boston, MA: Birkhäuser, 1992.
- [12] S. Donaldson, *Scalar Curvature and Projective Embeddings, I*, J. Diff. Geometry 59 (2001), 479-522.
- [13] S. Donaldson, *Scalar Curvature and Projective Embeddings, II*, Q. J. Math. 56 (2005), 345–356.
- [14] M. Engliš, *Berezin quantization and reproducing kernels on complex domains*, Trans. Amer. Math. Soc. 348 (1996), no. 2, 411–479.
- [15] M. Engliš, *Asymptotic behaviour of reproducing kernels of weighted Bergman spaces*, Trans. Amer. Math. Soc. 349 (1997), no. 9, 3717–3735.
- [16] M. Engliš, *A Forelli-Rudin construction and asymptotics of weighted Bergman kernels*, J. Funct. Anal. 177 (2000), no. 2, 257–281.
- [17] M. Engliš, *The asymptotics of a Laplace integral on a Kähler manifold*, J. Reine Angew. Math. 528 (2000), 1-39.
- [18] M. Engliš, *Weighted Bergman kernels and balanced metrics*, RIMS Kokyuroku 1487 (2006), 40–54.
- [19] M. Engliš, *Toeplitz operators and weighted Bergman kernels*, Journal of Functional Analysis 255 (2008) 1419-1457.
- [20] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. 26 (1974), 1-65.
- [21] T. Gramchev, A. Loi, *TYZ expansion for the Kepler manifold*, Comm. Math. Phys. 289, (2009), 825-840.
- [22] A. Greco, A. Loi, *Radial balanced metrics on the unit disk* J. Geom. Phys. 60 (2010), 53-59.
- [23] C. LeBrun, *Complete Ricci-flat Kähler metrics on  $\mathbb{C}^n$  need not be flat*, Proceedings of Symposia in Pure Mathematics, vol. 52 (1991), Part 2, 297-304.
- [24] A. Loi, *The Tian–Yau–Zelditch asymptotic expansion for real analytic Kähler metrics*, Int. J. of Geom. Methods Mod. Phys. 1 (2004), 253-263.
- [25] A. Loi, *A Laplace integral, the T-Y-Z expansion and Berezin's transform on a Kaehler manifold*, Int. J. of Geom. Methods Mod. Phys. 2 (2005), 359-371.
- [26] A. Loi, *Regular quantizations of Kähler manifolds and constant scalar curvature metrics*, J. Geom. Phys. 53 (2005), 354-364.
- [27] A. Loi, M. Zedda, *Kähler–Einstein submanifolds of the infinite dimensional projective space*, to appear in Mathematische Annalen.

- [28] A. Loi, M. Zedda, *Balanced metrics on Hartogs domains*, to appear in Abh. Math. Sem. Univ. Hamburg.
- [29] A. Loi, M. Zedda, *Balanced metrics on Cartan and Cartan–Hartogs domains*, to appear in Math. Zeitschrift.
- [30] A. Loi, F. Zuddas, *Symplectic maps of complex domains into complex space forms*, J. Geom. Phys. 58 (2008), 888–899.
- [31] A. Loi, F. Zuddas, *Engliš expansion for Hartogs domains*, Int. J. Geom. Methods Mod. Phys. Vol. 6, No. 2 (2009), 233–240.
- [32] Z. Lu, *On the lower terms of the asymptotic expansion of Tian–Yau–Zelditch*, Amer. J. Math. 122 (2000), 235–273.
- [33] Z. Lu and G. Tian, *The log term of Szegő Kernel*, Duke Math. J. Volume 125, No 2 (2004), 351–387.
- [34] X. Ma, G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, Birkhäuser.
- [35] I. P. Ramadanov, *A characterization of the balls in  $\mathbb{C}^n$  by means of the Bergman kernel*, C. R. Acad. Bulgare Sci., 34(7) (1981), 927–929.
- [36] J. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford (2), n. 28 (1977), 403–415.
- [37] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Diff. Geom. 32 (1990), no. 1, 99–130.
- [38] S. Zelditch, *Szegő Kernels and a Theorem of Tian*, Internat. Math. Res. Notices 6 (1998), 317–331.
- [39] F. Zheng, *Complex differential geometry*, Studies in Advanced Mathematics, Vol. 18, AMS (2000).

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