

# MOMENTS OF THE TRANSMISSION EIGENVALUES, PROPER DELAY TIMES AND RANDOM MATRIX THEORY I

F. MEZZADRI AND N. J. SIMM

ABSTRACT. We develop a method to compute the moments of the eigenvalue densities of matrices in the Gaussian, Laguerre and Jacobi ensembles for all the symmetry classes  $\beta \in \{1, 2, 4\}$  and finite matrix dimension  $n$ . The moments of the Jacobi ensembles have a physical interpretation as the moments of the transmission eigenvalues of an electron through a quantum dot with chaotic dynamics. For the Laguerre ensemble we also evaluate the finite  $n$  *negative* moments. Physically, they correspond to the moments of the proper delay times, which are the eigenvalues of the Wigner-Smith matrix. Our formulae are well suited to an asymptotic analysis as  $n \rightarrow \infty$ .

## CONTENTS

1. Introduction	2
1.1. Background	2
1.2. The Transmission Eigenvalues	3
1.3. The Wigner-Smith Matrix	4
Acknowledgements	5
2. Statement of Results	5
2.1. The Moments of Transmission Eigenvalues and of the Proper Delay Times	5
2.2. The Gaussian Ensembles	8
3. Unitary Ensembles	9
3.1. Jacobi Unitary Ensemble	12
3.2. Laguerre Unitary Ensemble	13
3.3. Gaussian Unitary Ensemble	14
4. Symplectic and Orthogonal Symmetries	14
5. Symplectic Ensembles	18
5.1. Jacobi Symplectic Ensemble	18
5.2. Laguerre Symplectic Ensemble	19
5.3. Gaussian Symplectic Ensemble	20
6. Orthogonal Ensembles	21
6.1. A Duality between $\beta = 1$ and $\beta = 4$	21
6.2. Incomplete Integrals — Positive Moments	22
6.3. Incomplete Integrals — Negative Moments	24
6.4. Gaussian Orthogonal Ensemble	25
Appendix A. The Coefficients $h_n$ , $e_n^{(1)}$ and $e_n^{(4)}$	28
Appendix B. The Generating Function $f_j(s)$	29
References	30

---

Research partially supported by EPSRC, grant no: EP/G019843/1, and by the Leverhulme Trust, Research Fellowship no: RF/4/RFG/2009/0092.

## 1. INTRODUCTION

**1.1. Background.** Over the past twenty years Random Matrix Theory (RMT) has provided a powerful tool to investigate quantum properties of electronic transport through ballistic cavities (quantum dots) [6–8, 15, 31].

The purpose of this paper is to compute averages of the form

$$(1) \quad M_{\mathcal{E}}^{(\beta)}(k, n) = \frac{1}{C} \int_I \cdots \int_I \left( \sum_{j=1}^n x_j^k \right) \prod_{j=1}^n w_{\beta}(x_j) \prod_{1 \leq j < k \leq n} |x_k - x_j|^{\beta} dx_1 \cdots dx_n$$

for finite  $n$  and  $k$  and for any value of  $\beta \in \{1, 2, 4\}$ . Here  $\mathcal{E}$  labels one of the Gaussian (G), Laguerre ( $L_b$ ) or Jacobi ( $J_{a,b}$ ) ensembles and the value of  $\beta$  corresponds to ensembles of real symmetric ( $\beta = 1$ ), complex hermitian ( $\beta = 2$ ) or quaternion self-dual matrices ( $\beta = 4$ ). The function  $w_{\beta}(x)$  is the weight of the ensemble:

$$(2) \quad w_{\beta}(x) = \begin{cases} e^{-\beta x^2/2}, & I = (-\infty, \infty), \quad \text{Gaussian ensembles,} \\ x^{\beta/2(b+1)-1} e^{-\beta x/2}, & I = [0, \infty), \quad \text{Laguerre ensembles,} \\ x^{\beta/2(b+1)-1} (1-x)^{\beta/2(a+1)-1}, & I = [0, 1], \quad \text{Jacobi ensembles,} \end{cases}$$

where  $C$  is a normalization constant which may vary at each occurrence. The averages (1) for the Jacobi ensembles correspond to the moments of the *transmission eigenvalues* of the electric current through a ballistic cavity; the *negative* moments of the Laguerre ensembles are the moments of the density of the eigenvalues of the Wigner-Smith time-delay matrix.

The physical dimensions of mesoscopic systems are such that the quantum nature of the electron becomes important and a classical treatment of its dynamics is not accurate anymore. Furthermore, at low temperature and voltage electron-electron interactions can be neglected; therefore, the electron scatters elastically inside the cavity, which is attached to two ideal leads connecting two reservoirs in equilibrium at zero temperature. If the leads support  $m$  and  $n$  quantum channels respectively, all the information on the electric transport is contained in the scattering matrix

$$(3) \quad S = \begin{pmatrix} r_{m \times m} & t'_{m \times n} \\ t_{n \times m} & r'_{n \times n} \end{pmatrix}.$$

The sub-blocks  $r_{m \times m}$  and  $t_{n \times m}$  are the reflection and transmission matrices through the left lead, while  $t'_{m \times n}$  and  $r'_{n \times n}$  are those through the right lead. Without loss of generality, throughout this paper we shall assume that  $m \geq n$ . Since the scattering is elastic  $S$  is unitary. This is known as the Landauer-Büttiker formalism.

The scattering matrix depends in a complicated way on macroscopic parameters like the energy of the electron and the shape of the cavity. If the classical dynamics inside the cavity is chaotic, then the fundamental assumption is that the electric current displays universal features; thus, it is natural to model the scattering matrix  $S$  with a random matrix drawn from one of Dyson's circular ensembles: the Circular Unitary Ensemble (CUE) when  $\beta = 2$ ; the Circular Orthogonal Ensemble (COE) when  $\beta = 1$ ; and the Circular Symplectic Ensemble (CSE) when  $\beta = 4$ . Let  $K$  denote a time reversal operator. If the dynamics is not time-reversal invariant then  $\beta = 2$ ; if it is time-reversal invariant then  $\beta = 1$  if  $K^2 = 1$  and  $\beta = 4$  if  $K^2 = -1$ .

In this paper we give a unified approach to compute the family of integrals (1) for all the  $\beta \in \{1, 2, 4\}$  and give particular emphasis to those connected to statistics of the electric current. Our formulae are exact for finite matrix dimension. Since experiments can now be performed in quantum dots with a number of channels arbitrarily small [46], recently there has been an increasing interest in computing finite  $n$  formulae [32, 44, 47, 54]. Some

of these integrals have never been computed before, others are already available in the literature [27, 30, 35, 44, 54]. In particular, most of the formulae for  $\beta = 1$  and  $\beta = 4$  are original. In §2 we will discuss in detail our results and specify which of the averages (1) are already known.

Our formulae have distinctive advantages. Firstly, we can compute ‘negative’ moments for the Laguerre ensemble. Since the joint probability density function (*j.p.d.f.*) of the *inverse delay times* coincides with that of the Laguerre ensemble [16], we obtain the moments of their density. Secondly, for positive moments the sums in our formulae extend to the order of the moments  $k$  and not to the dimension of the matrices  $n$ . The sums that express the negative moments in the Laguerre ensemble run to  $n$ , but their limit as  $n \rightarrow \infty$  can be computed with little effort. As a consequence, although still relatively involved, our expressions are simpler and more manageable than those in the literature. Furthermore, our formulae provide a bridge between finite  $n$  results and their asymptotics. Indeed, in the second part of this work [41], we compute the first three terms of the expansions as  $n \rightarrow \infty$  of the moments of the transmission eigenvalues and of the delay times. They agree with those recently obtained semiclassically [9–11].

**1.2. The Transmission Eigenvalues.** The eigenvalues  $T_1, \dots, T_n$  of the matrix  $tt^\dagger$  are the *transmission eigenvalues*. The unitarity of  $S$  implies that the  $T_1, \dots, T_n$  all lie in the interval  $[0, 1]$ . The dimensionless conductance at zero temperature is defined by

$$(4) \quad G := \text{Tr } tt^\dagger = \text{Tr } t't'^\dagger = T_1 + \dots + T_n.$$

Furthermore, if  $S$  belongs to one of Dyson’s circular ensembles, then the *j.p.d.f.* of  $T_1, \dots, T_n$  is

$$(5) \quad p^{(\beta)}(T_1, \dots, T_n) = \frac{1}{C} \prod_{j=1}^n T_j^\alpha \prod_{1 \leq j < k \leq n} |T_k - T_j|^\beta.$$

The parameter  $\alpha = \frac{\beta}{2}(m - n + 1) - 1$  measures the asymmetry of the quantum channels in the leads. Formula (5) was first computed when  $m = n$  by Baranger and Mello [6] and by Jalabert *et al* [31]; when  $m \neq n$  it was reported in this form by Beenakker [8], where it was attributed to unpublished work by Brouwer (1994); its general derivation appeared in the literature for the first time in an article by Forrester [23].

In a classic article Dyson [21] classified complex many-body systems according to their fundamental symmetries and proved that they correspond to the random matrix ensembles labelled  $\beta \in \{1, 2, 4\}$ . Zirnbauer [56] extended Dyson’s classification scheme to Cartan’s symmetric spaces and introduced new symmetry classes in Random Matrix Theory. Zirnbauer also argued that these non-standard ensembles appear in the stochastic modelling of ballistic cavities in contact with a superconductor. Such mesoscopic systems are called *Andreev quantum dots*. In his PhD thesis Dueñez [20] further generalized Zirnbauer’s classification. Furthermore, Altland and Zirnbauer [4] divided the symmetries of Andreev quantum dots into four fundamental classes. These ensembles are labelled by two integers  $(\beta, \delta)$ : as for Dyson’s ensembles,  $\beta$  takes values in  $\{1, 2, 4\}$ ; instead  $\delta \in \{-1, 1, 2\}$ . The four classes are  $(1, -1)$ ,  $(2, -1)$ ,  $(4, 2)$  and  $(2, 1)$ ; they correspond to different combinations of time-reversal and spin-rotation symmetries.

Our formalism applies to Andreev quantum dots too. Indeed, in a recent paper Dahlhaus *et al* [18] computed the *j.p.d.f.* of the transmission eigenvalues. It is obtained by deforming

the right-hand side in equation (5):

$$(6) \quad p^{(\beta, \delta)}(T_1, \dots, T_n) = \frac{1}{C} \prod_{j=1}^n T_j^\alpha (1 - T_j)^{\delta/2} \prod_{1 \leq j < k \leq n} |T_k - T_j|^\beta$$

As for the *j.p.d.f.* in (5)  $\alpha = \frac{\beta}{2}(m - n + 1) - 1$ .

Equations (5) and (6) are particular cases of the *j.p.d.f.* of the eigenvalues of matrices in the Jacobi ensembles, namely

$$(7) \quad p_{J_{a,b}}^{(\beta)}(x_1, \dots, x_n) = \frac{1}{C} \prod_{j=1}^n x_j^{\beta/2(b+1)-1} (1 - x_j)^{\beta/2(a+1)-1} \prod_{1 \leq j < k \leq n} |x_k - x_j|^\beta,$$

for  $0 \leq x_j \leq 1$ , ( $j = 1, \dots, n$ ). We recover (6) by setting

$$(8) \quad a = \frac{2}{\beta} \left( 1 + \frac{\delta}{2} \right) - 1 \quad \text{and} \quad b = m - n.$$

The moments of the density of the transmission eigenvalues are defined by

$$(9) \quad \langle \mathcal{T}_{k,n,m}^{(\beta, \delta)} \rangle := \langle \text{Tr}[(tt^\dagger)^k] \rangle = M_{J_{a,b}}^{(\beta)}(k, n),$$

where  $a$  and  $b$  are given in equation (8). From a physical point of view, they are important because they are connected to the cumulants  $\langle\langle \kappa_j \rangle\rangle$  of the charge transmitted over a finite interval of time by the generating function [36]

$$(10) \quad \sum_{j=1}^{\infty} \frac{x^j}{j!} \langle\langle \kappa_j \rangle\rangle = - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \langle \mathcal{T}_k \rangle (e^x - 1)^k.$$

(See also [12], appendix A.) For simplicity in this formula we have omitted the dependence on  $(\beta, \delta)$  and on the numbers of quantum channels  $m$  and  $n$ . The charge cumulants can be directly accessed in experiments [14]. Using the results in §2 and the generating function in (10) we can compute the cumulants to any given order. For example, the variance and skewness are given by

$$(11a) \quad \langle\langle \kappa_2 \rangle\rangle = \frac{nm \left( \frac{2+\delta}{\beta} - 1 + n \right) \left( \frac{2+\delta}{\beta} - 1 + m \right)}{\left( \frac{4+\delta}{\beta} - 1 + n + m \right) \left( \frac{2+\delta}{\beta} - 2 + n + m \right) \left( \frac{2+\delta}{\beta} - 1 + n + m \right)}$$

$$(11b) \quad \frac{\langle\langle \kappa_3 \rangle\rangle}{\langle\langle \kappa_2 \rangle\rangle} = - \frac{(n - m - \frac{2+\delta}{\beta} + 1)(n - m + \frac{2+\delta}{\beta} - 1)}{(n + m + \frac{2+\delta}{\beta} - 3)(n + m + \frac{6+\delta}{\beta} - 1)}.$$

The special case  $\delta = 0$  of equations (11) were computed by Savin *et al* [50] (see also [13]).

**1.3. The Wigner-Smith Matrix.** The Wigner-Smith time-delay matrix is defined as

$$(12) \quad Q = -i\hbar S^{-1} \frac{\partial S}{\partial E}.$$

The individual eigenvalues  $\tau_1, \dots, \tau_n$  of  $Q$  are called *proper delay times*, and their average

$$(13) \quad \tau_W = \frac{1}{n} \text{Tr} Q$$

is referred to as *Wigner delay time*. Here  $n$  is the total number of quantum channels in the leads. The Wigner delay time measures the extra time an electron spends in the cavity as a result of being scattered. If  $S$  belongs to one of the circular ensembles, then it was

shown by Brouwer *et al* [16] that the *j.p.d.f.* of the inverses  $\gamma_j = \tau_j^{-1}$  ( $j = 1, \dots, n$ ) of the proper delay times is

$$(14) \quad P_\beta(\gamma_1, \dots, \gamma_n) = \frac{1}{C} \prod_{j=1}^n \gamma_j^{n\beta/2} e^{-\beta\tau_H\gamma_j/2} \prod_{1 \leq j < k \leq n} |\gamma_k - \gamma_j|^\beta,$$

where  $\tau_H$  is the Heisenberg time. In our context  $\tau_H = n$ . In a sequence of papers Savin and collaborators [48, 49, 51] computed the probability distribution function of the proper delay times.

The moments of the density of the proper delay times are defined by

$$(15) \quad \langle \mathcal{D}_{k,n}^{(\beta)} \rangle = \frac{1}{n} \langle \text{Tr } Q^k \rangle = n^{k-1} M_{L_b}^{(\beta)}(-k, n), \quad k < n\beta/2 + 1,$$

where in this case  $b = n - 1 + 2/\beta$ , and the right-hand side of (15) denotes negative integer moments of the Laguerre ensemble.

The outline of the paper is the following: in §2 we present our main results; §3 is devoted to ensembles with unitary symmetry; in §4 we discuss the approach underlying the computations of the moments for ensembles with symplectic and orthogonal symmetries; §5 and §6 contain the proofs of the formulae for ensembles with symplectic and orthogonal symmetries, respectively.

In the final stages of the preparation of this article, and after the results in this paper had been presented at two workshops,<sup>1</sup> we received a preprint by Livan and Vivo [37], in which some of our formulae were derived with a different method and approach. Their expressions are different, but equivalent to ours.

#### ACKNOWLEDGEMENTS

We would like to express our gratitude to Peter Forrester, Yan Fyodorov, Jonathan Keating, Jack Kuipers, Marcel Novaes and Dmitry Savin for stimulating and helpful discussions.

## 2. STATEMENT OF RESULTS

### 2.1. The Moments of Transmission Eigenvalues and of the Proper Delay Times.

Since the transmission eigenvalues are distributed with the *j.p.d.f.* of the Jacobi ensemble, their moments are the moments of the eigenvalue density of this ensemble for finite matrix dimension. A 4-*th* order recurrence relation for the exact moments at  $\beta = 2$  were first reported Ledoux [34]. Explicit formulae were then obtained by Novaes [44] and by Vivo and Vivo [54]. Bai *et al* [5] computed the leading order term of the asymptotic expansion for  $\beta = 1$ .

From equation (14), computing the moments of the proper delay times is tantamount to calculating the negative moments of the eigenvalue density of the Laguerre ensemble for finite  $n$ . These negative moments have never been determined before, though positive integer moments were calculated by Hanlon *et al* [29] and Haagerup and Thorbjørnsen [28].

It is worth reminding the reader that the moments of the proper delay times exist only for

$$k < \frac{n\beta}{2} + 1,$$

because for larger  $k$  the integral  $M_{L_b}(-k, n)$ , with  $b = n - 1 + 2/\beta$ , diverges.

---

<sup>1</sup>Random Matrix Theory and Its Applications I, MSRI, 13–17 September 2010, Berkeley, USA; VI Brunel Workshop on Random Matrix Theory, 17–18 December 2010, Brunel University, UK.

The general formulae for moments of the Jacobi and Laguerre ensembles are reported in the main sections of the paper. Throughout, the notation  $(n)_{(k)} = \Gamma(n+k)/\Gamma(n)$  refers to the Pochhammer symbol; the binomial coefficient can take arbitrary complex arguments, *i.e.*

$$(16) \quad \binom{k}{j} = \frac{\Gamma(k+1)}{\Gamma(k-j+1)\Gamma(j+1)},$$

and is defined for negative integers by the limiting form

$$(17) \quad \binom{-k}{j} = (-1)^j \binom{k+j-1}{k-1}.$$

### 2.1.1. Broken Time Reversal ( $\beta = 2$ ).

**Theorem 2.1.** *The moments of the transmission eigenvalues and of the proper delay times for  $\beta = 2$  are*

$$(18) \quad \langle \mathcal{T}_{k,n,m}^{(2,\delta)} \rangle = \frac{nm}{\delta/2 + n + m} - \sum_{j=1}^{k-1} \frac{1}{j} \sum_{i=1}^{\min(j,n)} \binom{j}{i} \binom{j}{i-1} \mathcal{U}_{i,j}^{m,n,\delta}$$

and

$$(19) \quad \langle \mathcal{D}_{k,n}^{(2)} \rangle = \frac{n^{k-1}}{k} \sum_{j=0}^{n-1} \binom{k+j-1}{k-1} \binom{k+j}{k-1} \frac{(2n)_{(-k-j)}}{(n+1)_{(-j-1)}},$$

where

$$\mathcal{U}_{i,j}^{m,n,\delta} = \frac{(\delta/2 + m + n - 2i + j + 1) (\delta/2 + m)_{(j-i+1)} (m)_{(j-i+1)}}{(\delta/2 + m + n - i)_{(j+2)} (\delta/2 + m + n - i + 1)_{(j)} (\delta/2 + n + 1)_{(-i)} (n+1)_{(-i)}}.$$

**Remark 2.2.** The moments for the Laguerre Unitary Ensemble can be defined even when  $k$  is complex and have the following particularly simple expression:

$$(20) \quad M_{L_b}^{(2)}(k, n) = \frac{1}{k} \sum_{j=0}^n \binom{k}{j} \binom{k}{j-1} \frac{(b+n)_{(k-j+1)}}{(1+n)_{(-j)}},$$

of which (19) is a special case. If  $k$  is a positive integer, the sum in (20) consists of at most  $k$  terms.

**Remark 2.3.** The coefficients

$$(21) \quad N(k, j) = \frac{1}{k} \binom{k}{j} \binom{k}{j-1}$$

in formulae (18) and (20) appear frequently in enumerative combinatorics, where they are called *Narayana numbers*.

### 2.1.2. Conserved Time Reversal with $K^2 = -1$ ( $\beta = 4$ ).

**Theorem 2.4.** *The moments of the transmission eigenvalues are*

$$(22) \quad \langle \mathcal{T}_{k,n,m}^{(4,\delta)} \rangle = \frac{1}{2} \langle \mathcal{T}_{k,2n,2m}^{(2,\delta-2)} \rangle - \sum_{j=1}^{\min(\lfloor n \rfloor, \lfloor k/2 \rfloor)} \sum_{i=0}^{\min(k-2j, 2n-2j)} \binom{k}{i} \binom{k}{i+2j} \mathcal{S}_{i,j}^{\delta}(k, m, n)$$

The coefficient  $\mathcal{S}_{i,j}^\delta(k, m, n)$  is

$$\mathcal{S}_{i,j}^\delta(k, m, n) = 2^{4j-3} \frac{(\delta/2 + 2n - i - 2j)_{(i)} (2m)_{(k-i-2j+1)} (\delta/2 + 2m - 1)_{(k-i-2j+1)}}{(\delta/4 + n + 1/2)_{(-j)} (m)_{(1-j)} (\delta/4 + m - 1/2)_{(1-j)} (2n - 2j + 1)_{(-i)} (n + 1)_{(-j)}} \\ \times \frac{(\delta/2 + 2m + 2n - 4j) (\delta/2 + 2m + 2n - 2i - 4j + k)}{(\delta/2 + 2m + 2n - i - 2j)_{(1+k)} (\delta/2 + 2m + 2n - i - 4j)_{(1+k)}}.$$

Furthermore, the moments of the proper delay times for  $\beta = 4$  are given by

$$(23) \quad \langle \mathcal{D}_{k,n}^{(4)} \rangle = n^{k-1} M_{L_{n-1/2}}^{(4)}(-k, n),$$

where the moments of the Laguerre Symplectic Ensemble are

$$(24) \quad M_{L_b}^{(4)}(k, n) = 2^{-k-1} M_{L_{2b}}^{(2)}(k, 2n) \\ - \sum_{j=1}^{\lfloor n \rfloor} \sum_{i=0}^{2n-2j} \binom{k}{i} \binom{k}{i+2j} \frac{(2b+2n)_{(k-i-2j+1)} (2n-i-2j+1)_{(i)}}{2^{k-2j+2} (n+1)_{(-j)} (b+n)_{(1-j)}}.$$

The symbol  $\lfloor \cdot \rfloor$  denotes the integer part.

**Remark 2.5.** The order of the moments  $k$  in equations (22) and (23) is a positive integer. However, (24) holds even when  $k$  is complex. As for  $\beta = 2$ , if  $k$  is positive the sum in equation (24) contains only  $k$  terms.

**Remark 2.6.** Although  $n$  is typically an integer, as it denotes the dimension of a matrix, the expressions on the right-hand sides of equations (22) and (24) are well defined for any half-integer  $n$ . It is useful to generalize it, because the evaluation of the moments for  $\beta = 1$  requires moments for  $\beta = 4$  computed at half-integer  $n$ .

2.1.3. *Conserved Time Reversal with  $K^2 = 1$  ( $\beta = 1$ ).* For simplicity we assume that the outgoing lead supports an even number of open channels.

**Theorem 2.7.** *The moments of the transmission eigenvalues are*

$$(25) \quad \langle \mathcal{T}_{k,m,n}^{(1,\delta)} \rangle = 2 \langle \mathcal{T}_{k,(n-1)/2,(m-1)/2}^{(4,2\delta+4)} \rangle + \sum_{j=0}^{\min(n/2-1,k)} \binom{2k}{2j} I_j^{(1,\delta)}(k, m, n) + \phi_{k,n}^J,$$

where

$$(26a) \quad I_j^{(1,\delta)}(k, m, n) = 4^k \frac{(\delta + m + n - 4j + 2k)_{(\frac{1}{2}(\delta + m + 1))_{(k-j)} (\frac{1}{2}m)_{(k-j)}}{(\delta + m + n - 2j)_{(2k+1)} (\frac{1}{2}(\delta + n + 2))_{(-j)} (\frac{1}{2}(1+n))_{(-j)}}$$

and

$$(26b) \quad \phi_{k,n}^J = \sum_{j=1}^k \frac{2^{\delta+2} \Gamma(\frac{1}{2}(\delta + m - n + 2j + 1)) \Gamma(\frac{1}{2}(\delta + m + 2))}{\Gamma(\delta + 1 + m + j + k) \Gamma(j + k + 1 - n) \Gamma(\frac{1}{2}(m - n + 1 + 2j))} \\ \times \frac{\Gamma(\frac{1}{2}(\delta + n + 2)) \Gamma(m - n + k + j) \Gamma(j + k)}{\Gamma(\frac{m}{2}) \Gamma(\frac{\delta}{2} + 1) \Gamma(\frac{n}{2})}.$$

The moments of the proper delay times are

$$(27) \quad \langle \mathcal{D}_{k,n}^{(1)} \rangle = n^{k-1} 2^{1-k} M_{L_{(n+1)/2}}^{(4)}(-k, (n-1)/2) \\ + \left(\frac{n}{2}\right)^{k-1} \sum_{j=0}^{n/2-1} \binom{2k+2j-1}{2j} \frac{(n+1/2)_{(-j-k)}}{2(\frac{1}{2}(1+n))_{(-j)}} + n^{k-1} \phi_{-k,n}^L,$$

where

$$(28) \quad \phi_{-k,n}^L = \frac{\Gamma(n)}{\Gamma(n/2)\Gamma(2n)} \sum_{j=0}^{k-1} \frac{\Gamma(k+j+n)\Gamma(1+n-k-j)}{\Gamma(n/2+1-j)\Gamma(k+j+1)} 2^j.$$

**Remark 2.8.** Due to the term  $\Gamma(j+k+1-n)$  in the denominator of (26b), the  $\phi_{k,n}^J$ 's are identically zero for any  $n > 2k$ . By Stirling's formula, the  $\phi_{-k,n}^L$ 's in (28) decay exponentially fast as  $n \rightarrow \infty$ . Therefore, neither of these terms contribute to the asymptotics of the moments as  $n \rightarrow \infty$  at any finite algebraic order.

It is straightforward to compute the limit as  $n \rightarrow \infty$  of the formulae in this section. They differ fundamentally from most of the known exact results in the literature, whose asymptotic limit often involves many cancellations, which means that even the leading order term may be out of reach. This difficulty is discussed in some detail by Krattenthaler [33], where a solution is presented for  $\beta = 2$  (see also [17]).

Indeed, it is a simple exercise using our exact results to show that

$$(29) \quad \lim_{n,m \rightarrow \infty} \frac{1}{n} \langle \mathcal{T}_{k,m,n}^{(\beta,\delta)} \rangle = \left(1 + \frac{m}{n}\right) \sum_{j=0}^{k-1} \binom{k-1}{j} C_j (-1)^j \xi^{j+1},$$

where  $\xi$  is the variable  $\xi = \frac{nm}{(n+m)^2}$ , which remains finite as  $n, m \rightarrow \infty$ , and  $C_j = \frac{1}{j+1} \binom{2j}{j}$  is the  $j$ -th Catalan number. This formula agrees with the semiclassical computation of Berkolaiko *et al* [9] Furthermore, for the proper delay times we have

$$(30) \quad \lim_{n \rightarrow \infty} \langle \mathcal{D}_{k,n}^{(\beta)} \rangle = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} 2^j,$$

which is the  $k$ -th Schröder number (note the appearance of the Narayana numbers (21)). This limit was computed semiclassically by Berkolaiko and Kuipers [10], and can also be obtained from the Marčenko-Pastur distribution [38] (see, *e.g.*, [10, 16]). It is a simple consequence of (19) too.

Equation (29) was first computed using RMT by Novaes [43] (see also [5]), while (30) and (29) were recently rederived through combinatorial techniques [45]. Our exact results allow a simple derivation of these facts, while also consenting the investigation of  $\beta$ -dependent subleading corrections. We address these issues more thoroughly in the second part of this work [41], where we show that the first two subleading terms in the asymptotic expansions of the previous theorems agree with those obtained semiclassically by Berkolaiko and Kuipers [11].

**2.2. The Gaussian Ensembles.** Our techniques apply equally well to the Gaussian ensembles. Recursion formulae for the finite  $n$  moments of the density of the eigenvalues were derived by Harer and Zagier [30] for the Gaussian Unitary Ensemble (GUE), while Goulden and Jackson [27] derived explicit formulae for both the Gaussian Orthogonal Ensemble (GOE) and the GUE, while the GUE moment generating function was computed by Haagerup and Thorbjørnsen [28]. More recently, recursion formulae were obtained by Ledoux [35] for the GOE and GSE.

**Theorem 2.9.** *The moments of the eigenvalue density for the GUE are*

$$(31) \quad M_G^{(2)}(2k, n) = \frac{2^n \Gamma(n/2 + 1) \Gamma(n/2)}{\sqrt{\pi} (2k + 1) \Gamma(n)} \sum_{j=0}^{\min(n/2-1, k)} \binom{k}{j} \binom{k+1}{j+1} (n/2 - j)_{(k+1/2)}$$



for even  $n$ , and

$$(32) \quad M_G^{(2)}(2k, n) = \frac{2^n \Gamma((n+1)/2)^2}{\sqrt{\pi}(2k+1)\Gamma(n)} \sum_{j=0}^{\min((n-1)/2, k)} \binom{k}{j} \binom{k+1}{j} ((n+1)/2 - j)_{(k+1/2)}$$

for odd  $n$ . For the GSE we have

$$(33) \quad M_G^{(4)}(2k, n) = 2^{-k-1} M_G^{(2)}(2k, 2n) - \frac{\Gamma(n+1)\Gamma(n)}{2^k \sqrt{\pi} \Gamma(2n) 4^{1-n}} \sum_{j=1}^{\min(n, k)} \sum_{i=0}^{\min(n-j, k-j)} \binom{k}{i} \binom{k}{i+j} (n-i-j+1)_{(k-1/2)}$$

Let  $n$  be even. Then, the moments for the GOE are

$$M_G^{(1)}(2k, n) = M_G^{(2)}(2k, n-1) - \sum_{j=1}^{\min(\frac{n}{2}-1, k)} \sum_{i=0}^{\min(k, \frac{n}{2}-1-j)} \binom{k}{i} \binom{k}{i+j} \frac{(\frac{n}{2}-i-j)_{(k+1/2)}}{(\frac{n}{2}-j)_{(1/2)}} + \phi_{k, n}^G.$$

For  $n \leq 2k$  the quantity  $\phi_{k, n}^G$  is given by

$$\phi_{k, n}^G = 2^{n/2-k} \frac{(2k)!}{\Gamma(n/2)} \sum_{j=0}^{k-n/2} \sum_{i=0}^{n/2-1} \frac{\binom{n-1}{2i} 2^{-j-2i} (-1)^j}{(k-n/2-j)!} + \frac{(2k)!}{\Gamma(n/2)} \sum_{j=0}^{n/2-1} \sum_{i=0}^j \frac{(n/2-i-1)! \binom{n-1}{n-2i-1}}{(j-i)!(k-j)! 4^{k-i}}.$$

If  $n > 2k$  we have

$$(34) \quad \phi_{k, n}^G = (2k)! \sum_{j=0}^k \frac{(n/2 + 1/2 - j)_{(j)}}{2^{k-3j} (2j)! (k-j)!}.$$

### 3. UNITARY ENSEMBLES

We shall now compute the moments of the eigenvalues densities for the Jacobi, Laguerre and Gaussian ensembles when  $\beta = 2$ . For brevity we shall refer to these ensembles with the usual notation JUE, LUE and GUE. Except for the GUE, our expressions are valid for complex  $k$ . Theorem 2.1 and equations (31) and (32) of theorem 2.9 are corollaries of the results of this section.

For all the ensembles and symmetry classes that we consider the *j.p.d.f.* of the eigenvalues has the form

$$(35) \quad p_{\mathcal{E}}^{(\beta)}(x_1, \dots, x_n) = \frac{1}{C} \prod_{j=1}^n w_{\beta}(x_j) \prod_{1 \leq j < k \leq n} |x_k - x_j|^{\beta}.$$

The marginal probabilities are obtained by subsequent integrations of the right-hand side of (35); furthermore, since it is invariant under permutations of its arguments, it is irrelevant which variables are integrated over. Therefore, the probability density of the

eigenvalues is obtained by integrating out all but one variable. It follows that

$$(36) \quad \begin{aligned} \langle \text{Tr } X^k \rangle &= \int_I \cdots \int_I (x_1^k + \cdots + x_n^k) p_{\mathcal{E}}^{(\beta)}(x_1, \dots, x_n) dx_1 \cdots dx_n \\ &= \int_I x^k \rho_{\beta}(x) dx, \end{aligned}$$

where  $\rho_{\beta}(x)$  is the eigenvalue density normalized to  $n$  and  $I$  is the support of  $w_{\beta}(x)$ .

We develop effective techniques to compute the integral in the right-hand side of equation (36) using ideas first introduced by Haagerup and Thorbjørnsen [28] for  $\beta = 2$  and by Adler *et al* [2] for  $\beta = 1, 4$ .

When  $\beta = 2$  the density of the eigenvalues takes a particularly simple form (see, *e.g.*, [25], §5.1)

$$(37) \quad \rho_2(x) = \left\langle \sum_{j=1}^n \delta(x - x_j) \right\rangle = w_2(x) \sum_{j=0}^{n-1} \frac{P_j(x)^2}{h_j},$$

where the  $P_j(x)$ 's are orthogonal polynomials associated with the weight  $w_2(x)$  and  $j = 0, 1, \dots$  denotes their degree. In other words, we have

$$(38) \quad \int_I w_2(x) P_j(x) P_k(x) dx = h_j \delta_{jk}, \quad j, k = 0, 1, \dots$$

The system of orthogonal polynomials  $\{P_j(x)\}_{j=0}^{\infty}$  is unique up to multiplicative constants  $k_j$ , which we can take to be the coefficient of the monomial of highest degree. Orthogonal polynomials satisfy a recurrence relation of the form

$$(39) \quad P_{j+1}(x) = (\alpha_j + x\beta_j)P_j(x) - \gamma_j P_{j-1}(x), \quad j = 0, 1, \dots$$

where for convention  $P_{-1}(x) = 0$ . For the classical orthogonal polynomials the constants  $h_j$ ,  $k_j$ ,  $\alpha_j$ ,  $\beta_j$  and  $\gamma_j$  are tabulated in many books on special functions (see, *e.g.*, [1]). A consequence of (39) is

$$(40) \quad \rho_2(x) = w_2(x) \frac{k_{n-1}}{k_n h_{n-1}} (P'_n(x) P_{n-1}(x) - P_n(x) P'_{n-1}(x)),$$

which is a limiting case of the Christoffel-Darboux formula. (For the proofs of formulae (39) and (40) see, *e.g.*, [52], §3.2).

In the rest of this article we shall assume that  $k_j = 1$ , for  $j = 0, 1, \dots$ . In other words, we only consider *monic* orthogonal polynomials. In order to distinguish them from the way the classical polynomials are conventionally defined in the literature, we shall use the notation  $\mathcal{H}_n(x)$ ,  $\mathcal{L}_n^b(x)$  and  $\mathcal{P}_n^{a,b}(x)$  for the Hermite, Laguerre and Jacobi polynomials respectively. We shall denote the generic monic polynomial by  $p(x)$ . We tabulate the orthogonality constants  $h_j$  for the monic classical polynomials in appendix A. We shall also need the following differential equations (see [1], §22.6)

$$(41) \quad \begin{cases} \mathcal{H}_j''(x) - 2x\mathcal{H}_j'(x) + 2j\mathcal{H}_j(x) = 0, \\ x\mathcal{L}_j(x)'' + (b+1-x)\mathcal{L}_j^b(x)' + j\mathcal{L}_j^b(x) = 0, \\ x(1-x)\mathcal{P}_j^{a,b}(x)'' + (b+1-(a+b+2)x)\mathcal{P}_j^{a,b}(x)' \\ + j(a+b+j+1)\mathcal{P}_j^{a,b}(x) = 0. \end{cases}$$

Haagerup and Thorbjørnsen [28] computed the moment generating function

$$(42) \quad M(t) = \int_I \rho_2(x) e^{-tx} dx$$

in terms of hypergeometric functions for the GUE and LUE. They combined the differential equations (41) with (40) to obtain

$$(43) \quad \frac{d}{dx} (f(x)\rho_2(x)) = \begin{cases} -D_n^H e^{-x^2} \mathcal{H}_n(x) \mathcal{H}_{n-1}(x), & \text{Hermite,} \\ -D_n^L x^b e^{-x} \mathcal{L}_n^b(x) \mathcal{L}_{n-1}^b(x), & \text{Laguerre,} \end{cases}$$

where

$$(44) \quad D_n^H = \frac{2^n}{\sqrt{\pi}\Gamma(n)} \quad \text{and} \quad D_n^L = (\Gamma(b+n)\Gamma(n))^{-1}.$$

Furthermore,  $f(x) = 1$  for the Hermite polynomials, while  $f(x) = x$  for the Laguerre ones. We shall use similar ideas to compute the moments (9) for  $\beta = 2$ .

First we need the analogue of the identities (43) for the Jacobi polynomials.

**Lemma 3.1.** *Let  $\rho_2(x)$  be the mean eigenvalue density for the JUE. We have the following differential identity:*

$$(45) \quad \frac{d}{dx} (x(1-x)\rho_2(x)) = -D_n^J x^b (1-x)^a \mathcal{P}_n^{a,b}(x) \mathcal{P}_{n-1}^{a,b}(x),$$

where

$$(46) \quad D_n^J = \frac{\Gamma(a+b+2n+1)\Gamma(a+b+2n-1)}{\Gamma(a+n)\Gamma(b+n)\Gamma(a+b+n)\Gamma(n)}.$$

*Proof.* The normalization coefficient  $h_{n-1}$  associated with the polynomials  $\mathcal{P}_{n-1}^{a,b}(x)$  is

$$(47) \quad h_{n-1} = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+b+2n)\Gamma(a+b+2n-1)}.$$

Inserting (47) into the representation (40) and using the differential equation in (41) we obtain

$$(48) \quad x(1-x) \left( \frac{\rho_2(x)}{w_2(x)} \right)' + (1+b - (a+b+2)x) \left( \frac{\rho_2(x)}{w_2(x)} \right) = -D_n^J \mathcal{P}_n^{a,b}(x) \mathcal{P}_{n-1}^{a,b}(x),$$

where  $D_n^J$  is given in (46). Finally, since the weight associated with the Jacobi polynomials is  $w_2(x) = x^b(1-x)^a$  we arrive at

$$\begin{aligned} \frac{d}{dx} (x(1-x)\rho_2(x)) &= \frac{d}{dx} \left( x^{b+1}(1-x)^{a+1} \frac{\rho_2(x)}{w_2(x)} \right) \\ &= x^b(1-x)^a \left( ((b+1)(1-x) - x(a+1)) \frac{\rho_2(x)}{w_2(x)} + x(1-x) \left( \frac{\rho_2(x)}{w_2(x)} \right)' \right) \\ &= -D_n^J x^b(1-x)^a \mathcal{P}_n^{a,b}(x) \mathcal{P}_{n-1}^{a,b}(x). \end{aligned}$$

□

**Remark 3.2.** For our purposes it is not helpful to compute the moment generating function (42). Although in principle one can employ a type of fractional calculus to extract more general types of moments from (42), we will see in the following that moments for general  $k$  are directly accessible with our method.

**3.1. Jacobi Unitary Ensemble.** Lemma 3.1 allows us to compute the *difference* of the moments. Then, the moments themselves can be computed by adding all the differences. Finally, equation (18) of theorem 2.1 is obtained by setting  $a = \delta/2$  and  $b = m - n$ .

Let us define

$$(49) \quad \Delta M_{J_{a,b}}^{(2)}(k, n) = M_{J_{a,b}}^{(2)}(k, n) - M_{J_{a,b}}^{(2)}(k+1, n).$$

**Proposition 3.3.** *We have*

$$(50) \quad \Delta M_{J_{a,b}}^{(2)}(k, n) = \frac{1}{k} \sum_{j=0}^n \binom{k}{j} \binom{k}{j-1} U_{k,j}^{n,a,b},$$

where

$$(51) \quad U_{k,j}^{n,a,b} = \frac{(a+b+2n-2j+k+1)(a+b+n)_{(k-j+1)}(a+n-j+1)_{(j)}(b+n)_{(k-j+1)}}{(a+b+2n-j)_{(k+2)}(a+b+2n-j+1)_{(k)}(n+1)_{(-j)}}.$$

If  $k$  is a positive integer, equation (50) reduces to

$$(52) \quad \Delta M_{J_{a,b}}^{(2)}(k, n) = \frac{1}{k} \sum_{j=0}^{\min(n,k)} \binom{k}{j} \binom{k}{j-1} U_{k,j}^{n,a,b}.$$

*Proof.* Integrating by parts using equation (45) leads to

$$(53) \quad \int_0^1 x^k (1-x) \rho_2(x) dx = \frac{D_n^J}{k} \int_0^1 x^{k+b} (1-x)^a \mathcal{P}_n^{a,b}(x) \mathcal{P}_{n-1}^{a,b}(x) dx$$

Consider the identity

$$(54) \quad \mathcal{P}_n^{a,b}(x) = \sum_{j=0}^n \mathcal{C}_j^{k,n} \mathcal{P}_j^{a,b+k}(x),$$

where

$$(55) \quad \mathcal{C}_j^{k,n} = \binom{k}{j} \frac{(a+n+1-j)_{(j)}(a+b+2n+1)_{(-j)}}{(a+b+2n-2j+2+k)_{(j)}(n+1)_{(-j)}}$$

are the *connection coefficients*. Inserting this formula into (53) and evaluating the integrals using orthogonality leads to

$$(56) \quad \Delta M_{J_{a,b}}^{(2)}(k, n) = \frac{D_n^J}{k} \sum_{j=0}^n \mathcal{C}_j^{k,n} \mathcal{C}_{j-1}^{k,n-1} h_{n-j}^{a,b+k}.$$

Substituting the appropriate coefficients (see appendix A) gives immediately (50).

When  $k$  is a positive integer, the terms with  $j > k$  vanish because  $\binom{k}{j} = 0$  if  $j > k$ , leading immediately to (52).  $\square$

**Corollary 3.4.** *The integer moments of the level density for the JUE are*

$$(57) \quad M_{J_{a,b}}^{(2)}(k, n) = M_{J_{a,b}}^{(2)}(1, n) - \sum_{j=1}^{k-1} \frac{1}{j} \sum_{i=1}^{\min(j,n)} \binom{j}{i} \binom{j}{i-1} U_{j,i}^{n,a,b}.$$

where the first moment is

$$(58) \quad M_{J_{a,b}}^{(2)}(1, n) = \frac{n(b+n)}{a+b+2n}.$$

The first moment  $M_{J_{a,b}}^{(2)}(1, n)$  is an Aomoto integral. For its evaluation see, *e.g.*, [39], §17.3.

**3.2. Laguerre Unitary Ensemble.** Since the moments of the Wigner-Smith matrix (12) require the computation of the integral (36) for  $k < 0$ , we shall present formulae for the moments of the LUE for general complex  $k$ .

**Proposition 3.5.** *Suppose that neither  $b + n$  nor  $b + k$  are negative integers. Then one has*

$$(59) \quad M_{L_b}^{(2)}(k, n) = \frac{1}{k} \sum_{j=0}^n \binom{k}{j} \binom{k}{j-1} \frac{(b+n)_{(k-j+1)}}{(n+1)_{(-j)}}.$$

*Proof.* Integrating by parts the second equation in (43) gives

$$(60) \quad \int_0^\infty x^k \rho_2(x) dx = \frac{D_n^L}{k} \int_0^\infty x^{b+k} e^{-x} \mathcal{L}_n^b(x) \mathcal{L}_{n-1}^b(x) dx.$$

For the Laguerre polynomials the connection formula is [52]

$$(61) \quad \mathcal{L}_n^b(x) = \sum_{j=0}^n \mathcal{C}_j^{k,n} \mathcal{L}_j^{b+k}(x), \quad \text{where} \quad \mathcal{C}_j^{k,n} = \binom{k}{j} (n+1)_{(-j)}.$$

Inserting formula (61) into (60) gives

$$(62) \quad M_{L_b}^{(2)}(k, n) = \frac{D_n^L}{k} \sum_{j=0}^n \mathcal{C}_j^{k,n} \mathcal{C}_{j-1}^{k,n-1} h_{n-j}^{b+k},$$

where we evaluated the integrals using orthogonality. Using the appropriate connection coefficients and normalisation constants completes the proof.  $\square$

If  $k$  is a positive integer, the binomial coefficient  $\binom{k}{j} = 0$  if  $j > k$ , leaving only a sum with  $k$  terms. Negative moments are obtained simply by using the identity

$$\binom{-k}{j} = (-1)^j \binom{k+j-1}{k-1}.$$

**Corollary 3.6.** *Let  $k$  be a positive integer, then*

$$(63) \quad M_{L_b}^{(2)}(k, n) = \frac{1}{k} \sum_{j=0}^{\min(n,k)} \binom{k}{j} \binom{k}{j-1} \frac{(b+n)_{(k-j+1)}}{(n+1)_{(-j)}}.$$

Furthermore, if  $k < n + 1$  we have

$$(64) \quad M_{L_b}^{(2)}(-k, n) = \frac{1}{k} \sum_{j=0}^{n-1} \binom{k+j}{k-1} \binom{k+j-1}{k-1} \frac{(b+n)_{(-k-j)}}{(n+1)_{(-j-1)}}.$$

Equation (19) is a particular case of formula (64), where  $b = n$  and the scaling introduced by the Heisenberg time  $\tau_H = n$  has been taken into account.

**Remark 3.7.** The appearance of the Narayana coefficients in (63) anticipates the fact that its leading order term as  $n \rightarrow \infty$  is the  $k$ -th moment of the Marčenko-Pastur law [38].

**3.3. Gaussian Unitary Ensemble.** In this section we give the proof of equation (31) of theorem 2.9. The approach is the same as for the JUE and LUE. The proof for  $n$  odd is very similar and we omit the details.

Integrating by parts the first formula in (43) gives

$$(65) \quad M_G^{(2)}(2k, n) = \int_{-\infty}^{\infty} x^{2k} \rho_2(x) dx = \frac{D_n^H}{2k+1} \int_{-\infty}^{\infty} x^{2k+1} e^{-x^2} \mathcal{H}_n(x) \mathcal{H}_{n-1}(x) dx.$$

The integral (65) can be evaluated using the Laguerre polynomials since

$$(66) \quad \mathcal{H}_n(x) = \mathcal{L}_{n/2}^{-1/2}(x^2) \quad \text{and} \quad \mathcal{H}_{n-1}(x) = x \mathcal{L}_{n/2-1}^{1/2}(x^2).$$

A change of variables then leads to

$$(67) \quad M_G^{(2)}(2k, n) = \frac{D_n^H}{2k+1} \int_0^{\infty} x^{k+1/2} e^{-x} \mathcal{L}_{n/2-1}^{1/2}(x) \mathcal{L}_{n/2}^{-1/2}(x) dx.$$

This integral is of the same type as that one in the right-hand side of equation (60) and can be computed in the same way.

#### 4. SYMPLECTIC AND ORTHOGONAL SYMMETRIES

Very few non-perturbative results are available for the moments of the densities of the eigenvalues for  $\beta = 1$  and  $\beta = 4$ . Our goal here is to develop a novel approach that allows us to compute these moments for all the ensembles associated with the weights (2).

There are two possible ways of tackling this problem: the first is through the Selberg integral; the other one is a direct computation of the moments (36). The Selberg integral was very effective in computing the moments of the transmission eigenvalues for  $\beta = 2$  [44]; when  $\beta = 1$  it does not seem to produce explicit formulae [32]. It cannot be applied to  $\beta = 4$ .

Following an approach of Dyson [22], Mehta and Mahoux [40] expressed the densities for  $\beta = 1$  and  $\beta = 4$  in terms of skew-orthogonal polynomials. Since then several articles have attempted to improve their formulae [2, 26, 42, 53, 55]. Tracy and Widom [53] and Widom [55] succeeded to write such densities as sums of  $\rho_2(x)$  plus correction terms involving orthogonal polynomials. Building on the work of Adler and van Moerbeke [3], Adler *et al* [2] obtained integral representations of the correction terms.

Equation (36) presents one major challenge: for finite  $n$  it is a complicated sum involving all the orthogonal polynomials up to  $n - 1$ . Further integration would lead to cumbersome formulae whose asymptotics cannot be easily extracted. Our method relies on using the coefficients (54) and (61) to expand the orthogonal polynomials in a convenient basis, within which they are orthogonal with respect to the perturbed weight  $x^k w_2(x)$ . As when  $\beta = 2$ , this allows us to obtain positive moments involving sums that run to the order of the moments and not to the dimension of the ensemble. Another interesting feature of our results is that we are able to express the moments at  $\beta = 1$  in terms of the moments at  $\beta = 4$  plus a fairly simple correction term. Like for unitary ensembles, our formulae are sum of ratios of Gamma functions which may be studied in the limit  $n \rightarrow \infty$ .

Since our approach is based on the results in by Adler *et al* [2], we will discuss their formalism in detail. For  $\beta = 1$  and  $\beta = 4$  a special role is played by the skew-orthogonal polynomials. Recall that an inner product  $\langle A, B \rangle$  is referred to as *skew* if  $\langle A, B \rangle = -\langle B, A \rangle$ . A sequence of monic polynomials  $\{q_j(x)\}_{j=0}^{\infty}$  are called skew-orthogonal with respect to

$\langle A, B \rangle$  if

$$(68a) \quad \langle q_{2m}, q_{2n+1} \rangle = -\langle q_{2n+1}, q_{2m} \rangle = r_m \delta_{m,n}$$

$$(68b) \quad \langle q_{2m}, q_{2n} \rangle = \langle q_{2m+1}, q_{2n+1} \rangle = 0.$$

Let us introduce the potential  $V(x)$  by defining

$$(69) \quad w_2(x) = e^{-2V(x)},$$

where  $w_2(x)$  is the weight function of the associated unitary ensemble. We also assume that

$$(70) \quad 2V'(x) = \frac{g(x)}{f(x)}$$

is a rational function of  $x$  and take  $f(x)$  to be a monic polynomial. Now define the *modified potentials*:

$$(71a) \quad V_1(x) = V(x) + \frac{1}{2} \log f(x), \quad \beta = 1,$$

$$(71b) \quad V_4(x) = V(x) - \frac{1}{2} \log f(x), \quad \beta = 4.$$

Let us introduce the inner products

$$(72a) \quad \langle A, B \rangle_4 = \frac{1}{2} \int_I e^{-2V_4(x)} (A(x)B'(x) - B(x)A'(x)) dx$$

and

$$(72b) \quad \langle A, B \rangle_1 = \frac{1}{2} \int_I \int_I e^{-V_1(x) - V_1(y)} \operatorname{sgn}(y - x) A(x) B(y) dx dy.$$

Associated to these inner products are two systems of monic skew-orthogonal polynomials  $\{\tilde{q}_j^{(4)}(x)\}_{j=0}^\infty$  and  $\{\tilde{q}_j^{(1)}(x)\}_{j=0}^\infty$ . We shall denote their skew-norms as defined in (68) by  $\tilde{r}_j^{(4)}$  and  $\tilde{r}_j^{(1)}$  respectively. The tilde notation indicates that the weight has been perturbed by the transformations (71a) and (71b).

Because the skew-orthogonality relations (68) are invariant under the transformation  $\tilde{q}_{2m+1} \rightarrow \tilde{q}_{2m+1} + \alpha_{2m} \tilde{q}_{2m}$  for any  $\alpha_{2m} \in \mathbb{C}$ , a system of skew-orthogonal polynomials is not uniquely defined. However, they can be expressed in terms of the monic polynomials orthogonal with respect to  $w_2(x)$ :

$$(73a) \quad p_{2j+1}(x) = \tilde{q}_{2j+1}^{(4)}(x), \quad p_{2j}(x) = \tilde{q}_{2j}^{(4)}(x) - \frac{c_{2j-1}}{c_{2j-2}} \tilde{q}_{2j-2}^{(4)}(x),$$

$$(73b) \quad \tilde{q}_{2j}^{(1)}(x) = p_{2j}(x), \quad \tilde{q}_{2j+1}^{(1)}(x) = p_{2j+1}(x) - \frac{\gamma_{2j-1}}{\gamma_{2j}} p_{2j-1}(x).$$

For the classical orthogonal polynomials the constants in these equations are given by

$$(74) \quad c_n = h_{n+1} h_n \gamma_n,$$

where

$$h_n \gamma_n = \begin{cases} 1, & \text{Hermite,} \\ \frac{1}{2}, & \text{Laguerre,} \\ \frac{1}{2}(2n + a + b + 2), & \text{Jacobi.} \end{cases}$$

We point out that the numbers  $D_n^\mathcal{E}$  appearing in the differential identities (43) may be expressed in terms of  $\gamma_n$  via  $D_n^\mathcal{E} = 2\gamma_{n-1}$  for all the three ensembles. For each ensemble,

we denote the mean eigenvalue density by  $\tilde{\rho}_\beta(x)$ , where the tilde indicates the ensemble average (36) defined by the weight  $e^{-V_1(x)}$  for  $\beta = 1$  or  $e^{-2V_4(x)}$  for  $\beta = 4$ .

**Remark 4.1.** Before we proceed it is worth noting that the weights  $e^{-V_1(x)}$  turn out to be exactly equal to the weights  $w_1(x)$  in equation (2). The eigenvalue densities  $\tilde{\rho}_4(x)$ , however, correspond to the weights  $e^{-2V_4(x)}$ , which are

$$(75) \quad e^{-2V_4(x)} = \begin{cases} e^{-x^2}, & \text{Hermite,} \\ x^{b+1}e^{-x}, & \text{Laguerre,} \\ x^{b+1}(1-x)^{a+1} & \text{Jacobi.} \end{cases}$$

These are not quite the same as the weights in (2) for  $\beta = 4$ . We must make the substitution  $(a, b) \rightarrow (2a, 2b)$  for the Jacobi ensemble and  $b \rightarrow 2b$  for the Laguerre ensemble. In addition there is a missing factor of 2 in the exponentials which we take into account by multiplying our final results for the moments by the appropriate power of 2. This discrepancy arises because the symplectic ensembles are sets of self-dual  $n \times n$  quaternion matrices. Their representation in terms of complex matrices leads to Kramer's degeneracy, which is responsible for the normalisations in (75). Without loss of generality we shall still use the notation  $w_4(x)$ .

Let us introduce the  $\epsilon$ -transform of a suitable function  $f(t)$  by

$$(76) \quad \epsilon[f(t)](x) = \frac{1}{2} \int_I \text{sgn}(x-t) f(t) dt.$$

In the following,  $I = (m_1, m_2)$  will denote the interval of orthogonality. We have the following formulae [2]:

$$(77a) \quad \tilde{\rho}_4(x) = \frac{1}{2} \rho_2(x)_{n \rightarrow 2n} - \frac{1}{2} \gamma_{2n-1} e^{-V_1(x)} p_{2n}(x) \int_x^{m_2} e^{-V_1(t)} p_{2n-1}(t) dt$$

and (for  $n$  even)

$$(77b) \quad \tilde{\rho}_1(x) = \rho_2(x)_{n \rightarrow n-1} + \gamma_{n-2} e^{-V_1(x)} p_{n-1}(x) \epsilon[p_{n-2}(t) e^{-V_1(t)}](x).$$

For convenience we shall alter the representations (77a) and (77b) into a form which is more suitable for the evaluation of the integrals (36). The following proposition allows us to expand the integrals in (77a) and (77b) in terms of monic orthogonal polynomials.

**Proposition 4.2.** *Let  $\{p_j(x)\}_{j=0}^\infty$  be the system of monic polynomials orthogonal with respect to  $w_2(x)$ . We have the following identities:*

$$(78) \quad \epsilon[e^{-V_1(t)} p_{2n+2}(t)](x) = -e^{-V_4(x)} \sum_{j=0}^n e_{j,n}^{(1)} p_{2j+1}(x) + \eta_n^{(1)} \epsilon[e^{-V_1(t)}](x)$$

and

$$(79) \quad \int_x^{m_2} e^{-V_1(t)} p_{2n+1}(t) dt = -e^{-V_4(x)} \sum_{j=0}^n e_{j,n}^{(4)} p_{2j}(x),$$

where

$$(80) \quad e_{j,n}^{(4)} = \frac{h_{2n+1}}{c_{2n}} \prod_{i=j}^{n-1} \frac{c_{2i+1}}{c_{2i}}, \quad e_{j,n}^{(1)} = \frac{h_{2n+2}}{c_{2j+1}} \prod_{i=j+1}^n \frac{c_{2i}}{c_{2i+1}}, \quad \eta_n^{(1)} = \prod_{j=0}^n \frac{c_{2j} h_{2j+2}}{c_{2j+1} h_{2j}}.$$



*Proof.* We begin from the differential identities [2]

$$(81a) \quad \frac{d}{dx} \left( e^{-V_4(x)} \tilde{q}_{2n}^{(4)}(x) \right) = \frac{c_{2n}}{h_{2n+1}} e^{-V_1(x)} p_{2n+1}(x),$$

$$(81b) \quad \frac{d}{dx} \left( e^{-V_4(x)} \tilde{q}_{2n+1}^{(4)}(x) \right) = e^{-V_1(x)} \left( \frac{c_{2n}}{h_{2n}} p_{2n}(x) - \frac{c_{2n+1}}{h_{2n+2}} p_{2n+2}(x) \right).$$

We also need  $e^{-V_4(m_1)} = e^{-V_4(m_2)} = 0$ , which can be easily checked from (75).

We first derive (79). Integrating equation (81a) between  $x$  and  $m_2$  gives

$$(82) \quad \int_x^{m_2} e^{-V_1(t)} p_{2n+1}(t) dt = -\frac{h_{2n+1}}{c_{2n}} e^{-V_4(x)} q_{2n}^{(4)}(x) = -e^{-V_4(x)} \sum_{j=0}^n e_{j,n}^{(4)} p_{2j}(x),$$

where the last equality was obtained by iteratively solving equation (73a) for  $\tilde{q}_{2n}^{(4)}(x)$ .

In order to derive (78), we start by integrating equation (81b) between  $x$  and  $m_2$ :

$$(83) \quad \int_x^{m_2} e^{-V_1(t)} p_{2n+2}(t) dt = \frac{c_{2n} h_{2n+2}}{c_{2n+1} h_{2n}} \int_x^{m_2} e^{-V_1(t)} p_{2n}(t) dt - \frac{h_{2n+2}}{c_{2n+1}} e^{-V_4(x)} q_{2n+1}^{(4)}(x).$$

Integrating (81b) between  $m_1$  and  $x$  and subtracting the result from (83) gives an equation for the  $\epsilon$ -transform

$$(84) \quad \epsilon \left[ e^{-V_1(t)} p_{2n+2}(t) \right] = \frac{c_{2n} h_{2n+2}}{c_{2n+1} h_{2n}} \epsilon \left[ e^{-V_1(t)} p_{2n}(t) \right] - \frac{h_{2n+2}}{c_{2n+1}} e^{-V_4(x)} p_{2n+1}(x),$$

where we used that  $\tilde{q}_{2j+1}^{(4)} = p_{2j+1}(x)$ . Iterating this equation  $n$  times leads to (78).  $\square$

**Remark 4.3.** The coefficients  $e_{j,n}^{(4)}$ ,  $e_{j,n}^{(1)}$  and  $\eta_n^{(1)}$  are tabulated in appendix A for each ensemble.

**Corollary 4.4.** *We have the following representations for the eigenvalue densities:*

$$(85a) \quad \tilde{\rho}_4(x) = \frac{1}{2} \rho_2(x)_{n \rightarrow 2n} - \frac{1}{2} \gamma_{2n-1} e^{-2V(x)} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} p_{2j}(x) p_{2n}(x),$$

$$(85b) \quad \tilde{\rho}_1(x) = \rho_2(x)_{n \rightarrow n-1} - \gamma_{n-2} e^{-2V(x)} \sum_{j=0}^{n/2-2} e_{j,n/2-2}^{(1)} p_{2j+1}(x) p_{n-1}(x) \\ + \gamma_{n-2} e^{-V_1(x)} p_{n-1}(x) \eta_{n/2-2}^{(1)} \epsilon \left[ e^{-V_1(t)} \right] (x).$$

*Proof.* Substituting the integration identities (79) and (78) into (77a) and (77b) respectively, and using  $V_1(t) + V_4(t) = 2V(t)$ , gives (85a) and (85b).  $\square$

We will see in §5 and §6 that formulae (85) are particularly suited to our purposes. A key feature of these representations is that they are expressed solely in terms of the weight function  $e^{-2V(x)}$  and the corresponding monic orthogonal polynomials. For the orthogonal ensembles, there is an additional term involving the  $\epsilon$ -transform of the weight  $e^{-V_1(t)}$ , which is related to the error function, incomplete gamma function or incomplete beta function depending on the ensemble in question. We shall compute the moments for  $\beta = 1$  and  $\beta = 4$  by combining the representations (85) with a variant of the technique used for unitary ensembles.

## 5. SYMPLECTIC ENSEMBLES

The purpose of this section is to compute the integrals

$$(86) \quad \tilde{M}_{\mathcal{E}}^{(4)}(k, n) = \int_I x^k \tilde{\rho}_4(x) dx$$

for each ensemble  $\mathcal{E}$  defined by the weights (2).

Inserting the representation (85a) into equation (86) leads to two integrals: the first one contains the mean eigenvalue density of a unitary ensemble, which was computed in §3; the second one involves orthogonal polynomials. More explicitly it is given by

$$(87) \quad \tilde{S}_{\mathcal{E}}(k, n) = \gamma_{2n-1} \frac{1}{2} \sum_{j=0}^{n-1} e_{j, n-1}^{(4)} \int_I x^k e^{-2V(x)} p_{2j}(x) p_{2n}(x) dx.$$

We know how to evaluate these integrals as they are exactly of the type that appeared in equations (53), (60) and (65). We write the polynomials  $p_n(x)$  in a basis which is orthogonal with respect to the perturbed weight  $x^k e^{-2V(x)}$ ; then, we can use the orthogonality of the polynomials to write the integral in (87) as a single sum involving the connection coefficients (54) and (61). Eventually, the moments for  $\beta = 4$  become

$$(88) \quad \tilde{M}_{\mathcal{E}}^{(4)}(k, n) = M_{\mathcal{E}}^{(2)}(k, 2n) - \tilde{S}_{\mathcal{E}}(k, n).$$

As in §4 the tilde notation indicates quantities that differ from the integrals (1) by a factor discussed in remark 4.1.

**5.1. Jacobi Symplectic Ensemble.** We now compute  $S_{J_{a,b}}(k, n)$  for complex  $k$ . Equation (22) is then obtained by restricting  $k$  to be a positive integer and setting  $a = \delta/4 - 1/2$  and  $b = m - n$ .

**Proposition 5.1.** *We have*

$$(89) \quad S_{J_{a,b}}(k, n) = \sum_{j=1}^n \sum_{i=0}^{2n-2j} \binom{k}{i+2j} \binom{k}{i} S_{i,j}^{a,b}(k, n),$$

where the coefficient  $S_{i,j}^{a,b}(k, n)$  is given by

$$(90) \quad S_{i,j}^{a,b}(k, n) = \frac{2^{4j-3} (2a+2n-i-2j+1)_{(i)} (2b+2n)_{(k-i-2j+1)} (2a+2b+2n)_{(k-i-2j+1)}}{(2n-2j+1)_{(-i)} (n+1)_{(-j)} (a+n+1)_{(-j)} (b+n)_{(1-j)} (a+b+n)_{(1-j)}} \\ \times \frac{(2a+2b+4n-4j+1) (2a+2b+4n-2i-4j+k+1)}{(2a+2b+4n-i-2j+1)_{(1+k)} (2a+2b+4n-i-4j+1)_{(1+k)}}.$$

*Proof.* By (87) we have

$$(91) \quad \tilde{S}_{J_{a,b}}(k, n) = \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j, n-1}^{(4)} \int_0^1 x^{b+k} (1-x)^a \mathcal{P}_{2n}^{a,b}(x) \mathcal{P}_{2j}^{a,b}(x) dx.$$

Inserting the connection formula (54) into the integrand leads to

$$\begin{aligned}
\tilde{S}_{J_{a,b}}(k, n) &= \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \sum_{i=0}^{2j} \mathcal{C}_i^{k,2j} \sum_{p=0}^{2n} \mathcal{C}_p^{k,2n} \int_0^1 x^{b+k} (1-x)^a \mathcal{P}_{2n-p}^{a,b+k}(x) \mathcal{P}_{2j-i}^{a,b+k}(x) dx \\
(92) \quad &= \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \sum_{i=0}^{2j} \mathcal{C}_i^{k,2j} \sum_{p=0}^{2n} \mathcal{C}_p^{k,2n} h_{2j-i}^{a,b+k} \delta_{2n-p,2j-i} \\
&= \frac{\gamma_{2n-1}}{2} \sum_{j=1}^n \sum_{i=0}^{2n-2j} e_{n-j,n-1}^{(4)} h_{2n-2j-i}^{a,b+k} \mathcal{C}_i^{k,2n-2j} \mathcal{C}_{i+2j}^{k,2n}.
\end{aligned}$$

To obtain the above expression we have applied the orthogonality of the Jacobi polynomials and then rearranged the indices in the sum. Substituting the coefficients  $\mathcal{C}_j^{k,n}$ ,  $h_n^{a,b+k}$  and  $e_{j,k}^{(4)}$  (see appendix A) and replacing  $(a, b) \rightarrow (2a, 2b)$  completes the proof.  $\square$

**Remark 5.2.** The complexity of the expression (90) is mainly due to formula (54), which relates Jacobi polynomials of different weights. The Laguerre ensemble is slightly simpler, because the associated coefficients (61) and normalizations  $h_j$  are more concise.

## 5.2. Laguerre Symplectic Ensemble.

**Proposition 5.3.** *Suppose that neither  $2b + k$  nor  $b + n$  are negative integers. Then, we have*

$$(93) \quad S_{L_b}(k, n) = \sum_{j=1}^{\lfloor n \rfloor} \sum_{i=0}^{2n-2j} \binom{k}{i} \binom{k}{i+2j} \frac{(2b+2n)_{(k-i-2j+1)} (2n-i-2j+1)_{(i)}}{2^{k-2j+2} (n+1)_{(-j)} (b+n)_{(1-j)}}.$$

*Proof.* From (87) we have

$$(94) \quad \tilde{S}_{L_b}(k, n) = \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \int_0^\infty x^{b+k} e^{-x} \mathcal{L}_{2n}^b(x) \mathcal{L}_{2j}^b(x) dx.$$

Proceeding as in the proof of proposition 5.1 we obtain

$$\begin{aligned}
\tilde{S}_{L_b}(k, n) &= \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \sum_{i=0}^{2j} \mathcal{C}_i^{k,2j} \sum_{p=0}^{2n} \mathcal{C}_p^{k,2n} \int_0^\infty x^{b+k} e^{-x} \mathcal{L}_{2n-p}^{b+k}(x) \mathcal{L}_{2j-i}^{b+k}(x) dx \\
(95) \quad &= \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \sum_{i=0}^{2j} \mathcal{C}_i^{k,2j} \sum_{p=0}^{2n} \mathcal{C}_p^{k,2n} h_{2j-i}^{b+k} \delta_{2n-p,2j-i} \\
&= \frac{\gamma_{2n-1}}{2} \sum_{j=1}^n \sum_{i=0}^{2n-2j} e_{n-j,n-1}^{(4)} \mathcal{C}_i^{k,2n-2j} \mathcal{C}_{i+2j}^{k,2n} h_{2n-2j-i}^{b+k}
\end{aligned}$$

By replacing  $b \rightarrow 2b$  and multiplying both sides of this equation by  $2^{-k}$  gives the statement of the proposition.  $\square$

Both the propositions 5.1 and 5.3 hold for complex values of  $k$ , except where the integrals (1) diverge. In particular, the moments of the proper delay times (15) are expressed in terms of negative moments of the Laguerre ensemble and can be obtained from (93) using the identity

$$(96) \quad \binom{-k}{j} \binom{-k}{i+2j} = \binom{k+j-1}{k-1} \binom{k+i+2j-1}{k-1}$$

and setting  $b = n + 1$  in (93). Thus, we arrive at the following:

**Corollary 5.4.** *Let  $k$  be a positive integer. We have*

$$(97) \quad S_{J_{a,b}}(k, n) = \sum_{j=1}^{\min(\lfloor n \rfloor, \lfloor k/2 \rfloor)} \sum_{i=0}^{\min(2n-2j, k-2j)} \binom{k}{i+2j} \binom{k}{i} S_{i,j}^{a,b}(k, n)$$

$$(98) \quad S_{L_b}(k, n) = \sum_{j=1}^{\min(\lfloor n \rfloor, \lfloor k/2 \rfloor)} \sum_{i=0}^{\min(2n-2j, k-2j)} \binom{k}{i+2j} \binom{k}{i} \times \frac{(2b+2n)_{(k-i-2j+1)}(2n-i-2j+1)_{(i)}}{2^{k-2j+2}(n+1)_{(-j)}(n+b)_{(1-j)}}.$$

Furthermore, if  $k < 2n + 1$

$$(99) \quad S_{L_b}(-k, n) = \sum_{j=1}^{\lfloor n \rfloor} \sum_{i=0}^{2n-2j} \binom{k+j-1}{k-1} \binom{k+i+2j-1}{k-1} \times \frac{(2b+2n)_{(-k-i-2j+1)}(2n-i-2j+1)_{(i)}}{2^{-k-2j+2}(n+1)_{(-j)}(n+b)_{(1-j)}}.$$

**Remark 5.5.** The combinatorial aspect of the sums in this corollary arises directly from the connection coefficients  $\mathcal{C}_i$  and  $\mathcal{C}_{i+2j}$  appearing in (95) and (92), leading to the binomial coefficients. Due to these binomial coefficients, the sums (97) only go up to  $k$  and their complexity does not increase as  $n$  grows. This therefore permits an investigation of the asymptotics. The  $n \rightarrow \infty$  analysis of (99) leads to certain infinite sums which also turn out to be tractable. A similar remark holds for the GSE.

**5.3. Gaussian Symplectic Ensemble.** In the Gaussian case, only the even moments are different from zero. For simplicity we only state the results for integer  $k$ .

**Proposition 5.6.** *We have*

$$(100) \quad S_G(2k, n) = \frac{\Gamma(n+1)\Gamma(n)}{2^k \sqrt{\pi} \Gamma(2n) 4^{1-n}} \sum_{j=1}^{\min(n,k)} \sum_{i=0}^{\min(n-j, k-j)} \binom{k}{i} \binom{k}{i+j} (n-i-j+1)_{(k-1/2)}$$

*Proof.* The integrals (87) give

$$(101) \quad \tilde{S}_G(2k, n) = \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \int_{-\infty}^{\infty} x^{2k} e^{-x^2} \mathcal{H}_{2n}(x) \mathcal{H}_{2j}(x) dx.$$

Applying formula (66) leads to

$$(102) \quad \tilde{S}_G(2k, n) = \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \int_{-\infty}^{\infty} x^{2k} e^{-x^2} \mathcal{L}_n^{-1/2}(x^2) \mathcal{L}_j^{-1/2}(x^2) dx.$$

Changing variables and inserting the connection formula (61) results in

$$\tilde{S}_G(2k, n) = \frac{\gamma_{2n-1}}{2} \sum_{j=0}^{n-1} e_{j,n-1}^{(4)} \sum_{i=0}^j \mathcal{C}_i^{k,j} \sum_{p=0}^n \mathcal{C}_p^{k,n} \int_0^{\infty} x^{k-1/2} e^{-x} \mathcal{L}_{n-p}^{k-1/2}(x) \mathcal{L}_{j-i}^{k-1/2}(x) dx.$$

Using the orthogonality of the Laguerre polynomials gives the double sum

$$(103) \quad \tilde{S}_G(2k, n) = \frac{\gamma_{2n-1}}{2} \sum_{j=1}^n \sum_{i=0}^{n-j} e_{n-j,n-1}^{(4)} \mathcal{C}_i^{k,n-j} \mathcal{C}_{i+j}^{k,n} h_{n-i-j}^{k-1/2}.$$

It is worth emphasising that the coefficients  $C_i^{k,n-j}$ ,  $C_{i+j}^{k,n}$  and  $h_n$  are those of the Laguerre polynomials, while the coefficient  $e_{n-j,n-1}^{(4)}$  is related to the Hermite polynomials. Finally, multiplying equation (103) by  $2^{-k}$ , as discussed in remark 4.1, completes the proof.  $\square$

## 6. ORTHOGONAL ENSEMBLES

In this section we compute the moments for  $\beta = 1$ . Theorem 2.7 and equation (34) of Theorem 2.9 are corollaries of the results we prove here. For simplicity we assume that  $n$  is an even integer.

The main task is to compute the integral

$$(104) \quad M_{\mathcal{E}}^{(1)}(k, n) = \int_I x^k \tilde{\rho}_1(x) dx.$$

When  $\beta = 1$  the density  $\tilde{\rho}_1(x)$  coincides with  $\rho_1(x)$ , so the integrals (104) coincide with the averages (1).

Substituting the representation (85b) into (104), we are left to compute three integrals: the first one gives the moments of the corresponding unitary ensemble; the second one is closely related to the quantity  $\tilde{S}_{\mathcal{E}}(k, n)$  discussed in §5, namely

$$(105) \quad O_{\mathcal{E}}(k, n) = \gamma_{n-2} \sum_{j=0}^{n-1} e_{j,n/2-2}^{(1)} \int_I x^k e^{-2V(x)} \mathcal{P}_{2j+1}(x) \mathcal{P}_{n-1}(x) dx;$$

the last one arises from the  $\epsilon$ -transform in equation (85b), *i.e.*

$$(106) \quad I_{\mathcal{E}}(k, n) = \gamma_{n-2} \eta_{n/2-2}^{(1)} \int_I x^k \mathcal{P}_{n-1}(x) e^{-V_1(x)} \epsilon [e^{-V_1(t)}] (x) dx.$$

Therefore, the moments for  $\beta = 1$  may be expressed as

$$(107) \quad M_{\mathcal{E}}^{(1)}(k, n) = M_{\mathcal{E}}^{(2)}(k, n-1) - O_{\mathcal{E}}(k, n) + I_{\mathcal{E}}(k, n)$$

In this section we focus on the integrals  $O_{\mathcal{E}}(k, n)$  and  $I_{\mathcal{E}}(k, n)$ .

**6.1. A Duality between  $\beta = 1$  and  $\beta = 4$ .** We first discuss a remarkable duality between the quantities  $M_{\mathcal{E}}^{(2)}(k, n-1) - O_{\mathcal{E}}(k, n)$  and the moments of the symplectic ensembles  $M_{\mathcal{E}}^{(4)}(k, n)$ , where  $n$  now can assume half integer values. Such moments are well defined (see equation (88) and remark 2.6). Similar dualities have appeared in the literature before [19, 24, 34].

**Lemma 6.1.** *Let  $n$  be an even integer. We have the following dualities:*

$$(108a) \quad M_{L_b}^{(1)}(k, n) = 2^{1+k} M_{L_{b/2}}^{(4)}(k, (n-1)/2) + I_{L_b}(k, n),$$

$$(108b) \quad M_{J_{a,b}}^{(1)}(k, n) = 2M_{J_{a/2,b/2}}^{(4)}(k, (n-1)/2) + I_{J_{a,b}}(k, n).$$

*Proof.* Firstly, by equation (88) we observe that

$$(109) \quad 2^{1+k} M_{L_{b/2}}^{(4)}(k, (n-1)/2) = M_{L_b}^{(2)}(k, n-1) - 2\tilde{S}_{L_b}(k, (n-1)/2).$$

Thus, it is sufficient to check that  $2\tilde{S}_{L_{b/2}}(k, (n-1)/2) = O_{L_b}(k, n)$ .

A direct computation shows that

$$\begin{aligned}
(110) \quad O_{L_b}(k, n) &= \gamma_{n-2} \sum_{j=0}^{n/2-2} e_{j, n/2-2}^{(1)} \int_0^\infty x^{b+k} e^{-x} \mathcal{L}_{n-1}^b(x) \mathcal{L}_{2j+1}^b(x) dx \\
&= \gamma_{n-2} \sum_{j=0}^{n/2-2} e_{j, n/2-2}^{(1)} \sum_{i=0}^{2j+1} \mathcal{C}_i^{k, 2j+1} \sum_{p=0}^{n-1} \mathcal{C}_p^{k, n-1} h_{2j+1-i}^{b+k} \delta_{2j+1-i, n-1-p} \\
&= \gamma_{n-2} \sum_{j=1}^{n/2-1} \sum_{i=0}^{n-2j-1} e_{n/2-1-j, n/2-2}^{(1)} \mathcal{C}_i^{k, n-2j-1} \mathcal{C}_{i+2j}^{k, n-1} h_{n-1-2j-i}^{b+k}.
\end{aligned}$$

From (95) we see that

$$(111) \quad 2\tilde{S}_{L_b}(k, (n-1)/2) = \gamma_{n-2} \sum_{j=1}^{n/2-1} \sum_{i=0}^{n-2j-1} e_{n/2-1/2-j, n/2-3/2}^{(4)} \mathcal{C}_i^{k, n-2j-1} \mathcal{C}_{i+2j}^{k, n-1} h_{n-1-2j-i}^{b+k}.$$

From equation (A6a) we have that

$$(112) \quad e_{n/2-1/2-j, n/2-3/2}^{(4)} = e_{n/2-1-j, n/2-2}^{(1)}.$$

Thus, the right-hand sides of equations (110) and (111) coincide.

In the proof of the duality (108b) one has to show that

$$2\tilde{S}_{J_{a/2, b/2}}(k, (n-1)/2) = O_{J_{a, b}}(k, n).$$

The strategy is the same as for the Laguerre ensemble and we omit the computation.  $\square$

We are now left with the task of computing the integrals (106). When  $k$  is a positive integer, we find a single sum containing  $k$  terms for each ensemble; when  $k$  is negative the sums go up to order of the matrix dimension.

**6.2. Incomplete Integrals — Positive Moments.** We now assume that  $k$  is a positive integer and focus on the Laguerre and Jacobi ensembles.

**Lemma 6.2.** *We have*

$$(113) \quad I_{L_b}(k, n) = 2^k \sum_{j=0}^{\min(n/2-1, k)} \binom{2k}{2j} \frac{(\frac{1}{2}(b+n))_{(k-j)}}{(\frac{1}{2}(1+n))_{(-j)}} + \tilde{\phi}_{k, n}^L$$

and

$$\begin{aligned}
(114) \quad I_{J_{a, b}}(k, n) &= 4^k \sum_{j=0}^{\min(n/2-1, k)} \binom{2k}{2j} \frac{(a+b+2n-4j-1+2k)(\frac{1}{2}(a+b+n))_{(k-j)}(\frac{1}{2}(b+n))_{(k-j)}}{(a+b+2n-2j-1)_{(2k+1)}(\frac{1}{2}(a+n+1))_{(-j)}(\frac{1}{2}(1+n))_{(-j)}} \\
&\quad + \phi_{k, n}^J,
\end{aligned}$$

where

$$(115) \quad \tilde{\phi}_{k, n}^L = \frac{\Gamma(n/2 + b/2 - 1/2)}{\Gamma(n/2)\Gamma(b+n-1)} \sum_{j=1}^k \frac{\Gamma(j+k)\Gamma(b+j+k)2^{-j}}{\Gamma(j+k-n+1)\Gamma(b/2+1/2+j)}$$

and

$$(116) \quad \phi_{k,n}^J = \sum_{j=1}^k \frac{2^{a+1} \Gamma(a/2 + b/2 + j) \Gamma(a/2 + b/2 + 1/2 + n/2)}{\Gamma(a + b + n + j + k) \Gamma(j + k + 1 - n) \Gamma(b/2 + 1/2 + j)} \\ \times \frac{\Gamma(a/2 + 1/2 + n/2) \Gamma(b + k + j) \Gamma(j + k)}{\Gamma(n/2 + b/2) \Gamma(a/2 + 1/2) \Gamma(n/2)}.$$

Furthermore, if  $n > 2k$ ,  $\tilde{\phi}_{k,n}^L = \phi_{k,n}^J = 0$ .

*Proof.* We begin with the proof of formula (113). Equation (106) becomes

$$(117) \quad I_{L_b}(k, n) = \gamma_{n-2} \eta_{n/2-2}^{(1)} \int_0^\infty x^{k+(b-1)/2} \mathcal{L}_{n-1}^b(x) \epsilon \left[ t^{(b-1)/2} e^{-t/2} \right] (x) dx.$$

The  $\epsilon$ -transform appearing in the right-hand side is the difference of the two incomplete Gamma functions

$$(118) \quad \gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt \quad \text{and} \quad \Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt.$$

The main idea here is to expand them in a sum of incomplete Gamma functions whose weight has been perturbed by a factor  $t^k$ . To this end we insert

$$(119) \quad \epsilon \left[ t^{\frac{b-1}{2}} e^{-t/2} \right] (x) = \sum_{j=1}^k d_j^{(b+1)/2} x^{\frac{b-1}{2}+j} e^{-x/2} + \frac{1}{2} d_k^{(b+1)/2} \epsilon \left[ t^{\frac{b-1}{2}+k} e^{-t/2} \right] (x),$$

where  $d_j^b = 2^{1-j} \frac{\Gamma(b)}{\Gamma(b+j)}$ . When  $k = 1$  this identity can be found in [1], equations (6.5.21) and (6.5.23); the formula for general  $k$  is obtained by iteration.

This leads to two integrals. The first one is

$$(120) \quad \tilde{\phi}_{k,n}^L = \eta_{n/2-2}^{(1)} \gamma_{n-2} \sum_{j=1}^k \int_0^\infty d_j^{(b+1)/2} x^{b+j+k-1} e^{-x} \mathcal{L}_{n-1}^b(x) dx \\ = \eta_{n/2-2}^{(1)} \gamma_{n-2} \sum_{j=1}^k \sum_{i=0}^{j+k-1} \int_0^\infty \mathcal{C}_i^{j+k-1, n-1} d_j^{(b+1)/2} x^{b+j+k-1} e^{-x} \mathcal{L}_{n-1-i}^{b+j+k-1}(x) dx,$$

where we inserted the connection formula (61). Because of the orthogonality of the Laguerre polynomials the only contribution to the inner sum occurs at  $i = n - 1$ ; furthermore, since  $\max(i) = 2k - 1$ , we have that  $\tilde{\phi}_{k,n}^L = 0$  if  $n > 2k$ . If  $n \leq 2k$ , inserting the appropriate coefficients (see appendix A) and using the duplication formula (see [1], §6)

$$(121) \quad \Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$$

gives (115).

The remaining non-trivial integral is

$$(122a) \quad \psi_{k,n} = C \int_0^\infty x^{\frac{b-1}{2}+k} e^{-x/2} \mathcal{L}_{n-1}^b(x) \epsilon \left[ t^{\frac{b-1}{2}+k} e^{-t/2} \right] (x) dx \\ = C \sum_{j=0}^{\min(n-1, 2k)} \int_0^\infty x^{\frac{b-1}{2}+k} e^{-x/2} \mathcal{C}_j^{2k, n-1} \mathcal{L}_{n-1-j}^{b+2k}(x) \epsilon \left[ t^{\frac{b-1}{2}+k} e^{-t/2} \right] (x) dx$$

$$(122b) \quad = C \sum_{j=0}^{\min(n/2-1, k)} \mathcal{C}_{2j}^{2k, n-1} \int_0^\infty x^{\frac{b-1}{2}+k} e^{-x/2} \mathcal{L}_{n-2j-1}^{b+2k}(x) \epsilon \left[ t^{\frac{b-1}{2}+k} e^{-t/2} \right] (x) dx,$$

where  $C = \eta_{n/2-2}^{(1)} \gamma_{n-2} d_k^{(b+1)/2} / 2$ . To obtain (122a) we used the connection formula, while (122b) follows from the fact that the contributions to the sum with odd indices vanish due to the skew-orthogonality constraints (68).

Now we integrate (122b) by parts using the identity (79) and the formula

$$(123) \quad \frac{d}{dx} \epsilon \left[ t^{\frac{b-1}{2}+k} e^{-t/2} \right] (x) = -x^{\frac{b-1}{2}+k} e^{-x/2}.$$

This leads to

$$(124) \quad \begin{aligned} \psi_{k,n} &= C \sum_{j=0}^{\min(n/2-1,k)} \sum_{i=0}^{n/2-j-1} \mathcal{C}_{2j}^{2k,n-1} e_{i,n/2-j-1}^{(4),b+2k} \int_0^\infty x^{b+2k} e^{-x} \mathcal{L}_{2i}^{b+2k}(x) dx \\ &= C \sum_{j=0}^{\min(n/2-1,k)} \mathcal{C}_{2j}^{2k,n-1} e_{0,n/2-j-1}^{(4),b+2k} h_0^{b+2k}. \end{aligned}$$

Inserting all the relevant formulae for the orthogonality norms and connection coefficients gives (113).

We sketch the proof of (114) as it follows a similar pattern; we only emphasise the differences. For the Jacobi ensemble the  $\epsilon$ -transform is expressed in terms of incomplete Beta functions; thus, we replace (119) with the following identity:

$$(125) \quad \epsilon[t^b(1-t)^a](x) = \sum_{j=1}^k d_j^{a,b} x^{b+j} (1-x)^{a+1} + (a+b+k+1) d_k^{a,b} \epsilon[t^{b+k}(1-t)^a](x),$$

where

$$(126) \quad d_j^{a,b} = \frac{\Gamma(a+b+j+1)\Gamma(b+1)}{\Gamma(a+b+2)\Gamma(b+1+j)}.$$

Equation (125) can be obtained by iteration from formulae (26.5.15) and (25.5.16) in [1]. Proceeding as for the Laguerre ensemble and using formula (121) gives (114).  $\square$

**6.3. Incomplete Integrals — Negative Moments.** When the moments are negative, we focus only on the physically interesting case of the Laguerre ensemble, which leads to moments of the proper delay times.

When the moments are positive the correction terms  $\tilde{\phi}_{k,n}^L$  and  $\phi_{k,n}^J$  vanish if  $n > 2k$ . Now we have a similar contribution, which we shall denote  $\phi_{-k,n}^L$  and which is not zero for  $n > 2k$ ; however, it turns out that  $\phi_{-k,n}^L \rightarrow 0$  exponentially fast as  $n \rightarrow \infty$ .

**Lemma 6.3.** *Let  $k$  be a positive integer. One has*

$$(127) \quad I_{L_b}(-k, n) = 2^{-k} \sum_{j=0}^{n/2-1} \binom{2k+2j-1}{2j} \frac{(\frac{1}{2}(b+n))_{(-k-j)}}{(\frac{1}{2}(1+n))_{(-j)}} + \phi_{-k,n}^L,$$

where

$$(128) \quad \phi_{-k,n}^L = \frac{\Gamma(n/2 + b/2 - 1/2)}{\Gamma(n/2)\Gamma(b+n-1)} \sum_{j=0}^{k-1} \frac{\Gamma(k+j+n)\Gamma(b-k-j)2^j}{\Gamma(b/2 + 1/2 - j)\Gamma(k+j+1)}.$$

Furthermore, we have

$$(129) \quad \phi_{-k,n}^L = O(e^{-cn}), \quad n \rightarrow \infty, \quad c > 0.$$



*Proof.* The incomplete integral now becomes

$$(130) \quad I_{L_b}(-k, n) = \eta_{n/2-2}^{(1)} \gamma_{n-2} \int_0^\infty x^{\frac{b-1}{2}-k} e^{-x/2} \mathcal{L}_{n-1}^b(x) \epsilon \left[ t^{\frac{b-1}{2}} e^{-t/2} \right] (x) dx.$$

As for positive moments, we insert into (130) the identity

$$(131) \quad \epsilon \left[ t^{\frac{b-1}{2}} e^{-t/2} \right] (x) = - \sum_{j=0}^{k-1} d_j^{(b+1)/2} x^{\frac{b-1}{2}-j} e^{-x/2} + \frac{1}{2} d_k^{(b+1)/2} \epsilon \left[ t^{\frac{b-1}{2}-k} e^{-t/2} \right] (x),$$

where  $d_j^b = 2^{j+1} \frac{\Gamma(b)}{\Gamma(b-j)}$ . Formula (131) is obtained in the same way as equation (119). This gives two integrals: the first one is

$$(132) \quad \begin{aligned} \phi_{-k,n}^L &= -\eta_{n/2-2}^{(1)} \gamma_{n-2} \sum_{j=0}^{k-1} \int_0^\infty d_j^{(b+1)/2} x^{b-j-k-1} e^{-x} \mathcal{L}_{n-1}^b(x) dx \\ &= -\eta_{n/2-2}^{(1)} \gamma_{n-2} \sum_{j=0}^{k-1} \sum_{i=0}^{n-1} \int_0^\infty \mathcal{C}_i^{-k-j-1, n-1} d_j^{(b+1)/2} x^{b-j-k-1} e^{-x} \mathcal{L}_i^{b-j-k-1}(x) dx, \end{aligned}$$

where we inserted the connection coefficients  $\mathcal{C}_i$  for the Laguerre polynomials (61). Application of orthogonality implies that the only contribution to the inner sum occurs at  $i = 0$ , yielding equation (128). The remaining non-trivial integral is

$$(133) \quad \frac{1}{2} \eta_{n/2-2}^{(1)} \gamma_{n-2} d_k^{(b+1)/2} \int_0^\infty x^{\frac{b-1}{2}-k} e^{-x/2} \mathcal{L}_{n-1}^b(x) \epsilon \left[ t^{\frac{b-1}{2}-k} e^{-t/2} \right] (x) dx,$$

which can be computed in the same way as the right-hand side of equation (124).  $\square$

**Remark 6.4.** The sum

$$(134) \quad 2^{-k} \sum_{j=0}^{n/2-1} \binom{2k+2j-1}{2j} \frac{(n/2+b/2)_{(-k-j)}}{(n/2+1/2)_{(-j)}}$$

in equation (127) is  $O(n^{-k})$  and of subleading order compared to  $M_{L_b}^{(2)}(-k, n-1)$ , which gives the main contribution to the moments of the proper delay times for  $\beta = 1$ . However, it goes to zero much more slowly than the correction term  $\phi_{-k,n}^L$ .

**6.4. Gaussian Orthogonal Ensemble.** The treatment of the GOE by our method is slightly different from the LOE and JOE. We do not find a duality relation similar to lemma 6.1. However, the following proposition is the analogue of equation (100) for the GSE.

**Lemma 6.5.** *Let  $k$  be a positive integer and suppose  $n$  is even. Then, the integral (105) is explicitly given by*

$$(135) \quad O_G(2k, n) = \sum_{j=1}^{\min(n/2-1, k)} \sum_{i=0}^{\min(n/2-j-1, k-j)} \binom{k}{i} \binom{k}{i+j} \frac{(n/2-i-j)_{(k+1/2)}}{(n/2-j)_{(1/2)}}.$$

*Proof.* By changing variable of integration and using the relation (66) we obtain

$$(136) \quad \begin{aligned} O_G(2k, n) &= \int_{-\infty}^\infty \sum_{j=0}^{n/2-2} x^k \gamma_{n-2} e^{-x^2} e_{j, n/2-2}^{(1)} \mathcal{H}_{n-1}(x) \mathcal{H}_{2j+1}(x) dx \\ &= \int_0^\infty \sum_{j=0}^{n/2-2} x^{k+1/2} \gamma_{n-2} e^{-x} e_{j, n/2-2}^{(1)} \mathcal{L}_{n/2-1}^{1/2}(x) \mathcal{L}_j^{1/2}(x) dx. \end{aligned}$$

Inserting the connection formula (61) into (136) leads to

$$(137) \quad O_G(2k, n) = \gamma_{n-2} \sum_{j=0}^{n/2-2} e_{j, n/2-2}^{(1)} \sum_{i=0}^j \mathcal{C}_i^{k, j} \\ \times \sum_{p=0}^{n/2-1} \mathcal{C}_p^{k, n/2-1} \int_0^\infty x^{k+1/2} e^{-x} \mathcal{L}_{n/2-1-p}^{k+1/2}(x) \mathcal{L}_{j-i}^{k+1/2}(x) dx.$$

By using the orthogonality of the Laguerre polynomials and rearranging the indices we obtain

$$(138) \quad O_G(2k, n) = \gamma_{n-2} \sum_{j=1}^{n/2-1} e_{n/2-j-1, n/2-2}^{(1)} \sum_{i=0}^{n/2-j-1} \mathcal{C}_i^{k, n/2-j-1} \mathcal{C}_{i+j}^{k, n/2-1} h_{n/2-j-i-1}^{k+1/2}.$$

Inserting the appropriate constants from appendix A completes the proof. The coefficients  $\mathcal{C}_i^{k, n/2-j-1}$ ,  $\mathcal{C}_{i+j}^{k, n/2-1}$  and  $h_{n/2-j-i-1}^{k+1/2}$  are those for the Laguerre polynomials; the constants  $\gamma_{n-2}$  and  $e_{n/2-j-1, n/2-2}^{(1)}$  are those associated to the GOE.  $\square$

The remaining task is the evaluation of the integral  $I_G(k, n)$  in (106). As previously, we obtain slightly different expressions depending on whether  $n \leq 2k$  or  $n > 2k$ . For the latter inequality the formula simplifies considerably.

**Lemma 6.6.** *Let  $n$  be an even integer. If  $n \leq 2k$  we have*

$$\phi_{k, n}^G = I_G(2k, n) = 2^{n/2-k} \frac{(2k)!}{\Gamma(n/2)} \sum_{j=0}^{k-n/2} \sum_{i=0}^{n/2-1} \frac{\binom{n-1}{2i} \frac{2^{-j-2i} (-1)^j}{(2j+2i+1)j!}}{(k-n/2-j)!} \\ + \frac{(2k)!}{\Gamma(n/2)} \sum_{j=0}^{n/2-1} \sum_{i=0}^j \frac{(n/2-i-1)! \binom{n-1}{n-2i-1}}{(j-i)!(k-j)! 4^{k-i}}.$$

When  $n > 2k$  we obtain

$$(139) \quad \phi_{k, n}^G = (2k)! \sum_{j=0}^k \frac{(n/2 + 1/2 - j)_{(j)} 2^{3j-k}}{(2j)!(k-j)!}.$$

*Proof.* We have

$$(140) \quad I_G(2k, n) = C \int_{-\infty}^\infty x^{2k} e^{-x^2/2} \mathcal{H}_{n-1}(x) \int_{-\infty}^\infty e^{-t^2/2} \text{sgn}(x-t) dt dx$$

where  $C = \eta_{n/2-2}^{(1)} \gamma_{n-2} / 2 = (2\sqrt{\pi} \Gamma(n/2))^{-1}$ . Now, consider the generating function

$$\mathcal{M}_G(s) = C \int_{-\infty}^\infty e^{sx} e^{-x^2/2} \mathcal{H}_{n-1}(x) \int_{-\infty}^\infty e^{-t^2/2} \text{sgn}(x-t) dt dx.$$

Completing the square in the exponent and changing variables leads to

$$(141) \quad \mathcal{M}_G(s) = C \sum_{j=0}^{n-1} e^{s^2/2} \int_{-\infty}^\infty \mathcal{H}_j(u) s^{n-1-j} \binom{n-1}{j} \int_{-\infty}^\infty e^{-(v+s)^2/2} \text{sgn}(u-v) dv du,$$

where we have applied the connection formula [28]

$$\mathcal{H}_{n-1}(u+s) = \sum_{j=0}^{n-1} \binom{n-1}{j} \mathcal{H}_j(u) s^{n-1-j}.$$

Equation (141) motivates us to study the function

$$(142) \quad f_j(s) = \int_{-\infty}^{\infty} e^{-u^2/2} \mathcal{H}_j(u) \int_{-\infty}^{\infty} e^{-(v+s)^2/2} \text{sgn}(u-v) dv du.$$

In appendix B we compute  $f_j(s)$  in terms of the power series

$$(143) \quad f_{2j}(s) = \sum_{p=j}^{\infty} s^{2p+1} a_{p,j}^+, \quad \text{and} \quad f_{2j+1}(s) = 2j! \sqrt{\pi} + \sum_{p=j+1}^{\infty} s^{2p} a_{p,j}^-,$$

where

$$(144) \quad a_{p,j}^+ = \frac{2^{1-2p} \sqrt{\pi} (-1)^{j-p}}{(2p+1)\Gamma(p-j+1)}, \quad \text{and} \quad a_{p,j}^- = \frac{2^{2-2p} \sqrt{\pi} (-1)^{j-p}}{(2p)\Gamma(p-j)}.$$

Thus, we can write the decomposition

$$(145) \quad \mathcal{M}_G(s) = C(\mathcal{M}_G^+(s) + \mathcal{M}_G^-(s)),$$

where

$$(146) \quad \mathcal{M}_G^+(s) = \sum_{j=0}^{n/2-1} e^{s^2/2} \binom{n-1}{2j} s^{n-2j-1} f_{2j}(s),$$

$$(147) \quad \mathcal{M}_G^-(s) = \sum_{j=0}^{n/2-1} e^{s^2/2} \binom{n-1}{2j+1} s^{n-2j-2} f_{2j+1}(s).$$

Computing the Taylor expansions of these functions is a routine (though tedious) exercise. Eventually, we obtain

$$(148) \quad C \frac{d^{2k}}{ds^{2k}} \mathcal{M}_G^+(s) \Big|_{s=0} = 2^{n/2-k} \frac{(2k)!}{\Gamma(n/2)} \sum_{j=0}^{k-n/2} \sum_{i=0}^{n/2-1} \frac{\binom{n-1}{2i} \frac{2^{-j-2i} (-1)^j}{(2j+2i+1)j!}}{(k-n/2-j)!},$$

which vanishes if  $n > 2k$ . We also have

$$(149) \quad C \frac{d^{2k}}{ds^{2k}} \mathcal{M}_G^-(s) \Big|_{s=0} = \frac{(2k)!}{\Gamma(n/2)} \sum_{j=0}^{\min(n/2-1, k)} \sum_{i=0}^j \frac{(n/2-i-1)! \binom{n-1}{n-2i-1}}{(j-i)!(k-j)! 4^{k-i}},$$

which combined with (135) gives the equation (34). Under the assumption  $n > 2k$ , equation (149) can be simplified further, leading to

$$\begin{aligned} I_G(2k, n) &= \frac{(2k)!}{\Gamma(n/2)} \sum_{i=0}^k \sum_{j=i}^k \frac{4^{i-k} (n/2-i-1)! \binom{n-1}{n-2i-1}}{(k-j)!(j-i)!} \\ &= (2k)! \sum_{j=0}^k \frac{(n/2+1/2-j)_{(j)} 2^{3j-k}}{(2j)!(k-j)!}, \end{aligned}$$

where in the first equality we interchanged the order of summation, while in the second one we have used the formula

$$(150) \quad \sum_{j=p}^k \frac{1}{(k-j)!(j-p)!} = \frac{2^{k-p}}{(k-p)!}.$$

□

APPENDIX A. THE COEFFICIENTS  $h_n$ ,  $e_n^{(1)}$  AND  $e_n^{(4)}$

The orthogonality normalisations  $h_j$  defined in (38) for the classical orthogonal polynomials are tabulated in any standard reference on special functions [1, 52]. However, in this article we work with *monic* polynomials. This is not the standard normalization found in the literature. Therefore, for the reader's convenience, we report the coefficients  $h_n$ ,  $e_n^{(1)}$ ,  $e_n^{(4)}$  that we use throughout this paper.

We have

$$(A1) \quad \begin{cases} h_j^{a,b} = \frac{\Gamma(a+j+1)\Gamma(b+j+1)\Gamma(j+1)\Gamma(a+b+j+1)}{\Gamma(a+b+2j+1)\Gamma(a+b+2j+2)}, & \text{Jacobi,} \\ h_j^b = \Gamma(j+1)\Gamma(b+j+1), & \text{Laguerre,} \\ h_j = j!2^{-j}\sqrt{\pi}, & \text{Hermite.} \end{cases}$$

Given these normalisations, recall that

$$(A2) \quad c_n = h_{n+1}h_n\gamma_n,$$

where

$$(A3) \quad h_n\gamma_n = \begin{cases} 1, & \text{Hermite,} \\ \frac{1}{2}, & \text{Laguerre,} \\ \frac{1}{2}(2n+a+b+2), & \text{Jacobi.} \end{cases}$$

In §4 for the orthogonal and symplectic ensembles we introduced the quantities

$$(A4) \quad e_{j,n}^{(1)} = \frac{h_{2n+2}}{c_{2j+1}} \prod_{i=j+1}^n \frac{c_{2i}}{c_{2i+1}}, \quad e_{j,n}^{(4)} = \frac{h_{2n+1}}{c_{2n}} \prod_{i=j}^{n-1} \frac{c_{2i+1}}{c_{2i}}, \quad \eta_n^{(1)} = \prod_{j=0}^n \frac{c_{2j}h_{2j+2}}{c_{2j+1}h_{2j}}.$$

We have the following explicit formulae:

JOE and JSE,

$$(A5a) \quad e_{j,n}^{(1),a,b} = \frac{16^{n-j}2\Gamma(n+3/2)\Gamma(n+3/2+a/2)}{\Gamma(j+3/2)\Gamma(j+3/2+a/2)\Gamma(j+3/2+b/2)} \\ \times \frac{\Gamma(n+3/2+a/2+b/2)\Gamma(4j+4+a+b)}{\Gamma(j+3/2+a/2+b/2)\Gamma(4n+5+a+b)}$$

$$(A5b) \quad e_{j,n}^{(4),a,b} = \frac{16^{n-j}2\Gamma(n+1)\Gamma(n+a/2+1)}{\Gamma(j+1)\Gamma(j+1+a/2)} \\ \times \frac{\Gamma(n+b/2+1)\Gamma(n+a/2+b/2+1)\Gamma(4j+a+b+2)}{\Gamma(j+1+b/2)\Gamma(j+a/2+b/2+1)\Gamma(4n+a+b+3)}$$

$$(A5c) \quad \eta_n^{(1)} = \frac{\Gamma(a/2+b/2+3/2+n)\Gamma(b/2+3/2+n)\Gamma(a/2+3/2+n)}{2^{-4n-4-a-b}\pi\Gamma(a+b+4n+5)} \\ \times \frac{\Gamma(a/2+b/2+1)\Gamma(n+3/2)}{\Gamma(b/2+1/2)\Gamma(a/2+1/2)};$$

LOE and LSE,

$$(A6a) \quad e_{j,n}^{(1),b} = \frac{4^{n-j} 2\Gamma(n+3/2)\Gamma(n+3/2+b/2)}{\Gamma(j+3/2)\Gamma(j+3/2+b/2)}$$

$$(A6b) \quad e_{j,n}^{(4),b} = \frac{4^{n-j} 2\Gamma(n+1)\Gamma(n+b/2+1)}{\Gamma(j+1)\Gamma(j+1+b/2)}$$

$$(A6c) \quad \eta_n^{(1)} = 4^{n+1} \frac{\Gamma(n+3/2)\Gamma(n+b/2+3/2)}{\sqrt{\pi}\Gamma(b/2+1/2)};$$

GOE and GSE,

$$(A7) \quad e_{j,n}^{(1)} = \frac{\Gamma(n+3/2)}{\Gamma(j+3/2)}, \quad e_{j,n}^{(4)} = n!/j! \quad \text{and} \quad \eta_n^{(1)} = \frac{\Gamma(n+3/2)}{\sqrt{\pi}}.$$

## APPENDIX B. THE GENERATING FUNCTION $f_j(s)$

In the proof of lemma 6.6 we needed to study the generating function

$$(B1) \quad f_j(s) = \int_{-\infty}^{\infty} e^{-u^2/2} \mathcal{H}_j(u) \int_{-\infty}^{\infty} e^{-(v+s)^2/2} \text{sgn}(u-v) dv du.$$

The signed integral in the equation (B1) is closely related to the error function, for which the  $k^{\text{th}}$  derivative can be expressed in terms of the monic Hermite polynomial  $\mathcal{H}_{k-1}(u)$ . Differentiating under the integral, we find

$$(B2) \quad \frac{1}{(2k)!} \frac{d^{2k}}{ds^{2k}} f_{2j}(s) \Big|_{s=0} = \frac{-2^k}{(2k)!} \sqrt{2} \int_{-\infty}^{\infty} e^{-u^2} \mathcal{H}_{2j}(u) \mathcal{H}_{2k-1}(u/\sqrt{2}) du = 0,$$

which follows from the oddness of the integrand.

For the odd derivatives we get

$$(B3a) \quad \frac{1}{(2k+1)!} \frac{d^{2k+1}}{ds^{2k+1}} f_{2j}(s) \Big|_{s=0} = \frac{2^{k+1}}{(2k+1)!} \int_{-\infty}^{\infty} e^{-u^2} \mathcal{H}_{2j}(u) \mathcal{H}_{2k}(u/\sqrt{2}) du$$

$$= \frac{2^{k+1}}{(2k+1)!} \int_0^{\infty} e^{-u} u^{-1/2} \mathcal{L}_j^{-1/2}(u) \mathcal{L}_k^{-1/2}(u/2) du$$

$$(B3b) \quad = \sum_{i=0}^k \binom{k-1/2}{k-i} \frac{k!}{i!} \frac{2}{(2k+1)!} \int_0^{\infty} e^{-u} u^{-1/2} \mathcal{L}_j^{-1/2}(u) \mathcal{L}_i^{-1/2}(u) du$$

$$(B3c) \quad = \frac{2(-1)^{k-j} \sqrt{\pi} 4^{-k}}{(2k+1)(k-j)!}.$$

The expression in line (B3a) was obtained by means of the relations (66), while (B3b) follows from applying the connection formula

$$(B4) \quad \mathcal{L}_k^b(x/2) = 2^{-k} \sum_{i=0}^k \binom{b+k}{k-i} \frac{k!}{i!} (-1)^{i-k} \mathcal{L}_i^b(x).$$

The last line (B3c) then follows from the orthogonality (38). Thus, we have

$$(B5) \quad f_{2j}(s) = \sum_{p=j}^{\infty} s^{2p+1} \frac{2^{1-2p} \sqrt{\pi} (-1)^{j-p}}{(2p+1)\Gamma(p-j+1)}.$$

In a similar fashion we find

$$(B6) \quad f_{2j+1}(s) = 2j!\sqrt{\pi} + \sum_{p=j+1}^{\infty} s^{2p} \frac{2^{2-2p}\sqrt{\pi}(-1)^{j-p}}{(2p)\Gamma(p-j)},$$

which required the double integral

$$(B7) \quad \int_{-\infty}^{\infty} e^{-u^2/2} \mathcal{H}_{2j+1}(u) \int_{-\infty}^{\infty} e^{-v^2/2} \text{sgn}(u-v) dv du = 2j!\sqrt{\pi}.$$

#### REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover Publications Inc., New York, NY, 1972.
2. M. Adler, P. J. Forrester, T. Nagao, and P. van Moerbeke, *Classical skew orthogonal polynomials and random matrices*, J. Stat. Phys. **99** (2000), no. 1–2, 141–170.
3. M. Adler and P. van Moerbeke, *Toda versus pfaff lattice and related polynomials*, Duke Math. J. **112** (2002), no. 1, 1–58.
4. A. Altland and M. R. Zirnbauer, *Random matrix theory of a chaotic Andreev quantum dot*, Phys. Rev. Lett. **76** (1996), no. 18, 3420–3423.  
 ———, *Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures*, Phys. Rev. B **55** (1997), no. 2, 1142–1161.
5. Z. D. Bai, Y. Q. Yin, and P. R. Krishnaiah, *On the limiting empirical distribution function of the eigenvalues of a multivariate F matrix*, Teor. Veroyatnost. i Primenen. **32** (1987), no. 3, 537–548, English translation in Theory Probab. Appl. **32** (1987), no. 3, 490–500.
6. H. U. Baranger and P. A. Mello, *Mesoscopic transport through chaotic cavities: A random S-matrix theory approach*, Phys. Rev. Lett. **73** (1994), no. 1, 142–145.
7. C. W. J. Beenakker, *Universality in the random-matrix theory of quantum transport*, Phys. Rev. Lett. **70** (1993), no. 8, 1155–1158.
8. ———, *Random-matrix theory of quantum transport*, Rev. Mod. Phys. **69** (1997), no. 3, 731–808.
9. G. Berkolaiko, J. Harrison, and M. Novaes, *Full counting statistics of chaotic cavities from classical action correlations*, J. Phys. A: Math. Theor. **41** (2008), no. 36, 365102 (12pp).
10. G. Berkolaiko and J. Kuipers, *Moments of the Wigner delay times*, J. Phys. A: Math. Theor. **43** (2010), no. 3, 035101 (18pp).
11. ———, *Transport moments beyond the leading order*, New J. Phys. **13** (2011), no. 6, 063020 (40pp).
12. Ya. M. Blanter and M. Büttiker, *Shot noise in mesoscopic conductors*, Phys. Rep. **336** (2000), no. 1–2, 1–166.
13. Ya. M. Blanter, H. Schomerus, and C. W. J. Beenakker, *Effect of dephasing on charge-counting statistics in chaotic cavities*, Physica E **11** (2001), no. 1, 1–7.
14. Yu. Bomze, G. Gershon, D. Shovkun, L. S. Levitov, and M. Reznikov, *Measurement of counting statistics of electron transport in a tunnel junction*, Phys. Rev. Lett. **95** (2005), no. 17, 176601 (4pp).
15. P. W. Brouwer and C. W. J. Beenakker, *Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems*, J. Math. Phys. **37** (1996), no. 10, 4904–4934.
16. P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, *Quantum mechanical time-delay matrix in chaotic scattering*, Phys. Rev. Lett. **78** (1997), no. 25, 4737–4740.

17. C. Carré, M. Deneufchatel, J.-G. Luque, and P. Vivo, *Asymptotics of Selberg-like integrals: The unitary case and Newton's interpolation formula*, J. Math. Phys. **51** (2010), no. 12, 123516 (19pp).
18. J. P. Dahlhaus, B. Béri, and C. W. J. Beenakker, *Random-matrix theory of thermal conduction in superconducting quantum dots*, Phys. Rev. B **82** (2010), no. 1, 014536 (7pp).
19. P. Desrosiers, *Duality in random matrix ensembles for all  $\beta$* , Nucl. Phys. B **817** (2009), no. 3, 224–251.
20. E. Dueñez, *Random matrix ensembles associated to compact symmetric spaces*, Ph.D. thesis, Princeton University, 2001.  
 ———, *Random matrix ensembles associated to compact symmetric spaces*, Commun. Math. Phys. **244** (2004), no. 1, 29–61.
21. F. Dyson, *The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Math. Phys. **3** (1962), no. 6, 1199–1215.
22. ———, *Correlations between the eigenvalues of a random matrix*, Commun. Math. Phys. **19** (1970), no. 3, 235–250.
23. P. J. Forrester, *Quantum conductance problems and the Jacobi ensemble*, J. Phys. A: Math. Gen. **39** (2006), no. 22, 6861–6870.
24. ———, *A random matrix decimation procedure relating  $\beta = 2/(r+1)$  and  $\beta = 2(r+1)$* , Commun. Math. Phys. **285** (2009), no. 2, 653–672.
25. ———, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.
26. S. Ghosh and A. Pandey, *Skew-orthogonal polynomials and random-matrix ensembles*, Phys. Rev. E **65** (2002), 046221 (21pp).
27. I. P. Goulden and D. M. Jackson, *Maps in locally orientable surfaces and integrals over real symmetric matrices*, Canad. J. Math **49** (1997), no. 5, 865–882.
28. U. Haagerup and S. Thorbjørnsen, *Random matrices with complex gaussian entries*, Expo. Math. **21** (2003), no. 4, 293–337.
29. P. J. Hanlon, R. P. Stanley, and J. R. Stembridge, *Some combinatorial aspects of the spectra of normally distributed random matrices*, Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications, Contemporary Mathematics, vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 151–174.
30. J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485.
31. R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, *Universal quantum signatures of chaos in ballistic transport*, Europhys. Lett. **27** (1994), no. 4, 255–260.
32. B. A. Khoruzhenko, D. V. Savin, and H.-J. Sommers, *Systematic approach to statistics of conductance and shot-noise in chaotic cavities*, Phys. Rev. B **80** (2009), no. 12, 125301 (14pp).
33. C. Krattenthaler, *Asymptotic analysis of a Selberg-type integral via hypergeometrics*, eprint: arXiv:1004.3941, 2010.
34. M. Ledoux, *Differential operators and spectral distributions of invariant ensembles from the classical orthogonal polynomials. The continuous case*, Electron. J. Probab. **9** (2004), no. 7, 177–208.
35. ———, *A recursion formula for the moments of the Gaussian Orthogonal Ensemble*, Ann. Inst. H. Poincaré Probab. Stat. **45** (2009), no. 3, 754–769.
36. H. Lee, A. Yu. Yakovets, and L. S. Levitov, *Universal statistics of transport in disordered conductors*, Phys. Rev. B **51** (1995), no. 7, 4079–4083.

37. G. Livan and P. Vivo, *Moments of Wishart-Laguerre and Jacobi ensembles of random matrices: application to the quantum transport problem in chaotic cavities*, Acta Phys. Pol. B **42** (2011), no. 5, 1081–1104.
38. V. A. Marčenko and L. A. Pastur, *Distribution of eigenvalues for some sets of random matrices*, Mat. Sb. (N. S.) **72(114)** (1967), no. 4, 507–536, English translation in Math. USSR-Sb. **1** (1967), no. 4, 457–483.
39. M. L. Mehta, *Random matrices*, Elsevier Inc., San Diego, CA, 2004.
40. M. L. Mehta and G. Mahoux, *A method of integration over matrix variables: IV*, J. Phys. 1 (France) **1** (1991), 1093–1108.
41. F. Mezzadri and N. J. Simm, *Moments of the transmission eigenvalues, proper delay times and random matrix theory II*, eprint: [arXiv:1108.2859](https://arxiv.org/abs/1108.2859), 2011.
42. T. Nagao and M. Wadati, *Correlation functions of random matrix ensembles related to classical orthogonal polynomials*, J. Phys. Soc. Jap. **60** (1991), no. 10, 3298–3322.
43. M. Novaes, *Full counting statistics of chaotic cavities with many open channels*, Phys. Rev. B **75** (2007), no. 7, 073304 (4pp).
44. ———, *Statistics of quantum transport in chaotic cavities with broken time-reversal symmetry*, Phys. Rev. B **78** (2008), no. 3, 035337 (5pp).
45. ———, *Asymptotics of Selberg-like integrals by lattice path counting*, Ann. Phys. **326** (2010), no. 4, 828–838.
46. S. Oberholzer, E. V. Sukhorukov, C. Strunk, C. Schönenberger, T. Heinzl, and M. Holland, *Shot noise by quantum scattering in chaotic cavities*, Phys. Rev. Lett. **86** (2001), no. 10, 2114–2117.
47. V. A. Osipov and E. Kanzieper, *Integrable theory of quantum transport in chaotic cavities*, Phys. Rev. Lett. **101** (2008), no. 17, 176804 (4pp).  
 ———, *Statistics of thermal shot noise crossover in chaotic cavities*, J. Phys. A: Math. Theor. **42** (2009), no. 47, 475101 (14pp).
48. D. V. Savin, Y. V. Fyodorov, and H.-J. Sommers, *Reducing nonideal to ideal coupling in random matrix description of chaotic scattering: Application to the time-delay problem*, Phys. Rev. E **63** (2001), no. 3, 035202 (4pp).
49. D. V. Savin and H.-J. Sommers, *Delay times and reflection in chaotic cavities with absorption*, Phys. Rev. E **68** (2003), no. 3, 036211 (5pp).
50. D. V. Savin, H.-J. Sommers, and W. Wiczeorek, *Nonlinear statistics of quantum transport in chaotic cavities*, Phys. Rev. B **77** (2008), no. 12, 125332 (5pp).
51. H.-J. Sommers, D. V. Savin, and V. V. Sokolov, *Distribution of proper delay times in quantum chaotic scattering: A crossover from ideal to weak coupling*, Phys. Rev. Lett. **87** (2001), no. 9, 094101 (4pp).
52. G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc., New York, NY, 1939.
53. C. H. Tracy and H. Widom, *Correlation functions, cluster functions and spacing distributions for random matrices*, J. Stat. Phys. **92** (1998), no. 5, 809–835.
54. P. Vivo and E. Vivo, *Transmission eigenvalue densities and moments in chaotic cavities from random matrix theory*, J. Phys. A: Math. Theor. **41** (2008), no. 12, 122004 (10pp), *Corrigendum*, J. Math. Phys. A: Math. Theor. **41** (2008), no. 23, 239801 (1p).
55. H. Widom, *On the relation between orthogonal, symplectic and unitary matrix ensemble*, J. Stat. Phys. **94** (1998), no. 3-4, 347–363.
56. M. R. Zirnbauer, *Riemannian symmetric superspaces and their origin in random-matrix theory*, J. Math. Phys. **37** (1996), no. 10, 4986–5018.



SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL BS8 1TW, UK  
*E-mail address:* `f.mezzadri@bristol.ac.uk`

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL BS8 1TW, UK  
*E-mail address:* `n.simm@bristol.ac.uk`