A NOTE ON KASPAROV PRODUCTS

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Combining Kasparov's theorem of Voiculesu and Cuntz's description of KK-theory in terms of quasihomomorphisms, we give a simple construction of the Kasparov product. This will be used in a more general context of locally convex algebras in order to treat products of certain universal cycles.

1 Introduction

The goal of this note is to establish existence of the Kasparov product based on Kasparov's theorem of Voiculescu ([Kas80a]), and to examine how this construction is related to the one used by Kasparov.

In the first section, we interpret the connection condition and the existence of Kasparov product ([Kas80b]) as the existence of a certain extension of a quasihomomorphism ([Cun87]). Such extensions always exist, as can be seen by applying split exactness of KK to a certain algebra D_α that is a semidirect product of the domain and target of a quasihomomorphism. The resulting description of the Kasparov product already yields a useful way to construct the Kasparov product; it is particularly well adapted to generalisations of the bimodule-formalism to locally convex algebras, where it may be used to calculate products of certain "smooth" submodules, and is used in [Gre] in a crucial manner.

In the second section, it is shown that, without making use of split exactness of KK, one can, in case that Kasparov's theorem of Voiculscu is available, construct the product by using this interpretation. First we show how to reduce quasihomomorphisms to a single morphism and a unitary; and if an absorbing morphism is chosen, all classes of quasihomomorphisms are obtained from it by conjugation by a unitary. Applying this to a pair of composable quasihomomorphisms, we see that it suffices to extend quasihomomorphisms to just one "universal" algebra; if further the domain or target of the first quasihomomorphism is nuclear, there is a canonical way to extend quasihomomorphisms.

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2 The Kasparov product revisited

Quasihomomorphisms were introduced by Cuntz in [Cun83] and further developed in [Cun87].

Definition 1. *Let B be stable, $\hat{B}$ a C^*-algebra containing B as an ideal; then a quasihomomorphism from A to B is a pair of homomorphisms from A to $\hat{B}$ such that $\alpha(a) - \bar{\alpha}(a) \in B$ for all $a \in A$.*

For nonstable B, a quasihomomorphism from A to B is by definition a quasihomomorphism from A into the stabilisation $\mathbb{K} \otimes B$ of B.

Let (E, φ, F) be a Kasparov (A, B)-module with A and B trivially graded. If F is selfadjoint and invertible, then with respect to the grading:

$$\varphi = \begin{pmatrix} \varphi^{(0)} & \\ & \varphi^{(1)} \end{pmatrix} \text{ and } F = \begin{pmatrix} & T^{-1} \\ T & \end{pmatrix}$$

where the $\varphi^{(i)}$ are homomorphisms $A \to \mathbb{B}_B(E^{(i)})$ and T is by hypothesis a unitary in $\mathbb{B}_B(E^{(0)}, E^{(1)})$. Thence we obtain a quasihomomorphism $(\alpha, \bar{\alpha}) := (\varphi^{(0)}, T^{-1}\varphi^{(1)}T)$ from A to $\mathbb{K}_B(E^{(0)})$ simply by identifying $E^{(0)}$ and $E^{(1)}$ via T, and where we view the latter as a subalgebra of $\mathbb{K} \otimes B$ via the stabilization-theorem.

We may always reduce to this case by using the standard simplifications in KK-theory, and therefore we can define an associated quasihomomorphism $Qh(x)$ to every Kasparov module.

The original construction of the Kasparov product from [Kas80b] was quite technical. We will use the version based on the notion of connection introduced by Connes and Skandalis. We fix the following setting: Let E_1 be a graded Hilbert B-module, E_2 a graded Hilbert C-module, $\varphi : B \to \mathbb{B}_C(E_2)$ a $*$-homomorphism and F an odd selfadjoint operator on E_2. We set $E_{12} := E_1 \otimes_B E_2$, and define for every $x \in E_1$ an operator $T_x : E_2 \to E_1 \otimes_B E_2,\ y \mapsto x \otimes y$. Note that the adjoint of T_x is given by the mapping $E_{12} \to E_2,\ y \otimes z \mapsto \varphi(\langle x|y\rangle)z$, and $T_{x'}T_x^* = \theta_{x',x} \otimes \mathrm{id}_{E_2}$.

Definition 2. *An E_1-connection for an odd operator F is an odd salfadjoint operator G such that for all homogeneous $x \in E_1$*

$$T_x F - (-1)^{\partial x} G T_x \in \mathbb{K}_C(E_2, E_{12}) \text{ and } F T_x^* - (-1)^{\partial x} T_x^* G \in \mathbb{K}_C(E_{12}, E_2).$$

As a consequence of the stabilisation theorem, such connections exist in case we deal with Kasparov modules; more precisely:

Proposition 3. *If E_1 is countably generated and $[F, b]$ is compact for all $b \in B$, then there exists an odd E_1 connection for F.*

If (E_1, φ_1, F_1) is a Kasparov (A, B)-module, (E_2, φ_2, F_2) a Kasparov (B, C)-module, G an F_2 connection for E_1, then $(E_{12}, \mathrm{id}_{\mathbb{K}_B(E_1)} \otimes 1, G)$ is a Kasparov $(\mathbb{K}_B(E_1), C)$-module.

The existence statement stems from [CS84]; the second fact was stated in [Ska84], Proposition 9.

The composition product is given in terms of the representatives of the cycles involved: If (φ_1, E_1, F_1) is a Kasparov (A, B)-module and (φ_2, E_2, F_2) a Kasparov (B, C)-module, then a Kasparov (A, C)-module $(E_1 \otimes_B E_2, \varphi_1 \otimes 1, F_{12})$ is called a product of (E_1, φ_1, F_1) and (E_2, φ_2, F_2) if

(i) F_{12} is an E_1 connection for F_2 (connection condition)

(ii) For all $a \in A$, $\varphi_1(a) \otimes 1[F_1 \otimes 1, F_{12}]\varphi_1(a)^* \otimes 1$ is positive in the quotient $\mathbb{B}_C(E_{12})/\mathbb{K}_C(E_{12})$ (positivity condition).

The set of operators F_{12} satisfying the above conditions will be denoted $F_1 \sharp F_2$.

Using Kasparov's technical theorem, one can show that a product as above always exists if A is separable, is unique up to operator homotopy, and passes to homotopy classes (cf. [Ska84]).

Recall also that a Hilbert B-module E is called full if the linear span of $\langle E|E \rangle$ is dense in B.

Definition 4. *Let A and B be graded C^*-algebras. A graded Morita(-Rieffel) equivalence between A and B is given by a graded full Hilbert B-module E, called the equivalence bimodule, and a graded isomorphism $\varphi : A \to \mathbb{K}_B(E)$.*

We identify A with $\mathbb{K}(E)$ and drop the isomorphism φ. If E is a graded Morita equivalence bimodule from $A = \mathbb{K}_B(E)$ to B, then we define the $(B, \mathbb{K}(E))$-module $E^* := \mathbb{K}(E, B)$. The $\mathbb{K}(E)$-valued scalar product is simply $\langle T|S \rangle := R^* S$, and this makes E^* into a graded Hilbert $\mathbb{K}(E)$-module.

Let A and B be separable. Then the class $[(E, \mathrm{id}_A, 0)]$ of the equivalence bimodule yields a KK equivalence from A to B with inverse $[(E^*, \mathrm{id}_B, 0)]$.

Conversely, any given full Hilbert B-module E may be viewed as a graded Morita equivalence from $\mathbb{K}_B(E)$ to B.

If $y = (E_2, \varphi_2, F_2) \in \mathbb{E}(B, C)$, E_1 is a Hilbert B-module, and w denotes the Kasparov module defined by the Morita equivalence determined by E_1, then the operator G in a product $w \cap x$ is exactly an E_1-connection for F_2, as the positivity condition is trivially satisfied.

If $x = (E_1, \varphi_1, F_1)$ and $v = (E_1^*, \mathrm{id}_B, 0)$ is the inverse of w, then the product $x \cap v$ is represented by $(\mathbb{K}_B(E_1), \varphi_1, F_1)$, where the bounded operators on E are considered to act on the Hilbert $\mathbb{K}_B(E_1)$-module by multiplication. This is easily seen by using the explicit form of the isomorphism $U : E_1 \hat{\otimes}_B E_1^* \to \mathbb{K}_B(E_1)$ given above, as $UTU^{-1}(|\xi\rangle\langle\eta|) = |T\xi\rangle\langle\eta|$ for all $T \in \mathbb{B}_B(E)$. Hence compact operators on E act again by compact operators on $\mathbb{K}_B(E)$, and therefore $(\mathbb{K}_B(E_1), \varphi, F_1)$ does indeed define a cycle, the connection condition is obvious, and positivity follows from $a[F_1, F_1]a^* = a(2F_1^2)a^* = aa^*$ modulo compacts.

We fix two Kasparov bimodules $(E_1, \varphi_1, F_1) \in \mathbb{E}(A, B)$ and $(E_2, \varphi_2, F_2) \in \mathbb{E}(B, C)$, and denote their classes in KK by x and y. The module E_1 is seen as a Morita equivalence from $\mathbb{K}_B(E_1)$ to B, whose class in KK we denote by w, and its inverse by v. Let $(\alpha, \bar{\alpha}) : A \rightrightarrows D \trianglerighteq \mathbb{K}_B(E_1)$ be the quasihomomorphism associated to $x' := x \cap v$, and recall that $y' := w \cap y$ may be viewed as the class of the Kasparov module defined via an E_1 connection for F_2. If we define D_α as the sub-C^*-algebra of $A \oplus D$ generated by $(a, \alpha(a))$ and $0 \oplus B$, $a \in A$, we obtain the double split short exact sequence

$$0 \longrightarrow \mathbb{K}_B(E_1) \xrightarrow{\iota} D_\alpha \underset{\mathrm{id}_A \oplus \alpha}{\overset{\mathrm{id}_A \oplus \bar{\alpha}}{\rightleftarrows}} A \longrightarrow 0$$

which in turn, by split exactness of KK, yields a long exact sequence

$$0 \longrightarrow KK(A, C) \longrightarrow KK(D_\alpha, C) \xrightarrow{\iota^*} KK(\mathbb{K}_B(E_1), C) \longrightarrow 0\,.$$

We may thus assume that $y' = \iota^* z$ for some $z \in KK(D_\alpha, C)$. We claim that $\alpha^*(z) - \bar{\alpha}^*(z) = y \cap x$. This follows as $KK((\alpha, \bar{\alpha}), C)$ is multiplication by x' on the left, and therefore

$$x \cap y = x' \cap y' = KK((\alpha, \bar{\alpha}), C)(y') = (\alpha^* - \bar{\alpha}^*)(\iota^*)^{-1}\iota^*(z) = (\alpha^* - \bar{\alpha}^*)(z).$$

Calculating a representative for the last expression, we have thus proved:

Theorem 5. *Let $x \in KK(A, B)$, $y = [(E_2, \varphi_2, F_2)] \in KK(B, C)$. Then the Kasparov product of x and y may be defined by* *(i)* *representing x as a quasihomomorphism $(\alpha, \bar{\alpha}) : A \rightrightarrows \mathbb{B}_B(E_1) \trianglerighteq \mathbb{K}_B(E_1)$.*

(ii) *choosing an E_1 connection G for F_2*

(iii) *lifting the Kasparov $(\mathbb{K}_B(E_1), C)$-module $(E_1 \otimes_B E_2, \mathrm{id}_{\mathbb{K}_B(E_1)} \otimes 1, G)$ along the canonical inclusion of $\mathbb{K}_B(E_1) \to D_\alpha$ to a Kasparov (D_α, C)-module $(\tilde{\varphi}, \tilde{E}, \tilde{G})$,*

(iv) *and setting*

$$x \cap y := \left[\left(\begin{pmatrix} \tilde{\varphi} \circ \alpha & \\ & \tilde{\varphi} \circ \bar{\alpha} \circ \varepsilon \end{pmatrix}, \tilde{E} \oplus \tilde{E}^{op}, \begin{pmatrix} \tilde{G} & \\ & -\tilde{G} \end{pmatrix} \right) \right] \in KK(A, C)$$

where E^{op} denotes the Hilbert B-module E with inversed grading, and ε the grading operator on E.

Here (iii) means exactly that the quasihomomorphism

$$Qh(E_1 \otimes_B E_2, \mathrm{id}_{\mathbb{K}_B(E_1)} \otimes 1, G)$$

extends to a quasihomomorphism on the larger algebra D_α; note that the class of the cycle $x \cap y$ as defined above is independent of the choice of the extension.

3 Reduction of quasihomomorphisms and a construction of the Kasparov product

For a given linear map $\varphi : A \to \mathbb{B}_B(E)$, where E is a Hilbert B-module, we define $E^\infty := \bigoplus_{n=1}^\infty E$, and $\varphi^\infty : A \to \mathbb{B}_B(E^\infty)$ as the diagonal action of φ.

Proposition 6. *The class of every quasihomomorphism is represented by a quasihomomorphism of the form $(\alpha, \mathrm{Ad}_U \circ \alpha)$, where U is a unitary.*

Proof. Let $(\alpha, \bar{\alpha}) : A \rightrightarrows \hat{B} \trianglerighteq B$ be a quasihomomorphism. We may assume that $\hat{B} = \mathbb{B}_B(E)$ and $B = \mathbb{K}_B(E)$ for some Hilbert B-module E. We may replace $(\alpha, \bar{\alpha})$ by

$$(\alpha \oplus \alpha^\infty \oplus \bar{\alpha}^\infty, \alpha \oplus \alpha^\infty \oplus \bar{\alpha}^\infty) : A \to \mathbb{B}_B(E \oplus E^\infty \oplus E^\infty) \trianglerighteq \mathbb{K}_B(E \oplus E^\infty \oplus E^\infty)$$

because $(\alpha^\infty \oplus \bar{\alpha}^\infty, \alpha^\infty \oplus \bar{\alpha}^\infty)$ is degenerate.

Now let U be the unitary on $E \oplus E^\infty \oplus E^\infty$ that maps

$$(\xi_0, (\xi_1, \xi_2, \ldots), (\eta_1, \eta_2, \ldots)) \to (\xi_1, (\xi_2, \xi_3, \ldots), (\xi_0, \eta_1, \eta_2, \ldots)).$$

Then

$$(\alpha(a) \oplus \alpha^\infty(a) \oplus \bar{\alpha}^\infty(a))U = U(\bar{\alpha}(a) \oplus \alpha^\infty(a) \oplus \bar{\alpha}^\infty(a)).$$

□

Definition 7. *Let A and B be C^*-algebras, $\beta : A \to \mathbb{B}(\mathcal{H}_B)$ a $*$-homomorphism such that for every $*$-homomorphism $\alpha : A \to \mathbb{B}(\mathcal{H}_B)$ there exists a unitary U with $\alpha \oplus \beta = U^*\beta U$ modulo compact operators. Then β will be called absorbing.*

The following theorem was proved in [Kas80a]:

Theorem 8 (Kasparov-Voiculescu)**.** *Let A and B be separable C^*-algebras and $\beta_0 : A \to \mathbb{B}(\mathcal{H})$ a faithful representation of A such that $(\tilde{\beta}_0)^{-1}(\mathbb{K}(\mathcal{H})) = \{0\}$. We denote by β the inclusion of A into $\mathbb{B}_B(\mathcal{H}_B)$ obtained from β_0 by viewing $\mathbb{B}(\mathcal{H})$ as a subalgebra of $\mathbb{B}_B(\mathcal{H}_B)$. If either A or B is nuclear, then β is absorbing.*

In general, there is a result of Thomsen from [Tho01], Theorem 2.7, which shows that for A and B separable, there is an absorbing homomorphism from A into the stable multiplier algebra $\mathcal{M}(B \otimes \mathbb{K})$ of B.

Lemma 9. *Let $(\alpha, \alpha^U) : A \rightrightarrows \hat{B} \trianglerighteq B$ be a quasihomomorphism, and $\beta : A \to \hat{B}$ a homomorphism such that $\alpha(a) - \beta(a) \in B$ for all a. Then (β, β^U) is a quasihomomorphism equivalent to (α, α^U).*

Proof. Using the usual rotation matrices, we obtain a path of unitaries

$$U_t := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} U & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Reparametrizing, we get a homotopy

$$(\alpha \oplus \beta, \mathrm{Ad}_{U_t} \circ \alpha \oplus \beta)$$

of the quasihomomorphisms $(\alpha, \mathrm{Ad}_U \circ\alpha) \oplus (\beta, \beta)$ and $(\alpha, \alpha) \oplus (\beta, \mathrm{Ad}_U \circ\beta)$ □

Proposition 10. *Let $\beta : A \to \mathbb{B}(\mathcal{H}_B)$ be absorbing. Then every element of $KK(A, B)$ is represented by a quasihomomorphism of the form*

$$(\beta, \mathrm{Ad}_U \circ\beta) : A \to \mathbb{B}_B(\mathcal{H}_B) \trianglerighteq \mathbb{K} \otimes B,$$

where $U \in \mathbb{B}_B(\mathcal{H}_B)$ is a unitary.

Proof. By Proposition 6, we may assume that we are given a quasihomomorphism $(\alpha, \alpha^U) : A \rightrightarrows \mathbb{B}_B(\mathcal{H}_B) \trianglerighteq \mathbb{K}_B(\mathcal{H}_B)$, where U is a unitary in $\mathbb{B}_B(\mathcal{H}_B)$. Let V be a unitary such that $\alpha \oplus \beta = V^*\beta V$. Then we get

$$(\alpha, \alpha^U) \sim (\alpha \oplus \beta, \alpha^U \oplus \beta) \sim (\beta^V, \beta^{V(U\oplus 1)}) \sim (\beta, \beta^{V(U\oplus 1)V^*})$$

by the above Lemma. □

Corollary 11. *Let A, B, C be separable C^*-algebras, β as in the above Proposition absorbing, $(\gamma, \bar\gamma)$ a quasihomomorphism from B to C. Then it suffices to find an extension of $(\gamma, \bar\gamma)$ to the one algebra D_β, in order to calculate explicitly all products of $(\gamma, \bar\gamma)$ with elements from $KK(A, B)$ (as in 5).*

One can use these ideas to construct the Kasparov product in good cases:

Let $(\alpha, \bar\alpha) : A \rightrightarrows \mathbb{B}(B \otimes \mathcal{H}) \trianglerighteq B \otimes \mathbb{K}(\mathcal{H})$ be a quasihomomorphism, where $\bar\alpha = 1 \otimes \pi$ is induced by a representation π of A on $\mathcal{H}$ with $\pi(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$. Let $(\beta, \bar\beta) : B \rightrightarrows \hat{C} \trianglerighteq C$ be another quasihomomorphism. We may extend $(\beta, \bar\beta)$ to a quasihomomorphism

$$(\beta', \bar\beta') : 1 \otimes \mathbb{B}(\mathcal{H}) + B \otimes \mathbb{K}(\mathcal{H}) \to \mathcal{M}(\hat{C} \otimes \mathbb{K}(\mathcal{H})) \trianglerighteq C \otimes \mathbb{K}(\mathcal{H})$$

by first stabilizing and then setting $\beta'(1 \otimes T + x) := 1 \otimes T + \beta \otimes \mathrm{id}_{\mathbb{K}}(x)$. Because $D_{\bar\alpha} \subseteq 1 \otimes \mathbb{B}(\mathcal{H}) + B \otimes \mathbb{K}(\mathcal{H})$, we have constructed a product. Note further that because $(\beta', \bar\beta')$ represents zero on the image of $\bar\alpha$, the product has a very simple form:

$$[\alpha, \bar\alpha]\,[\beta', \bar\beta'] = [\beta' \circ \alpha, \bar\beta' \circ \alpha].$$

In particular, if we have any two quasihomomorphisms $(\alpha, \bar\alpha)$ from A to B and $(\beta, \bar\beta)$ from B to C and either A or B is nuclear, then by Proposition 10 we may assume that $\bar\alpha$ is obtained from a faithful representation A whose image is disjoint from the compacts, and then apply the construction as above. More generally, one may construct on this way the Kasparov product for the functor KK_{nuc} from [Ska88].

This construction of the product coincides with the one by Kasparov by the preceeding section.

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