

WEB WORLDS, WEB-COLOURING MATRICES, AND WEB-MIXING MATRICES

MARK DUKES, EINAN GARDI, EINAR STEINGRÍMSSON, AND CHRIS D. WHITE

ABSTRACT. We introduce a new combinatorial object called a *web world* that consists of a set of *web diagrams*. The diagrams of a web world are generalizations of graphs, and each is built on the same underlying graph. Instead of ordinary vertices the diagrams have *pegs*, and edges incident to a peg have different heights on the peg. The web world of a web diagram is the set of all web diagrams that result from permuting the order in which endpoints of edges appear on a peg. The motivation comes from particle physics, where web diagrams arise as particular types of Feynman diagrams describing scattering amplitudes in non-Abelian gauge (Yang-Mills) theories. To each web world we associate two matrices called the *web-colouring matrix* and *web-mixing matrix*. The entries of these matrices are indexed by ordered pairs of web diagrams (D_1, D_2) , and are computed from those colourings of the edges of D_1 that yield D_2 under a transformation determined by each colouring.

We show that colourings of a web diagram (whose constituent indecomposable diagrams are all unique) that lead to a reconstruction of the diagram are equivalent to order-preserving mappings of certain partially ordered sets (posets) that may be constructed from the web diagrams. For web worlds whose web graphs have all edge labels equal to 1, the diagonal entries of web-mixing and web-colouring matrices are obtained by summing certain polynomials determined by the descents in permutations in the Jordan-Hölder set of all linear extensions of the associated poset. We derive tri-variate generating functions for the number of web worlds according to three statistics and enumerate the number of different web diagrams in a web world. Three special web worlds are examined in great detail, and the traces of the web-mixing matrices calculated in each case.

1. INTRODUCTION

In this paper we study combinatorial objects called *web worlds*. A web world consists of a set of diagrams that we call *web diagrams*. The motivation for introducing these comes from particle physics, where web diagrams arise as particular types of Feynman diagrams describing scattering amplitudes of quantum fields in non-Abelian gauge (Yang-Mills) theories. These structures, which do not seem to have been studied previously from the purely combinatorial point of view, may be thought of as generalisations of simple graphs where each edge has a height associated to each of its endpoints. To avoid confusion with proper graphs, the vertices of our web diagrams will be referred to as *pegs*. The different edges that connect onto a given peg are strictly ordered by their heights as illustrated in the example in Figure 1. The *web world* of a web diagram is the set of all web diagrams that result from permuting the order in which endpoints of edges appear on a peg, that is, the heights of the edges incident with that peg. To each web world we associate two matrices, $\mathfrak{M}(x)$ and \mathfrak{R} , which are called the *web-colouring matrix* and *web-mixing matrix*, respectively. The entries of these matrices are indexed by ordered pairs of web diagrams, and are computed from those colourings of the edges of one of the two web diagrams that yield the other one under a certain transformation (Definition 2.11) determined by each colouring.

Web diagrams, web worlds and their web-mixing matrices play an important role in the study of scattering amplitudes in quantum chromodynamics (QCD) [7, 10, 9, 8]. While not essential for the combinatorial aspects dealt with here, we briefly describe the physics context in which they arise.

Key words and phrases. web diagram, web world, edge colouring, reconstruction, poset.

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- In general, in perturbative quantum field theory, Feynman diagrams provide means of computing scattering amplitudes: each diagram corresponds to an algebraic expression depending on the momenta and other quantum numbers of the external particles. Specifically in QCD these diagrams describe the collisions of energetic quarks and gluons (generically referred to as partons, the constituents of protons and neutrons); these carry a matrix-valued Yang-Mills charge belonging respectively to the fundamental or adjoint representations of the $SU(N)$ group. These matrices do not commute, hence the name non-Abelian gauge theory.
- The perturbation expansion of a scattering amplitude, formally a power series in the coupling constant (the strength of the interaction) amounts to a loop expansion: the leading term is the sum of all diagrams having a tree topology connecting all external particles, while an n -th order term in the loop expansion corresponds to the sum of all diagrams involving n loops, which is obtained by dressing a tree diagram by n additional gluon exchanges. In a non-Abelian theory gluons may connect to each other via 3 and 4 gluon vertices.
- A salient feature of scattering amplitudes is that starting at one loop they involve long-distance (“soft”) singularities. In order to study the structure of these singularities one may consider an *Eikonal amplitude*, which is a simplified version of the QCD scattering amplitude that fully captures its singularities. This simplification is ultimately a consequence of the fact that all singularities arise due to “soft” (low energy) fields, and the quantum-mechanical incoherence of “soft” and “hard” (high energy) fields. Eikonal amplitudes are formed by representing each external parton by an Eikonal line (or “Wilson line”), a single semi-infinite line extending from the origin to infinity in the direction fixed by the momentum of the external parton, and which carries the same matrix-valued Yang-Mills charge.
- Eikonal amplitudes *exponentiate*: they can be written as an exponential where the exponent takes a particularly simple form. Web diagrams arise as a direct Feynman–diagrammatic description of this exponent [7, 10], namely, they define the coefficients in the loop expansion of the exponent of the Eikonal amplitude. In these diagrams the Eikonal lines are the *pegs* introduced above. Loops are then formed by connecting additional gluons between these Eikonal lines: these are the *edges* mentioned above. Physically, the latter represent “soft” fields whose energy is small compared to energies of the external partons. Since each gluon emission along a given Eikonal line is associated with a non-commuting $SU(N)$ matrix, the order of these emissions (corresponding to the *height* of the edge on the peg in the above terminology) is important, and distinguishes between different web diagrams belonging to the same web world.

Following [8] we consider in this paper a particular subset of the web diagrams, those generated by any number of single gluon exchanges between the Eikonal lines. We thus exclude from the present discussion any web diagrams that includes 3 or 4 gluon vertices, as well as ones where a gluon connects an Eikonal line to itself. The generalization to these cases is interesting and will be addressed by the authors in a future paper.

The present paper is a first combinatorial study of these web worlds and proves several general results. One of our results provides a rich connection between those colourings of a web diagram that lead to a unique reconstruction of the diagram on one hand, and order-preserving mappings of certain posets (partially ordered sets) on the other. This connection is then used to show that the diagonal entries of a web-mixing matrix are obtained by summing certain polynomials determined by the descents in permutations in the Jordan–Hölder set of all linear extensions of the associated poset.

As for results on the enumeration of web worlds we give two tri-variate generating functions, recording statistics that keep track of the number of pegs, the number of edges and the number of pairs of pegs that are joined by some edge. We give a generating function for the number of proper

web worlds in terms of three different statistics. We also obtain an expression for the number of different web diagrams in a given web world, in terms of entries of a matrix that represents the web world.

Three special cases of web worlds are examined in great detail, and the traces of the web-mixing matrices calculated in each case. The first of these cases concerns web worlds whose web diagrams are uniquely encoded by permutations, because each peg, except for the last one, contains the left endpoint of a unique edge and all right endpoints are on the last peg. In this case we give exact values for entries of the web-mixing matrices in terms of a permutation statistic on the pair of permutations that index each entry. The other two cases are very different from the first, but quite similar to each other. Namely, these are web worlds where each pair of adjacent pegs has a unique edge between them and there are no other edges. They differ in that in one case we look at the sequence of pegs cyclically and demand that one edge join the first and the last peg. Despite the small difference in definition, the cyclic setup changes the trace of the resulting web-mixing matrices greatly.

Generalising the last two cases just mentioned we define a class of web worlds that we call *transitive web worlds*. Remarkably, the set of transitive web worlds is in one-to-one correspondence with the $(2 + 2)$ -free posets. It seems likely that this connection will engender some interesting results, given the fast growing literature on these posets (see [1, 2, 4, 6, 5] for references).

The outline of this paper is as follows. Section 2 defines web diagrams, web worlds and operations on web diagrams such as colourings and how the partitioning of a web diagram induced by a colouring describes the construction of another web diagram. This section also defines the web-colouring, $\mathfrak{M}(x)$, and web-mixing, \mathfrak{R} , matrices. Section 3 looks at colourings of a web diagram whose corresponding reconstructions result in the same web diagram. Section 4 gives generating functions for web worlds in terms of three statistics; the number of pegs, the number of edges, and the number of pairs of pegs that have at least one edge between them. It also gives an expression for the number of different web diagrams arising from a web world. Section 5 is an interlude on some material that will be used for determining web mixing matrices for particular classes of web worlds in Sections 6 to 8

In Section 6 we study a certain web world on $n + 1$ pegs whose diagrams are uniquely encoded by permutations of the set $\{1, \dots, n\}$. Sections 7 and 8 look at two web worlds with vastly different properties from that of Section 6, but closely related to one another, where each pair of adjacent pegs is connected by a unique edge. Section 9 defines *transitive web worlds* which are shown to be in one-to-one correspondence with $(2 + 2)$ -free posets and thus with a host of other families of combinatorial structures. The paper ends with some challenging open problems in Section 10.

2. WEB-DIAGRAMS

Intuitively, a web diagram consists of a sequence of pegs and a set of edges, each connecting two pegs, as illustrated in the example in Figure 1. In the following formal definition of a web diagram D , a 4-tuple $(x_j, y_j, a_j, b_j) \in D$ will represent an edge whose left vertex has height a_j on peg x_j and whose right vertex has height b_j on peg y_j .

Definition 2.1. A *web diagram* on n pegs having L edges is a collection $D = \{e_j = (x_j, y_j, a_j, b_j) : 1 \leq j \leq L\}$ of 4-tuples that satisfy the following properties:

- (i) $1 \leq x_j < y_j \leq n$ for all $j \in \{1, \dots, L\}$.
- (ii) For $i \in \{1, \dots, n\}$ let $p_i(D)$ be the number of j such that x_j or y_j equals i , that is, the number of edges in D incident with peg i . Then

$$\{b_j : y_j = i\} \cup \{a_j : x_j = i\} = \{1, 2, \dots, p_i(D)\}.$$

We write $\text{Pegs}(D) = (p_1(D), \dots, p_n(D))$. Condition (ii) says that the labels of the $p_i(D)$ vertices on peg i , when read, say, from top to bottom, are a permutation of the set $\{1, \dots, p_i(D)\}$.

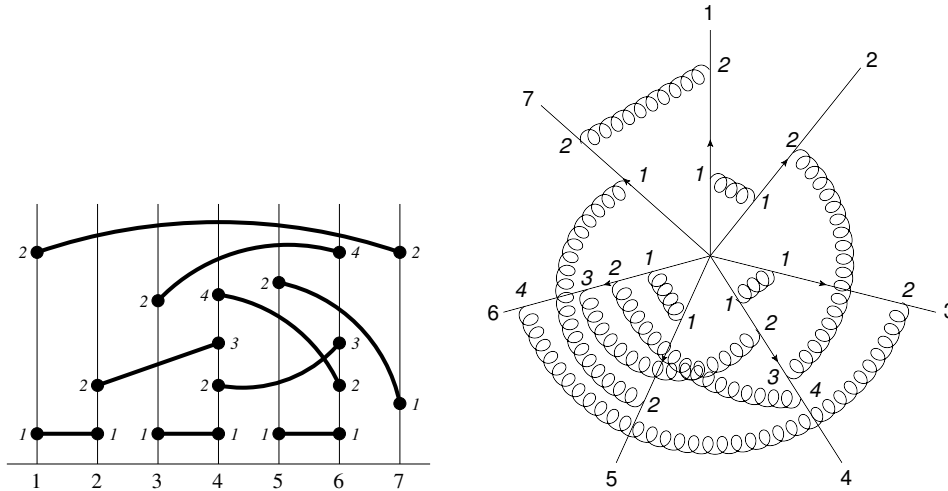


FIGURE 1. In the diagram on the left the indices of the pegs are shown at the bottom. The heights of the endpoints of the edges are shown in italics at each endpoint. The unique edge between pegs 3 and 6 is represented by the 4-tuple $(3, 6, 2, 4)$ since the left endpoint of the edge (on peg 3) has height 2 and the right endpoint of the edge (on peg 6) has height 4. The diagram on the right is the Feynman diagram illustration of the web diagram.

Given a web diagram $D = \{e_j = (x_j, y_j, a_j, b_j) : 1 \leq j \leq L\}$, let

$$\begin{aligned} \text{PegSet}(D) &= \{x_1, \dots, x_L, y_1, \dots, y_L\}, \\ \text{EdgeSet}(D) &= D, \\ \text{PegpairsSet}(D) &= \{(x_1, y_1), \dots, (x_L, y_L)\}. \end{aligned}$$

Example 2.2. The web diagram given in Figure 1 is

$$D = \{(1, 2, 1, 1), (1, 7, 2, 2), (2, 4, 2, 3), (3, 4, 1, 1), (3, 6, 2, 4), (4, 6, 2, 3), (4, 6, 4, 2), (5, 6, 1, 1), (5, 7, 2, 1)\}.$$

The number of vertices on each peg is $\text{Pegs}(D) = (2, 2, 2, 4, 2, 4, 2)$. Also,

$$\begin{aligned} \text{PegSet}(D) &= \{1, 2, 3, 4, 5, 6, 7\}, \\ \text{PegpairsSet}(D) &= \{(1, 2), (1, 7), (2, 4), (3, 4), (3, 6), (4, 6), (5, 6), (5, 7)\}. \end{aligned}$$

Notice that in this particular example $\text{PegpairsSet}(D)$ has size one less than $\text{EdgeSet}(D)$ since there are two edges that connect pegs 4 and 6.

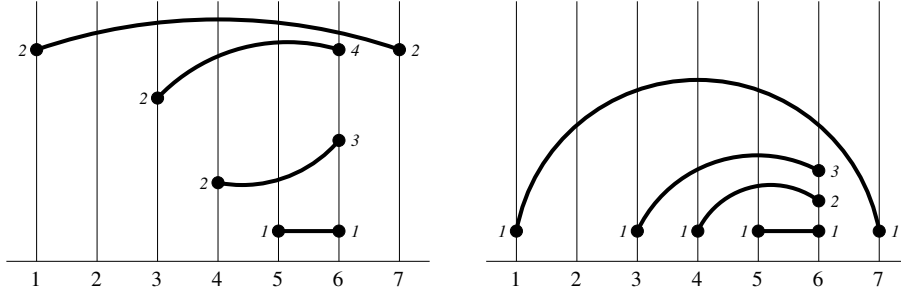
We now define the sum of two web diagrams.

Definition 2.3. Let $D = \{e_j = (x_j, y_j, a_j, b_j) : 1 \leq j \leq L\}$ and $D' = \{e'_j = (x'_j, y'_j, a'_j, b'_j) : 1 \leq j \leq L'\}$ be two web diagrams with $\text{PegSet}(D), \text{PegSet}(D') \subseteq \{1, \dots, n\}$. The *sum* $D \oplus D'$ is the web diagram obtained by placing the diagram D' on top of D ;

$$D \oplus D' = D \cup \{(x'_j, y'_j, a'_j + p_{x'_j}(D), b'_j + p_{y'_j}(D)) : 1 \leq j \leq L'\}.$$

If there exist two non-empty web diagrams E and F such that $D = E \oplus F$ then we say that D is *decomposable*. Otherwise we say that D is *indecomposable*.

Example 2.4. Consider the following two web diagrams: $D_1 = \{(1, 4, 1, 1), (2, 6, 1, 2), (2, 6, 2, 1)\}$ and $D_2 = \{(1, 2, 1, 1), (3, 5, 1, 1), (5, 6, 2, 1)\}$. For D_1 we have $(p_1(D_1), \dots, p_6(D_1)) = (1, 2, 0, 1, 0, 2)$


 FIGURE 2. The transformation $X \rightarrow \text{rel}(X)$.

and so

$$\begin{aligned}
 & D_1 \oplus D_2 \\
 &= \{(1, 4, 1, 1), (2, 6, 1, 2), (2, 6, 2, 1)\} \cup \{(1, 2, 1 + p_1(D_1), 1 + p_2(D_1)), \\
 &\quad (3, 5, 1 + p_3(D_1), 1 + p_5(D_1)), (5, 6, 2 + p_5(D_1), 1 + p_6(D_1))\} \\
 &= \{(1, 4, 1, 1), (2, 6, 1, 2), (2, 6, 2, 1)\} \cup \{(1, 2, 1 + 1, 1 + 2), (3, 5, 1 + 0, 1 + 0), (5, 6, 2 + 0, 1 + 2)\} \\
 &= \{(1, 4, 1, 1), (2, 6, 1, 2), (2, 6, 2, 1), (1, 2, 2, 3), (3, 5, 1, 1), (5, 6, 2, 3)\}.
 \end{aligned}$$

Definition 2.5. Let D be a web diagram and $X \subseteq D$. Let $\text{rel}(X)$ be the web diagram that results from re-labeling the third and fourth entries of every element of X so that the set of labels of points on a peg i of $\text{rel}(X)$ are $\{1, 2, \dots, \ell_i\}$ for some ℓ_i , and for all pegs. We call $\text{rel}(X)$ a *subweb diagram* of D .

Note that we do not remove empty pegs. The reason we do not remove them is that we will be defining an operation which combines subweb diagrams of a diagram. This will be made clear in Definitions 2.11 and 2.12.

Example 2.6. Let D be the web diagram in Figure 1. Let

$$X = \{(1, 7, 2, 2), (3, 6, 2, 4), (4, 6, 2, 3), (5, 6, 1, 1)\}.$$

Then $\text{rel}(X) = \{(1, 7, 1, 1), (3, 6, 1, 3), (4, 6, 1, 2), (5, 6, 1, 1)\}$. See Figure 2.

Definition 2.7. Suppose that $D = \{(x_j, y_j, a_j, b_j) : 1 \leq j \leq L\}$ is a web diagram on n pegs. Let $\text{Pegs}(D) = (p_1(D), \dots, p_n(D))$. Let $\pi = (\pi^{(1)}, \dots, \pi^{(n)})$ be a sequence of permutations where $\pi^{(i)}$ is a permutation of the set $\{1, \dots, p_i(D)\}$. Let $\pi(D)$ be the web diagram that results from moving the vertex at height j on peg i to height $\pi^{(i)}(j)$, for all j and i . Finally let $W(D) = \{\pi(D) : \pi \in \text{Pegs}(D)\}$, the set of all possible web diagrams that can be obtained from D . We call $W(D)$ the *web world* of D .

Example 2.8. If $D = \{(1, 2, 1, 2), (1, 2, 2, 1)\}$ then $W(D) = \{D, E\}$ where $E = \{(1, 2, 1, 1), (1, 2, 2, 2)\}$.

In terms of physics, a particular subset of web worlds are of interest.

Definition 2.9. Let W be a web world and $D = \{(x_i, y_i, a_i, b_i) : 1 \leq i \leq L\} \in W$. Let $G(W) = (V, E, \ell)$ be the edge-labeled simple graph where $V = \text{PegSet}(D)$,

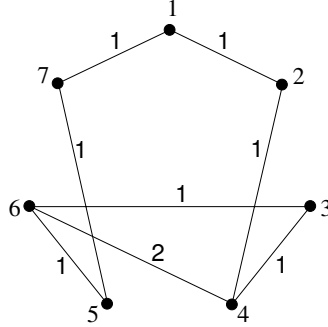
$$E = \{\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_L, y_L\}\}$$

and

$$\ell(\{x, y\}) = |\{1 \leq i \leq L : x_i = \min(x, y) \text{ and } y_i = \max(x, y)\}|$$

for all $\{x, y\} \in E$. We call $G(W)$ the *web graph* of the web world W .

The web graph is the graph that results from ‘forgetting’ the heights of all endpoints of edges in a web diagram. Another way to say this is that this is the graph one sees by looking down onto a web diagram, so that the pegs appear as points, and where each edge of $G(W)$ is weighted by the number of edges between the two pegs containing its endpoints in the original diagram. For example, the web graph corresponding to the web world generated by the diagram in Figure 1 is:



Definition 2.10. A web world W is called *proper* if the web graph $G(W)$ is connected.

Proper web worlds have been called ‘webs’ in references [7, 8, 9, 10]. Their significance is that they contribute to the exponent of an Eikonal amplitude, while web worlds corresponding to disconnected web graphs do not. The three web worlds we look at later on in this paper are proper web worlds. We now introduce colouring and reconstruction operations on our web diagrams.

Definition 2.11. Suppose that $D = \{e_i = (x_i, y_i, a_i, b_i) : 1 \leq i \leq L\}$ is a web diagram on n pegs, and $\ell \leq L$ a positive integer. A *colouring* c of D is a surjective function $c : \{1, \dots, L\} \rightarrow \{1, \dots, \ell\}$. Let $D_c(j) = \{e_i \in D : c(i) = j\}$ for all $1 \leq j \leq \ell$, the set of all those edges of D that are coloured j . The *reconstruction* $\text{Recon}(D, c) \in W(D)$ of D according to the colouring c is the web diagram

$$\text{Recon}(D, c) = \text{rel}(D_c(1)) \oplus \text{rel}(D_c(2)) \oplus \dots \oplus \text{rel}(D_c(\ell)).$$

Definition 2.12. Let D be a web diagram on n pegs, and let c be an ℓ -colouring of D . We say that the colouring c is *self-reconstructing* if $\text{Recon}(D, c) = D$.

Given $W = W(D)$ for some web diagram D on n pegs, and $D_1, D_2 \in W(D)$, let

$$F(D_1, D_2, \ell) = \{\ell\text{-colourings } c \text{ of } D_1 : \text{Recon}(D_1, c) = D_2\}$$

and $f(D_1, D_2, \ell) = |F(D_1, D_2, \ell)|$. Let $\mathfrak{M}^{(W)}(x)$ be the matrix whose (D_1, D_2) -entry is

$$\mathfrak{M}_{D_1, D_2}^{(W)}(x) = \sum_{\ell \geq 1} x^\ell f(D_1, D_2, \ell).$$

We call this matrix the *web-colouring matrix*. From a physics perspective, another matrix is of more immediate interest. Let $\mathfrak{R}^{(W)}$ be the matrix whose (D_1, D_2) entry is

$$\mathfrak{R}_{D_1, D_2}^{(W)} = \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} f(D_1, D_2, \ell).$$

We call $\mathfrak{R}^{(W)}$ the *web-mixing matrix* of W . It is straightforward to show that the (D_1, D_2) entries of $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$ are related via the following formula:

$$\mathfrak{R}_{D_1, D_2}^{(W)} = \int_{-1}^0 \frac{\mathfrak{M}_{D_1, D_2}^{(W)}(x)}{x} dx = - \int_0^1 \frac{\mathfrak{M}_{D_1, D_2}^{(W)}(-x)}{x} dx. \quad (1)$$

In the present paper, the basic problems we consider are as follows: Given a web world W ,

- (i) What can we say about the matrices $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$, their entries, trace and rank?
- (ii) Can we determine the entries of $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$ for special cases?

In Gardi and White [9] it was shown that

Theorem 2.13. [9] *Let W be a web world.*

- (i) *The row sums of $\mathfrak{R}^{(W)}$ are all zero.*
- (ii) *$\mathfrak{R}^{(W)}$ is idempotent.*

In the next section we look at diagonal entries of the matrices $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$. The following theorem is the \mathfrak{M} -analogue to Theorem 2.13(i). We omit the proof as it is rather elementary.

Theorem 2.14. *Let D be a web diagram with $m = |D|$ edges. The row sums $\mathfrak{M}^{(W)}(x)$ are all the same and given by the ordered Bell polynomials*

$$\mathfrak{N}_m(x) = \sum_{\ell=1}^m x^\ell \left\{ \begin{matrix} m \\ \ell \end{matrix} \right\} \ell!$$

where $\left\{ \begin{matrix} m \\ \ell \end{matrix} \right\}$ are the Stirling numbers of the 2nd kind.

Using the recursion $\left\{ \begin{matrix} m+1 \\ \ell \end{matrix} \right\} = \left\{ \begin{matrix} m \\ \ell-1 \end{matrix} \right\} + \left\{ \begin{matrix} m \\ \ell \end{matrix} \right\}$ in the definition of $\mathfrak{N}_{m+1}(x)$ above, one finds that the polynomials $\mathfrak{N}_m(x)$ satisfy the differential equation $x(d/dx)((x+1)\mathfrak{N}_m(x)) = \mathfrak{N}_{m+1}(x)$. Equivalently,

$$\sum_{m \geq 0} \mathfrak{N}_m(x) t^m = \frac{1}{1 - \frac{xt}{(x+1)t}} \cdot \frac{1}{1 - \frac{2xt}{2(x+1)t}} \cdot \frac{1}{1 - \dots}$$

3. SELF-RECONSTRUCTING COLOURINGS AND ORDER-PRESERVING MAPS

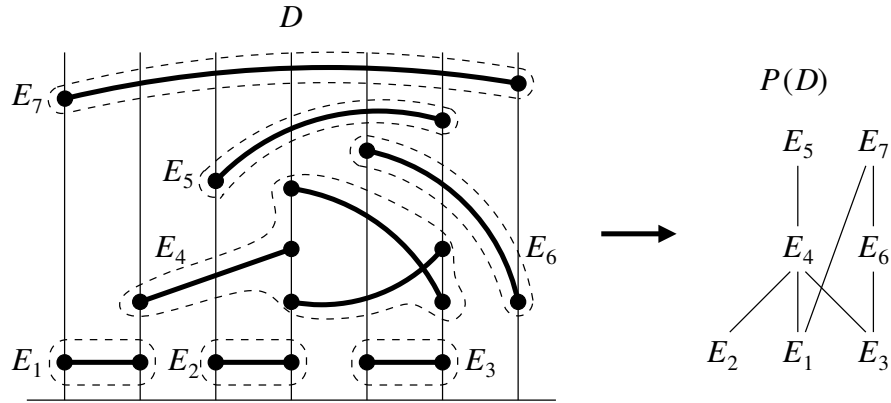
In this section we will study those colourings of a web diagram D that reconstruct D . This is to gain insight into the diagonal entries of the matrices $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$ and thus their traces. Since $\mathfrak{R}^{(W)}$ is idempotent (Theorem 2.13), its trace is also its rank, which plays an important role in the physics context [7]. In what follows we will see that every web diagram D can be decomposed and written as a sum of indecomposable subweb diagrams. A poset (partially ordered set) P on the set of these indecomposable subweb diagrams is then formed. Self-reconstructing colourings of D are then shown to correspond to linear extensions of P , to be explained below.

Definition 3.1. Let W be a web world and $D \in W$. Suppose that $D = E_1 \oplus E_2 \oplus \dots \oplus E_k$ where each E_i is an indecomposable web diagram. Define the partial order $P = (P, \preceq)$ as follows: $P = (E_1, \dots, E_k)$ and $E_i \preceq E_j$ if

- (a) $i < j$, and
- (b) there is an edge $e = (x, y, a, b)$ in E_i and an edge $e' = (x', y', a', b')$ in E_j such that an endpoint of e is below an endpoint of e' on some peg.

We call $P(D)$ the *decomposition poset* of D .

Example 3.2. Consider the diagram D given in Figure 1. The poset $P(D)$ we get from this diagram is illustrated as follows:



Note that $E_1 = \{(1, 2, 1, 1)\}$, $E_2 = \{(3, 4, 1, 1)\}$, $E_3 = \{(5, 6, 1, 1)\}$, $E_4 = \{(2, 4, 2, 3), (4, 6, 2, 3), (4, 6, 4, 2)\}$, $E_5 = \{(3, 6, 2, 4)\}$, $E_6 = \{(5, 7, 2, 1)\}$, and $E_7 = \{(1, 7, 2, 2)\}$.

Before we explain the relation between the linear extensions of $P(D)$ and self-reconstructing colourings we need some background. Given any two posets P and Q , a map $f : P \rightarrow Q$ is *order-preserving* if, for all x and y in P , $x \leq_P y$ implies $f(x) \leq_Q f(y)$. Let p be the number of elements in a poset P . A *linear extension* of P is an order-preserving bijection $f : P \rightarrow [p]$ where $[p] = \{1, \dots, p\}$ is equipped with the usual order on the integers. Each linear extension of P can be represented by a permutation of the elements of P , where the first element in the permutation is that element x of P for which $f(x) = 1$ and so on. The set of these permutations is called the *Jordan-Hölder set of P* , and denoted $\mathcal{L}(P)$.

Recall that a descent in a permutation $\pi = a_1 a_2 \dots a_n$ is an i such that $a_i > a_{i+1}$, and let $\text{des}(\pi)$ be the number of descents in π . The first part of the following lemma is from [11, Theorem 3.15.8]. The second part follows from the first part using the inclusion-exclusion principle.

Lemma 3.3. *Let P be a poset with p elements, and let $\Omega(P, m)$ be the number of order preserving maps $\sigma : P \rightarrow [m]$. Then*

$$\sum_{m \geq 0} \Omega(P, m) x^m = \frac{1}{(1-x)^{p+1}} \sum_{\pi \in \mathcal{L}(P)} x^{1+\text{des}(\pi)}.$$

Let $\Theta(P, m)$ be the number of surjective order-preserving maps from P to $[m]$. Define $\Theta(P, 0) = \Omega(P, 0) = 0$. Then we have

$$\Omega(P, m) = \sum_k \binom{m}{k} \Theta(P, k), \quad \Theta(P, m) = \sum_k \binom{m}{k} (-1)^{m-k} \Omega(P, k).$$

From now on, we will assume that, as in Example 2.4, we have labeled the elements of $P = P(D)$ *naturally*, that is, so that if $E_i < E_j$ in P then $i < j$. In a permutation $\pi = E_{i_1} E_{i_2} \dots E_{i_p}$ in $\mathcal{L}(P)$, declare k to be a descent if and only if $i_k > i_{k+1}$.

Theorem 3.4. *Let D be a web diagram with $D = E_1 \oplus \dots \oplus E_k$ where the entries of the sum are all indecomposable web diagrams. Let $P = P(D)$ and $p = |P(D)|$. If every member of the sequence (E_1, \dots, E_k) is distinct then*

$$\mathfrak{M}_{D,D}^{(W)}(x) = \sum_{\pi \in \mathcal{L}(P)} x^{1+\text{des}(\pi)} (1+x)^{p-1-\text{des}(\pi)} \quad (2)$$

and

$$\mathfrak{R}_{D,D}^{(W)} = \sum_{\pi \in \mathcal{L}(P)} \frac{(-1)^{\text{des}(\pi)}}{p^{\binom{p-1}{\text{des}(\pi)}}}. \quad (3)$$

Proof. The number $f(D, D, \ell)$ defined in Section 2 is the number of surjective order-preserving maps from $P(D)$ to $\{1, \dots, \ell\}$. By Lemma 3.3 we get

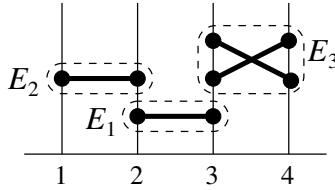
$$\sum_{m \geq 0} \Theta(P, m)x^m = (1+x)^p \sum_{\pi \in \mathcal{L}(P)} \left(\frac{x}{1+x} \right)^{1+\text{des}(\pi)} = \mathfrak{M}_{D,D}^{(W)}(x).$$

Let $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ be the beta function. For the diagonal terms of the web mixing matrix we have

$$\begin{aligned} \mathfrak{R}_{D,D}^{(W)} &= \sum_{\pi \in \mathcal{L}(P)} \int_{-1}^0 dx \left(x^{\text{des}(\pi)} (1+x)^{p-1-\text{des}(\pi)} \right) \\ &= \sum_{\pi \in \mathcal{L}(P)} (-1)^{\text{des}(\pi)} B(\text{des}(\pi) + 1, p - \text{des}(\pi)) \\ &= \sum_{\pi \in \mathcal{L}(P)} \frac{(-1)^{\text{des}(\pi)}}{p \binom{p-1}{\text{des}(\pi)}}. \end{aligned} \quad \square$$

By Theorem 3.4, computing the diagonal entries of our matrices is thus equivalent to computing the descent statistic on the Jordan-Hölder set of the corresponding poset.

Example 3.5. Let D be the following web diagram:



Since each of the web diagrams (E_1, E_2, E_3) are distinct, Theorem 3.4 may be applied. The poset $P = P(D)$ is the poset on $\{E_1, E_2, E_3\}$ with relations $E_1 < E_2, E_3$. We find that $\mathcal{L}(P) = \{E_1 E_2 E_3, E_1 E_3 E_2\}$, with $\text{des}(E_1 E_2 E_3) = 0$ and $\text{des}(E_1 E_3 E_2) = 1$. Consequently we have

$$\mathfrak{M}_{D,D}^{(W(n))}(x) = x(1+x)^2 + x^2(1+x) = x + 3x^2 + 2x^3$$

$$\text{and } \mathfrak{R}_{D,D}^{(W)} = (-1)^0/3 + (-1)^1/3 \binom{2}{1} = 1/6.$$

Given a web world W , let $\text{AllPosets}(W) = \{P(D) : D \in W\}$. For $P \in \text{AllPosets}(W)$, let $\text{multip}_W(P) = |\{D \in W : P(D) = P\}|$. Using this notation we have

Corollary 3.6. Suppose that W is a web world whose graph $G(W) = (V, E, \ell)$ is such that $\ell(e) = 1$ for all $e \in E$. Then

$$\begin{aligned} \text{trace}(\mathfrak{M}^{(W)}(x)) &= \sum_{D \in W} \mathfrak{M}_{D,D}^{(W)}(x) \\ &= \sum_{P \in \text{AllPosets}(W)} \text{multip}_W(P) \sum_{\pi \in \mathcal{L}(P)} x^{1+\text{des}(\pi)} (1+x)^{|P|-1-\text{des}(\pi)} \end{aligned} \quad (4)$$

and

$$\begin{aligned} \text{trace}(\mathfrak{R}^{(W)}) &= \sum_{D \in W} \mathfrak{R}_{D,D}^{(W)} \\ &= \sum_{P \in \text{AllPosets}(W)} \text{multip}_W(P) \sum_{\pi \in \mathcal{L}(P)} \frac{(-1)^{\text{des}(\pi)}}{p \binom{p-1}{\text{des}(\pi)}} \end{aligned} \quad (5)$$

Note that, owing to the idempotence of web mixing matrices (Theorem 2.13), the trace of a web mixing matrix is equal to its rank. A corollary is that the trace must be a positive integer number. In the physics context of reference [7] this invariant represents the number of independent contributions to the exponent of an Eikonal scattering amplitude from the corresponding web world.

We illustrate the above calculations in the following two examples. In the first example posets on a different number of elements emerge. However, in the second example only posets on three elements emerge. The second example is a special case of the web world that will be discussed later in Case 2 in Section 7.

Example 3.7. *The web world $W(D)$ of the web diagram*

$$D = \{(1, 2, 1, 1), (2, 3, 2, 1), (3, 4, 2, 1)\}$$

contains four web diagrams. $\text{AllPosets}(W)$ contains three posets; the chain $\mathbf{3}$ arises twice; the wedge poset \wedge arises once, as does the V-shaped poset \vee . Thus $\text{multip}_W(\mathbf{3}) = 2$ and $\text{multip}_W(\wedge) = \text{multip}_W(\vee) = 1$. We have $\mathcal{L}(\mathbf{3}) = \{(1, 2, 3)\}$ and $\mathcal{L}(\vee) = \mathcal{L}(\wedge) = \{(1, 2, 3), (1, 3, 2)\}$. Thus,

$$\text{trace}(\mathfrak{M}^{(W)}(x)) = 2(x^1(1+x)^2) + 1(x^1(1+x)^2 + x^2(1+x)) + 1(x^1(1+x)^2 + x^2(1+x)) = 6x^3 + 10x^2 + 4x$$

and

$$\text{trace}(\mathfrak{R}^{(W)}) = 2(1/3) + 1(1/6) + 1(1/6) = 1.$$

(See Figure 3.)

4. THE NUMBER OF WEB WORLDS

In this section we present some results on the number of web worlds with prescribed sizes of the sets PegSet , EdgeSet , and PegpairsSet . If W is a web world and D_1, D_2 are two web diagrams in W , then $\text{PegSet}(D_1) = \text{PegSet}(D_2)$, $\text{EdgeSet}(D_1) = \text{EdgeSet}(D_2)$, and $\text{PegpairsSet}(D_1) = \text{PegpairsSet}(D_2)$. We will use $\text{PegSet}(W)$, $\text{EdgeSet}(W)$, and $\text{PegpairsSet}(W)$ to refer to these numbers without having to refer to diagrams of the web world.

A web world W is uniquely specified by its web graph $G(W) = (V, E, \ell)$. An equivalent specification is by a square matrix $\text{Represent}(W)$ of integers whereby

$$\text{Represent}(W)_{i,j} = \begin{cases} \ell(i, j) & \text{if } i < j \text{ and } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.3 gives the generating function for the number of web worlds according to these three statistics. Web-worlds which contain pegs that have no incident edges, equivalent to isolated vertices when one talks of graphs, have little relevance in the corresponding physics model. Theorem 4.4 gives the generating function for the number of web worlds with no isolated pegs, according to the three statistics. Theorem 4.6 gives the generating function for the number of proper web worlds according to the same three statistics as Theorem 4.4. Theorem 4.7 gives an expression for the number of different diagrams in a web world in terms of the values in the matrix $\text{Represent}(W)$.

Let W be a web world and D a web diagram in W . Define the *web diagram matrix* of a web diagram D to be a matrix $\text{WDM}(D)$ whose entries are sets:

$$(a_j, b_j) \in \text{WDM}(D)_{x_j, y_j} \text{ for all } (x_j, y_j, a_j, b_j) \in D.$$

Example 4.1. *Consider the web diagram D in Figure 1. The set of 4-tuples for that web diagram is $\{(1, 2, 1, 1), (1, 7, 2, 2), (2, 4, 2, 3), (3, 4, 1, 1), (3, 6, 2, 4), (4, 6, 2, 3), (4, 6, 4, 2), (5, 6, 1, 1), (5, 7, 2, 1)\}$.*

Diagram D	Poset $P(D)$	$\mathcal{L}(P(D))$
	$ \begin{array}{c} E_3 \\ \\ E_2 = \mathbf{3} \\ \\ E_1 \end{array} $	$\{(1, 2, 3)\}$
	$ \begin{array}{c} E_3 \\ / \quad \backslash \\ E_1 \quad E_2 \end{array} $	$\{(1, 2, 3), (1, 3, 2)\}$
	$ \begin{array}{c} E_3 \\ \\ E_2 = \mathbf{3} \\ \\ E_1 \end{array} $	$\{(1, 2, 3)\}$
	$ \begin{array}{c} E_2 \quad E_3 \\ \backslash \quad / \\ E_1 \end{array} $	$\{(1, 2, 3), (1, 3, 2)\}$

FIGURE 3. The four web diagrams in the web world generated from D in Example 3.7.

The web diagram matrix $\text{WDM}(D)$ is:

$$\text{WDM}(D) = \begin{pmatrix}
 \emptyset & \{(1, 1)\} & \emptyset & \emptyset & \emptyset & \emptyset & \{(2, 2)\} \\
 \emptyset & \emptyset & \emptyset & \{(2, 3)\} & \emptyset & \emptyset & \emptyset \\
 \emptyset & \emptyset & \emptyset & \{(1, 1)\} & \emptyset & \{(2, 4)\} & \emptyset \\
 \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{(2, 3), (4, 2)\} & \emptyset \\
 \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{(1, 1)\} & \{(2, 1)\} \\
 \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
 \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset
 \end{pmatrix}.$$

Every web world W can be uniquely specified by a matrix $\text{Represent}(W)$ since all permutations of line end-points on pegs form diagrams in the same web world, and it is only the *number* of such lines that defines the web world. A matrix $A = \text{Represent}(W)$ is the matrix of a web world iff $a_{ij} = 0$ for all $1 \leq j \leq i \leq m$ and the other entries are non-negative integers. These matrices are also obtained by simply taking the cardinality of the entries of the matrices WDM .

Example 4.2. Let W be the web world given in Example 4.1. Then

$$\text{Represent}(W) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\text{nww}(m, t, n) = \left| \left\{ \text{web worlds } W : \begin{array}{l} \text{PegSet}(W) \subseteq \{1, \dots, m\}, \\ |\text{EdgeSet}(W)| = t, \\ |\text{PegpairsSet}(W)| = n \end{array} \right\} \right|,$$

the number of web worlds with the three prescribed properties. The generating function $\text{NWW}(u, z, y)$ for these numbers is

$$\text{Theorem 4.3. } \text{NWW}(u, z, y) = \sum_{\substack{m \geq 2 \\ t, n \geq 0}} \text{nww}(m, t, n) u^m z^t y^n = \sum_{m \geq 2} \left(1 + \frac{yz}{1-z}\right)^{\binom{m}{2}} u^m.$$

The above generating function allows pegs which have no incident edges. We now consider web diagrams which do not contain such pegs. Let

$$\text{nwwnip}(m, t, n) = \left| \left\{ \text{web worlds } W : \begin{array}{l} \text{PegSet}(W) = \{1, \dots, m\}, \\ |\text{EdgeSet}(W)| = t, \\ |\text{PegpairsSet}(W)| = n \end{array} \right\} \right|,$$

the number of web worlds having no isolated pegs and the three prescribed properties. This collection of web worlds corresponds to matrices $A = \text{Represent}(W)$ with the added property that the ‘hook’ at position (i, i) in A is nonempty, i.e. for all $1 \leq i \leq m$,

$$a_{1,i} + \dots + a_{i-1,i} + a_{i,i} + a_{i,i+1} + \dots + a_{im} > 0.$$

Let

$$\text{NWWNIP}(u, z, y) = \sum_{\substack{m \geq 2 \\ t, n \geq 1}} \text{nwwnip}(m, t, n) u^m z^t y^n.$$

We have

$$\text{Theorem 4.4. } \text{NWWNIP}(u, z, y) = - \left(\frac{u}{1+u}\right)^2 + \frac{1}{1+u} \sum_{m \geq 2} \left(1 + \frac{yz}{1-z}\right)^{\binom{m}{2}} \left(\frac{u}{1+u}\right)^m.$$

Proof. Every matrix that is counted by $\text{nwwnip}(m, t, n)$ gives rise to $\binom{m+i}{i}$ matrices of dimension $m+i$ where we have inserted i zeros on the diagonal in appropriate places. Fill in the hooks of these new zeros on the diagonal with zeros. The resulting matrices are simply the collection of matrices which have integer entries, and whose diagonal and lower diagonal entries are all zero. Therefore

$$\sum_{\substack{m \geq 2 \\ t, n \geq 1}} \sum_{i \geq 0} \binom{m+i}{i} \text{nwwnip}(m, t, n) x^{m+i} z^t y^n = \sum_{\substack{m \geq 2 \\ n, t \geq 1}} \text{nww}(m, t, n) x^m z^t y^n.$$

By Theorem 4.3 the right hand side can be written as

$$\sum_{m \geq 2} x^m \left(\left(1 + \frac{yz}{1-z}\right)^{\binom{m}{2}} - 1 \right).$$

The left hand side of the above equation is $\frac{1}{1-x} \text{NWWNIP}(x/(1-x), z, y)$. Change variable by setting $u = x/(1-x)$ and the result follows. \square

Using standard techniques to extract the coefficient of $u^a z^b y^c$ in Theorem 4.4 we have the following corollary.

Corollary 4.5. $\text{nwwnip}(a, b, c) = \binom{b-1}{c-1} \sum_k \binom{a}{k} \binom{\binom{k}{2}}{c} (-1)^{a-k}$.

Let

$$\text{npww}(m, t, n) = \left| \left\{ \begin{array}{l} \text{web worlds } W : \text{PegSet}(W) = \{1, \dots, n\}, \\ \text{EdgeSet}(W) = m, \\ \text{PegpairsSet}(W) = t \end{array} \right\} \right|,$$

the number of proper web worlds with the three prescribed properties. Let

$$C(x, q, z) = \sum_{m, t, n} \text{npww}(m, t, n) x^m q^t \frac{z^n}{n!}.$$

Theorem 4.6. $C(x, q, z) = \log \left(1 + \sum_{n \geq 1} \left(1 + \frac{qx}{1-x} \right) \binom{\binom{n}{2}}{2} \frac{z^n}{n!} \right)$.

Proof. Using the exponential formula, the bivariate generating function $C(q, z) = \sum_{n, k} c(k, n) q^k \frac{z^n}{n!}$ where $c(k, n)$ is the number of connected simple graphs on n labeled vertices having k edges is seen to satisfy

$$\exp C(q, z) = 1 + \sum_{n \geq 1} (1+q) \binom{\binom{n}{2}}{2} \frac{z^n}{n!}.$$

One may now replace q by $q(x + x^2 + x^3 + \dots)$ so that the power of x records the sum of weights on labeled edges between vertices. Therefore

$$\exp C(x, q, z) = 1 + \sum_{n \geq 1} (1 + q(x + x^2 + \dots)) \binom{\binom{n}{2}}{2} \frac{z^n}{n!}.$$

Taking logarithms of both sides we have: $C(x, q, z) = \log \left(1 + \sum_{n \geq 1} \left(1 + \frac{qx}{1-x} \right) \binom{\binom{n}{2}}{2} \frac{z^n}{n!} \right)$. \square

Next we present a different type of enumerative result. Rather than giving another theorem for the number of web worlds according to various statistics, we now present a formula for the number of web diagrams contained in a given web world.

Theorem 4.7. *Let W be a web world on n pegs and $A = \text{Represent}(W)$. The number of different diagrams $D \in W$ is*

$$|W| = \prod_{i=1}^n (a_{i*} + a_{*i})! / \prod_{1 \leq i < j \leq n} a_{ij}!$$

where a_{i*} (resp. a_{*i}) is the sum of entries in column (resp. row) i of A .

Proof. The number of diagrams $D \in W$ is the number of ways to form an $n \times n$ matrix $X = \text{WDM}(D)$ whose entries consist of sets of ordered pairs that satisfy the following hook property for all $1 \leq i \leq n$: The set $A \cup B$ is a permutation of the set $\{1, \dots, x_{1i} + \dots + x_{ii} + \dots + x_{in}\}$ where A is the set of all first elements of pairs in the sets $X_{i, i+1}, \dots, X_{i, n}$ and B is the set of all second elements of pairs in the sets $X_{1, i}, X_{2, i}, \dots, X_{i-1, i}$.

The sets in each matrix entry can be replaced with the sequence of pairs wherein the pairs of elements are listed in increasing order of their first entry.

The entries of the matrix $A = \text{Represent}(W)$ tells us the cardinalities of the entries of X , i.e. $a_{ij} = |X_{ij}|$. We count the number of ways to fill the entries with respect to the hook $X_{1,i}, \dots, X_{i,i}, \dots, X_{i,n}$. Entries in the vertical part of the hook get inserted as second entries of the pairs. Entries of the horizontal part of the hook get inserted as the first entries of pairs. There are a total of $(a_{1,i} + \dots + a_{i,i} + \dots + a_{i,n})! = (a_{*i} + a_{i*})!$ ways to do this. However we must divide by $a_{i,i+1}! \dots a_{i,n}!$ as the sequence of first entries at each matrix entry must be increasing. The result follows by multiplying together the number of ways of filling each of the n hooks on the diagonal. \square

5. A BRIEF INTERLUDE ON COLOURING SEQUENCES

Before looking at three examples of web worlds we collect here some notation and terminology that will be used in subsequent sections. Most of the notation and terminology concern colourings and we remind the reader that constructing new web diagrams from a given web diagram by means of a colouring was defined in Definition 2.11.

Let

$$\text{Colors}(n, k) = \{(c(1), \dots, c(n)) : \{c(1), \dots, c(n)\} = \{1, \dots, k\}\}.$$

The set $\text{Colors}(n, k)$ is the set of all colour sequences of length n where each of the colours $\{1, \dots, k\}$ appears at least once. When performing calculations with respect to colour sequences, we will find it useful to decompose a colour sequence as a list of colours as they appear from left to right in the colour sequence (so that no two adjacent colours are the same) along with a sequence which gives the multiplicity of each unique colour.

Let $\text{Keys}(n, k) = \{\vec{c} \in \text{Colors}(n, k) : c(i) \neq c(i+1) \text{ for all } 1 \leq i < n\}$, the set of all colour sequences of length n which contain no two adjacent colours. Define

$$\text{Keys}^=(n, k) = \{c \in \text{Keys}(n, k) : c(1) = c(n)\}$$

$$\text{Keys}^\neq(n, k) = \{c \in \text{Keys}(n, k) : c(1) \neq c(n)\}.$$

This partition of $\text{Keys}(n, k)$ will be of interest in Section 8 when we consider the colouring of an edge between the first peg and last peg in a web diagram.

Lemma 5.1. *Let $n, k \in \mathbb{N}$ with $1 \leq k \leq n$. Then*

$$\begin{aligned} |\text{Keys}(n, k)| &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i(i-1)^{n-1} \\ |\text{Keys}^\neq(n, k)| &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} ((i-1)^n + (i-1)(-1)^n) \\ |\text{Keys}^=(n, k)| &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} ((i-1)^{n-1} + (i-1)(-1)^{n-1}). \end{aligned}$$

Proof. Since $n(n-1)^{k-1} = \sum_{i=0}^k \binom{k}{i} |\text{Keys}(n, i)|$ for all $0 \leq k \leq n$, inclusion-exclusion gives $|\text{Keys}(n, k)| = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i(i-1)^{n-1}$ for all $0 \leq k \leq n$. Define $\text{NoEquals}_{n,k} = \{(a_1, \dots, a_n) : a_1 \leq a_2, a_2 \neq a_3, \dots, a_m \neq a_1 \text{ and } a_i \in \{1, \dots, k\}\}$. This set is enumerated $|\text{NoEquals}_{n,k}| = (k-1)^n + (k-1)(-1)^n$. We have

$$\text{Keys}^\neq(n, k) = \{a \in \text{NoEquals}_{n,k} : \{a_1, \dots, a_n\} = \{1, \dots, k\}\}.$$

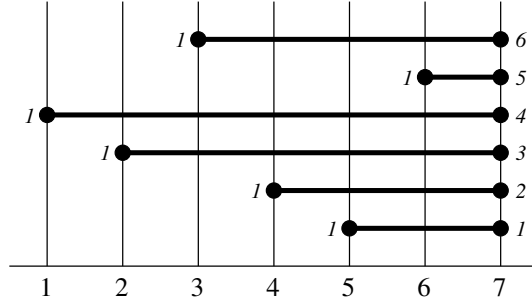


FIGURE 4. Example of a web diagram in a $\text{Pegs}(D) = (1, 1, 1, 1, 1, 1, 6)$ web world. This is represented by D_π in Equation (6) with $\pi = (5, 4, 2, 1, 6, 3)$.

Since $|\text{NoEquals}_{n,k}| = \sum_{\ell} \binom{k}{\ell} |\text{Keys}^\neq(n, \ell)|$ for all $0 \leq k \leq n$, inclusion-exclusion yields

$$|\text{Keys}^\neq(n, k)| = \sum_i (-1)^{k-i} \binom{k}{i} |\text{NoEquals}_{n,i}| = \sum_i (-1)^{k-i} \binom{k}{i} ((i-1)^n + (i-1)(-1)^n),$$

hence the second equation. Subtracting the second equation from the first equation in the statement of the Lemma yields the third equation. \square

Every $c \in \text{Colors}(n, k)$ can be uniquely expressed as a pair $\langle w, A \rangle$, where $A = \{2 \leq i \leq n : c(i) = c(i-1)\} \subseteq [2, n]$, and $w = (c(i))_{\substack{i \geq 1 \\ i \notin A}} \in \text{Keys}(m, k)$. For example, if $c = (4, 2, 1, 1, 1, 2, 3, 3, 3, 4, 4, 1)$ then we have $A = (4, 5, 8, 9, 11)$ and $w = (4, 2, 1, 2, 3, 4, 1)$. Notice that if $c = \langle w, A \rangle$ where $c \in \text{Colors}(n, k)$ where $|A| = n - m$ and $w \in \text{Keys}(m, k)$, then

$$\text{equ}(c) := |\{1 \leq i < n : c(i) = c(i+1)\}| = n - m.$$

6. CASE 1: A WEB WORLD WITH $\text{Pegs}(D) = (1, 1, \dots, 1, n)$

Fix $n \in \mathbb{N}$ and let $W(n)$ be the web world of all diagrams

$$D = D_\pi = \{e_i = (\pi(i), n+1, 1, i) : 1 \leq i \leq n\} \quad (6)$$

where π is a permutation of $\{1, \dots, n\}$. An example of such a web diagram for $n = 6$ is illustrated in Figure 4.

Definition 6.1. Given $c \in \text{Colors}(n, k)$, let α_c be the lexicographically smallest permutation in \mathfrak{S}_n such that

$$c(\alpha_c(1)) \leq c(\alpha_c(2)) \leq \dots \leq c(\alpha_c(n)).$$

Another way to read α_c from c is to first list the positions (in increasing order) at which 1 appears in v , then do the same for 2, and so on.

Example 6.2. If $c = (3, 1, 1, 2, 1, 2, 2, 3) \in \text{Colors}(8, 3)$ then $\alpha_c = (2, 3, 5, 4, 6, 7, 1, 8)$.

Lemma 6.3. Given $D_\pi \in W(n)$ and $c \in \text{Colors}(n, k)$, $\text{Recon}(D_\pi, c) = D_{\pi \circ \alpha_c}$.

Proof. Let $D_\pi \in W(n)$ and $c \in \text{Colors}(n, k)$. Notice that edges can only share an endpoint on peg $n+1$. For this reason we have

$$D_\pi = (\pi(1), n+1, 1, 1) \oplus (\pi(2), n+1, 1, 1) \oplus \dots \oplus (\pi(n), n+1, 1, 1).$$

For all $j \in \{1, \dots, k\}$ let $X_j = \{i : c(i) = j\}$ and define Y_j to be the sequence of elements of X_i written in increasing order. Let Z_j be the sequence of elements of $Y_1 : Y_2 : \dots : Y_j$ where :

denotes sequence concatenation and define $Z_0 = \emptyset$. From Definitions 2.5 and 2.11 we have for all $j \in \{1, \dots, k\}$,

$$\text{rel}(D_c(j)) = \bigoplus_{i=|Z_{j-1}|}^{|Z_j|} (\pi(\alpha_c(i)), n+1, 1, 1).$$

Thus

$$\text{Recon}(D_\pi, c) = \bigoplus_{i=1}^{|Z_k|} (\pi(\alpha_c(i)), n+1, 1, 1) = D_{\pi \circ \alpha_c}. \quad \square$$

Example 6.4. Given $\pi = (2, 8, 5, 4, 1, 3, 7, 6) \in \mathfrak{S}_8$ and $c = (3, 1, 1, 2, 1, 2, 2, 3) \in \text{Colors}(8, 3)$, we have $\alpha_c = (2, 3, 5, 4, 6, 7, 1, 8)$ and $\pi \circ \alpha_c = (8, 5, 1, 4, 3, 7, 2, 6)$.

For every pair D_π and D_σ of diagrams in $W(n)$, there always exists a colouring of D_π which will yield D_σ . There is a unique colouring with a minimal number of colours which achieves this and is described as follows:

Definition 6.5. Let $\text{Minimal}(\pi, \sigma) = (X_1, \dots, X_\ell)$ be the sequence of sequences obtained by repeatedly reading π from left to right until all the letters have been read, in the order in which they appear in σ . Define $\text{minimal}(\pi, \sigma) = \ell$ and if $\pi_j \in X_i$ then set $c(j) = i$.

Example 6.6. If $\pi = (2, 8, 5, 4, 1, 3, 7, 6)$ and $\sigma = (8, 5, 1, 4, 3, 7, 2, 6)$ then we have $\text{Minimal}(\pi, \sigma) = ((8, 5, 1), (4, 3, 7), (2, 6))$. This means $\text{minimal}(\pi, \sigma) = 3$ and the unique colouring having fewest colours which transforms D_π into D_σ is $c = (1, 3, 2, 2, 1, 3, 2, 1)$.

Lemma 6.7. Let $D_\pi, D_\sigma \in W(n)$ with $\text{minimal}(\pi, \sigma) = m$. Given k with $m \leq k \leq n$, the number of $c \in \text{Colors}(n, k)$ for which $\pi \circ \alpha_c = \sigma$ is $f(D_\pi, D_\sigma, k) = \binom{n-m}{k-m}$.

Proof. Let $D_\pi, D_\sigma \in W(n)$ with $\text{minimal}(\pi, \sigma) = m$ and $\text{Minimal}(\pi, \sigma) = (X_1, \dots, X_m)$. Let $x_i = |X_i|$. Once the minimal colouring for converting π to σ is given, all other colourings easily follow from this. All elements in X_i have colour i and σ is the permutation $X_1 X_2 \cdots X_m$. We may partition each of these sequences X_i individually, to (Y_1, \dots, Y_k) whereby $Y_1 \cdots Y_{\beta(1)} = X_1$, $Y_{\beta(1)+1} \cdots Y_{\beta(2)} = X_2$, and so on. The number of ways of doing this is $\binom{(x_1-1)+\dots+(x_m-1)}{(k-1)-(m-1)}$ since there are $m-1$ positions already specified from the minimal colouring. This value is $\binom{n-m}{k-m}$. \square

Theorem 6.8. Suppose that $D_\pi, D_\sigma \in W(n)$ with $m = \text{minimal}(\pi, \sigma)$. Then

$$\mathfrak{M}_{D_1, D_2}^{(W(n))}(x) = x^m (1+x)^{n-m} \quad \text{and} \quad \mathfrak{R}_{D_\pi, D_\sigma}^{(W(n))} = \frac{(-1)^{m-1}}{n \binom{n-1}{m-1}}. \quad (7)$$

Consequently,

$$\begin{aligned} \mathfrak{R}_{D_\pi, D_\pi}^{(W(n))} &= 1/n, & \text{trace}(\mathfrak{R}^{(W(n))}) &= (n-1)! \\ \mathfrak{M}_{D_\pi, D_\pi}^{(W(n))}(x) &= x(1+x)^{n-1}, & \text{trace}(\mathfrak{M}^{(W(n))}(x)) &= n!x(1+x)^{n-1}. \end{aligned}$$

Proof. Let $m = \text{minimal}(\pi, \sigma)$. Using Lemma 6.7 we have

$$\mathfrak{M}_{D_1, D_2}^{(W(n))}(x) = \sum_{k=m}^n x^k f(D_\pi, D_\sigma, k) = \sum_{k=m}^n x^k \binom{n-m}{k-m} = x^m (1+x)^{n-m}. \quad (8)$$

Using Equation (1) we have

$$\begin{aligned}
 \mathfrak{R}_{D_\pi, D_\sigma}^{(W(n))} &= - \int_0^1 \frac{\mathfrak{M}_{D_\pi, D_\pi}^{(W(n))}(-x)}{x} dx \\
 &= (-1)^{m-1} \int_0^1 x^{m-1} (1-x)^{n-m} dx \\
 &= (-1)^{m-1} B(m, n-m+1), \\
 &= \frac{(-1)^{m-1}}{n \binom{n-1}{m-1}},
 \end{aligned}$$

where B is the beta function. Since $\text{minimal}(\pi, \pi) = 1$ we have that $\mathfrak{R}_{D_\pi, D_\pi}^{(W(n))} = 1/n$ and $\mathfrak{M}_{D_\pi, D_\pi}^{(W(n))}(x) = x(1+x)^{n-1}$. Therefore $\text{trace}(\mathfrak{R}^{(W(n))}) = n!/n = (n-1)!$ and $\text{trace}(\mathfrak{M}^{(W(n))}(x)) = n!x(1+x)^{n-1}$. This result is consistent with Equation (3) as there is only one linear extension of the poset in this setting. \square

Let $\sigma \in \mathfrak{S}_n$. It is straightforward to check that $\text{minimal}(\text{id}, \sigma) = 1 + \text{des}(\sigma)$. Thus the number of σ in the top row of $\mathfrak{M}^{(W(n))}(x)$ for which $\text{minimal}(\text{id}, \sigma) = k$ is the Eulerian number $\langle n \rangle_k$. The same observation yields that for any $\pi \in \mathfrak{S}_n$, the number of $\sigma \in \mathfrak{S}_n$ such that $\text{minimal}(\pi, \sigma) = k$ is the Eulerian number $\langle n \rangle_k$. This fact now accounts for value multiplicities in the rows and columns of $\mathfrak{M}^{(W(n))}(x)$ and $\mathfrak{R}^{(W(n))}$.

7. CASE 2: A WEB WORLD WITH $\text{Pegs}(D) = (1, 2, \dots, 2, 1)$

Fix $n \in \mathbb{N}$ and let $W(n)$ be the collection of all web diagrams

$$D = \{e_i = (i, i+1, x_i, y_{i+1}) : 1 \leq i \leq n+1\} \quad (9)$$

where $x_1 = y_{n+2} = 1$ and $\{x_i, y_i\} = \{1, 2\}$ for all $2 \leq i \leq n+1$. $W(n)$ is the set of all web diagrams consisting of $n+2$ pegs, $n+1$ lines whose endpoints are on adjacent pegs, and the only pegs that have one endpoint rather than two endpoints are pegs 1 and $n+2$. An example of a web diagram in $W(4)$ is illustrated in Figure 5.

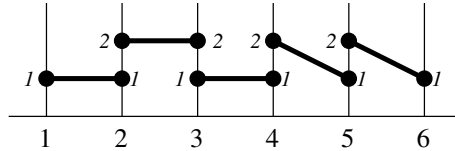


FIGURE 5. An example of a web diagram of the form given in Equation (9).

Every diagram $D \in W(n)$ can be uniquely encoded as a sequence $\pi \in (\pm 1)^n$ where

$$\pi(i) = \begin{cases} +1 & \text{if } y_{i+1} = 1 \text{ and } x_{i+1} = 2 \\ -1 & \text{if } y_{i+1} = 2 \text{ and } x_{i+1} = 1. \end{cases}$$

See Figure 6 for an illustration of this. The example in Figure 5 is encoded by $\pi = (+1, -1, +1, +1)$. We denote by D_π the diagram that is encoded by the sequence π .

A colouring of D_π is a sequence $c \in \text{Colors}(n+1, k)$ where $c(i)$ is the colour of edge e_i . Given $c \in \text{Colors}(n+1, k)$ let $\text{DEA}(c) = (\text{Des}(c), \text{Equ}(c), \text{Asc}(c))$ where

$$\begin{aligned}
 \text{Des}(c) &= \{1 \leq i \leq n : c(i) > c(i+1)\}, \\
 \text{Equ}(c) &= \{1 \leq i \leq n : c(i) = c(i+1)\}, \text{ and} \\
 \text{Asc}(c) &= \{1 \leq i \leq n : c(i) < c(i+1)\}.
 \end{aligned}$$

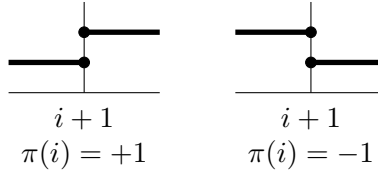


FIGURE 6. The value of $\pi(i)$ depends on how the edges meet at peg $i + 1$.

We refer to $\text{Des}(c)$, $\text{Equ}(c)$ and $\text{Asc}(c)$ as the *descent*, *plateau*, and *ascent sets*, respectively. Let $\text{WordEuler}_{n,k}(X_1, X_2, X_3) = \{c \in \text{Colors}(n+1, k) : \text{DEA}(c) = (X_1, X_2, X_3)\}$, the set of colourings of $\text{Colors}(n, k)$ whose descent, plateau and ascent sets are given by X_1 , X_2 and X_3 , respectively. Set $\text{wordeuler}_{n,k}(X_1, X_2, X_3) = |\text{WordEuler}_{n,k}(X_1, X_2, X_3)|$.

Given diagrams $D_\pi, D_\sigma \in W(n)$, let

$$Y(\pi, \sigma) = (Y_1(\pi, \sigma), Y_2(\pi, \sigma), Y_3(\pi, \sigma), Y_4(\pi, \sigma))$$

be an ordered partition of $\{1, \dots, n\}$ where

$$Y_1(\pi, \sigma) = \{1 \leq i \leq n : \pi_i = +1 \text{ and } \sigma_i = -1\}$$

$$Y_2(\pi, \sigma) = \{1 \leq i \leq n : \pi_i = -1 \text{ and } \sigma_i = -1\}$$

$$Y_3(\pi, \sigma) = \{1 \leq i \leq n : \pi_i = +1 \text{ and } \sigma_i = +1\}$$

$$Y_4(\pi, \sigma) = \{1 \leq i \leq n : \pi_i = -1 \text{ and } \sigma_i = +1\}.$$

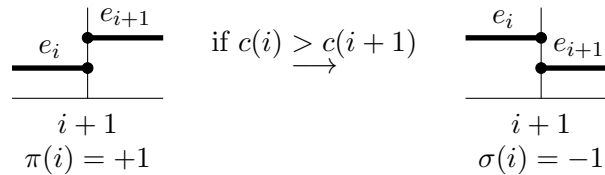
(We can summarise these more compactly as $i \in Y_{(5+2\sigma_i-\pi_i)/2}(\pi, \sigma)$ for all $1 \leq i \leq n$.)

Theorem 7.1. *Let $k \in \mathbb{N}$. Let $D_\pi, D_\sigma \in W(n)$ and suppose that $Y(\pi, \sigma) = (Y_1, Y_2, Y_3, Y_4)$. Then*

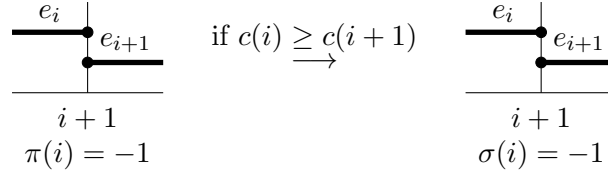
$$F(D_\pi, D_\sigma, k) = \bigcup_{\substack{A \subseteq Y_2 \\ B \subseteq Y_3}} \text{WordEuler}_{n,k}(Y_1 \cup A, Y_2 \cup Y_3 - (A \cup B), Y_4 \cup B),$$

$$f(D_\pi, D_\sigma, k) = \sum_{\substack{A \subseteq Y_2 \\ B \subseteq Y_3}} \text{wordeuler}_{n,k}(Y_1 \cup A, Y_2 \cup Y_3 - (A \cup B), Y_4 \cup B).$$

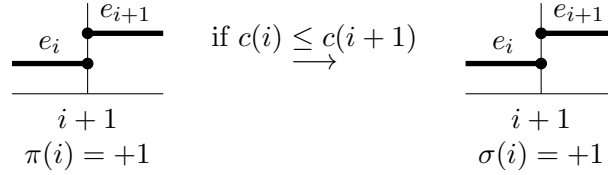
Proof. Consider how changes occur at peg $i + 1$ with respect to a colouring $c \in \text{Colors}(n+1, k)$. If $\pi_i = +1$ and $\sigma_i = -1$, i.e., $i \in Y_1$, then the colour $c(i)$ must be greater than the colour $c(i+1)$ in order for the edges incident with peg $i + 1$ to change their relative orders. So $c(i) > c(i+1)$ which means $i \in \text{Des}(c)$ and we must have $Y_1 \subseteq \text{Des}(c)$. This is illustrated in the following diagram:



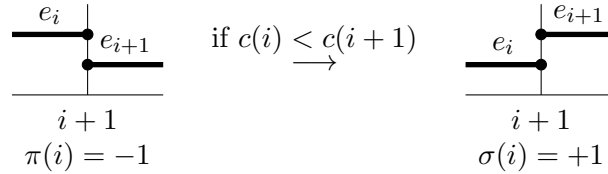
If $\pi_i = -1$ and $\sigma_i = -1$ (i.e. $i \in Y_2$) then the colour $c(i)$ must be greater than or equal to colour $c(i+1)$ in order for the edges incident with peg $i + 1$ to remain as they are. So $c(i) \geq c(i+1)$ which means $i \in \text{Des}(c) \cup \text{Equ}(c)$ and we must have $Y_2 \subseteq \text{Des}(c) \cup \text{Equ}(c)$. This is illustrated in the following diagram:



If $\pi_i = +1$ and $\sigma_i = +1$ (i.e. $i \in Y_3$) then the colour $c(i)$ must be less than or equal to colour $c(i+1)$ in order for the edges incident with peg $i+1$ to remain as they are. So $c(i) \leq c(i+1)$ which means $i \in \text{Asc}(c) \cup \text{Equ}(c)$ and we must have $Y_3 \subseteq \text{Asc}(c) \cup \text{Equ}(c)$. This is illustrated in the following diagram:



If $\pi_i = -1$ and $\sigma_i = +1$ (i.e. $i \in Y_4$) then the colour $c(i)$ must be less than colour $c(i+1)$ in order for the edges incident with peg $i+1$ to change their relative orders. So $c(i) < c(i+1)$ which means $i \in \text{Asc}(c)$ and we must have $Y_4 \subseteq \text{Asc}(c)$. This is illustrated in the following diagram:



Combining these, we see that a colouring $c \in \text{Colors}(n+1, k)$ transforms D_π to D_σ iff $Y_1(\pi, \sigma) \subseteq \text{Des}(c)$, $Y_2(\pi, \sigma) \subseteq \text{Des}(c) \cup \text{Equ}(c)$, $Y_3(\pi, \sigma) \subseteq \text{Asc}(c) \cup \text{Equ}(c)$, and $Y_4(\pi, \sigma) \subseteq \text{Asc}(c)$. These four conditions are equivalent to $\text{Des}(c) = Y_1(\pi, \sigma) \cup A$, $\text{Asc}(c) = Y_4(\pi, \sigma) \cup B$, and $\text{Equ}(c) = Y_2(\pi, \sigma) \cup Y_3(\pi, \sigma) - A - B$ for some sets A and B . \square

The quantities $\mathfrak{M}_{D_\pi, D_\sigma}^{(W(n))}(x)$ and $\mathfrak{R}_{D_\pi, D_\sigma}^{(W(n))}$ can now be calculated using the expression in Theorem 7.1.

Notice that for $D_\pi \in W(n)$ we have $Y(\pi, \pi) = (\emptyset, Y_2, \{1, \dots, n\} - Y_2, \emptyset)$.

Corollary 7.2. *Let $D_\pi \in W(n)$ with $Y(\pi, \pi) = (\emptyset, Y_2, \{1, \dots, n\} - Y_2, \emptyset)$. Then*

$$\mathfrak{R}_{D_\pi, D_\pi}^{(W(n))} = \sum_k \frac{(-1)^{k-1}}{k} \sum_{\substack{A \subseteq Y_2 \\ B \subseteq \{1, \dots, n\} - Y_2}} \text{wordeuler}_{n,k}(A, \{1, \dots, n\} - A - B, B).$$

The following lemma is needed for the proofs of the two theorems that follow it.

Lemma 7.3. $\frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} (i+1)^n = \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}.$

Proof. Expand $(i+1)^n$ on the left-hand side and swap the order of summation to get

$$\frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} (i+1)^n = \sum_{j=0}^n \binom{n}{j} \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^j = \sum_{j=0}^n \binom{n}{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\},$$

where the last identity follows from a well known expression for the Stirling numbers of the second kind (see [11, Equation 1.94a]). The last sum here is the number of ways to choose a partition of a

j -element subset of $\{1, \dots, n\}$ into k blocks, for any j . This is the same as the number of ways to partition the set $\{1, \dots, n+1\}$ into $k+1$ blocks, as the block which contains $n+1$ represents those elements that were not chosen to be in any of the k blocks. That number is therefore $\binom{n+1}{k+1}$. \square

Theorem 7.4. *For all $n \geq 1$, $\text{trace}(\mathfrak{R}^{(W(n))}) = 1$.*

Proof. Using Corollary 7.2 we have:

$$\begin{aligned}
\text{trace}(\mathfrak{R}^{(W(n))}) &= \sum_{\pi \in (\pm 1)^n} \mathfrak{R}_{D_\pi, D_\pi}^{(W(n))} \\
&= \sum_{\pi \in (\pm 1)^n} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{A \subseteq Y_2 = Y_2(\pi, \pi) \\ B \subseteq [n] - Y_2}} \text{wordeuler}_{n,k}(A, [n] - A - B, B) \\
&= \sum_{Y_2 \subseteq [n]} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{A \subseteq Y_2 \\ B \subseteq [n] - Y_2}} \text{wordeuler}_{n,k}(A, [n] - A - B, B) \\
&= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{A \subseteq [n] \\ B \subseteq [n] - A}} \sum_{Y_2: A \subseteq Y_2 \subseteq [n] - B} \text{wordeuler}_{n,k}(A, [n] - A - B, B) \\
&= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{A \subseteq [n] \\ B \subseteq [n] - A}} \text{wordeuler}_{n,k}(A, [n] - A - B, B) 2^{n-|A|-|B|}.
\end{aligned}$$

The inner sum is

$$\begin{aligned}
\sum_{\substack{A \subseteq [n] \\ B \subseteq [n] - A}} \text{wordeuler}_{n,k}(A, [n] - A - B, B) 2^{n-|A|-|B|} &= \sum_{c = \langle w, A \rangle \in \text{Colors}(n+1, k)} 2^{\text{equ}(c)} \\
&= \sum_{m=0}^n \binom{n}{n-m} 2^{n-m} |\text{Keys}(m+1, k)| \\
&= \sum_{m=0}^n \binom{n}{n-m} 2^{n-m} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i(i-1)^m \\
&= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i(i+1)^n.
\end{aligned}$$

Thus

$$\begin{aligned}
\text{trace}(\mathfrak{R}^{(W(n))}) &= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \sum_{v=1}^k (-1)^{k-v} v \binom{k}{v} (v+1)^n \\
&= - \sum_{v=1}^{n+1} (-1)^v (v+1)^n \sum_{k=v}^{n+1} \binom{k-1}{v-1} \\
&= - \sum_{v=1}^{n+1} (-1)^v (v+1)^n \binom{n+1}{v} \\
&= 1,
\end{aligned}$$

by using Lemma 7.3 with $k = n+1$. \square

Theorem 7.5. For all $n \geq 1$, $\text{trace}(\mathfrak{M}^{(W(n))}(x)) = \sum_{k=1}^{n+1} x^k k! \left(\left\{ \begin{matrix} n+2 \\ k+1 \end{matrix} \right\} - \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \right)$.

Proof. As in the proof of Theorem 7.4,

$$\begin{aligned} \text{trace}(\mathfrak{M}^{(W(n))}(x)) &= \sum_{\pi \in (\pm 1)^n} \mathfrak{M}_{D_\pi, D_\pi}^{(W(n))}(x) \\ &= \sum_{k=1}^{n+1} x^k \sum_{\substack{A \subseteq [n] \\ B \subseteq [n]-A}} \text{wordeuler}_{n,k}(A, [n] - A - B, B) 2^{n-|A|-|B|}. \end{aligned}$$

The inner sum may be written

$$\begin{aligned} &\sum_{\substack{A \subseteq [n] \\ B \subseteq [n]-A}} \text{wordeuler}_{n,k}(A, [n] - A - B, B) 2^{n-|A|-|B|} \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i(i+1)^n \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} (i+1)^{n+1} - \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} (i+1)^n \\ &= k! \left(\left\{ \begin{matrix} n+2 \\ k+1 \end{matrix} \right\} - \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \right), \end{aligned}$$

where the last identity follows from Lemma 7.3. Thus

$$\text{trace}(\mathfrak{M}^{(W(n))}(x)) = \sum_{k=1}^{n+1} x^k k! \left(\left\{ \begin{matrix} n+2 \\ k+1 \end{matrix} \right\} - \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \right). \quad \square$$

8. CASE 3: A WEB WORLD WITH $\text{Pegs}(D) = (2, 2, \dots, 2)$

In this section we consider a variant of the case in the previous section. This variant consists of inserting one more edge from peg 1 to peg n . We will encode configurations as we did in the previous section. Fix $n \in \mathbb{N}$ and let $W(n)$ be the collection of all web diagrams

$$D = \{e_i = (i, i+1, x_i, y_{i+1}) : 1 \leq i < n\} \cup \{e_n = (1, n, y_1, x_n)\} \quad (10)$$

where $\{x_i, y_i\} = \{1, 2\}$ for all $1 \leq i \leq n$. So $W(n)$ is the collection of web diagrams on n pegs, consisting of n edges whose endpoints are on adjacent pegs, and with an edge between peg 1 and peg n . Every such diagram can be encoded by a sequence $\pi = (\pi(1), \dots, \pi(n)) \in (\pm 1)^n$ where $\pi(i) = x_i - y_i$ for all $1 \leq i \leq n$.

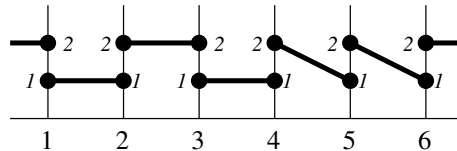


FIGURE 7. An example of a web diagram in $W(6)$ which is of the form given in Equation (10). The edge $e_6 = (1, 6, 2, 2)$ from peg 1 to peg 6 is illustrated in two parts on either extremities of the diagram. This diagram is encoded by the sequence $\pi = (-1, +1, -1, +1, +1, +1)$.

Given $c \in \text{Colors}(n, k)$ and $D_\pi \in W(n)$, let $\text{Recon}(D_\pi, c)$ be the web diagram that results by colouring edge e_i with colour $c(i)$, for all i , and reconstructing. Given $c \in \text{Colors}(n, k)$, let $\text{DEA}^*(c) = (\text{Des}^*(c), \text{Equ}^*(c), \text{Asc}^*(c))$ where

$$\begin{aligned}\text{Des}^*(c) &= \{1 \leq i \leq n : c(i) > c(i+1) \text{ with } c(n+1) := c(1)\}, \\ \text{Equ}^*(c) &= \{1 \leq i \leq n : c(i) = c(i+1) \text{ with } c(n+1) := c(1)\}, \text{ and} \\ \text{Asc}^*(c) &= \{1 \leq i \leq n : c(i) < c(i+1) \text{ with } c(n+1) := c(1)\}.\end{aligned}$$

Let $\text{des}^*(c)$, $\text{equ}^*(c)$ and $\text{asc}^*(c)$ be the cardinalities of the sets $\text{Des}^*(c)$, $\text{Equ}^*(c)$ and $\text{Asc}^*(c)$, respectively. Let $\text{WordEuler}_{n,k}^*(X_1, X_2, X_3) = \{c \in \text{Colors}(n, k) : \text{DEA}^*(c) = (X_1, X_2, X_3)\}$, the set of colourings of $\text{Colors}(n, k)$ whose descent, plateau and ascents sets are given by X_1 , X_2 and X_3 , respectively. Set $\text{wordeuler}_{n,k}^*(X_1, X_2, X_3) = |\text{WordEuler}_{n,k}^*(X_1, X_2, X_3)|$.

Given diagrams $D_\pi, D_\sigma \in W(n)$, let $Y(\pi, \sigma) = (Y_1(\pi, \sigma), Y_2(\pi, \sigma), Y_3(\pi, \sigma), Y_4(\pi, \sigma))$ be an ordered partition of $\{1, \dots, n\}$ where the Y 's are as defined in Case 2.

Theorem 8.1. *Let $k \in \mathbb{N}$. Let $D_\pi, D_\sigma \in W(n)$ and suppose that $Y(\pi, \sigma) = (Y_1, Y_2, Y_3, Y_4)$. Then*

$$\begin{aligned}F(D_\pi, D_\sigma, k) &= \bigcup_{\substack{A \subseteq Y_2 \\ B \subseteq Y_3}} \text{WordEuler}_{n,k}^*(Y_1 \cup A, Y_2 \cup Y_3 - (A \cup B), Y_4 \cup B) \\ f(D_\pi, D_\sigma, k) &= \sum_{\substack{A \subseteq Y_2 \\ B \subseteq Y_3}} \text{wordeuler}_{n,k}^*(Y_1 \cup A, Y_2 \cup Y_3 - (A \cup B), Y_4 \cup B).\end{aligned}$$

We omit the proof of Theorem 8.1 because it is almost identical to the proof of Theorem 7.1. The only places in which they differ is in taking account of the colouring of the solitary edge between peg 1 and peg n .

Theorem 8.2. *For all $n \geq 1$, $\text{trace}(\mathfrak{R}^{(W(n))}) = n + 1$.*

$$\begin{aligned}\text{Proof. } \text{trace}(\mathfrak{R}^{(W(n))}) &= \sum_{\pi \in (\pm 1)^n} \mathfrak{R}_{D_\pi, D_\pi}^{(W(n))} \\ &= \sum_{\pi \in (\pm 1)^n} \sum_k \frac{(-1)^{k-1}}{k} \sum_{\substack{A \subseteq Y_2 = Y_2(\pi, \pi) \\ B \subseteq [n] - Y_2}} \text{wordeuler}_{n,k}^*(A, [n] - A - B, B) 2^{n-|A|-|B|}.\end{aligned}$$

The inner sum is

$$\begin{aligned}\sum_{\substack{A \subseteq Y_2(\pi, \pi) \\ B \subseteq [n] - Y_2(\pi, \pi)}} \text{wordeuler}_{n,k}^*(A, [n] - A - B, B) 2^{n-|A|-|B|} &= \sum_{c \in \text{Colors}(n, k)} 2^{\text{equ}^*(c)} \\ &= \sum_{\substack{c \in \text{Colors}(n, k) \\ c(1) \neq c(n)}} 2^{\text{equ}(c)} + \sum_{\substack{c \in \text{Colors}(n, k) \\ c(1) = c(n)}} 2^{1+\text{equ}(c)}.\end{aligned}$$

The first sum is

$$\begin{aligned}\sum_{\substack{c \in \text{Colors}(n, k) \\ c(1) \leq c(n)}} 2^{\text{equ}(c)} &= \sum_{m=k}^n \sum_{w \in \text{Keys}^\neq(m, k)} 2^{n-m} \binom{n-1}{m-1} \\ &= \sum_{m=k}^n 2^{n-m} \binom{n-1}{m-1} |\text{Keys}^\neq(m, k)|,\end{aligned}$$

and the second sum is

$$\begin{aligned} \sum_{\substack{c \in \text{Colors}(n,k) \\ c(1)=c(n)}} 2^{1+\text{equ}(c)} &= \sum_m \sum_{w \in \text{Keys}^=(m,k)} 2^{1+n-m} \binom{n-1}{m-1} \\ &= \sum_m 2^{1+n-m} \binom{n-1}{m-1} |\text{Keys}^=(m,k)|. \end{aligned}$$

Adding the two equations above, and using the expressions given in Lemma 5.1, we have

$$\sum_{\substack{A \subseteq Y_2(\pi, \pi) \\ B \subseteq [n] - Y_2(\pi, \pi)}} \text{wordeuler}_{n,k}^*(A, [n] - A - B, B) 2^{n-|A|-|B|} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} ((i+1)^n + (i-1)).$$

Replacing this into the equation for the trace

$$\begin{aligned} \text{trace}(\mathfrak{R}^{(W(n))}) &= \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} ((i+1)^n + (i-1)) \\ &= \sum_{i=1}^n \frac{(-1)^{i+1}}{i} \binom{n}{i} ((i+1)^n + i - 1). \end{aligned}$$

Since $((i+1)^n + i - 1)/i = 1 + \sum_{j=1}^n \binom{n}{j} i^{j-1}$,

$$\begin{aligned} \text{trace}(\mathfrak{R}^{(W(n))}) &= \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} + \sum_{i,j=1}^n (-1)^{i+1} \binom{n}{i} \binom{n}{j} i^{j-1} \\ &= 1 + \sum_{j=1}^n \binom{n}{j} \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} i^{j-1} \\ &= 1 + \sum_{j=1}^n \binom{n}{j} \left(-0^{j-1} (-1)^n + \sum_{i'=0}^n (-1)^{i'} \binom{n}{i'} (n-i')^{j-1} \right) \\ &= 1 + \sum_{j=1}^n \binom{n}{j} \left(-0^{j-1} (-1)^n + n! \left\{ \begin{matrix} j-1 \\ n \end{matrix} \right\} \right). \end{aligned}$$

As $j-1 < n$, the Stirling number $\left\{ \begin{matrix} j-1 \\ n \end{matrix} \right\}$ is 0. The other term in the parentheses is also zero except when $j=1$. Thus

$$\text{trace}(\mathfrak{R}^{(W(n))}) = 1 + \binom{n}{1} (-1)^{n+1} (-1(-1)^n) = 1 + n. \quad \square$$

Theorem 8.3. For all $n \geq 1$, $\text{trace}(\mathfrak{M}^{(W(n))}(x)) = x + \sum_{k=1}^{n+1} x^k k! \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}$.

Proof. As in the proof of the previous theorem,

$$\begin{aligned}
\text{trace}(\mathfrak{M}^{(W(n))}(x)) &= \sum_{\pi \in (\pm 1)^n} \mathfrak{M}_{D_\pi, D_\pi}^{(W(n))}(x) \\
&= \sum_{k=1}^{n+1} x^k \sum_{\substack{A \subseteq [n] \\ B \subseteq [n]-A}} \text{wordeuler}_{n,k}^*(A, [n] - A - B, B) 2^{n-|A|-|B|}. \\
&= \sum_{k=1}^{n+1} x^k \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} ((i+1)^n + (i-1)^n) \\
&= x + \sum_{k=1}^{n+1} x^k k! \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\},
\end{aligned}$$

by Lemma 7.3. □

9. TRANSITIVE WEB WORLDS

In this section we consider a special class of web worlds that we term *transitive web worlds*. The defining property of these web worlds is the simple condition (in terms of the web diagram) that every peg (other than the first and last) is the right endpoint of at least one edge and the left endpoint of at least one edge.

Definition 9.1. Let W be a web world and $D \in W$ with $\text{PegSet}(D) = \{1, \dots, n\}$. We will call W *transitive* if each of the following hold;

- (i) $(1, k) \in \text{PegpairsSet}(D)$ for some $1 < k \leq n$.
- (ii) $(k, n) \in \text{PegpairsSet}(D)$ for some $1 \leq k < n$.
- (iii) For all k with $1 < k < n$ there exists z and z' such that $1 \leq z < k < z' \leq n$ such that (z, k) and (k, z') are both in $\text{PegpairsSet}(D)$.

The condition for W to be transitive is equivalent to $\text{Represent}(W)$ being such that the only rows and columns of all zeros are the leftmost column and the bottom row.

Example 9.2. Consider all web worlds with exactly 3 edges. There are 30 of these. Only 5 are transitive, represented by the following matrices:

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices which represent transitive web worlds are in one-to-one correspondence with the upper triangular matrices introduced in [6] (to go from the former to the latter remove the leftmost column and the bottom row). A striking property of the latter class of matrices is their rich connection to several other combinatorial objects, namely $(2+2)$ -free partially ordered sets, pattern avoiding permutations, Stoimenow matchings, and sequences of integers called ascent sequences [1, 2, 4].

Cases 2 and 3, in Sections 7 and 8, are examples of transitive web worlds. Results concerning the above structures are readily applicable to transitive web worlds. For example, the generating function for the number of transitive web worlds according to the number of edges, number of pegs, and number of distinct pairs of pegs connected by edges, is given in [5, Theorem 11].

10. FURTHER COMMENTS AND OPEN PROBLEMS

Results presented in this paper will be explained in a physics context in the forthcoming paper [3]. There are many questions outstanding in understanding the properties of these new structures. We list here a small collection of these questions, each of whose answers would represent a significant step forward in this new area. Let W be a web world and let D and E be web diagrams in W .

- (1) In Theorem 3.4, in order to apply this theorem we must have that the web diagram D , when written as a sum of indecomposable web diagrams $D = E_1 \oplus \dots \oplus E_k$, is such that all of its constituent indecomposable web diagrams (E_1, \dots, E_k) are different. How can this theorem be extended to the case for when there are repeated constituent web diagrams. The simplest such to analyse is the web diagram $D = \{(1, 2, 1, 1), (1, 2, 2, 2)\} = \{(1, 2, 1, 1)\} \oplus \{(1, 2, 1, 1)\}$.
- (2) Can the off-diagonal terms $\mathfrak{M}_{D,E}^{(W)}(x)$ and $\mathfrak{R}_{D,E}^{(W)}$ be calculated in terms of mappings between the posets $P(D)$ and $P(E)$?
- (3) Find a direct way to perform the calculations in Equations (4) and (5) using $\text{Represent}(W)$.
- (4) Can the web worlds W with $\text{trace}(\mathfrak{R}^{(W)}) = 1$ be characterized?
- (5) Can one find other exactly solvable examples of web worlds than those presented here? By exactly solvable we mean that it is possible to give an expression for every entry of the matrices $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$.
- (6) Can the relationship between the representing matrices $\text{Represent}(W)$ for transitive web diagrams and $(2+2)$ -free posets (or other combinatorial objects in bijection with these) be exploited in order to simplify calculations related to the trace?
- (7) From the physics perspective (see [7]) the main question is the determination of the left eigenvectors of the mixing matrix \mathfrak{R} . The first step in this direction is to determine how many such eigenvectors have eigenvalue 1, which is the number of independent contributions to the exponent of the Eikonal amplitude from the given web world. It is given by the rank of \mathfrak{R} (which equals the trace of \mathfrak{R}). This task was achieved here for certain classes of webs. The natural question is what is the general formula for this invariant given the entries of $\text{Represent}(W)$.

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