

Dual models of the neutrino mass spectrum

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Abstract

We show that any model with a homogeneous relationship among elements of the neutrino mass matrix with one mass hierarchy yields predictions for the oscillation parameters and Majorana phases similar to those given by a model with the same homogeneous relationship among cofactors of the neutrino mass matrix with the opposite mass hierarchy, except when the lightest mass is of order 20 meV or less.

The study of neutrino physics has greatly progressed in the last twenty years. On the one hand, there have been many experimental discoveries and breakthroughs that have culminated in determinations of the two mass-squared differences between the three neutrinos and the three neutrino mixing angles. On the theoretical side, neutrino models have been built to explain the experimental results and provide guidance for the next generation of experiments.

Many models predict a relationship among the elements of the light neutrino mass matrix, while other models predict the same relationship among the cofactors of the light neutrino mass matrix. In Refs. [1, 2], we found that there are strong similarities between single texture zero models with one mass hierarchy and single cofactor zero models with the opposite mass hierarchy if the lightest mass in each case is not too small. This curious feature was also discussed in Ref. [3]. The phenomenon is not unique – models with two equalities between mass matrix elements are similar to models with two equalities between cofactors with the opposite mass hierarchy, as noted in Ref. [4]. We find [5] this similarity to also exist between models with two texture zeros in the light neutrino mass matrix [6] and models with two cofactor zeros in the light neutrino mass matrix [7].

In this article we generalize this correspondence by showing that any model with a homogeneous relationship among elements of the light neutrino mass matrix with one mass hierarchy predicts oscillation parameters and Majorana phases similar to those of models with the same homogeneous relationship among cofactors of the mass matrix with the opposite mass hierarchy. Since the neutrino mass hierarchy remains undetermined, two such models are indistinguishable using current data. The allowed oscillation parameters are nearly identical when the masses are quasi-degenerate, but can differ in some cases when the lightest neutrino mass is of order 20 meV or less.

Comparison between element and cofactor models. The neutrino mass matrix can be written as

$$M = V^* \text{diag}(m_1, m_2, m_3) V^\dagger, \quad (1)$$

where $V = U \text{diag}(1, e^{i\phi_2/2}, e^{i\phi_3/2})$ is unitary and [8]

$$U = \begin{bmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix}. \quad (2)$$

Either a normal mass hierarchy (NH, $m_1 < m_2 < m_3$) or an inverted mass hierarchy (IH, $m_3 < m_1 < m_2$) are allowed, following the convention that the mass-squared difference $m_2^2 - m_1^2$ is responsible for the oscillation of solar neutrinos.

Suppose a model imposes a relationship among *elements* of the mass matrix M given by

$$f(M_{\alpha\beta}) = 0, \quad (3)$$

where $\alpha, \beta = e, \mu, \tau$ and f is a homogeneous function of the $M_{\alpha\beta}$. We take the coefficients in the homogeneous function to be real, as in most models. Then from Eq. (1), $M_{\alpha\beta} = m_1 V_{\alpha 1}^* V_{\beta 1} + m_2 V_{\alpha 2}^* V_{\beta 2} + m_3 V_{\alpha 3}^* V_{\beta 3}$ and Eq. (3) becomes

$$f(m_1 V_{\alpha 1}^* V_{\beta 1} + m_2 V_{\alpha 2}^* V_{\beta 2} + m_3 V_{\alpha 3}^* V_{\beta 3}) = 0. \quad (4)$$

Since the coefficients in the homogeneous function are real, the complex conjugate of the above equation is

$$f(m_1 V_{\alpha 1} V_{\beta 1} + m_2 V_{\alpha 2} V_{\beta 2} + m_3 V_{\alpha 3} V_{\beta 3}) = 0 \quad (\text{element condition}). \quad (5)$$

Now consider a model that imposes the same homogeneous relationship among *cofactors* of the light neutrino mass matrix, i.e.,

$$f(C_{\alpha\beta}) = 0, \quad (6)$$

where $C_{\alpha\beta}$ is the (α, β) cofactor of M , given by $(M^{-1})_{\alpha\beta} = \frac{1}{\det M} C_{\beta\alpha}$. Since the mass matrix is symmetric and f is a homogeneous function, we have $f((M^{-1})_{\alpha\beta}) = 0$. Then since $M^{-1} = V \text{diag}(m_1^{-1}, m_2^{-1}, m_3^{-1}) V^T$, we can write the condition as

$$f(m_1^{-1} V_{\alpha 1} V_{\beta 1} + m_2^{-1} V_{\alpha 2} V_{\beta 2} + m_3^{-1} V_{\alpha 3} V_{\beta 3}) = 0 \quad (\text{cofactor condition}). \quad (7)$$

To compare the element NH case with the cofactor IH case, we divide the argument in Eq. (5) by m_3 , multiply the argument in Eq. (7) by m_3 , and use the properties of homogeneous

functions to write the condition for the element NH case as

$$f\left(\frac{m_1}{m_3}V_{\alpha 1}V_{\beta 1} + \frac{m_2}{m_3}V_{\alpha 2}V_{\beta 2} + V_{\alpha 3}V_{\beta 3}\right) = 0, \quad (8)$$

and the condition for the cofactor IH case as

$$f\left(\frac{m_3}{m_1}V_{\alpha 1}V_{\beta 1} + \frac{m_3}{m_2}V_{\alpha 2}V_{\beta 2} + V_{\alpha 3}V_{\beta 3}\right) = 0. \quad (9)$$

In the quasi-degenerate regime ($m_1 \simeq m_2 \simeq m_3$), all the mass ratios are approximately unity so that the three mixing angles and three phases allowed by the constraints are nearly identical for the two hierarchies.

For masses lighter than about 100 meV, the leading term in each argument is the third term, and they are identical. The only differences in the sub-leading terms are the two mass ratios. In the figure we plot the fractional difference between the two mass ratios with opposite hierarchies versus the lightest mass using the recent best-fit values [9], $\delta m^2 \equiv m_2^2 - m_1^2 = 7.54 \times 10^{-5} \text{ eV}^2$ and $\Delta m^2 \equiv |m_3^2 - (m_1^2 + m_2^2)/2| = 2.43 \times 10^{-3} \text{ eV}^2$ for the normal hierarchy and $2.42 \times 10^{-3} \text{ eV}^2$ for the inverted hierarchy. The percentage difference between $(\frac{m_1}{m_3})_{\text{NH}}$ and $(\frac{m_3}{m_1})_{\text{IH}}$ is very small and always less than 1.7% for any value of the lightest mass. The percentage difference between $(\frac{m_2}{m_3})_{\text{NH}}$ and $(\frac{m_3}{m_2})_{\text{IH}}$ becomes less than 10% (5%) {2%} if the lightest mass is larger than 19 (27) {42} meV. Hence except for conditions with α and β such that $V_{\alpha 3}V_{\beta 3}$ is small compared to $V_{\alpha 1}V_{\beta 1}$ and $V_{\alpha 2}V_{\beta 2}$, the two conditions are almost the same for masses that are not nearly degenerate. Even in some extreme cases, such as $\alpha = \beta = e$, for which the θ_{13} -dependent leading term is relatively small, the two conditions are almost identical if the lightest mass is larger than about 20 meV, with the percentage difference between the two mass ratios less than 10%.

To compare the element IH case with the cofactor NH case, we divide the argument in Eq. (5) by m_1 and multiply the argument in Eq. (7) by m_1 to obtain

$$f\left(V_{\alpha 1}V_{\beta 1} + \frac{m_2}{m_1}V_{\alpha 2}V_{\beta 2} + \frac{m_3}{m_1}V_{\alpha 3}V_{\beta 3}\right) = 0, \quad (10)$$

for the element IH case and

$$f\left(V_{\alpha 1}V_{\beta 1} + \frac{m_1}{m_2}V_{\alpha 2}V_{\beta 2} + \frac{m_1}{m_3}V_{\alpha 3}V_{\beta 3}\right) = 0, \quad (11)$$

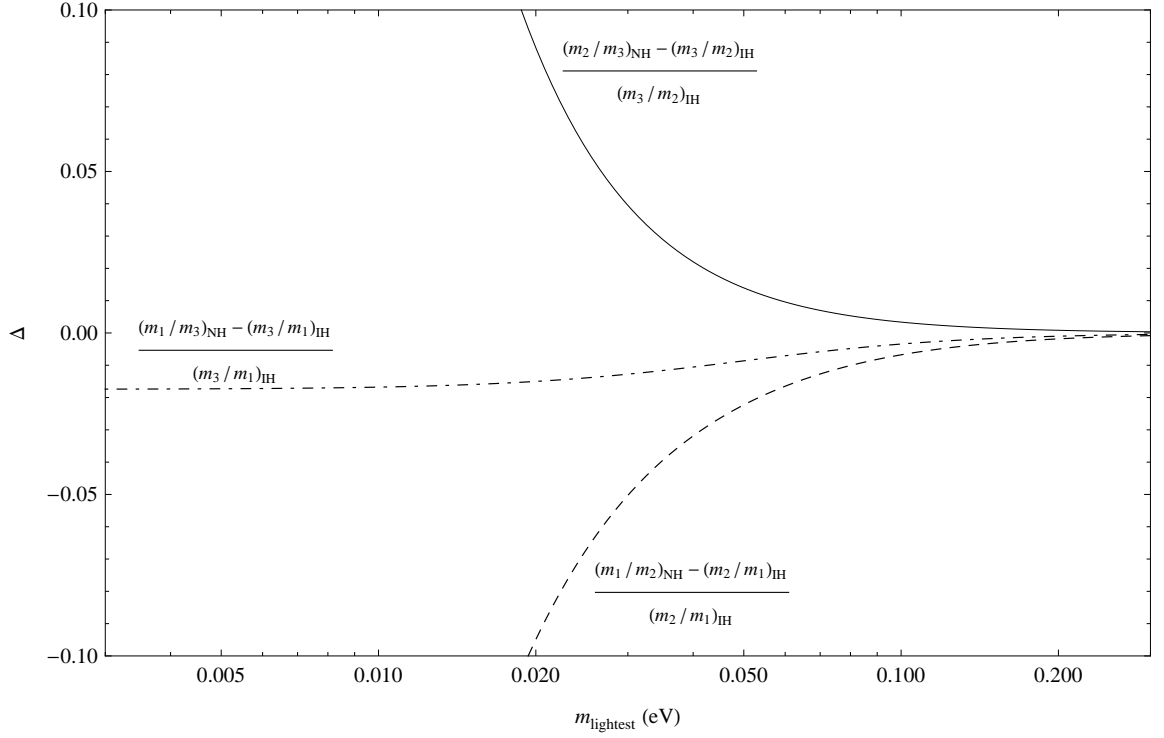


Figure 1: Fractional differences in mass ratios Δ for the two mass hierarchies as a function of the lightest neutrino mass. We set $\delta m^2 \equiv m_2^2 - m_1^2 = 7.54 \times 10^{-5} \text{ eV}^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2 = 2.43 \times 10^{-3} \text{ eV}^2$ for the normal hierarchy and $2.42 \times 10^{-3} \text{ eV}^2$ for the inverted hierarchy.

for the cofactor NH case. Again, in the quasi-degenerate regime the two conditions are nearly identical, and the allowed values of the mixing angles and phases are almost equal in the two models.

For masses lighter than about 100 meV, the leading terms in each argument are the first two terms if the lightest mass is not very small. The percentage difference between $(\frac{m_2}{m_1})_{\text{IH}}$ and $(\frac{m_1}{m_2})_{\text{NH}}$ is less than 10% (5%) {2%} if the lightest mass is larger than 19 (30) {53} meV. Hence, if the lightest mass is not very small, the two conditions are almost identical. The sub-leading terms are always close to each other because the percentage difference between $(\frac{m_1}{m_3})_{\text{NH}}$ and $(\frac{m_3}{m_1})_{\text{IH}}$ is always less than 1.7% for any value of the lightest mass.

In the above analysis we only considered real coefficients in the homogeneous functions. For complex coefficients, the complex conjugate of Eq. (4) does not give Eq. (5). However, if the cofactor-based model has coefficients that are the complex conjugate of the coefficients in the element-based model, then the cofactor-based model is dual to the corresponding element-based model.

Although our proof used only one condition, the same arguments can easily be applied to multiple conditions. The only requirement is that there be two models with the same homogeneous conditions for the elements and cofactors, respectively. A consequence is that a model with two texture zeros yields predictions for the oscillation parameters and Majorana phases similar to those of the corresponding model with two cofactor zeros. Likewise, as noted in Ref. [4], models with two equalities between mass matrix elements are similar to models with two equalities between cofactors.

Application to neutrino model building. The homogeneous relationships in Eqs. (3) and (6) are quite common in neutrino mass models, such as texture zero models [3, 6, 10], cofactor zero models [7, 11], scaling models [12], and models in which two mass matrix elements or cofactors are equal [4]. The latter includes the $\mu - \tau$ symmetric models that impose $|M_{e\mu}| = |M_{e\tau}|$ and $|M_{\mu\mu}| = |M_{\tau\tau}|$. However, the existence of an element/cofactor duality requires models that have the same homogeneous relationship among elements in one model and cofactors in a second model. While models with conditions on the elements are common, models with conditions on cofactors are not so common. However, models with the same homogeneous relationships among cofactors can be defined. In particular, the

existence of the inverse of the right-handed neutrino mass matrix in the conventional seesaw mechanism [13], with $M = M_D^T M_R^{-1} M_D$, provides a good motivation for the corresponding cofactor models, as we discuss below.

M_D is proportional to the unit matrix. A simple example arises when $M_D = m_D I$, so that inverting the seesaw formula gives $M^{-1} = M_R/m_D^2$. Since $M^{-1} = C^T/\text{Det}(M)$, it follows that $M_R \propto C^T$. Now since M_R is symmetric, any homogeneous relationship among the elements of the right-handed neutrino mass matrix M_R will be equivalent to the same homogeneous relationship among the cofactors of the light neutrino mass matrix. Thus a dual cofactor model can be obtained by having the same homogeneous conditions on M_R in one model as there are on M in the dual element model; the cofactor conditions on M are inherited from M_R .

This leads to an even more ambitious conclusion: any model consistent with the observed mixing angles (and phases) for the light neutrinos will have a dual model with the opposite mass hierarchy. We can build the dual model by choosing M_R to be proportional to M , and according to our argument above (with M_D proportional to the unit matrix), the model generated by M_R would have a light neutrino cofactor matrix that is proportional to M . Thus the cofactors are related to each other in the same way the elements of M are related to each other, so the model generated by M_R with the opposite mass hierarchy will be dual to the model represented by M , and we cannot distinguish these two models without knowing the mass hierarchy.

M_D is diagonal. Next we relax the condition that the Dirac mass matrix is proportional to the unit matrix and consider the case where it is diagonal. If we assume that the same homogeneous relationship holds for both the light neutrino mass matrix and the right-handed neutrino mass matrix, under what conditions will the cofactor matrix of M have the same homogeneous relationship as the elements of M ?

Defining $M_D = \text{diag}(c_1, c_2, c_3)$ and $(M_R)_{ij} = R_{ij}$, since M_R is symmetric, the cofactor matrix for M becomes

$$C = (\text{Det}M)M_D^{-1}M_R^T(M_D^T)^{-1} = (\text{Det}M) \begin{bmatrix} R_{11}/c_1^2 & R_{12}/c_1c_2 & R_{13}/c_1c_3 \\ R_{12}/c_1c_2 & R_{22}/c_2^2 & R_{23}/c_2c_3 \\ R_{13}/c_1c_3 & R_{23}/c_2c_3 & R_{33}/c_3^2 \end{bmatrix}. \quad (12)$$

We see that a texture zero in M_R still translates to a cofactor zero for M [2, 14]. However, in general a more complicated homogeneous relationship among elements in M_R , such as those involving more than one element, will not be inherited by the corresponding cofactor matrix unless there is a special relationship among the c_i . For example, $R_{\mu\mu} = R_{\tau\tau}$ does not imply $C_{\mu\mu} = C_{\tau\tau}$ unless $c_2 = c_3$.

Conclusion. We have shown that if a model has a homogeneous relationship among *elements* of the light neutrino mass matrix, it will yield predictions for the oscillation parameters and Majorana phases similar to those of another model with the opposite mass hierarchy that has the same homogeneous relationship among *cofactors* of the mass matrix, except when the lightest neutrino mass is of order 20 meV or less. Many existing models have one or more homogeneous relationships among mass matrix elements, but there are fewer models that are constructed by imposing homogeneous relationships among cofactors. However, any model that fits current neutrino data will have a dual model with the opposite mass hierarchy. We have shown that if the Dirac mass matrix is proportional to the identity matrix, a dual cofactor-based model can be generated via the seesaw mechanism if the right-handed neutrino mass matrix has the same homogeneous relationships as the light mass matrix elements in an element-based model. Since the mass hierarchy has not been experimentally determined, we cannot currently distinguish these dual models from each other.

Current global fits to oscillation data have almost identical best-fit values for the neutrino mixing angles and mass-squared differences for the two hierarchies. However, different allowed regions for the oscillation parameters for different mass hierarchies can lead to a breakdown of duality. In fact, the 2σ allowed regions are somewhat different, especially for the value of θ_{23} , where second octant values ($\theta_{23} > \pi/4$) are allowed only for the inverted hierarchy [9]. Due to this difference, the 2σ allowed regions for dual models differ even in the quasi-degenerate regime in a few cases we have studied.

The dual model ambiguity can be resolved by experiments that distinguish between the normal and inverted hierarchies, such as long baseline neutrino experiments (T2K [15], NO ν A [16], and LBNE [17]), atmospheric neutrino experiments (PINGU [18], and INO [19]) and medium baseline reactor experiments (Daya Bay II/JUNO [20, 21]), or a combination

of these [22]. Also, tritium beta decay, neutrinoless double beta decay ($0\nu\beta\beta$), and structure formation in our universe depend on the nature of the neutrino mass spectrum, and in principle can be used to distinguish between dual models. The 95% C.L. sensitivity of the KATRIN experiment [23] to the effective neutrino mass $m_\beta = (\sum_i |V_{ei}|^2 m_i^2)^{1/2}$ is 0.35 eV with an uncertainty of 0.08 eV^2 on m_β^2 , which is insufficient to break the duality. The effective Majorana mass measured by $0\nu\beta\beta$ experiments is constrained to be smaller than 140-380 meV at the 90% C.L. [24], which cannot break the duality. The future sensitivity of $0\nu\beta\beta$ experiments is expected to be 50 meV or lower [25], which would provide a partial but strongly model-dependent resolution of the dual model ambiguity [1, 2]. The current 95% C.L. upper bound on Σm_i from cosmology is 0.66 eV [26], which permits a quasi-degenerate spectrum, so that the duality is unbroken. In the future, lensing measurements will probe Σm_i as low as 0.05 eV [27], which will help distinguish between dual models.

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References

- [1] J. Liao, D. Marfatia and K. Whisnant, Phys. Rev. D **87**, 073013 (2013) [arXiv:1302.2372 [hep-ph]].
- [2] J. Liao, D. Marfatia and K. Whisnant, Phys. Rev. D **88**, 033011 (2013) [arXiv:1306.4659 [hep-ph]].
- [3] E. I. Lashin and N. Chamoun, Phys. Rev. D **85**, 113011 (2012) [arXiv:1108.4010 [hep-ph]].

- [4] S. Dev, R. R. Gautam and L. Singh, Phys. Rev. D **87**, 073011 (2013) [arXiv:1303.3092 [hep-ph]].
- [5] J. Liao, D. Marfatia and K. Whisnant, arXiv:1311.2639 [hep-ph].
- [6] P. H. Frampton, S. L. Glashow and D. Marfatia, Phys. Lett. B **536**, 79 (2002) [hep-ph/0201008]; Z.-z. Xing, Phys. Lett. B **530**, 159 (2002) [hep-ph/0201151]; Z.-z. Xing, Phys. Lett. B **539**, 85 (2002) [hep-ph/0205032]; H. Fritzsch, Z.-z. Xing and S. Zhou, JHEP **1109**, 083 (2011) [arXiv:1108.4534 [hep-ph]].
- [7] L. Lavoura, Phys. Lett. B **609**, 317 (2005) [hep-ph/0411232]; E. I. Lashin and N. Chamoun, Phys. Rev. D **78**, 073002 (2008) [arXiv:0708.2423 [hep-ph]]; S. Dev, S. Gupta, R. R. Gautam and L. Singh, Phys. Lett. B **706**, 168 (2011) [arXiv:1111.1300 [hep-ph]].
- [8] J. Beringer *et al.* [Particle Data Group Collaboration], Phys. Rev. D **86**, 010001 (2012).
- [9] G. L. Fogli, E. Lisi, A. Marrone, D. Montanino, A. Palazzo and A. M. Rotunno, Phys. Rev. D **86**, 013012 (2012) [arXiv:1205.5254 [hep-ph]].
- [10] Z.-z. Xing, Phys. Rev. D **69**, 013006 (2004) [hep-ph/0307007]; A. Merle and W. Rodejohann, Phys. Rev. D **73**, 073012 (2006) [hep-ph/0603111].
- [11] E. I. Lashin and N. Chamoun, Phys. Rev. D **80**, 093004 (2009) [arXiv:0909.2669 [hep-ph]]; S. Dev, S. Verma, S. Gupta and R. R. Gautam, Phys. Rev. D **81**, 053010 (2010) [arXiv:1003.1006 [hep-ph]].
- [12] R. N. Mohapatra and W. Rodejohann, Phys. Lett. B **644**, 59 (2007) [hep-ph/0608111]; A. Blum, R. N. Mohapatra and W. Rodejohann, Phys. Rev. D **76**, 053003 (2007) [arXiv:0706.3801 [hep-ph]]; A. S. Joshipura and W. Rodejohann, Phys. Lett. B **678**, 276 (2009) [arXiv:0905.2126 [hep-ph]]; B. Adhikary, M. Chakraborty and A. Ghosal, Phys. Rev. D **86**, 013015 (2012) [arXiv:1205.1355 [hep-ph]]; M. Yasue, Phys. Rev. D **86**, 116011 (2012) [arXiv:1210.7448 [hep-ph]].

- [13] P. Minkowski, Phys. Lett. B **67** (1977) 421; T. Yanagida, in *Proceedings of the Workshop on the Unified Theory and the Baryon Number in the Universe*, eds. O. Sawada et al., (KEK Report 79-18, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond and R. Slansky, in *Supergravity*, eds. P. van Nieuwenhuizen et al., (North-Holland, 1979), p. 315; S.L. Glashow, in *Quarks and Leptons*, Cargèse, eds. M. L évy et al., (Plenum, 1980), p. 707; R. N. Mohapatra and G. Senjanović, Phys. Rev. Lett. **44** (1980) 912.
- [14] E. Ma, Phys. Rev. D **71**, 111301 (2005) [hep-ph/0501056].
- [15] K. Abe *et al.* [T2K Collaboration], Nucl. Instrum. Meth. A **659**, 106 (2011) [arXiv:1106.1238 [physics.ins-det]].
- [16] R. B. Patterson [NOvA Collaboration], Nucl. Phys. Proc. Suppl. **235-236**, 151 (2013) [arXiv:1209.0716 [hep-ex]].
- [17] T. Akiri *et al.* [LBNE Collaboration], arXiv:1110.6249 [hep-ex].
- [18] PINGU Collaboration, arXiv:1306.5846 [astro-ph.IM].
- [19] N. K. Mondal [INO Collaboration], Pramana **79**, 1003 (2012).
- [20] Y. F. Wang. talk given in NuFact12, Williamsburg, VA, USA (2012).
<https://www.jlab.org/indico/materialDisplay.py?contribId=16>
- [21] S. Kettell, J. Ling, X. Qian, M. Yeh, C. Zhang, C. -J. Lin, K. -B. Luk and R. Johnson *et al.*, arXiv:1307.7419 [hep-ex].
- [22] R. N. Cahn, D. A. Dwyer, S. J. Freedman, W. C. Haxton, R. W. Kadel, Y. .G. Kolomen-sky, K. B. Luk and P. McDonald *et al.*, arXiv:1307.5487 [hep-ex].
- [23] A. Osipowicz *et al.* [KATRIN Collaboration], hep-ex/0109033.
- [24] M. Auger *et al.* [EXO Collaboration], Phys. Rev. Lett. **109**, 032505 (2012) [arXiv:1205.5608 [hep-ex]].
- [25] W. Rodejohann, J. Phys. G **39**, 124008 (2012) [arXiv:1206.2560 [hep-ph]].

- [26] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1303.5076 [astro-ph.CO].
- [27] R. de Putter, O. Zahn and E. V. Linder, Phys. Rev. D **79**, 065033 (2009) [arXiv:0901.0916 [astro-ph.CO]].