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KINETIC THEORY OF EQUILIBRIUM AXISYMMETRIC COLLISIONLESS PLASMAS IN OFF-EQUATORIAL TORI AROUND COMPACT OBJECTS

Claudio Cremaschini, Jiří Kovář, Petr Slaný and Zdeněk Stuchlík
Institute of Physics, Faculty of Philosophy and Science, Silesian University in Opava, Bezručovo nám.13, CZ-74601 Opava, Czech Republic

Vladimír Karas
Astronomical Institute, Academy of Sciences, Boční II, CZ-14131 Prague, Czech Republic
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ABSTRACT

The possible occurrence of equilibrium off-equatorial tori in the gravitational and electromagnetic fields of astrophysical compact objects has been recently proved based on non-ideal MHD theory. These stationary structures can represent plausible candidates for the modelling of coronal plasmas expected to arise in association with accretion discs. However, accretion disc coronae are formed by a highly diluted environment, and so the fluid description may be inappropriate. The question is posed of whether similar off-equatorial solutions can be determined also in the case of collisionless plasmas for which treatment based on kinetic theory, rather than fluid one, is demanded. In this paper the issue is addressed in the framework of the Vlasov-Maxwell description for non-relativistic multi-species axisymmetric plasmas subject to an external dominant spherical gravitational and dipolar magnetic field. Equilibrium configurations are investigated and explicit solutions for the species kinetic distribution function are constructed, which are expressed in terms of generalized Maxwellian functions characterized by isotropic temperature and non-uniform fluid fields. The conditions for the existence of off-equatorial tori are investigated. It is proved that these levitating systems are admitted under general conditions when both gravitational and magnetic fields contribute to shaping the spatial profiles of equilibrium plasma fluid fields. Then, specifically-kinetic effects carried by the equilibrium solution are explicitly provided and identified here with diamagnetic, energy-correction and electrostatic contributions. It is shown that these kinetic terms characterize the plasma equation of state by introducing non-vanishing deviations from the assumption of thermal pressure.

Subject headings: accretion discs, off-equatorial tori, collisionless plasmas, kinetic theory

1. INTRODUCTION

This paper concerns the Vlasov-Maxwell description of collisionless magnetized plasmas (Galeev et al. 1983; Eliezer et al. 1989; Kulsrud 2005; Swanson 2008) related to axisymmetric discs arising in the combined gravitational and electromagnetic (EM) fields of astrophysical compact objects. In particular, in this work an application of the kinetic theory developed by Cremaschini et al. (2010, 2011a,b, 2012a, 2013a,b) is considered, to the treatment of equilibrium and possibly non-neutral equatorial as well as off-equatorial plasma tori. Remarkably, the existence of equilibrium levitating (i.e., off-equatorial) structures has been recently pointed out by Slaný et al. (2013), based on a fluid non-ideal magnetohydrodynamics (MHD) description. Indication of existence of structures of this type occurring in collisionless plasmas is also supported by other recent works based on test-particle approach (Kovář et al. 2008, 2010; Kopáček et al. 2010). In addition, in the context of charge-separated pulsar magnetospheres, Neukirch (1993) found an indication of the possible development of stable off-equatorial tori that can be revealed in his early numerical simulations of magnetized collisionless plasmas. Indeed, from the physical point of view, off-equatorial plasma configurations are intrinsically different from the case of equatorial disc systems. The astrophysical relevance of these structures lies in the possibility of modelling coronal plasmas characterized by low-density and high-temperature conditions (Uzdensky et al. 2008; Goodman et al. 2008). In this regard, two different issues arise, which deserve a detailed investigation. First, under these conditions fluid descriptions may become inappropriate, requiring in principle the adoption of a kinetic treatment. However, it still remains to be ascertained whether the levitating structures can be recovered as equilibrium solutions in the framework of a kinetic description. Second, "a priori" it is not obvious whether the kinetic theory developed previously for equatorial plasmas can be extended to the description of off-equatorial tori, how this can be achieved in practice and what are the physical implications as far as their occurrence in real systems is concerned.

The statistical description of plasma dynamics can be carried out in terms of either fluid or kinetic treatments, with the choice of the appropriate framework depending on the plasma phenomenology to be addressed and the relevant features of the phenomena to be studied (Ichimaru 1973; Swanson 2008). The majority of fluid approaches are based on hydrodynamic or MHD treatments (Goedbloed et al. 2004). In the case of collisionless plasmas, when these are formulated independently of an underlying kinetic theory, some limitations can arise. First, it is well known that the set of fluid equations may not be closed, requiring in principle the prescription of arbitrary higher-order fluid fields and closure conditions, including in particular the equation of state (EoS) or the pressure tensor (Swanson 2008). Second, in these approaches typically no account is given of microscopic phase-space particle dynamics together with phase-space plasma collective phenomena. On the other hand, only in the context of kinetic theory these difficulties can be consistently overcome, as this treatment permits to obtain well-defined constitutive equations for the relevant fluid fields describing the plasma state and to solve at the same time the closure problem (Cremaschini et al. 2011b). These issues become relevant in the case of collisionless or weakly-collisional multi-species plasmas subject to EM and gravitational fields where phase-space particle dynamics is expected to play a dominant role. In particular, kinetic theory is essential for studying both stationary configurations and dynamical evolution of plasmas when kinetic effects are relevant, such as ones associated with conservation of particle adiabatic invariants (Cremaschini et al. 2013a), temperature and pressure anisotropies, diamagnetic and finite Larmor-radius (FLR) effects as well as energy-correction contributions (Cremaschini et al. 2010, 2011a,b, 2013a).

The problem of formulating a kinetic theory appropriate for the description of collisionless plasmas in quasi-stationary (i.e., equilibrium) configurations of astrophysical accreting systems and laboratory scenarios has been presented in series of works, based on the non-relativistic Vlasov-Maxwell description. In our context, several issues have been treated, ranging from laboratory plasmas occurring in Tokamak devices (Cremaschini et al. 2011a), axisymmetric accretion disc plasmas characterized by locally-nested magnetic surfaces (Cremaschini et al. 2010, 2011b, 2012a) and current-carrying magnetic loops (Cremaschini et al. 2013b) around compact objects as well as spatially non-symmetric systems in astrophysical and laboratory contexts (Cremaschini et al. 2013a). It was shown that consistent solutions of the Vlasov equation can be determined for the species kinetic distribution function (KDF) describing collisionless plasmas, based on the identification of the relevant single-particle invariants. The equilibrium KDFs were expressed in terms of generalized bi-Maxwellian distributions, characterized by temperature anisotropy, non-uniform fluid fields and local plasma flows. Chapman-Enskog representations of these equilibria were obtained by developing a suitable perturbative kinetic theory, which in turn made possible the analytical calculation of the fluid fields and the identification of the relevant kinetic effects included in the corresponding MHD description. As a basic consequence, it was shown that these solutions can exhibit non-vanishing current densities which can also support a kinetic dynamo mechanism for the self-generation of EM fields in which the plasma is immersed (Cremaschini et al. 2010, 2011b). Finally, more recently the kinetic theory has been extended to describe axisymmetric plasmas characterized by strong shear-flow and/or supersonic velocities (Cremaschini et al. 2013c), while Cremaschini et al. (2012b) reports a kinetic analysis of the stability properties of particular equilibrium solutions with respect to axisymmetric EM perturbations.

A notable feature of the kinetic equilibria mentioned here is the unique prescription of the functional dependences of an appropriate set of fluid fields, carried by the species KDFs, which are related to physical observables of the system. These fields are referred to as *structure functions* and are denoted as $\{\Lambda_s\}$ (see in particular Cremaschini et al. (2010, 2011a,b, 2013a) and the definition below), with the subscript "s" being the species index. Depending on the kinetic regime being considered, according to the classification scheme presented by Cremaschini et al. (2012a), these dependences are expressed in terms of the poloidal flux function ψ of the magnetic field and/or the effective potential Φ_s^{eff} defined as

$$\Phi_s^{\mathrm{eff}} = \Phi + \frac{M_s}{Z_s e}\Phi_G, \qquad (1)$$

where Φ_G and Φ are the gravitational and electrostatic (ES) potentials respectively, with M_s and $Z_s e$ denoting the species particle mass and charge. Thus, in the general case kinetic theory requires that $\Lambda_s = \Lambda_s\left(\psi, \Phi_s^{\mathrm{eff}}\right)$. It must be stressed that, behind the apparent simplicity of the result, for practical applications one ultimately needs obtaining a representation of Λ_s in terms of spatial coordinates, e.g. cylindrical ones (R, φ, z). Assuming an axisymmetric configuration where ψ and Φ_s^{eff} are generic functions of both (R, z), the representation $\Lambda_s = \Lambda_s\left(\psi, \Phi_s^{\mathrm{eff}}\right) = \overline{\Lambda}_s(R, z)$ applies. Hence, the complete solution of the problem actually requires determining the explicit representation of the potentials (ψ, Φ_G, Φ) in terms of the spatial cylindrical coordinates. This can be a very difficult task, since in the general case the plasma itself contributes to the generation of the gravitational filed, through its non-vanishing mass-density, and, more important, of the EM fields by means of non-vanishing charge and current densities. In the present discussion we ignore, however, the self-generation of the gravitational field, focusing only on the generation of EM fields. It follows that, in order to obtain explicitly the relationship between the EM potentials and the coordinate system, one has necessarily to solve (numerically) the coupled set of Vlasov-Maxwell equations to determine $\psi = \psi(R, z)$ and $\Phi = \Phi(R, z)$, where the source terms of the fields are prescribed functions of the potentials. On the other hand, such a solution is also demanded for the following additional reasons: a) in order to calculate explicitly the characteristic kinetic effects which enter the equilibrium KDF and are associated with diamagnetic-FLR and energy-correction effects (see Cremaschini et al. (2010, 2011a,b, 2012a, 2013a) and the discussion in Section 7); b) in order to establish the diffeomorphism $\mathcal{J} : (R, z) \leftrightarrow (\psi, \vartheta)$ which relates cylindrical (and similarly, spherical coordinates) to local magnetic coordinates, where ϑ is an angle-like coordinate defined on equipotential magnetic surfaces $\psi = const$. The complexity of the theory, as far as this issue is concerned, might represent a possible limit for the practical realization of kinetic equilibria of this type for configurations of astrophysical interest. This may be relevant especially when demanding a comparison between kinetic and fluid treatments based on analytical solutions. Therefore, the question arises of whether such a difficulty can be actually encompassed in some scenarios, possibly by invoking suitable asymptotic kinetic orderings to be imposed on the system. In particular, this concerns the existence of configurations which allow for the construction of the diffeomorphism $\mathcal{J}$ based on analytical solutions of the potentials (ψ, Φ_G, Φ) in such a way to afford an explicit treatment of the spatial dependences of the equilibrium fluid fields carried by the KDF according to the constraints posed by the Vlasov equation.

A second point of crucial importance concerns the determination of the EoS that characterizes disc plasmas in the collisionless state. As mentioned above, this is usually realized by prescribing the form of the scalar pressure, or more generally the components of the pressure tensor, which represent the closure condition for the Euler momentum equation in MHD treatments. Since collisionless plasmas are intrinsically characterized by the occurrence of phase-space anisotropies which cause the equilibrium KDF to generally deviate from a simple isotropic Maxwellian distribution, the knowledge of the correct form of the EoS is not a trivial task. In fact, one can only prescribe the pressure tensor in consistent way on the base of the kinetic theory (kinetic closure conditions). In this regard, the problem consists in the identification of the kinetic effects which must be included in the pressure tensor, the understanding of their physical origin and the way they influence the macroscopic configuration of the plasma system. Among the relevant ones, contributions associated with ES corrections can play a central role, since they arise from microscopic charge interactions and carry information about local ES fields generated by the validity of quasi-neutrality condition in rotating plasmas or deviations away from it in non-neutral systems.

Finally, from the astrophysical point of view a further motivation is represented by Slaný et al. (2013), where the discovery of the possible occurrence of stationary configurations of disc plasmas in off-equatorial tori is reported. In that work, a Newtonian axisymmetric model of non-conductive, charged and perfect fluid tori orbiting in the combined gravitational and dipolar magnetic fields generated by a central compact object is presented. The result is obtained in the framework of a non-ideal MHD description and shows that the interplay between gravitational and magnetic fields can effectively enhance vertically extended structures in a plasma torus, which may correspond to localized concentrations of matter above and under the equatorial plane. The importance of this conclusion lies in the possibility of interpreting these off-equatorial tori as forming the surrounding material usually invoked to explain the spectral emission/absorption features in accretion-disc systems. Possible examples of this type are provided by coronal halos consisting of non-neutral ion-electron plasmas or by obscuring dusty-plasma tori that are believed to be produced in galactic nuclei (Sargsyan et al. 2012; Goulding et al. 2012; Dorodnitsyn et al. 2011; Mor et al. 2011; Kawaguchi et al. 2011; Oyabu et al. 2011; Hatziminaoglou et al. 2009; Mor et al. 2009; Fabian et al. 2008; Hönig et al. 2007).

The complete understanding of the physical properties of these structures is far from being satisfactory and deserves further investigations, both theoretical and observational. In particular, here we consider the possibility of a low-density and high-temperature coronal plasma for which the collisionless state applies. Following the arguments discussed above, under these conditions the proper framework for the description of these plasmas is represented by the kinetic theory. In particular, the question is posed of whether kinetic equilibria can be proved to exist for collisionless magnetized plasmas which exhibit off-equatorial maxima in the matter distribution and how these solutions can be possibly related to fluid MHD ones. This amounts at identifying the appropriate kinetic regimes which meet these conditions and determining the spatial dependences of the corresponding fluid fields, including the pressure tensor and the intrinsically-kinetic effects contained in the EoS. To be successful, a program of this type must make possible an analytical approach in accordance with the considerations presented above, to extend the kinetic theory developed in Cremaschini et al. (2010, 2011a,b, 2012a, 2013a,b) and to permit its practical application to gain insights into an astrophysical issue of notable importance connected with accretion-disc phenomenology.

To conclude, we should comment on the fact that the reference works cited above on kinetic theory and off-equatorial tori as well as the present investigation are carried out in the framework of a non-relativistic description (both with respect to the treatment of the gravitational field and the plasma velocities). However, the problem of general-relativistic solutions could in principle be posed, since there is a number of situations in which the formation of accretion discs is due to strong gravitational fields, e.g. around black holes, and general-relativistic corrections must be considered. Although a complete theory of this type is still missing, non-relativistic treatments can nevertheless provide the reference framework for the inclusion of some relevant features characteristic of general-relativistic theories. This can be achieved for example by the adoption of pseudo-Newtonian potentials for the description of spherically-symmetric gravitational fields (Paczynsky et al. 1980; Stuchlík et al. 2008). It has been shown, in fact, that the precision of the pseudo-Newtonian description of stationary general-relativistic phenomena can be very high (Stuchlík et al. 2009). An alternative consists in the inclusion of post-Newtonian corrections. In this reference, a kinetic theory of self-gravitating collisionless gases adopting such a technique can be found in Agón et al. (2011) for spherical solutions, and in Ramos-Caro et al. (2012) for the case of axially-symmetric solutions.

2. GOALS AND SCHEME OF THE PAPER

In view of the considerations presented above, the purpose of this paper is the formulation of a kinetic theory appropriate for the analytical treatment of collisionless disc plasmas in axisymmetric off-equatorial tori (see Fig. 1 for a schematic view of the configuration geometry). The results of the investigation are the following ones:

1) The identification of a physically-realizable astrophysical configuration in which the spatial profiles of the potentials $\left(\psi, \Phi_s^{\mathrm{eff}}\right)$ can be analytically prescribed, when suitable plasma orderings apply. In the framework of an asymptotic theory, this permits to decouple the Vlasov equation from the Maxwell equations, at least to leading-order, with the possibility of an explicit treatment of the spatial dependences contained in the equilibrium plasma fluid fields.

2) The construction of species equilibrium KDFs which are consistent with the kinetic constraints imposed by microscopic phase-space conservation laws for the single

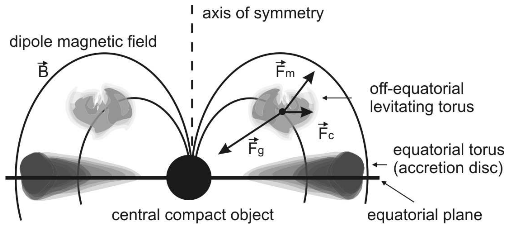

Figure 1. Schematic illustration of a poloidal section across the equilibrium axially symmetric distribution of electrically charged matter subject to a dipolar magnetic field and surrounding a gravitating body in the center. Directions of gravitational ($\vec{F}_{\rm g}$), magnetic ($\vec{F}_{\rm m}$) and centrifugal ($\vec{F}_{\rm c}$) forces acting on a generic fluid element belonging to the levitating torus and moving along the circular orbit are displayed.

particle dynamics. These are proved to be expressed in terms of generalized Maxwellian KDFs characterized by non-uniform fluid fields and isotropic temperature, both to be associated with a finite set of structure functions.

3) The identification of the relevant kinetic regimes which can in principle arise in the configuration determined at point 1) above and the prescription of the corresponding functional dependences on the structure functions.

4) The development of a perturbative theory which makes possible a representation of the equilibrium KDFs in terms of a Chapman-Enskog series. As a result, this permits first the analytical evaluation of the plasma fluid fields, and second to distinguish the characteristic kinetic effects which enter the solution and their main physical properties.

5) The proof that the gravitationally-bound and magnetized plasma regime provides the most general functional dependences in the structure functions carried by the equilibrium KDF, which are consistent both with the analytical treatment of the fluid fields and the occurrence of off-equatorial tori. As an illustration of the technique, explicit calculation of the leading-order species number density and velocity profiles is provided for two configurations of physical interest.

6) The calculation of the EoS for the equilibrium plasma, to be expressed in terms of the species pressure tensor associated with the KDF. By making use of the perturbative treatment, this includes also the identification of both the kinetic effects and the ES corrections that can effectively contribute to the EoS. In particular, the latter are shown to determine non-trivial deviations from the assumption of having thermal pressure for the collisionless plasma.

7) To determine the constraints to be imposed on the kinetic solution for the validity of the theory and the physical configuration realized, which arise from the Maxwell equations for the self-generated equilibrium EM fields. These represent necessary conditions which must be verified "a posteriori" for the complete solution of the Vlasov-Maxwell problem and the explicit treatment of ES corrections in comparison with fluid-based approaches.

The scheme of the paper is as follows. In Section 3 the model assumptions and the fundamental EM orderings are presented. Section 4 deals with the definition of plasma orderings and the introduction of corresponding kinetic regimes which characterize collisionless plasmas treated here. In Section 5 the equilibrium species KDFs are explicitly determined and their realizations are given for three different plasma regimes. In Section 6 a suitable perturbative theory is developed, which allows for the analytical treatment of the equilibrium KDFs and the corresponding fluid fields. The conditions for the occurrence of off-equatorial tori are then investigated in the case of collisionless plasmas belonging to the gravitationally-bound and magnetized plasma regime. In Section 7 the expression of the kinetic corrections which characterize the kinetic equilibria is provided, while the corresponding contributions in the EoS are computed in Section 8. Section 9 contains analysis of the Poisson and the Ampere equations and the constraints which they pose on the kinetic solution. Finally, concluding remarks are summarized in Section 10.

3. MODEL ASSUMPTIONS

For the construction of kinetic equilibria, we ignore the possible existence of weakly-dissipative effects (Coulomb collisions and turbulence) and EM radiation effects (Cremaschini et al. 2013d,e). It is assumed that the KDF and the EM fields associated with the plasma obey the system of Vlasov-Maxwell equations, with Maxwell's equations being considered in the quasi-static approximation. For definiteness, we shall consider here a plasma consisting of s-species of charged particles which are characterized by proper mass $M_{\rm s}$ and total charge $Z_{\rm s}e$. In particular, given a generic species KDF $f_{\rm s} = f_{\rm s}(\mathbf{r}, \mathbf{v}, t)$ defined in the phase-space $\Gamma = \Gamma_{\mathbf{r}} \times \Gamma_{\mathbf{v}}$, with $\Gamma_{\mathbf{r}}$ and $\Gamma_{\mathbf{v}}$ being the configuration and velocity space respectively, the Vlasov equation determines the dynamical evolution of $f_{\rm s}$ and is given by

$$\frac{\mathrm{d}}{\mathrm{dt}} f_{\rm s}(\mathbf{r}, \mathbf{v}, t) = 0. \tag{2}$$

The plasma is taken to be: a) non-relativistic, in the sense that the flow velocities of all species are small compared to the speed of light c, that the gravitational field can be treated within the classical Newtonian theory, and that the non-relativistic Vlasov kinetic equation is used as the dynamical equation for the KDF; b) collisionless, so that the mean free path of the plasma particles is much longer than the largest characteristic scale length of the plasma; c) axisymmetric, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the azimuthal angle φ, when referred to a set of either cylindrical coordinates (R, φ, z) or spherical coordinates (r, φ, θ). Thanks to the axisymmetry assumption, as a shortway in the following we shall denote with $\mathbf{x}$ the configuration state vector, where $\mathbf{x}$ denotes either $\mathbf{x} = (R, z)$ or $\mathbf{x} = (r, \theta)$.

We are concerned here with quasi-stationary configurations, namely solutions which are slowly-varying in time. This condition is also referred to as equilibrium configuration. For a generic physical quantity G that depends on spatial coordinates $\mathbf{x}$ and time t, the quasi-stationarity is expressed by letting in the following $G = G(\mathbf{x}, \lambda^k t)$, with $\lambda \ll 1$ being a small dimensionless parameter to be suitably defined (see below) and $k \geq 1$ an integer. Similar considerations apply for the equilibrium KDF $f_{\rm s}$, which is denoted in the following as $f_{\rm s} = f_{\rm s}(\mathbf{x}, \mathbf{v}, \lambda^k t)$.

From the symmetry properties introduced here one can immediately derive the fundamental quantities which are conserved for the single-particle dynamics. In particular, under the assumptions of axisymmetry the canonical momentum conjugate to the azimuthal angle φ is an integral of motion. This is given by:

$$P_{\varphi s} = M_s R\mathbf{v}\cdot\mathbf{e}_\varphi + \frac{Z_s e}{c}\psi \equiv \frac{Z_s e}{c}\psi_{*s}. \quad (3)$$

Furthermore, from the condition of quasi-stationarity, the total particle energy

$$E_s = \frac{M_s}{2}v^2 + Z_s e\Phi_s^{\mathrm{eff}}(\mathbf{x},\lambda^k t) \equiv Z_s e\Phi_{*s}, \quad (4)$$

represents an adiabatic invariant of prescribed order, with Φ_s^{eff} being defined in Eq. (1). Following the discussion in Cremaschini et al. (2010, 2011b), here we recall that a generic quantity $P = P(\mathbf{x},\mathbf{v},\lambda^n t)$ defined in the phase-space is an adiabatic invariant of order n with respect to λ when it satisfies the condition $\frac{1}{\Omega_{cs}}\frac{\mathrm{d}}{\mathrm{dt}}\ln P = 0 + O(\lambda^{n+1})$, where $n \geq 0$ is a suitable integer and $\Omega_{cs} \equiv \frac{Z_s eB}{M_s c}$ is the cyclotron frequency. This means that adiabatic invariants are conserved in asymptotic sense, namely up to a prescribed order of accuracy determined by the parameters λ and n.

We consider solutions of the equilibrium magnetic field $\mathbf{B}$ which admit, at least locally, a family of nested axisymmetric toroidal magnetic surfaces $\{\psi(\mathbf{x},\lambda^k t)\} \equiv \{\psi(\mathbf{x},\lambda^k t) = const.\}$, where ψ denotes the poloidal magnetic flux of $\mathbf{B}$. The magnetic surfaces can be either locally closed (Cremaschini et al. 2010) or locally open (Cremaschini et al. 2011b) in the configuration domain occupied by the plasma. In both cases a set of magnetic coordinates (ψ,φ,ϑ) can be defined locally, where ϑ is a curvilinear angle-like coordinate on the magnetic surfaces $\psi(\mathbf{x},\lambda^k t) = const.$ By construction, magnetic coordinates are related to cylindrical or spherical coordinates by a diffeomorphism J which must be consistently determined, as discussed above. Each relevant physical quantity $G(\mathbf{x},\lambda^k t)$ can then be conveniently expressed either in terms of the set $\mathbf{x}$ or as a function of the magnetic coordinates, i.e. $G(\mathbf{x},\lambda^k t) = \overline{G}(\psi,\vartheta,\lambda^k t)$.

Consistent with these assumptions, we require the EM field to be slowly-varying in time, i.e., of the form

$$\begin{aligned}\mathbf{E} &= \mathbf{E}(\mathbf{x},\lambda^k t),\\ \mathbf{B} &= \mathbf{B}(\mathbf{x},\lambda^k t).\end{aligned} \quad (5)$$

In particular, we assume the magnetic field to be represented as

$$\mathbf{B} \equiv \nabla\times\mathbf{A} = \mathbf{B}^{\mathrm{self}}(\mathbf{x},\lambda^k t) + \mathbf{B}^{\mathrm{ext}}(\mathbf{x},\lambda^k t), \quad (6)$$

where $\mathbf{B}^{\mathrm{self}}$ and $\mathbf{B}^{\mathrm{ext}}$ denote the self-generated magnetic field produced by the plasma and a finite external axisymmetric magnetic field (vacuum field). For definiteness, in this treatment both contributions are assumed to exhibit only non-vanishing poloidal components, to be denoted in the following as $\mathbf{B}_P$. Notice that, for what concerns the self-field, this assumption must be verified "a posteriori" to be consistent with the constraints placed on the kinetic solution by the Maxwell equations. Hence, the two fields are represented as

$$\mathbf{B}^{\mathrm{ext}} = \nabla\psi_{\mathrm{ext}}(\mathbf{x},\lambda^k t)\times\nabla\varphi, \quad (7)$$

$$\mathbf{B}^{\mathrm{self}} = \nabla\psi_{\mathrm{self}}(\mathbf{x},\lambda^k t)\times\nabla\varphi, \quad (8)$$

so that the total poloidal magnetic field takes the form

$$\mathbf{B} \equiv \mathbf{B}_P = \nabla\psi(\mathbf{x},\lambda^k t)\times\nabla\varphi, \quad (9)$$

with $\psi \equiv \psi_{\mathrm{ext}} + \psi_{\mathrm{self}}$. In particular, for the purpose of the present work, the external magnetic field is taken to coincide with a dipolar field. In such a case, the flux function ψ_{ext} is written in terms of the spherical coordinates (r,θ) as

$$\psi_{\mathrm{ext}} = \mathcal{M}_0\frac{\sin^2\theta}{r}, \quad (10)$$

with $\mathcal{M}_0$ being the magnitude of the dipole magnetic moment.

Charged particles are also assumed to be subject to the effective potential $\Phi_s^{\mathrm{eff}} \equiv \Phi_s^{\mathrm{eff}}(\mathbf{x},\lambda^k t)$ defined by Eq. (1). In principle, both the ES potential $\Phi(\mathbf{x},\lambda^k t)$ and the gravitational potential $\Phi_G(\mathbf{x},\lambda^k t)$ can be produced by the plasma itself and by external sources. However, in the following it is assumed that $\Phi(\mathbf{x},\lambda^k t)$ is uniquely generated by the plasma charge density, while we shall neglect the self contribution of the plasma to Φ_G. Hence, for an axisymmetric disc, the gravitational potential is taken as being stationary and to coincide identically with the potential generated by the central compact object. The latter is expressed here in terms of the spherically-symmetric Newtonian potential as

$$\Phi_G(\mathbf{x}) = -\frac{G_N M_+}{r}, \quad (11)$$

where G_N is the Newton gravitational constant and M_+ is the mass of the compact object.

Given validity of these assumptions, in order to address the first issue posed in the Section 2 we proceed introducing the following fundamental orderings for the self EM fields:

1) The self component of the equilibrium magnetic field $\mathbf{B}^{\mathrm{self}}$ is ordered with respect to the external field $\mathbf{B}^{\mathrm{ext}}$ as

$$\frac{|\mathbf{B}^{\mathrm{self}}|}{|\mathbf{B}^{\mathrm{ext}}|} \sim O(\lambda^j), \quad (12)$$

with $j \geq 1$, and in general $j \neq k$.

2) The equilibrium ES potential Φ satisfies the ordering assumption

$$\left|\frac{Z_s e\Phi}{M_s\Phi_G}\right| \sim O(\lambda^j), \quad (13)$$

with $j \geq 1$. This means that the ES potential energy is small with respect to the gravitational potential energy, where λ will be properly defined below. In the following the case $j = 1$ in Eqs. (12) and (13) will be considered.

It is important to remark that the two conditions (12) and (13) pose strong constraints from the physical point of view on the realizability of the equilibrium configuration for the collisionless disc plasma. In particular, they must be verified "a posteriori" in order to warrant the consistency of the kinetic equilibrium solution for the species KDF with the validity of the Maxwell equations.

Eqs. (12) and (13) are the fundamental EM orderings which permit the analytical treatment of the spatial dependences of the equilibrium fluid fields as prescribed by the kinetic solution to be determined below. In fact, when these orderings hold, to leading-order the magnetic flux function ψ and the effective potential Φ_s^{eff} coincide respectively with the vacuum fields, namely the external dipolar flux function $\psi_{\rm ext}$ in Eq. (10) and with the gravitational potential $\Phi_{\rm G}$ given by Eq. (11). Hence, at this order it is possible to construct explicitly the diffeomorphism which relates the spherical coordinates (r,θ) with the potentials $\left(\psi,\Phi_{\rm s}^{\rm eff}\right)\cong(\psi_{\rm ext},\Phi_{\rm G})$. This is expressed by letting

$$r=\left|\frac{G_{\rm N}M_{+}}{\Phi_{\rm G}}\right|, \quad (14)$$

$$\sin^{2}\theta=\left|\frac{G_{\rm N}M_{+}}{\mathcal{M}_{0}}\frac{\psi_{\rm ext}}{\Phi_{\rm G}}\right|. \quad (15)$$

The corresponding relationship holding for cylindrical coordinates can then be obtained from the spherical ones, giving in particular

$$R=r\sin\theta=\left|\frac{G_{\rm N}M_{+}}{\Phi_{\rm G}}\right|\sqrt{\left|\frac{G_{\rm N}M_{+}}{\mathcal{M}_{0}}\frac{\psi_{\rm ext}}{\Phi_{\rm G}}\right|}. \quad (16)$$

4. PLASMA ORDERINGS AND KINETIC REGIMES

In this section we introduce the relevant orderings for collisionless plasmas, which permit to identify corresponding kinetic regimes characterized by different equilibrium solutions and to verify their consistency with the existence of off-equatorial tori. To this aim we determine a classification scheme based on the method outlined in Cremaschini et al. (2012a) and suitable for the treatment of the physical setting indicated in the previous section.

The first step is the definition of the dimensionless species-dependent parameters $\varepsilon_{\rm M,s}$, $\varepsilon_{\rm s}$ and $\sigma_{\rm s}$. These are prescribed in such a way to be all independent of single-particle velocity and at the same time to be related to the characteristic species thermal velocities. Both perpendicular and parallel thermal velocities (defined with respect to the magnetic field local direction) must be considered. These are defined respectively by $v_{\perp\rm ths}=\left\{T_{\perp\rm s}/M_{\rm s}\right\}^{1/2}$ and $v_{\parallel\rm ths}=\left\{T_{\parallel\rm s}/M_{\rm s}\right\}^{1/2}$, with $T_{\perp\rm s}$ and $T_{\parallel\rm s}$ denoting here the species perpendicular and parallel temperatures. In detail, the first parameter is defined as $\varepsilon_{\rm M,s}\equiv\frac{r_{\rm Ls}}{L}$, where $r_{\rm Ls}=v_{\perp\rm ths}/\Omega_{\rm cs}$ is the species Larmor radius, with L being the characteristic scale-length of the spatial variations of all of the fluid fields associated with the KDF and of the EM fields. The second parameter $\varepsilon_{\rm s}$ is related to the particle canonical momentum $P_{\varphi\rm s}$. By denoting $v_{\rm ths}\equiv\sup\left\{v_{\parallel\rm ths},v_{\perp\rm ths}\right\}$, $\varepsilon_{\rm s}$ is identified with $\varepsilon_{\rm s}\equiv\left|\frac{M_{\rm s}Rv_{\rm ths}}{\frac{Z_{\rm s}e}{c}\psi}\right|$. Hence, $\varepsilon_{\rm s}$ effectively measures the ratio between the toroidal angular momentum $L_{\varphi\rm s}\equiv M_{\rm s}Rv_{\varphi}$ and the magnetic contribution to the toroidal canonical momentum, for all particles in which v_{φ} is of the order $v_{\varphi}\sim v_{\rm ths}$ while ψ is assumed as being non-vanishing. In particular, here the magnetic flux can be estimated as $\psi\sim B_{\rm p}RL_{1}$, with L_{1} denoting the characteristic length-scale of flux variations and $B_{\rm p}$ the magnitude of the poloidal magnetic field. Note that, by definition, $L\leq L_{1}$, but, in principle, we can also have $L\ll L_{1}$ locally.

We also notice that the present definition of $\varepsilon_{\rm s}$ is not the only possibility, as one could also define the parameter $\varepsilon_{\rm s}$ such that it is related to the azimuthal flow velocity, namely letting $v_{\varphi}\sim V_{\varphi\rm s}=\Omega_{\rm s}R$, with $\Omega_{\rm s}$ being the corresponding angular frequency. Finally, $\sigma_{\rm s}$ is related to the particle total energy $E_{\rm s}$ and is prescribed in this work as $\sigma_{\rm s}\equiv\left|\frac{\frac{M_{\rm s}}{2}v_{\rm ths}^{2}+Z_{\rm s}e\Phi}{M_{\rm s}\Phi_{\rm G}}\right|$. It follows that $\sigma_{\rm s}$ measures the ratio between particle kinetic and ES potential energy with respect to the gravitational potential energy, for all particles having velocity v of the order $v\sim v_{\rm ths}$, with $\Phi_{\rm G}$ being assumed as non-vanishing. We notice that the definition of $\sigma_{\rm s}$ differs from that used in previous works (Cremaschini et al. 2011a,b, 2012a) while it is consistent with the ordering assumption (13). In the following we shall denote as thermal subset of velocity space the subset of the Euclidean velocity space in which the asymptotic conditions $\frac{v}{v_{\rm ths}}\sim\frac{v_{\varphi}}{v_{\rm ths}}\sim O(1)$ holds.

A comment is in order regarding the role of the magnetic field in the two parameters $\varepsilon_{\rm s}$ and $\varepsilon_{\rm M,s}$. In the first case, the magnetic field is represented by means of the poloidal flux ψ which contributes to the toroidal canonical momentum $P_{\varphi\rm s}$, while $\varepsilon_{\rm M,s}$ depends on the magnitude of the total magnetic field. Invoking the definitions for $\varepsilon_{\rm s}$ and $\varepsilon_{\rm M,s}$ given above, it follows that $\varepsilon_{\rm s}\sim\varepsilon_{\rm M,s}\frac{L}{L_{1}}\frac{B}{B_{\rm p}}$, where L and L_{1} are respectively the characteristic scale-lengths of equilibrium fluid and EM fields and of the poloidal flux. In general, the two quantities should be considered as independent, with $L\leq L_{1}$ and $B_{\rm p}\leq B$ (in the present context one has identically $B_{\rm p}=B$ from Eq. (9)). Indeed, the parameter $\varepsilon_{\rm s}$ determines the particle spatial excursions from a magnetic flux surface, while $\varepsilon_{\rm M,s}$ measures the amplitude of the Larmor radius with respect to the inhomogeneities of the background fluid fields. These two effects correspond to different physical magnetic-related processes, due respectively to the Larmor-radius and magnetic-flux surface confinement mechanisms.

In this work we assume that the ordering condition

$$\varepsilon_{\rm M,s}\ll 1 \quad (17)$$

holds for the collisionless plasma considered here. This amounts at requiring that the Larmor radius remains small with respect to the scale-length L, which, as shown in Cremaschini et al. (2012a), represents a condition that is expected to be easily verified in accretion-disc systems. Hence, one can consistently identify the small parameter λ introduced above with $\lambda=\sup\left\{\varepsilon_{\rm M,s}\right\}$.

The classification that is introduced in this work is based on the magnitude of the two parameters $\varepsilon_{\rm s}$ and $\sigma_{\rm s}$. In detail, plasma species will be distinguished as belonging to the following regimes:

1) *Gravitationally-bound* if $\sigma_{\rm s}\ll 1$ and $\varepsilon_{\rm s}\gtrsim 1$.
2) *Magnetized* if $\varepsilon_{\rm s}\ll 1$ and $\sigma_{\rm s}\lesssim 1$.
3) *Gravitationally-bound and magnetized* if both $\sigma_{\rm s}\ll 1$ and $\varepsilon_{\rm s}\ll 1$.

In the case of regimes 1 and 3 the following asymptotic expansion holds for the total particle energy $Z_{\rm s}e\Phi_{*\rm s}$:

$$\Phi_{*\rm s}=\frac{M_{\rm s}}{Z_{\rm s}e}\Phi_{\rm G}\left[1+O\left(\sigma_{\rm s}\right)\right]. \quad (18)$$

Similarly, for regimes 2 and 3 the particle canonical momentum $\frac{Z_{\rm s}e}{c}\psi_{*\rm s}$ admits the expansion

$$\psi_{*\rm s}=\psi\left[1+O\left(\varepsilon_{\rm s}\right)\right]. \quad (19)$$

It is instructive to analyze the main features of these regimes and the physical conditions for their occurrence. The action of some energy non-conserving mechanisms is required for the establishment of the case of gravitationally-bound plasmas. In particular, plausible physical mechanisms that can be responsible for the decrease of the single-particle kinetic energy, in both collisionless and collisional AD plasmas, are EM interactions (e.g., binary Coulomb collisions among particles and particle-wave interactions, such as Landau damping) and/or radiation emission (radiation-reaction). These can in principle be ascribed also to the occurrence of EM instabilities and EM turbulence. For single particles these processes can be dissipative. As a consequence, these particles tend to move towards regions with higher gravitational potential (in absolute value). After multiple interactions of this type, the process can ultimately reach an equilibrium state which corresponds to the gravitationally-bound regimes. As far as the magnetic-field based classification, we notice that the requirement $\varepsilon_{\mathrm{s}} \ll 1$ (regimes 2 and 3) means that a particle trajectory remains close to the same magnetic surface $\psi = const.$, while satisfying the ordering (17).

Finally, for greater generality, in the rest of the treatment we shall assume that, in the regimes in which $\sigma_{\mathrm{s}} \ll 1$ and/or $\varepsilon_{\mathrm{s}} \ll 1$, the orderings $\sigma_{\mathrm{s}} \sim \varepsilon_{\mathrm{M,s}}$ and $\varepsilon_{\mathrm{s}} \sim \varepsilon_{\mathrm{M,s}}$ apply.

5. EQUILIBRIUM SPECIES KDF

In this section we proceed with the construction of the species equilibrium KDF and its characterization to the plasma regimes identified above. We consider both exact as well as asymptotic representations for the solution, the latter being expressed in terms of a Chapman-Enskog series. To reach the goal, here we adopt the solution technique developed in Cremaschini et al. (2010, 2011a,b, 2013a,b), which consists in the construction of solutions of the Vlasov equation of the form $f_{\mathrm{s}} = f_{*\mathrm{s}}$, where $f_{*\mathrm{s}}$ is a suitable adiabatic invariant. This amounts at requiring that $f_{*\mathrm{s}}$ is necessarily a function of particle adiabatic invariants. In view of the model assumptions introduced above, it follows that the general form of the equilibrium KDF in the present context is given by

$$f_{*\mathrm{s}} = f_{*\mathrm{s}}\left(E_{\mathrm{s}}, P_{\varphi\mathrm{s}}, \Lambda_{*\mathrm{s}}, \lambda^k t\right), \tag{20}$$

with $k \geq 1$ and where slow-time dependences are assumed to be uniquely associated with the particle energy. Here $\Lambda_{*\mathrm{s}}$ denotes the so-called structure functions, i.e., functions which depend implicitly on the particle state $(\mathbf{x}, \mathbf{v})$. In order for $f_{*\mathrm{s}}$ to be an adiabatic invariant, $\Lambda_{*\mathrm{s}}$ must also be functions of the adiabatic invariants. This restriction is referred to here as a kinetic constraint. The precise form of the functional dependences of $\Lambda_{*\mathrm{s}}$ is characteristic of each plasma regime, as discussed below.

In order to determine an explicit representation of $f_{*\mathrm{s}}$ according to Eq. (20), we impose the following requirements:

1) The KDF must be characterized by species-dependent non-uniform fluid fields, azimuthal flow velocity and isotropic temperature, to be suitably prescribed in terms of the structure functions.

2) Open, locally nested magnetic flux surfaces: the magnetic field is taken to allow quasi-stationary solutions with magnetic flux lines belonging to locally nested and generally open magnetic surfaces.

3) Kinetic constraints: suitable functional dependences are imposed on the structure functions $\Lambda_{*\mathrm{s}}$ which depend on the regime being considered and such to warrant $f_{*\mathrm{s}}$ to be an adiabatic invariant.

4) In all regimes, $f_{*\mathrm{s}}$ is required to be asymptotically "close" (in a suitable sense to be defined below) to a local Maxwellian KDF. This requires the possibility of determining "a posteriori" a perturbative representation of the KDF equivalent to the Chapman-Enskog expansion for the analytical treatment of implicit phase-space dependences contained in the structure functions, with the consistent inclusion of ES corrections, FLR-diamagnetic and/or energy-corrections contributions.

5) The KDF $f_{*\mathrm{s}}$ must be a strictly-positive real function and it must be summable, in the sense that the velocity moments of the form

$$\Xi_{\mathrm{s}}(\mathbf{x}, \lambda^k t) = \int_{\Gamma_{\mathbf{v}}} d\mathbf{v} K_{\mathrm{s}}(\mathbf{x}, \mathbf{v}, \lambda^k t) f_{*\mathrm{s}} \tag{21}$$

must exist for a suitable ensemble of weight functions $\left\{K_{\mathrm{s}}(\mathbf{x}, \mathbf{v}, \lambda^k t)\right\}$, to be prescribed in terms of polynomials of arbitrary degree defined with respect to components of the velocity vector field $\mathbf{v}$.

Then, following Cremaschini et al. (2010, 2011a,b, 2013a,b), we express the equilibrium KDF $f_{*\mathrm{s}}$ as

$$f_{*\mathrm{s}} = \frac{\eta_{*\mathrm{s}}}{\left(2\pi/M_{\mathrm{s}}\right)^{3/2} T_{*\mathrm{s}}^{3/2}} \exp\left\{-\frac{E_{\mathrm{s}} - \Omega_{*\mathrm{s}} P_{\varphi\mathrm{s}}}{T_{*\mathrm{s}}}\right\}, \tag{22}$$

which is referred to as the *Generalized Maxwellian KDF*. Here the structure functions are identified with the set $\Lambda_{*\mathrm{s}} \equiv (\eta_{*\mathrm{s}}, T_{*\mathrm{s}}, \Omega_{*\mathrm{s}})$, where $\eta_{*\mathrm{s}}$, $T_{*\mathrm{s}}$ and $\Omega_{*\mathrm{s}}$ are related to the species number density, isotropic temperature and azimuthal angular velocity respectively. Invoking the definitions (3) and (4), Eq. (22) can also be written as

$$f_{*\mathrm{s}} = \frac{\eta_{*\mathrm{s}} \exp\left[\frac{X_{*\mathrm{s}}}{T_{*\mathrm{s}}}\right]}{\left(2\pi/M_{\mathrm{s}}\right)^{3/2} T_{*\mathrm{s}}^{3/2}} \exp\left\{-\frac{M_{\mathrm{s}}\left(\mathbf{v} - \mathbf{V}_{*\mathrm{s}}\right)^2}{2T_{*\mathrm{s}}}\right\}, \tag{23}$$

where $\mathbf{V}_{*\mathrm{s}} = R\Omega_{*\mathrm{s}}\mathbf{e}_{\varphi}$ and

$$X_{*\mathrm{s}} \equiv M_{\mathrm{s}} \frac{\left|\mathbf{V}_{*\mathrm{s}}\right|^2}{2} + \frac{Z_{\mathrm{s}} e}{c} \psi \Omega_{*\mathrm{s}} - Z_{\mathrm{s}} e \Phi_s^{\mathrm{eff}}. \tag{24}$$

It is worth pointing out that the form of the solution (22) holds for all the plasma kinetic regimes identified in the previous section. The difference in the three cases concerns the prescription of the kinetic constraints to be imposed on $\Lambda_{*\mathrm{s}}$. In particular, consistent with the requirements listed above, these are assigned as follows:

1) For gravitationally-bound plasmas it is required that the functional dependence of $\Lambda_{*\mathrm{s}}$ is of the type

$$\Lambda_{*\mathrm{s}} \equiv \Lambda_{*\mathrm{s}}\left(\Phi_{*\mathrm{s}}\right), \tag{25}$$

for which Eq. (18) applies, while implicit dependences with respect to $\psi_{*\mathrm{s}}$ remain excluded in such a case.

2) For magnetized plasmas, the kinetic constraint is realized by imposing

$$\Lambda_{*\mathrm{s}} \equiv \Lambda_{*\mathrm{s}}\left(\psi_{*\mathrm{s}}\right), \tag{26}$$

for which Eq. (19) applies, while implicit dependences with respect to $\Phi_{*\mathrm{s}}$ are excluded.

3) For gravitationally-bound and magnetized plasmas both Eqs. (18) and (19) hold, so that the general form

of the kinetic constraint is given by

$$\Lambda_{*\mathrm{s}} \equiv \Lambda_{*\mathrm{s}}\left(\psi_{*\mathrm{s}}, \Phi_{*\mathrm{s}}\right). \tag{27}$$

The connection between the realization of these regimes and the occurrence of off-equatorial tori will be investigated in the next section. Here it must be noticed that, because of the constraints (25)−(27), at this stage the structure functions cannot be regarded as fluid fields, since they are defined in the phase-space, namely they depend on the single particle velocity via the particle energy E_{s} and the canonical momentum $P_{\varphi \mathrm{s}}$. Instead, the fluid fields associated with $f_{*\mathrm{s}}$ must be properly computed as velocity moments according to Eq. (21) and they are unique once the precise form of the structure functions is explicitly prescribed in $f_{*\mathrm{s}}$.

6. OFF-EQUATORIAL TORI: DENSITY AND VELOCITY PROFILES

In this section we first proceed determining a Chapman-Enskog representation for $f_{*\mathrm{s}}$ that makes possible the treatment of the implicit phase-space functional dependences carried by the structure functions as well as the analytical evaluation of the equilibrium fluid fields and the associated kinetic contributions. We then apply the result to prove the validity of the kinetic theory developed here as far as the description of off-equatorial toroidal structures is concerned. This task can be achieved by implementing an appropriate perturbative theory for $f_{*\mathrm{s}}$ which was first developed in Cremaschini et al. (2010, 2011b) and which is based on a Taylor expansion of $\Lambda_{*\mathrm{s}}$ with respect to the dimensionless parameters σ_{s} and ε_{s}.

It is understood that the basic feature of such a kinetic perturbative technique is that it is strictly applicable only in localized subsets of velocity space (thermal subsets), namely to particles whose velocity satisfies the asymptotic ordering (18) and/or (19). A notable consequence of such an approach is that, for each kinetic regime, quasi-stationary, self-consistent, asymptotic solutions of the Vlasov-Maxwell equations (kinetic equilibria) can be explicitly determined by means of suitable Taylor expansions of $f_{*\mathrm{s}}$. In particular, it is found that Maxwellian-like KDFs can be obtained locally in phase-space, where the appropriate convergence conditions hold. This procedure provides also the correct constitutive equations of the leading-order fluid fields as well as the precise form of the ES, FLR-diamagnetic and/or energy-correction contributions to the KDF.

In detail, invoking Eqs. (18) and (19), a linear asymptotic expansion for the structure functions can be obtained. In the general case, neglecting corrections of $O\left(\varepsilon_{\mathrm{s}}\sigma_{\mathrm{s}}\right)$, as well as of $O\left(\varepsilon_{\mathrm{s}}^{k}\right)$ and $O\left(\sigma_{\mathrm{s}}^{k}\right)$, with $k \geq 2$, this is given by

$$\begin{aligned}\Lambda_{*\mathrm{s}} \cong & \Lambda_{\mathrm{s}} + \left(\psi_{*\mathrm{s}} - \psi\right)\left[\frac{\partial \Lambda_{*\mathrm{s}}}{\partial \psi_{*\mathrm{s}}}\right]_{\psi_{*\mathrm{s}}=\psi, \Phi_{*\mathrm{s}}=\frac{M_{\mathrm{s}}}{Z_{\mathrm{s}}e}\Phi_{\mathrm{G}}} \\ & + \left(\Phi_{*\mathrm{s}} - \frac{M_{\mathrm{s}}}{Z_{\mathrm{s}}e}\Phi_{\mathrm{G}}\right)\left[\frac{\partial \Lambda_{*\mathrm{s}}}{\partial \Phi_{*\mathrm{s}}}\right]_{\psi_{*\mathrm{s}}=\psi, \Phi_{*\mathrm{s}}=\frac{M_{\mathrm{s}}}{Z_{\mathrm{s}}e}\Phi_{\mathrm{G}}}\end{aligned} \tag{28}$$

where Λ_{s} is the leading-order term which uniquely follows for each regime (see below). When Eq. (28) is applied to the equilibrium KDF $f_{*\mathrm{s}}$ and the ordering (13) is also invoked, the following Chapman-Enskog representation is found:

$$f_{*\mathrm{s}} = f_{\mathrm{M,s}}\left[1 + \varepsilon_{\mathrm{s}} h_{\mathrm{s}}^{(1)} + \sigma_{\mathrm{s}} h_{\mathrm{s}}^{(2)} + \lambda h_{\mathrm{s}}^{(3)}\right], \tag{29}$$

where the leading-order contribution $f_{\mathrm{M,s}}$ coincides with a drifted isotropic Maxwellian KDF carrying non-uniform number density, azimuthal differential flow velocity and isotropic temperature. In detail:

$$f_{\mathrm{M,s}} = \frac{n_{\mathrm{s}}}{\left(2\pi/M_{\mathrm{s}}\right)^{3/2} T_{\mathrm{s}}^{3/2}} \exp\left\{-\frac{M_{\mathrm{s}}\left(\mathbf{v} - \mathbf{V}_{\mathrm{s}}\right)^{2}}{2T_{\mathrm{s}}}\right\}, \tag{30}$$

where $\mathbf{V}_{\mathrm{s}} = R\Omega_{\mathrm{s}}\mathbf{e}_{\varphi}$ is the leading-order drift velocity carried by $f_{\mathrm{M,s}}$. Here n_{s} represents the leading-order species number density and is given by

$$n_{\mathrm{s}} \equiv \eta_{\mathrm{s}} \exp\left[\frac{\frac{M_{\mathrm{s}}}{2}R^{2}\Omega_{\mathrm{s}}^{2} + \frac{Z_{\mathrm{s}}e}{c}\psi\Omega_{\mathrm{s}} - M_{\mathrm{s}}\Phi_{\mathrm{G}}}{T_{\mathrm{s}}}\right], \tag{31}$$

with η_{s} being referred to as the pseudo-density. The leading-order structure functions Λ_{s} coincide now with the set of functions $\Lambda_{\mathrm{s}} \equiv \left(\eta_{\mathrm{s}}, T_{\mathrm{s}}, \Omega_{\mathrm{s}}\right)$ which are defined in the configuration space, with T_{s} and Ω_{s} being respectively the leading-order species temperature and azimuthal rotation angular frequency. In addition, the quantities $h_{\mathrm{s}}^{(1)}$, $h_{\mathrm{s}}^{(2)}$ and $h_{\mathrm{s}}^{(3)}$ represent first-order kinetic corrections. In particular, $h_{\mathrm{s}}^{(1)}$ is referred to as FLR-diamagnetic contribution, $h_{\mathrm{s}}^{(2)}$ carries energy-correction contributions (with respect to both kinetic and ES potential energies), while $h_{\mathrm{s}}^{(3)}$ represents a purely ES term.

The precise expression of these functions will be given below in a separate section, where we discuss the relevance of these kinetic effects and their physical meaning in the framework of the present perturbative theory. For the moment, it is sufficient to say that all the first-order corrections are part of the kinetic equilibrium, and cannot be neglected for the consistent formulation of the solution. It must be also stressed here that Eq. (29) is very general: while $h_{\mathrm{s}}^{(3)}$ is non-vanishing for all the kinetic regimes considered above, the existence of $h_{\mathrm{s}}^{(1)}$ and $h_{\mathrm{s}}^{(2)}$ depends instead on the type of kinetic constraints. In particular, we distinguish the following features:

1) For gravitationally-bound plasmas (regime 1) $h_{\mathrm{s}}^{(1)} = 0$ and Λ_{s} is subject to the constraint

$$\Lambda_{\mathrm{s}} = \Lambda_{\mathrm{s}}\left(\Phi_{\mathrm{G}}\right). \tag{32}$$

2) For magnetized plasmas (regime 2) $h_{\mathrm{s}}^{(2)} = 0$ and the functional dependence of Λ_{s} becomes

$$\Lambda_{\mathrm{s}} = \Lambda_{\mathrm{s}}\left(\psi\right). \tag{33}$$

3) For gravitationally-bound and magnetized plasmas (regime 3) in general both $h_{\mathrm{s}}^{(1)} \neq 0$ and $h_{\mathrm{s}}^{(2)} \neq 0$, while for Λ_{s} one has in this case

$$\Lambda_{\mathrm{s}} = \Lambda_{\mathrm{s}}\left(\psi, \Phi_{\mathrm{G}}\right). \tag{34}$$

We notice that, to the leading-order, the equilibrium solution determined here does not depend on the ES potential, but only on the gravitational potential Φ_{G} and the magnetic flux $\psi \cong \psi_{\mathrm{ext}}$, which are assigned and known functions of the spatial coordinates.

The perturbative theory developed here represents the starting point for the application of the kinetic theory to the modelling of off-equatorial plasma tori in axisymmetric disc systems. This is based on the analysis of the spatial dependences which characterize the leading-order kinetic solution and that can be dealt with analytically thanks to the fundamental EM ordering assumptions introduced in Section 3. In particular, for each of the three regimes considered here, the proof that the kinetic equilibria admit off-equatorial solutions follows by analyzing the spatial profile of the leading-order number density defined by Eq. (31) under the requirement of having maxima out of the equatorial plane, namely for $\theta \neq \frac{\pi}{2}$. This requires preliminary to assign the functional form of the structure functions Λ_{s} characterizing Eq. (31) in terms of the potentials ψ and/or Φ_{G}. A detailed discussion of this type for all of the three plasma regimes is beyond the scope of this work and will be addressed separately in future studies. For the purpose of the present investigation, it is sufficient to consider here the case of regime 3. In fact, this provides the most general conditions for the occurrence of levitating structures, while regimes 1 and 2 can be viewed as special realizations of regime 3. In particular, we notice that the latter is expected to represent also the most plausible realization in real systems, in which both the gravitational and magnetic fields contribute to determine the profiles of the fluid fields.

Let us then discuss the case of plasmas belonging to regime 3. The number density profile is prescribed according to Eqs. (31) and (34). We notice that, thanks to the analytical relationships (14) and (15) and the orderings (12)−(13), any function of $(\psi, \Phi_{\mathrm{G}})$ can be equivalently expressed in terms of (r, θ). Because of this, the rhs of Eq. (31) becomes now a generic function of (r, θ), namely of the form $n_{\mathrm{s}} = n_{\mathrm{s}}(r, \theta)$. This represents the most general kind of spatial dependence which is admitted by the kinetic equilibrium. Hence, in the general case and in the absence of other particular restrictions, suitable prescriptions of $n_{\mathrm{s}}(r, \theta)$ can be determined for regime 3 plasmas, according to the real system to be studied, which admit maxima out of the equatorial plane. This conclusion has a general character of validity and assures the consistency of the kinetic theory presented here for collisionless axisymmetric plasmas with the possible occurrence of levitating tori in the external gravitational and magnetic fields of the type prescribed above.

We can now explore in more detail the present conclusion by considering explicitly two possible physical realizations of this type of solution:

Case A: In this first example we assume that both η_{s} and T_{s} in Eq. (31) are constant. From the physical point of view, the requirement $\eta_{\mathrm{s}} = const.$ means that the spatial variations of the number density are uniquely determined by the exponential term (Maxwellian factor), which in turn depends on the leading-order plasma temperature as well as on the rotational frequency, gravitational potential and magnetic flux ψ. This choice is consistent with previous literature (see for example Schartmann et al. (2005); Szuszkiewicz et al. (1997, 2001)).

Concerning the condition $T_{\mathrm{s}} = const.$, this corresponds to a leading-order plasma isothermal profile that is consistent with the kinetic constraints that characterize the solution (see Section 7). We remark that in the present framework, the isothermal condition can only be satisfied to the leading-order, while for non-uniform plasmas, the full temperature profile is generally non-constant because of higher-order kinetic effects. These issues will be discussed in detail in Sections 7 and 8. In validity of the prescription of constant η_{s} and T_{s}, the only freedom left concerns the functional dependence of Ω_{s}, which is considered of the form (34).

Under this assumption, the number density still remains of the type $n_{\mathrm{s}}(r, \theta)$. In this situation, it is convenient to prescribe $n_{\mathrm{s}}(r, \theta)$ consistent with the requirement of exhibiting maxima out of the equatorial plane, independently of its actual representation given by Eq. (31). The prescription of a physically-acceptable profile of $n_{\mathrm{s}}(r, \theta)$ must be done in such a way to reproduce observational or experimental data. Once the profile of $n_{\mathrm{s}}(r, \theta)$ is set, since also η_{s} and T_{s} are constant in this example, then Eq. (31) can be inverted and used to uniquely derive the expression of the corresponding species angular frequency $\Omega_{\mathrm{s}} = \Omega_{\mathrm{s}}(r, \theta)$ which determines the levitating structure. In particular, the latter is obtained by solving the quadratic algebraic equation

$$\frac{M_{\mathrm{s}}}{2} R^2 \Omega_{\mathrm{s}}^2 + \frac{Z_{\mathrm{s}} e}{c} \psi \Omega_{\mathrm{s}} - M_{\mathrm{s}} \Phi_{\mathrm{G}} - T_{\mathrm{s}} \ln \frac{n_{\mathrm{s}}}{\eta_{\mathrm{s}}} = 0. \tag{35}$$

Hence, under these conditions, it is possible to introduce a density profile which is in agreement with physical configurations and the existence of off-equatorial tori. For leading-order isothermal systems this also prescribes the form of the corresponding plasma rotation frequency according to Eq. (35). The extension of this solution method to the case of a non-isothermal plasma species recquires the additional prescription of the temperature profile, namely $T_{\mathrm{s}} = T_{\mathrm{s}}(r, \theta)$, while the frequency Ω_{s} can still be obtained from Eq. (35). In both cases we notice that the kinetic equilibrium thus determined generally allows for the existence of two separate roots for Ω_{s}. If both are real, they should correspond to two different admissible equilibria with opposite direction of plasma rotation with respect to the external dipolar magnetic field orientation.

Case B: As a second example, we assume the validity of the kinetic constraint Eq. (34) for all the three structure functions. In particular, here we treat the situation in which the condition

$$n_{\mathrm{s}}(r, \theta) \equiv \eta_{\mathrm{s}}(r, \theta) \tag{36}$$

is identically satisfied in the configuration domain occupied by the collisionless plasma species. From the physical point of view, Eq. (36) means that the number density profile n_{s} is not modified by the Maxwellian exponential factor and coincides with the pseudo-density $\eta_{\mathrm{s}}(r, \theta)$. This requirement is satisfied when the exponential factor in Eq. (31) is one. Hence, this condition is met for the species angular frequency satisfying the algebraic quadratic equation

$$\frac{M_{\mathrm{s}}}{2} R^2 \Omega_{\mathrm{s}}^2 + \frac{Z_{\mathrm{s}} e}{c} \psi \Omega_{\mathrm{s}} - M_{\mathrm{s}} \Phi_{\mathrm{G}} = 0. \tag{37}$$

We notice that again Eq. (37) generates two roots for the frequency Ω_{s}, as for the case A discussed above. In addition, Eq. (37) holds for both isothermal and non-isothermal plasmas. Finally, in validity of the $\sigma_{\rm s}-$ordering and the ordering (13), when $\ln\frac{n_{\rm s}}{\eta_{\rm s}}\sim O(1)$, one can infer that the solutions of $\Omega_{\rm s}$ from Eq. (37) are asymptotically close to those from Eq. (35), although the number density and the temperature are not necessarily so.

To conclude this section, it is useful to make a qualitative comparison of the results obtained here with those presented in Ref. Slaný et al. (2013), where the existence of off-equatorial structures has been proved on the basis of a fluid non-ideal MHD description. This involves in particular the inspection of Eq. (41) for the pressure profile given in Ref. Slaný et al. (2013), which can give rise to off-equatorial maxima for suitable choices of the coefficients entering the same equation (see discussions in sections 3 and 4 in the same reference). Indeed, pressure and density profiles are proportional (at least to the leading-order) when the condition $T_{\rm s}=const.$ applies (see also Eq. (46) below). In such a case, it is immediate to verify that the rhs of Eq. (41) in Ref. Slaný et al. (2013) can be effectively expressed as a function of ψ and $\Phi_{\rm G}$ only, in agreement with the prescription holding for regime 3 plasmas. Although the two treatments (i.e., the present one and Ref. Slaný et al. (2013)) consider different physical conditions for the levitating plasma, the consistency pointed out here establishes a notable result. In fact, first it shows that, as anticipated in the Introduction, the kinetic theory developed in this paper and its analytical formulation allow for direct comparisons with previous literature works based on fluid approaches. Second, it proves that fluid results can in principle be reproduced consistently on the basis of a kinetic treatment, thus extending their validity to a wider class of plasma regimes. Third, in turn it reinforces the statement given above concerning the general character of the present kinetic theory for regime 3 plasmas in providing a suitable mathematical and physical framework for the investigation of off-equatorial structures.

7. KINETIC CORRECTIONS

In this section we provide the explicit representation of the kinetic corrections $h_{\rm s}^{(1)}$, $h_{\rm s}^{(2)}$ and $h_{\rm s}^{(3)}$ introduced in the Chapman-Enskog representation of the equilibrium KDF given by Eq. (29). The inclusion of these contributions is necessary for the complete solution of the equilibrium problem in the framework of Vlasov-Maxwell description of collisionless plasmas. In particular, these terms represent the deviations of the KDF from a Maxwellian distribution and arise because of the constraints imposed by single-particle phase-space conservation laws on the solution itself. The precise definition of the first-order terms $h_{\rm s}^{(1)}$, $h_{\rm s}^{(2)}$ and $h_{\rm s}^{(3)}$ is also required to distinguish the solution among the three kinetic regimes pointed out above.

In detail, the first-order corrections $h_{\rm s}^{(1)}$ and $h_{\rm s}^{(2)}$ originate from the perturbative treatment of the implicit phase-space dependences carried by the structure-functions $\Lambda_{*{\rm s}}$ entering the equilibrium KDF $f_{*{\rm s}}$. They are found to be given by:

$$h_{\rm s}^{(1)}\equiv\frac{cM_{\rm s}}{Z_{\rm s}e}R\left[A_{1{\rm s}}+\frac{p_{\varphi{\rm s}}\Omega_{\rm s}}{T_{\rm s}}A_{2{\rm s}}\right]v_{\varphi}+\frac{cM_{\rm s}}{Z_{\rm s}e}R\left[\frac{E_{\rm s}-\Omega_{\rm s}p_{\varphi{\rm s}}}{T_{\rm s}}-\frac{3}{2}\right]A_{3{\rm s}}v_{\varphi}, \quad (38)$$

$$h_{\rm s}^{(2)}\equiv\left[\frac{E_{\rm s}-\Omega_{\rm s}p_{\varphi{\rm s}}}{T_{\rm s}}-\frac{3}{2}\right]C_{3{\rm s}}\left(\frac{1}{2}v^{2}+\frac{Z_{\rm s}e}{M_{\rm s}}\Phi\right)+\left[C_{1{\rm s}}+\frac{p_{\varphi{\rm s}}\Omega_{\rm s}}{T_{\rm s}}C_{2{\rm s}}\right]\left(\frac{1}{2}v^{2}+\frac{Z_{\rm s}e}{M_{\rm s}}\Phi\right), \quad (39)$$

where the following definitions have been introduced:

$$A_{1{\rm s}}\equiv\frac{\partial\ln\eta_{\rm s}}{\partial\psi},A_{2{\rm s}}\equiv\frac{\partial\ln\Omega_{\rm s}}{\partial\psi},A_{3{\rm s}}\equiv\frac{\partial\ln T_{\rm s}}{\partial\psi}, \quad (40)$$

$$C_{1{\rm s}}\equiv\frac{\partial\ln\eta_{\rm s}}{\partial\Phi_{\rm G}},C_{2{\rm s}}\equiv\frac{\partial\ln\Omega_{\rm s}}{\partial\Phi_{\rm G}},C_{3{\rm s}}\equiv\frac{\partial\ln T_{\rm s}}{\partial\Phi_{\rm G}}. \quad (41)$$

Hence, $h_{\rm s}^{(1)}$ and $h_{\rm s}^{(2)}$ are polynomial functions of the particle velocity which contain diamagnetic and energy-correction contributions and depend on the so-called thermodynamic forces $\frac{\partial\Lambda_{\rm s}}{\partial\psi}$ and $\frac{\partial\Lambda_{\rm s}}{\partial\Phi_{\rm G}}$. The latter represent the gradients of the structure functions across equipotential magnetic and gravitational surfaces and arise in collisionless plasmas characterized by non-uniform fluid fields. Consistency of these expressions with the $\varepsilon_{\rm s}$ and $\sigma_{\rm s}$ ordering assumptions requires that

$$\frac{cM_{\rm s}}{Z_{\rm s}e}R\left[\left(\frac{E_{\rm s}-\Omega_{\rm s}p_{\varphi{\rm s}}}{T_{\rm s}}-\frac{3}{2}\right)A_{3{\rm s}}\right]v_{\varphi}\lesssim O(\varepsilon_{\rm s}), \quad (42)$$

$$\left[\frac{E_{\rm s}-\Omega_{\rm s}p_{\varphi{\rm s}}}{T_{\rm s}}-\frac{3}{2}\right]C_{3{\rm s}}\left(\frac{1}{2}v^{2}+\frac{Z_{\rm s}e}{M_{\rm s}}\Phi\right)\lesssim O(\sigma_{\rm s}), \quad (43)$$

which implies that $T_{\rm s}$ must actually be of the form $T_{\rm s}=T_{\rm s}\left(\varepsilon_{\rm s}^{k}\psi,\sigma_{\rm s}^{k}\Phi_{\rm G}\right)$, with $k\geq1$, i.e. at most slowly-dependent on ψ and $\Phi_{\rm G}$. This conclusion motivates the choice done in Section 6 to treat isothermal plasmas (to leading-order). As pointed out above, the contribution $h_{\rm s}^{(1)}$ is null for gravitationally-bound plasmas, while $h_{\rm s}^{(2)}$ vanishes for magnetized plasmas. Instead, both terms are present in the equilibrium solution for plasmas belonging to regime 3. We also notice that the $\sigma_{\rm s}-$expansion generates contributions in $h_{\rm s}^{(2)}$ which are proportional to both particle kinetic energy and ES potential. In particular, terms which depend on Φ contribute to the occurrence of ES corrections to the kinetic solution and the corresponding fluid fields.

Finally, the last contribution $h_{\rm s}^{(3)}$ originates from the validity of the $\lambda-$ordering (13) when this is taken into account in the expression for the leading-order number density, and results in the following term

$$h_{\rm s}^{(3)}\equiv\frac{Z_{\rm s}e\Phi}{M_{\rm s}\Phi_{\rm G}}. \quad (44)$$

It must be stressed that the ES contributions arising in $h_{\rm s}^{(2)}$ and $h_{\rm s}^{(3)}$ originate from different perturbative treatments. In fact, $h_{\rm s}^{(3)}$ follows from the $\lambda-$ordering and is common to all the regimes considered here when Eq. (13) applies. Instead the terms in $h_{\rm s}^{(2)}$ can only be included when the σ_s-ordering applies (regimes 1 and 3).

To conclude the section, it is worth pointing out that the treatment of the first-order kinetic corrections displayed here requires the following preliminary steps:

1) The precise identification of the appropriate plasma collisionless kinetic regime.
2) The prescription of the leading-order spatial profiles of the structure functions, consistent with the kinetic constraints for each regime.
3) The evaluation of the thermodynamic forces and the explicit calculation of the ES potential.
In particular, the existence of the equilibria determined here is subject to the validity of the Maxwell equations, i.e. the Poisson equation for the ES potential Φ and Ampere's equation (see Section 9).

8. EQUATION OF STATE

In this section we proceed with the calculation of the EoS corresponding to the kinetic equilibrium determined here. This requires in particular to compute the species pressure tensor $\underline{\underline{\Pi}}_{\mathrm{s}}$ carried by the KDF $f_{*\mathrm{s}}$ and defined as

$$\underline{\underline{\Pi}}_{\mathrm{s}} \equiv M_{\mathrm{s}} \int_{\Gamma_{\mathbf{v}}} d\mathbf{v}\left(\mathbf{v}-\mathbf{V}_{\mathrm{s}}\right)\left(\mathbf{v}-\mathbf{V}_{\mathrm{s}}\right) \mathrm{f}_{*\mathrm{s}}, \tag{45}$$

where $\Gamma_{\mathbf{v}}$ denotes the velocity domain of integration. Since $f_{*\mathrm{s}}$ is isotropic with respect to quadratic particle velocity dependences, one can immediately infer that each species pressure tensor is isotropic and of the form $\underline{\underline{\Pi}}_{\mathrm{s}} = p_{\mathrm{s}}^{\mathrm{tot}} \underline{\underline{\mathbf{I}}}$, where $p_{\mathrm{s}}^{\mathrm{tot}} = n_{\mathrm{s}}^{\mathrm{tot}} T_{\mathrm{s}}^{\mathrm{tot}}$ denotes the thermal scalar pressure, with $n_{\mathrm{s}}^{\mathrm{tot}}$ and $T_{\mathrm{s}}^{\mathrm{tot}}$ being respectively the species total number density and temperature associated with $f_{*\mathrm{s}}$. The calculation of $p_{\mathrm{s}}^{\mathrm{tot}}$ can be carried out analytically for thermal particles when the Chapman-Enskog representation (29) applies. In the following we consider this case. Furthermore, consistent with the ε_{s} and σ_{s} orderings, in the first-order terms $h_{\mathrm{s}}^{(1)}$ and $h_{\mathrm{s}}^{(2)}$ we approximate the canonical momentum $p_{\varphi\mathrm{s}}$ and the energy E_{s} respectively with $\frac{Z_{\mathrm{s}}e}{c}\psi$ and $M_{\mathrm{s}}\Phi_{\mathrm{G}}$. Hence, under these assumptions, one can prove that the scalar pressure can be represented as

$$p_{\mathrm{s}}^{\mathrm{tot}} = n_{\mathrm{s}} T_{\mathrm{s}}\left[1+\sigma_{\mathrm{s}} \Delta p_{\mathrm{s}}^{(2)}+\lambda h_{\mathrm{s}}^{(3)}\right], \tag{46}$$

where $p_{\mathrm{s}} \equiv n_{\mathrm{s}} T_{\mathrm{s}}$ is the leading-order term, with n_{s} being defined by Eq. (31). In addition we notice that the term $h_{\mathrm{s}}^{(1)}$ associated with the $\varepsilon_{\mathrm{s}}-$expansion does not contribute to the EoS because it is odd in the azimuthal component of particle velocity. Instead, $h_{\mathrm{s}}^{(3)}$ does not depend explicitly on particle velocity and therefore it is not affected when the integral (45) is computed on $\Gamma_{\mathbf{v}}$. Hence, its contribution in Eq. (46) is simply proportional to p_{s} and represents part of the ES corrections which enter the definition of the total pressure $p_{\mathrm{s}}^{\mathrm{tot}}$. Finally, invoking Eq. (39), explicit calculation gives for $\Delta p_{\mathrm{s}}^{(2)}$ the following result:

$$\Delta p_{\mathrm{s}}^{(2)} = \left(2 \frac{Z_{\mathrm{s}} e}{M_{\mathrm{s}}} \Phi+4 \frac{T_{\mathrm{s}}}{M_{\mathrm{s}}}\right) Y_{\mathrm{s}}, \tag{47}$$

where

$$Y_{\mathrm{s}} \equiv C_{1\mathrm{s}}+\frac{\frac{Z_{\mathrm{s}} e}{c} \psi \Omega_{\mathrm{s}}}{T_{\mathrm{s}}} C_{2\mathrm{s}}+\left(\frac{M_{\mathrm{s}} \Phi_{\mathrm{G}}-\Omega_{\mathrm{s}} \frac{Z_{\mathrm{s}} e}{c} \psi}{T_{\mathrm{s}}}-\frac{3}{2}\right) C_{3\mathrm{s}}. \tag{48}$$

From this result it is interesting to point out that, although to the leading-order the species pressure coincides with the thermal pressure, the first-order corrections introduce deviations in the EoS that are distinctive for collisionless plasmas. In particular, the energy-correction contributions which enter through $\Delta p_{\mathrm{s}}^{(2)}$ are associated with the gradients of structure functions across gravitational equipotential surfaces and include also ES corrections proportional to Φ. These terms however vanish in uniform collisionless plasmas. On the other hand, the ES correction to the EoS carried by $h_{\mathrm{s}}^{(3)}$ is independent and follows uniquely from the $\lambda-$ordering introduced above between ES and gravitational potential energy. Clearly, all the first-order contributions in the EoS arise as specifically-kinetic effects, which characterize the kinetic treatment of collisionless plasmas.

9. THE MAXWELL EQUATIONS

In this section we analyze the constraints which are posed by the Maxwell equations on the kinetic equilibria. These concern in particular the validity of the orderings (12) and (13) and for this reason they apply to all the plasma regimes identified above.

We consider first the Poisson equation for the ES potential, which is written as

$$\nabla^2 \Phi = -4\pi \sum_{\mathrm{s}} Z_{\mathrm{s}} e n_{\mathrm{s}}^{\mathrm{tot}}. \tag{49}$$

In the general case the solution is non-trivial, because the total number density n_s^{tot} depends both implicitly and explicitly on the ES potential itself (see for example Cremaschini et al. (2011a,b)). However, the solution simplifies in validity of the sub-ordering expansion (13) introduced above. In fact in this case the ES potential enters the kinetic solution only through the first-order corrections to the equilibrium KDF. Therefore, consistent with the orderings introduced in the present work and the perturbative theory developed here, one can obtain an asymptotic solution for Φ by considering only the leading-order contribution to the species number density. Thus, when the said sub-ordering applies, neglecting corrections of $O\left(\sigma_{\mathrm{s}}\right)$ and $O\left(\zeta_{\mathrm{s}}\right)$ and invoking Eq. (31), the Poisson equation becomes to this accuracy:

$$\nabla^2 \Phi = S\left(\mathbf{x}\right), \tag{50}$$

where the source term $S\left(x\right)$ is defined as

$$S\left(\mathbf{x}\right) \equiv -4\pi \sum_{\mathrm{s}} Z_{\mathrm{s}} e \eta_{\mathrm{s}} \exp\left[\frac{\frac{M_{\mathrm{s}}}{2} R^2 \Omega_{\mathrm{s}}^2+\frac{Z_{\mathrm{s}} e}{c} \psi \Omega_{\mathrm{s}}-M_{\mathrm{s}} \Phi_{\mathrm{G}}}{T_{\mathrm{s}}}\right]. \tag{51}$$

Here $S\left(\mathbf{x}\right)$ does not depend on Φ, and therefore the ES potential can be readily obtained by integrating Eq. (50) yielding

$$\Phi\left(\mathbf{x}\right) = \int d\mathbf{x}' \mathrm{G}\left(\mathbf{x}-\mathbf{x}'\right) \mathrm{S}\left(\mathbf{x}'\right), \tag{52}$$

with $G\left(\mathbf{x}-\mathbf{x}'\right)$ being the corresponding Green function. For the consistency of the theory, the solution for Φ given by the previous equation must be checked "a posteriori" to satisfy the initial ordering (13). In particular, this can represent a constraint condition for the magnitude of the species number densities which contribute to the ES potential through the system charge density (51). Manifestly, the validity of the ordering (13) is necessary for the present theory to apply, and for this reason the calculation of Φ represents the ultimate step to be done in order to warrant the consistency of the treatment.

Similar considerations apply to the Ampere equation, which determines the self-generation of magnetic field by the equilibrium collisionless plasma. The Ampere equation is written as

$$\nabla \times \mathbf{B}^{\mathrm{self}} = \frac{4\pi}{c}\mathbf{J}^{\mathrm{tot}}, \tag{53}$$

where $\mathbf{B}^{\mathrm{self}}$ is defined in Eq. (8) and $\mathbf{J}^{\mathrm{tot}}$ is the total current density, which is given by

$$\mathbf{J}^{\mathrm{tot}} \equiv \sum_{\mathrm{s}} \mathbf{J}_{\mathrm{s}}^{\mathrm{tot}} = \sum_{\mathrm{s}} Z_{\mathrm{s}} e n_{\mathrm{s}}^{\mathrm{tot}} \mathbf{V}_{\mathrm{s}}^{\mathrm{tot}}, \tag{54}$$

with $\mathbf{V}_{\mathrm{s}}^{\mathrm{tot}}$ being the species flow velocity. It is immediate to prove that, in the present formulation, $\mathbf{V}_{\mathrm{s}}^{\mathrm{tot}}$ is purely azimuthal at equilibrium; in fact additional components of the velocity can only arise in the presence of temperature anisotropy, see for example Cremaschini et al. (2010, 2011a,b). Again, consistent with the perturbative treatment presented here, one can retain only the leading-order contributions to $\mathbf{J}^{\mathrm{tot}}$ in Eq. (53). Under this assumption Eq. (53) becomes

$$\nabla \times \mathbf{B}^{\mathrm{self}} = \frac{4\pi}{c} \sum_{\mathrm{s}} Z_{\mathrm{s}} e n_{\mathrm{s}} \mathbf{V}_{\mathrm{s}}, \tag{55}$$

where n_{s} is given by Eq. (31) and $\mathbf{V}_{\mathrm{s}} = R\Omega_{\mathrm{s}}\mathbf{e}_{\varphi}$ (see Eq. (30)). Eq. (55) represents a generalized Grad-Shafranov equation for the poloidal magnetic flux ψ_{self} in which the source term on the rhs depends only on explicitly known quantities. The solution for $\mathbf{B}^{\mathrm{self}}$ which results from Eq. (55) must then be checked "a posteriori" to verify the ordering condition (12) introduced above, which is necessary in order to warrant the validity of the theory and its analytical development. In this case Eq. (12) can represent a constraint for the magnitude of the species rotation angular frequencies which contribute to the system charge current.

10. CONCLUSIONS

In this paper, a theoretical investigation of equilibrium configurations for collisionless non-relativistic and axisymmetric plasmas has been presented, taking into account the role of a central spherically-symmetric gravitational field. The formulation is based on a multi-species kinetic approach developed in the framework of the Vlasov-Maxwell description. The case of astrophysical plasmas arising in the gravitational field of compact objects and in the presence of both an external dipolar magnetic field and self electromagnetic fields has been treated.

Three different plasma regimes have been identified which are charactetrized by distinctive kinetic orderings. It has been proved that in all cases consistent kinetic equilibria can be determined, with the kinetic distribution function being expressed in terms of generalized Maxwellian functions. It has been shown that the three regimes differ by the form of the kinetic constraints which are imposed on the equilibrium solutions and which uniquely follow from phase-space single-particle conservation laws.

By imposing appropriate orderings on the self electromagnetic fields and by developing a suitable perturbative theory, an analytical treatment of the equilibria has been proposed. In terms of this, several issues have been addressed. First, the conditions of existence of equilibrium structures corresponding to off-equatorial tori have been investigated. It has been shown that these systems can generally arise for the regime which has been referred to here as magnetized and gravitationally-bound plasmas. This analysis can be important from the astrophysical point of view, since off-equatorial tori may represent a physically-realizable model of magnetized coronal plasmas which are believed to characterize accretion discs. In addition, the treatment based on kinetic theory can pose the basis for comparison with analogous fluid results carried out in terms of MHD theory.

As a second application, the plasma equation of state has been determined analytically and expressed in terms of the pressure tensor. It has been shown that the latter exhibits deviations from the thermal pressure characteristic of collisional plasmas because of the existence of specifically-kinetic effects. These have been identified with diamagnetic, energy-correction and electrostatic contributions which apply in combination with the occurrence of non-uniform fluid fields.

Finally, the validity of the Poisson and Ampere equations have been addressed, showing that they can introduce non-trivial constraints on the magnitude of the plasma number density and flow velocity for the consistency with the orderings introduced in the theory developed here.

The conclusions established in this work can be relevant for future investigations of astrophysical plasmas in equilibrium configurations, with particular focus on collisionless plasmas in accretion discs and off-equatorial tori associated with compact objects.

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