

Perturbative series and the $1/N$ expansion for the QED β -function

S.A. Larin

Institute for Nuclear Research of the Russian Academy of Sciences,
60th October Anniversary Prospect 7a, Moscow 117312, Russia

Abstract

A comparison of the perturbative series and the $1/N$ expansion for the QED renormalization group β -function in the Minimal Subtraction scheme is performed. The good agreement between two expansions is found which proves that the MS β -function is under perfect perturbative control.

1. The nature of perturbative series in Quantum Electrodynamics still remains an unresolved question, although one can believe that they are asymptotic sign-alternating series. Then one can hope that the error of the truncated series of this type is estimated by the value of the first truncated (or the last included) term of the expansion. Since QED is the cornerstone of modern Quantum Field Theory it is rather important to obtain as much information as possible concerning its perturbative expansions.

Quite recently the 5-loop approximation for the QED renormalization group β -function in different renormalization schemes was obtained, first for one active lepton [1] and then for an arbitrary number N_F of flavors [2, 3]. These results are obtained after more than twenty years since the calculation of the 4-loop order [4].

On the other hand there is a calculation [5] of the first nontrivial leading term of the $1/N_F$ expansion for the QED β -function in the MS-scheme. It is quite interesting to compare the available 5-loop perturbative series and the $1/N_F$ series. This is the purpose of the present letter.

2. Let us first cite the $1/N_F$ result for the β -function from the work [5]

$$\begin{aligned}
\beta(K) &= \frac{2}{3}K + \frac{1}{N_F} \frac{1}{2}K \int_{-K/3}^0 dx \frac{\Gamma(4+2x)(1+2x)(1+2x/3)(1-x)}{[\Gamma(2+x)]^2 \Gamma(3+x)\Gamma(1-x)} + O(1/N_F^2) \\
&= \frac{2}{3}K + \frac{K^2}{2N_F} \left[1 - \frac{11}{2} \frac{1}{3} \left(\frac{K}{3}\right) - \frac{7 \cdot 11}{2^2 \cdot 3^2} \frac{1}{3} \left(\frac{K}{3}\right)^2 \right. \\
&\quad + \left(\frac{107}{2^3 \cdot 3^2} + 2\zeta(3) \right) \frac{1}{4} \left(\frac{K}{3}\right)^3 + \left(\frac{251}{2^4 \cdot 3^2} - \frac{11}{3}\zeta(3) + 3\zeta(4) \right) \frac{1}{5} \left(\frac{K}{3}\right)^4 \\
&\quad + \left(\frac{67}{2^5} - \frac{7 \cdot 11}{2 \cdot 3^2}\zeta(3) - \frac{11}{2}\zeta(4) + 2 \cdot 3\zeta(5) \right) \frac{1}{6} \left(\frac{K}{3}\right)^5 \\
&\quad + \left(\frac{5 \cdot 7 \cdot 41}{2^6 \cdot 3^2} + \frac{107}{2^2 \cdot 3^2}\zeta(3) - \frac{7 \cdot 11}{2^2 \cdot 3}\zeta(4) - 11\zeta(5) \right. \\
&\quad \left. + 2 \cdot 5\zeta(6) - 2\zeta^2(3) \right) \frac{1}{7} \left(\frac{K}{3}\right)^6 + \dots \left. \right] + O(1/N_F^2),
\end{aligned} \tag{1}$$

here $K \equiv \alpha N_F / \pi$ is the coupling which has to be held fixed in the large N_F limit, α being the fine structure constant.

The function $\beta(K)$ is defined as

$$\alpha\beta(K) = \mu \frac{d}{d\mu} \alpha(\mu), \tag{2}$$

where μ is the renormalization scale.

In the numerical form the result of the equation (1) reads

$$\begin{aligned} \beta(K) = & \frac{2}{3}K + \frac{K^2}{2N_F} \left(1 - 3.055555556 \cdot 10^{-1}K - 7.921810700 \cdot 10^{-2}K^2 \right. \\ & + 3.602060109 \cdot 10^{-2}K^3 + 1.438230317 \cdot 10^{-3}K^4 - 1.906442773 \cdot 10^{-3}K^5 \\ & \left. + 1.521260392 \cdot 10^{-4}K^6 + 3.588903124 \cdot 10^{-5}K^7 + \dots \right) + O(1/N_F^2). \end{aligned}$$

The radius of convergence of the $\beta(K)$ expansion is $K = 15/2$. The authors of the work [5] checked numerically that the $1/N_F$ term has the only zero at $K = 0$ and is positive in the convergence region. They also found that for the physical value $N_F = 3$ the $1/N_F$ term is never larger than 15% of the leading term $2K/3$.

Let us now cite the 5-loop result for the β -function in the MS-scheme from the work [2]. In the normalization of eq.(2) it is

$$\begin{aligned} \beta = & 8\pi \frac{1}{\alpha} \left\{ N_F \left[\frac{4A^2}{3} \right] + 4N_F A^3 - A^4 \left[2N_F + \frac{44}{9}N_F^2 \right] \right. \\ & + A^5 \left[-46N_F + \frac{760}{27}N_F^2 - \frac{832}{9}\zeta(3)N_F^2 - \frac{1232}{243}N_F^3 \right] \\ & + A^6 \left(N_F \left[\frac{4157}{6} + 128\zeta(3) \right] + N_F^2 \left[= \frac{7462}{9} - 992\zeta(3) + 2720\zeta(5) \right] \right. \\ & \left. \left. + N_F^3 \left[-\frac{21758}{81} + \frac{16000}{27}\zeta(3) - \frac{416}{3}\zeta(4) - \frac{1280}{3}\zeta(5) \right] + N_F^4 \left[\frac{856}{243} + \frac{128}{27}\zeta(3) \right] \right) \right\} \end{aligned} \quad (3)$$

where $A \equiv \frac{\alpha}{4\pi}$.

The numerical form of the above equation for the value $N_F = 3$ is

$$\beta = 0.63662\alpha + 0.151982\alpha^2 - 0.050393\alpha^3 - 0.0819407\alpha^4 + 0.0412278\alpha^5, \quad (4)$$

this is the monotonically increasing function for $\alpha > 0$.

We will compare eq.(1) and eq.(3) for $N_F = 3$. For $\alpha = 1/137$ we have $\beta = 0.00465494$ for both equations; for $\alpha = 0.1$ the result is $\beta = 0.0651364$ for eq.(1) and $\beta = 0.0651236$ for eq.(3); for $\alpha = 0.2$ one gets 0.133032 and 0.132882 correspondingly; for $\alpha = 1$ one gets 0.737883 for eq.(1) and 0.697496 for eq.(3).

We see that even for $\alpha = 1$ when the convergence of the series (4) is quite questionable both results agree within 5%. Thus two different expansions

(the usual perturbative series and the $1/N_F$ series) give numerically very close values for the QED β -function in the wide interval of α . It definitely indicates that both expansions give good approximations for $\beta(\alpha)$.

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