

Study of Possible Ultraviolet Zero of the Beta Function in Gauge Theories with Many Fermions

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We study the possibility of an ultraviolet (UV) zero in the n -loop beta function of U(1) and non-Abelian gauge theories with N_f fermions for large N_f . The effect of scheme transformations on the coefficients of different powers of N_f in the n -loop term in the beta function is calculated. A general criterion is given for determining whether or not the n -loop beta function has a UV zero for large N_f . We compare the results with exact integral representations of the leading terms in the beta functions for the respective Abelian and non-Abelian theories in the limit $N_f \rightarrow \infty$ limit with $N_f\alpha$ finite. As part of this study, new analytic and numerical results are presented for certain coefficients, denoted $b_{n,n-1}$, that control the large- N_f behavior at n -loop order in the beta function. We also investigate various test functions incorporating a power-law and essential UV zero in the beta function and determine their manifestations in series expansions in powers of coupling and in powers of $1/N_f$.

I. INTRODUCTION

The dependence of the running coupling constant in a (four-dimensional, zero-temperature) gauge theory on the Euclidean momentum scale μ is of fundamental field-theoretic interest. This dependence is determined by the beta function of the theory [1]. In this paper we will consider a U(1) gauge theory with N_f fermions of a given charge and a non-Abelian gauge theory with gauge group G and N_f fermions transforming according to a representation R of G , in the limit of large N_f . The fermions are assumed to have masses that are zero or negligibly small relative to the relevant range of scales μ . Both of these theories have positive beta functions for small gauge coupling where they are perturbatively calculable, so they are infrared-free in this region of coupling.

We present several new results here. First, we investigate the question of a possible ultraviolet zero in a U(1) gauge theory further, extending our recent study in [2]. We analyze zeros of the n -loop β function as a function of α for a large range of N_f . We use our computations to test and confirm an approximate analytic solution to the equation for a UV zero of the n -loop β function. Using the results of [2, 3], we calculate the effect of scheme transformations on the coefficients of different powers of N_f in the n -loop term in the beta function for U(1) and non-Abelian gauge theories. We deduce a general criterion for the existence of a UV zero in the n -loop β function for large N_f applicable for both Abelian and non-Abelian gauge theories. We compare the results with implications from an exact integral representation for the leading term, denoted F_1 , in the beta function of a U(1) theory at large- N_f . Up to loop order $n = 24$ to which we probe, we do not find evidence for a stable UV zero in the U(1) beta function that would be reached from small coupling as μ is increased. We provide some insight into this by showing that the coefficients, denoted $b_{n,n-1}$, of the leading part of the n -loop term, b_n , in the beta func-

tion show a scatter of positive and negative signs, while the $b_{n,n-1}^{(d)}$ that correspond to the part of F_1 that might be responsible for a UV zero are all negative. Additional insight is obtained from calculations of series expansions obtained from various illustrative test functions for β . We also carry out a similar analysis for a non-Abelian gauge theory with fermions in various representations, in the large- N_f limit, reaching a similar conclusion.

We recall some relevant background and previous work. If a beta function of a quantum field theory is positive near zero coupling, the coupling grows as μ increases. However, the beta function may have an ultraviolet zero, so that the coupling approaches a constant as $\mu \rightarrow \infty$. An explicit example of this occurs in the $O(N)$ nonlinear σ model in $d = 2 + \epsilon$ spacetime dimensions. From an exact solution of this model in the limit $N \rightarrow \infty$ (involving a sum of an infinite number of Feynman diagrams that dominate in this limit), one finds that the beta function for small ϵ has the form [4]

$$\beta(\lambda) = \epsilon\lambda\left(1 - \frac{\lambda}{\lambda_c}\right), \quad (1.1)$$

where λ is the effective coupling and $\lambda_c = 2\pi\epsilon/N$. Thus, assuming that λ is small for small μ , it follows that as μ increases, λ approaches the UV fixed point at λ_c as $\mu \rightarrow \infty$.

It was observed early in the history of work on quantum electrodynamics (QED) that the property that the beta function of QED is positive for small couplings, where it is perturbatively calculable, implies that the theory is free as $\mu \rightarrow 0$ in the infrared (IR) [5]. The positive β function means that as μ increases toward the ultraviolet, the gauge coupling grows in strength. Indeed, integrating the one-loop renormalization group (RG) equation would yield a pole at finite μ in the ultraviolet, the Landau pole. Of course, one cannot reliably use the β function, perturbatively calculated to one-loop or even higher-loop order, for values of μ where the coupling gets

large, so there is no rigorous implication that the theory would, in fact, have a Landau pole. However, this led early researchers to inquire whether the β function of a U(1) gauge theory might exhibit a UV zero away from the origin. If such a UV zero of the beta function could be demonstrated reliably, then as the Euclidean scale μ increased, the gauge coupling would approach a finite value rather than continuing to increase, i.e., this would be a UV fixed point of the renormalization group. A necessary condition for the analysis to be reliable would be that the UV zero must occur at a reasonably small value of the gauge coupling. The calculation of the two-loop term in the β function for this theory [6] found that it was positive, like the one-loop term, and hence excluded the existence of a UV zero of β at this loop order.

The approach to the analysis of a possible UV zero in the beta function for the electromagnetic U(1)_{em} gauge theory changed after the development of the SU(2)_L \otimes U(1)_Y electroweak sector of the Standard Model (SM), since in the SM, the photon field $A_\mu^{(\gamma)}$ arises upon electroweak symmetry breaking as the linear combination $A_\mu^{(\gamma)} = \cos\theta_W B_\mu + \sin\theta_W A_\mu^3$, where \vec{A}_μ and B_μ are the gauge bosons for the SU(2)_L and weak hypercharge U(1)_Y factor groups, and θ_W is the weak mixing angle. Above the electroweak symmetry breaking scale, and hence for considerations of asymptotic ultraviolet behavior, the Abelian gauge interaction that one naturally analyzes is U(1)_Y. Although this is chiral, as contrasted with the vectorial U(1)_{em} interaction, it shares the property of being non-asymptotically free. Moreover, if the U(1)_Y gauge group is embedded in an asymptotically free simple gauge group at some mass scale M_{GUT} , as in grand unified theories, then one need not worry about the asymptotic behavior of the U(1)_Y gauge interaction in the UV at scales above M_{GUT} .

Nevertheless, the renormalization-group properties of a vectorial U(1) gauge theory have continued to be of abstract field-theoretic interest. In particular, higher-loop terms in the beta function of such a theory have been calculated [7]-[11]. For our analysis of an Abelian gauge theory, we will focus on this type of vectorial theory here. A number of studies of possible nonperturbative properties of a U(1) gauge theory have been carried out over the years using approximate solutions of Schwinger-Dyson equations and other methods [12]-[17]. In particular, lattice studies of U(1) gauge theories (with dynamical staggered fermions) have been performed in [14-16]. A useful result is an exact calculation of the coefficient, $F_1(y)$, of the leading $1/N_f$ correction to the β function in the large- N_f limit [18], where y is proportional to a product of N_f times the squared gauge coupling. An analogous exact large- N_f calculation for a non-Abelian gauge theory was given in [19] (related exact large- N_f results for the anomalous dimension of the fermion bilinear were reported in [20]).

An interesting analysis using the exact $N_f \rightarrow \infty$ results for the beta functions of the U(1) and non-Abelian gauge theories to explore possible zeros of the beta func-

tion has been carried out by Holdom in [21] (see also [22]). Holdom observed that $F_1(y)$ diverges logarithmically through negative values as y approaches a certain value ($y = 15/8$) from below, and hence the beta function, calculated to order $1/N_f$, which is proportional to $(1 + F_1(y)/N_f)$, has a UV zero. However, as he noted, terms of higher order in $1/N_f$ could modify this, so that the beta function might very well not, in fact, have a UV zero.

Most studies of the renormalization-group behavior of non-Abelian gauge theories were motivated by the approximate Bjorken scaling observed in deep inelastic scattering and the property of asymptotic freedom that explains this, as part of the theory of quantum chromodynamics (QCD) [23]. The beta function for a vectorial non-Abelian gauge theory has been calculated up to four-loop order [23]-[28]. A convenient scheme which has been widely used is the $\overline{\text{MS}}$ scheme [29, 30]. Calculating higher-order terms in the beta function of a gauge theory to three-loop and higher-loop level is useful because the results give a quantitative measure of the accuracy of the two-loop calculation. Indeed, in QCD, the value of the higher-loop calculations has been amply demonstrated by their use in fitting data on the $Q^2 = \mu^2$ dependence of the strong coupling $\alpha_s(\mu)$ [31].

For a given non-Abelian gauge group G and fermion representation R , if the number, N_f , of fermions is small, the theory is confining, with spontaneous chiral symmetry breaking. As N_f increases further, the theory exhibits an approximate or exact infrared zero in the beta function [25, 32]. For sufficiently large N_f , this occurs at a small value of the coupling, so that one expects the theory to evolve from the ultraviolet to a deconfined, chirally symmetric Coulombic phase in the infrared. This IR zero has been studied at higher-loop level in [33]-[36]. Effects of scheme transformations on the position of the IR zero have been investigated recently in [2, 3, 37] (see also [38, 39]). These can also be applied to the analysis of a UV zero in the beta function. As N_f increases sufficiently, the sign of the leading, one-loop term is reversed, and the theory becomes infrared-free rather than ultraviolet-free. In this regime, one can then examine the non-Abelian theory for a possible UV zero in the beta function. Since in this regime (as reviewed below) the one-loop and two-loop terms have the same sign, the beta function does not have such UV zero at the maximal scheme-independent level for general N_f . This is the same situation as for the U(1) theory, and as in the Abelian case, one may then investigate higher-loop terms in the beta function to see if such a UV zero might appear. Furthermore, one may study how such a UV zero, if present, relates to the exact results in the $N_f \rightarrow \infty$ limit. We address this question here for both U(1) and non-Abelian gauge theories.

This paper is organized as follows. In Sect. II we investigate a possible UV zero in the n -loop beta function of a U(1) gauge theory. In Sect. III we discuss the general structure of the beta function and calculate the effect of a

scheme transformation on the coefficients of the various powers of N_f in the n -loop term. Sect. III presents our analysis of a possible UV zero in the n -loop beta function, in particular, for large N_f . In Sect. IV the results of this analysis are compared with implications from an exact calculation of the coefficient of the leading $1/N_f$ correction term in an appropriately rescaled beta function in the large- N_f limit. In Sect. V we carry out a study of various test functions incorporating a UV zero in the beta function, of both a power-law and essential-zero form, to determine the manifestations that they produce in both kind of series expansions, namely an expansion in small $1/N_f$ for fixed $N_f\alpha$ and an expansion in α for fixed large N_f . Sect. VI contains corresponding results for a non-Abelian gauge theory. Our conclusions are given in Sect. VII. Some relevant formulas are given in appendices A-E.

II. UV ZERO OF THE n -LOOP BETA FUNCTION OF A GAUGE THEORY

A. General

In this section we discuss some general features of the beta function for a gauge theory. The discussion in the first subsection applies to both an Abelian and a non-Abelian gauge theory; we comment on relevant differences at appropriate points subsequently. The Abelian U(1) theory contains N_f fermions of a given charge q , while the non-Abelian theory has N_f fermions transforming according to a representation R of the gauge group G . The fermions are assumed to have masses that are negligibly small relative to the Euclidean momentum scale, μ . We denote the running gauge coupling as $g(\mu)$. In addition to the standard notation $\alpha(\mu) = g(\mu)^2/(4\pi)$, it will be convenient to use the quantity

$$a(\mu) = \frac{g(\mu)^2}{16\pi^2} = \frac{\alpha(\mu)}{4\pi}. \quad (2.1)$$

The scale μ will often be suppressed in the notation. In the Abelian case, with no loss of generality, we absorb the factor q into a rescaling of the coupling g and hence set $q = 1$. The dependence of α on μ is given by the β function

$$\beta \equiv \beta_\alpha \equiv \frac{d\alpha}{dt}, \quad (2.2)$$

where $dt = d \ln \mu$. This has the series expansion [40]

$$\beta_\alpha = 2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = 2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell, \quad (2.3)$$

where ℓ denotes the number of loops involved in the calculation of b_ℓ and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. The one-loop and two-loop coefficients b_1 and b_2 in the beta function are independent of the scheme used for regularization and

renormalization, while the coefficients at higher-loop order $n \geq 3$ are scheme-dependent [28]. The n -loop ($n\ell$) β function, denoted $\beta_{\alpha,n\ell}$, is defined by (2.3) with the upper limit on the summation over loop order ℓ given by $\ell = n$ rather than $n = \infty$.

The U(1) gauge theory is infrared-free and, for the regime of large N_f in which we are interested, the non-Abelian theory is also infrared-free. A zero of the beta function that is reached by renormalization-group evolution from the vicinity of zero coupling is thus a UV zero. The condition that the n -loop β function vanishes is the polynomial equation

$$\sum_{\ell=1}^n b_\ell a^{\ell-1} = 0 \quad (2.4)$$

of degree $n-1$ in a or equivalently, α . The $n-1$ roots of this equation depend on $n-1$ ratios of the n coefficients, which can be taken to be b_ℓ/b_n with $\ell = 1, \dots, n-1$. The real positive root nearest to the origin (if it exists) is the UV zero of $\beta_{\alpha,n\ell}$. We will denote this zero as $\alpha_{UV,n\ell} = \alpha_{UV,n\ell}/(4\pi)$. For later purposes, let us assume that for a given N_f , $\beta_{\alpha,n\ell}$ has a zero at $\alpha_{UV,n\ell}$, and let us define the interval $I_{\alpha,n\ell}$ as

$$I_{\alpha,n\ell} : 0 \leq \alpha \leq \alpha_{UV,n\ell}. \quad (2.5)$$

We will assume that $\alpha(\mu)$ is sufficiently small for small values of μ in the IR that perturbation theory is reasonably reliable. Then as μ increases from the IR to the UV, $\alpha(\mu)$ increases from these small values and approaches the UV zero of β_α at $\alpha_{UV,n\ell}$.

If a UV zero of β_α calculated at a given loop order occurs at a value of $\alpha \sim O(1)$, it is important to take account of higher-loop contributions to determine how they affect the position of this UV zero. We performed such a study for U(1) with various values of N_f in [2]. Here we generalize this to larger N_f and investigate how these calculations using a small- α expansion at fixed N_f relate to results obtained as $N_f \rightarrow \infty$ for fixed $N_f\alpha$. We will also explore a UV zero for non-Abelian gauge theories at large N_f . To distinguish quantities in a U(1) and a non-Abelian (NA) gauge theory, we will use the symbols β_α, b_ℓ , etc. to refer to the U(1) theory and $\beta_\alpha^{(NA)}, b_\ell^{(NA)}$, etc. to refer to the non-Abelian theory.

B. Effect of Scheme Transformations on b_n

We discuss here the structure of the n -loop terms in the U(1) beta function as polynomials in N_f and calculate how this structure changes under a scheme transformation. The one-loop and two-loop coefficients in the beta function are both proportional to N_f , and we write

$$b_1 = b_{1,1}N_f, \quad b_2 = b_{2,1}N_f. \quad (2.6)$$

where the values of $b_{\ell,1}$ are given in Eqs. (3.1) and (3.2) below. Although the higher-loop coefficients, b_ℓ with

$\ell \geq 3$, are scheme-dependent, one can make some general statements about their dependence on N_f from the structure of the Feynman diagrams that contribute to these higher-loop coefficients. For $\ell \geq 2$, b_ℓ is a polynomial in N_f in which the term of lowest degree in N_f has degree 1 and the term of highest degree in N_f has degree $\ell - 1$. That is, these coefficients have the structural form

$$b_\ell = \sum_{k=1}^{\ell-1} b_{\ell,k} N_f^k \quad \text{for } \ell \geq 2. \quad (2.7)$$

As defined in this manner, the $b_{\ell,k}$ are independent of N_f . For later purposes, we will formally extend the range of the index k in $b_{\ell,k}$ to allow $k = \ell$ and set $b_{\ell,\ell} = 0$ for $\ell \geq 2$.

We next investigate the effect of scheme transformations on the $b_{\ell,k}$. A scheme transformation (ST) can be expressed as a mapping between α and α' , or equivalently, a and a' :

$$a = a' f(a'). \quad (2.8)$$

To keep the UV properties the same, one requires $f(0) = 1$. We consider scheme transformations here that are analytic about $a = a' = 0$ [41] and hence can be expanded in the form

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s, \quad (2.9)$$

where the k_s are constants, $\bar{k}_s = k_s / (4\pi)^s$, and s_{max} may be finite or infinite. The beta function in the transformed scheme is

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha} \frac{d\alpha}{dt}. \quad (2.10)$$

This has the expansion

$$\beta_{\alpha'} = 2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = 2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell, \quad (2.11)$$

where $\bar{b}'_\ell = b'_\ell / (4\pi)^\ell$. Explicit expressions for the b'_ℓ in terms of the k_s determining the transformation function were calculated in [3] and studied further in [2]. In our discussion below on effects of scheme transformations on the coefficients b_ℓ to produce the b'_ℓ , we implicitly restrict to b'_ℓ with $\ell \geq 3$, since there is no change in the b_ℓ for $\ell = 1, 2$. A general property that was found in [3] is that b'_ℓ is a linear combination of b_n with $1 \leq n \leq \ell$, of the form

$$b'_\ell = b_\ell + \sum_{m=1}^{\ell-1} h_{\ell,\ell-m} b_{\ell-m}, \quad (2.12)$$

where the $h_{\ell,n}$ are functions of the k_s that determine the scheme transformation, as given in Eq. (2.9). For example, for $\ell = 3$ and $\ell = 4$, we calculated [3]

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \quad (2.13)$$

and

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1, \quad (2.14)$$

so that $h_{3,2} = k_1$, $h_{3,1} = k_1^2 - k_2$, and so forth for higher ℓ . For the range $\ell \geq 3$ where the b_ℓ change under a scheme transformation, $h_{\ell,\ell-1} = (\ell - 2)k_1$.

To determine the effect of a scheme transformation on the coefficients $b_{\ell,k}$, we apply the results above. A general scheme transformation changes the b_ℓ for $\ell \geq 3$ and, in particular, the $b_{\ell,k}$ for $\ell \geq 3$ and $1 \leq k \leq \ell - 1$. Since we are interested in a specific limit, $N_f \rightarrow \infty$, we will restrict ourselves here to scheme transformations that are independent of N_f . (We will sometimes emphasize this with the symbol NFI, standing for “ N_f -independent”.) Substituting Eq. (2.7) into Eq. (2.12), we obtain the result that b'_ℓ again has the same general polynomial form in terms of powers of N_f as the b_ℓ in Eq. (2.7), viz.,

$$b'_\ell = \sum_{k=1}^{\ell-1} b'_{\ell,k} N_f^k. \quad (2.15)$$

Explicitly, we have, for the values $\ell \geq 3$ where the b_ℓ change under a scheme transformation,

$$\sum_{k=1}^{\ell-1} b'_{\ell,k} N_f^k = \sum_{k=1}^{\ell-1} b_{\ell,k} N_f^k + \sum_{n=1}^{\ell-1} h_{\ell,n} t_n \quad (2.16)$$

where $t_1 = b_{1,1} N_f$ and

$$t_n = \sum_{r=1}^{n-1} b_{n,r} N_f^r \quad \text{if } 2 \leq n \leq \ell - 1. \quad (2.17)$$

The highest power of N_f in the second term in Eq. (2.16) is $r = n - 1 = \ell - 2$, so the only term of the form $N_f^{\ell-1}$ on the right-hand side of Eq. (2.16) arises from the first term and is $b_{\ell,\ell-1} N_f^{\ell-1}$. Therefore,

$$b'_{\ell,\ell-1} = b_{\ell,\ell-1}. \quad (2.18)$$

Thus, although the b_ℓ coefficients with $\ell \geq 3$ that we use for our calculations were calculated in the \overline{MS} scheme, in each b_ℓ , the term with the highest power of N_f , namely $b_{\ell,\ell-1} N_f^{\ell-1}$, is invariant under NFI scheme transformations. This has an important consequence, namely that although each term b_ℓ with $\ell \geq 3$ is scheme-dependent, the term in b_ℓ that dominates in the $N_f \rightarrow \infty$ limit is NFI scheme-independent. In effect, this limit picks out the NFI scheme-independent part of the beta function to arbitrarily high loop order ℓ . The same type of argument holds for a non-Abelian gauge theory, so (for $\ell \geq 3$ where the $b_\ell^{(NA)}$ change under an NFI scheme transformation)

$$b_{\ell,\ell-1}^{(NA)'} = b_{\ell,\ell-1}^{(NA)}. \quad (2.19)$$

Equating equal powers of N_f on the left-hand and right-hand sides of Eq. (2.16), from $N_f^{\ell-2}$ down to N_f ,

we obtain a relations between the $b'_{\ell,k}$ and $b_{\ell,k}$ involving the coefficients in Eq. (2.12) with $1 \leq k \leq \ell - 2$, namely

$$b'_{\ell,k} = b_{\ell,k} + \sum_{m=1}^{\ell-k-1} h_{\ell,\ell-m} b_{\ell-m,k}, \quad (2.20)$$

and, for $k = 1$,

$$b'_{\ell,1} = b_{\ell,1} + \sum_{m=1}^{\ell-1} h_{\ell,\ell-m} b_{\ell-m,1}. \quad (2.21)$$

We will use these relations below.

III. UV ZERO OF THE n -LOOP BETA FUNCTION OF A U(1) GAUGE THEORY

Having discussed the general structure of the beta function and the condition for a zero in this function, we next proceed to our actual calculations for the U(1) gauge theory. The coefficients of the one-loop and two-loop terms of β_α in this U(1) theory are [1, 6]

$$b_1 = \frac{4N_f}{3} \quad (3.1)$$

and

$$b_2 = 4N_f. \quad (3.2)$$

As noted above, since these coefficients have the same sign, the two-loop β_α function, $\beta_{\alpha,2\ell}$, does not have a UV zero [42]. For later purposes it will be convenient to extract the coefficients of powers of N_f in these $b_{\ell s}$, defining

$$b_{1,1} \equiv \frac{b_1}{N_f} = \frac{4}{3} \quad (3.3)$$

and

$$b_{2,1} \equiv \frac{b_2}{N_f} = 4. \quad (3.4)$$

The coefficient of the three-loop term in the β_α function of the U(1) gauge theory, calculated in the $\overline{\text{MS}}$ scheme, is [7, 8]

$$b_3 = -2N_f \left(1 + \frac{22N_f}{9} \right). \quad (3.5)$$

This is evidently negative for all N_f . Therefore, in addition to the IR zero at $\alpha = 0$, in the $\overline{\text{MS}}$ scheme, the three-loop β_α function, $\beta_{\alpha,3\ell}$, has a UV zero, namely,

$$\alpha_{UV,3\ell} = 4\pi a_{UV,3\ell} = \frac{4\pi [9 + \sqrt{3(45 + 44N_f)}]}{9 + 22N_f}. \quad (3.6)$$

In [2] we calculated values of $\alpha_{UV,3\ell}$ as a function of N_f from 1 to 10. For our present large- N_f study, we extend this to $N_f = 10^4$ and present results in Table I. In addition to the scheme-dependence, one must note that for moderate N_f , the value of $\alpha_{UV,3\ell}$ in Eq. (3.6) is too large for the perturbative three-loop calculation to be very reliable.

The coefficient of the four-loop term in β_α , again calculated in the $\overline{\text{MS}}$ scheme, is [9]

$$b_4 = N_f \left[-46 + \left(\frac{760}{27} - \frac{832\zeta(3)}{9} \right) N_f - \frac{1232}{243} N_f^2 \right], \quad (3.7)$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Numerically,

$$b_4 = -N_f [46 + 82.97533N_f + 5.06996N_f^2]. \quad (3.8)$$

As was the case with b_3 , the coefficient b_4 is negative for all $N_f > 0$. The condition that $\beta_{\alpha,4\ell} = 0$ for $\alpha \neq 0$, is the cubic equation $b_1 + b_2a + b_3a^2 + b_4a^3 = 0$. This equation has a physical root, $a_{UV,4\ell} = \alpha_{UV,4\ell}/(4\pi)$, as well as an unphysical pair of complex-conjugate values of a . We list values of $\alpha_{UV,4\ell}$ for N_f up to 10^4 in Table I. As was true of $\alpha_{UV,3\ell}$, in this $\overline{\text{MS}}$ scheme, $\alpha_{UV,4\ell}$ is a monotonically decreasing function of N_f .

The coefficient of the five-loop term in β_α , b_5 , is, in the $\overline{\text{MS}}$ scheme [10, 11]

$$b_5 = N_f \left[\frac{4157}{6} + 128\zeta(3) + \left(-\frac{7462}{9} - 992\zeta(3) + 2720\zeta(5) \right) N_f \right. \\ \left. + \left(-\frac{21758}{81} + \frac{16000\zeta(3)}{27} - \frac{208\pi^4}{135} - \frac{1280\zeta(5)}{3} \right) N_f^2 + \left(\frac{856}{243} + \frac{128\zeta(3)}{27} \right) N_f^3 \right]. \quad (3.9)$$

Note that here and below, terms involving $\zeta(m)$ with even $m = 2r$ are evaluated using the identity (B9) in Appendix B. Numerically,

$$b_5 = N_f (846.6966 + 798.8919N_f - 148.7919N_f^2$$

$$+ 9.22127N_f^3). \quad (3.10)$$

This is positive for all non-negative N_f . The condition that $\beta_{\alpha,5\ell}$ vanishes away from the origin is given by Eq.

(2.4 with $n = 5$, which is a quartic equation in a . In [2] we calculated this for $1 \leq N_f \leq 10$ (see Table II of [2]). For $1 \leq N_f \leq 4$, $\beta_{\alpha,5\ell}$ has no UV zero. For $5 \leq N_f \leq 19$, $\beta_{\alpha,5\ell}$ does have a UV zero, but as N_f (analytically continued to real values [43]) increases through $N_f \simeq 19.67$, this zero disappears, and for larger values of N_f , the roots of the quartic equation above consist of two complex-conjugate pairs.

In general, insofar as a perturbative expansion of a given quantity in a field theory is reliable, one would expect that calculating this quantity to higher-loop order should not change its qualitative properties. Indeed, one would expect that the fractional change in the quantity going from n to $n + 1$ loops should become progressively smaller as the loop order n increases. According to this expectation, having calculated the UV zero in β at three-loop and four-loop order, one would expect that the result of a five-loop calculation would be a small shift in the zero of β . As is evident from our results in Table I, we do not, in general, find that the UV zeros that we have calculated at $n = 3$, $n = 4$, and $n = 5$ loop order for a large range of N_f values satisfy this necessary condition. In the small interval of N_f from about 6 to roughly 10, the behavior of the UV zero is reasonably stable. But for larger N_f , the behavior is not stable. Indeed, for $N_f \geq 20$, the five-loop beta function (calculated in the $\overline{\text{MS}}$ scheme) does not even have a UV zero. Thus, as far as it has been calculated, the perturbative beta function of this U(1) gauge theory does not exhibit a stable UV zero for large N_f . Since this is a statement about the perturbative beta function calculated to a finite-loop level, it does not contradict the possible existence of a UV zero of the beta function in the limit $N_f \rightarrow \infty$ obtained from a summation over a leading set of diagrams up to infinite-loop order. We will discuss this $N_f \rightarrow \infty$ limit next.

IV. BETA FUNCTION OF THE U(1) GAUGE THEORY IN THE LARGE- N_f LIMIT

A. Large- N_f Limit and Rescaled Beta Function β_y

For a U(1) gauge theory with N_f fermions, we consider the limit

$$N_f \rightarrow \infty \quad \text{with} \quad y(\mu) \equiv N_f a(\mu) = \frac{N_f \alpha(\mu)}{4\pi}, \quad (4.1)$$

where the function $y(\mu)$ is a finite function of the Euclidean scale μ in the $N_f \rightarrow \infty$. We denote this as the LNF (large- N_f) limit. We will use the same term, LNF, below for the corresponding limit (6.2) in the case of a non-Abelian gauge theory. For notational simplicity, we will often suppress the argument μ and write $y(\mu)$ as y . The LNF limit is useful because certain Feynman diagrams, namely iterated fermion vacuum polarization insertions on the Abelian gauge boson propagator lines, give the dominant contribution to β in this limit, and this

contribution can be calculated exactly in terms of an integral representation, to be discussed below. Although for definiteness we phrase our discussion of the $N_f \rightarrow \infty$ limit here in terms of the U(1) gauge theory, it will also apply, with obvious changes, to the non-Abelian gauge theory to be analyzed later in the paper.

From the structural formula (2.7), it follows that

$$b_\ell \propto N_f^{\ell-1} \quad \text{for } \ell \geq 2 \quad \text{as } N_f \rightarrow \infty. \quad (4.2)$$

The asymptotic relation (4.2) may be contrasted with the dependence of the b_ℓ coefficients in an $\text{SU}(N_c)$ theory in the large- N_c limit, which is

$$b_\ell \propto N_c^\ell \quad \forall \ell \quad \text{as } N_c \rightarrow \infty. \quad (4.3)$$

The large- N_f dependence of the coefficients b_ℓ in β_α motivates the definition of rescaled coefficients that have finite limits as $N_f \rightarrow \infty$, namely $b_1/N_f = b_{1,1} = 4/3$, as given above, and

$$\check{b}_\ell \equiv \frac{b_\ell}{N_f^{\ell-1}} \quad \text{for } \ell \geq 2. \quad (4.4)$$

Combining these definitions with Eq. (2.7), we have

$$\lim_{N_f \rightarrow \infty} \check{b}_\ell = b_{\ell,\ell-1} \quad \text{for } \ell \geq 2. \quad (4.5)$$

For compact notation, we also define

$$\acute{b}_\ell \equiv \frac{\check{b}_\ell}{b_{1,1}} = \frac{3}{4} \check{b}_\ell \quad (4.6)$$

and

$$\acute{b}_{\ell,k} \equiv \frac{b_{\ell,k}}{b_{1,1}} = \frac{3}{4} b_{\ell,k}. \quad (4.7)$$

Given that $b_1 \propto N_f$ and the structural relation (2.7) for the b_ℓ with $\ell \geq 2$, one may construct a rescaled beta function β_y that is finite in the LNF limit by defining

$$\beta_y \equiv \beta_\alpha N_f. \quad (4.8)$$

Since

$$\beta_\alpha = 8\pi a^2 \left[b_{1,1} N_f + \sum_{\ell=2}^{\infty} \check{b}_\ell y^{\ell-1} \right], \quad (4.9)$$

one has

$$\beta_y = 8\pi b_{1,1} y^2 \left[1 + \frac{1}{N_f} \sum_{\ell=2}^{\infty} \acute{b}_\ell y^{\ell-1} \right]. \quad (4.10)$$

The n -loop rescaled beta function, $\beta_{y,n\ell}$, is given by Eq. (4.10) with the upper limit on the summation over loop order ℓ set to $\ell = n$ rather than $\ell = \infty$. We will next use Eq. (4.10) below for a general analysis of a UV zero of β_y in the large- N_f limit.

The fact that b_1 scales differently in the LNF limit from b_ℓ for $\ell \geq 2$, gives rise to significant differences between the large- N_c limit in an $SU(N_c)$ gauge theory and the large- N_f limit in either a $U(1)$ or $SU(N_c)$ theory. Thus, in contrast to an equation for a vanishing β function in the large- N_c limit in QCD, which can be written completely in terms of the variable $\xi = N_c\alpha$ (with infrared zeros that have been analyzed to higher-loop order in [36]), Eq. (4.11) below contains explicit dependence on N_f .

B. Criterion and Approximate Solution for UV Zero of β_y in the Large- N_f Limit

The condition that the n -loop β_y function vanishes away from the origin $y = 0$ is the algebraic equation (of degree $n - 1$ in y)

$$1 + \frac{1}{N_f} \sum_{\ell=2}^n \check{b}_\ell y^{\ell-1} = 0. \quad (4.11)$$

Consequently, in the LNF limit, of the $n - 1$ roots of Eq. (4.11), the relevant one has the approximate form

$$y_{UV,n\ell} \sim \left(-\frac{N_f}{\check{b}_{n,n-1}} \right)^{\frac{1}{n-1}} = \left(-\frac{b_{1,1}N_f}{b_{n,n-1}} \right)^{\frac{1}{n-1}}. \quad (4.12)$$

In terms of the coupling $a = y/N_f$, or equivalently, α , this approximate solution is

$$a_{UV,n\ell} = \frac{\alpha_{UV,n\ell}}{4\pi} \sim \left(-\frac{b_{1,1}}{b_{n,n-1}} \right)^{\frac{1}{n-1}} N_f^{-\frac{n-2}{n-1}}. \quad (4.13)$$

This is proved as follows. For fixed n , in the LNF limit, taking account of the prefactor $(N_f)^{-1}$ multiplying the sum, the ℓ 'th term in Eq. (4.11), where $2 \leq \ell \leq n$, behaves like

$$(N_f)^{-1} \check{b}_\ell y^{\ell-1} \sim \check{b}_\ell (-\check{b}_n)^{-\frac{\ell-1}{n-1}} (N_f)^{-\frac{n-\ell}{n-1}}. \quad (4.14)$$

Hence, for fixed n , all of the $n-2$ terms with $2 \leq \ell \leq n-1$ vanish in the LNF limit, and the equation reduces to just

$$1 + \dots + (N_f)^{-1} \check{b}_\ell y^{n-1} = 0, \quad (4.15)$$

where the \dots refer to the negligibly small terms in this limit. The solution to Eq. (4.15) is the result in (4.12), or equivalently, (4.13). Since b_1 and, for NFI scheme transformations, also $b_{n,n-1}$, are scheme-independent, these solutions for $y_{UV,n\ell}$ in Eq. (4.12) and for $a_{UV,n\ell}$ in Eq. (4.13) are also NFI scheme-independent. The fact that this solution for $N_f \rightarrow \infty$ is a reasonably accurate approximation at large but finite N_f can be demonstrated as follows. For values of n and N_f for which $\beta_{n\ell}$ has a UV zero and for which b_n is negative, we define

$$\kappa_{n\ell} \equiv a_{UV,n\ell} \left(-\frac{\check{b}_n}{b_{1,1}} \right)^{\frac{1}{n-1}} N_f^{\frac{n-2}{n-1}}. \quad (4.16)$$

Recalling Eq. (4.5) and using the actual solutions to Eq. (4.11), we then check to see that, for a fixed n , $\kappa_{n\ell}$ is consistent with an approach to 1 in the LNF limit. We display illustrative numerical results extending from $N_f = 1$ to $N_f = 10^4$ in Table II, which show that, indeed, this is the case. Note that at the five-loop level, even for the interval of values $5 \leq N_f \leq 19$ where Eq. (4.11) has a solution for a UV zero, no physical $\kappa_{5\ell}$ is defined because b_5 is positive.

From Eq. (4.12) or (4.13), we derive the following criterion for the existence of a UV zero in the beta function of a $U(1)$ gauge theory for large N_f :

As $N_f \rightarrow \infty$, $\beta_{\alpha,n\ell}$ has a UV zero if and only if $b_{n,n-1} < 0$. (4.17)

and similarly with β_y . Since the analysis leading to this result extends to the non-Abelian gauge theory also, with obvious changes (replacement of N_f by $T_f N_f$ and $y = N_f a$ by $\eta = T_f N_f a$; see Eq. (6.2)), the criterion (4.17) also applies to a non-Abelian gauge theory. We have shown above that $b_{n,n-1}$ is NFI scheme-independent, and hence, so is this criterion.

The asymptotic solution (4.12) for $y_{UV,n\ell}$ has some important implications for relating calculations of a UV zero in the n -loop beta function, $\beta_{\alpha,n\ell}$ at large N_f with results obtained by summing over Feynman diagrams up to infinitely many loops in the LNF limit (4.1). Let us assume that for some n , $b_{n,n-1} < 0$, so that $\beta_{\alpha,n\ell}$ does have a UV zero for large N_f . From Eq. (4.12), it follows that, for fixed loop order n and $N_f \rightarrow \infty$,

$$\lim_{N_f \rightarrow \infty} y_{UV,n\ell} = \infty. \quad (4.18)$$

This property prevents one from simply matching the solution for $y_{UV,n\ell}$ (for a given n for which $b_{n,n-1} < 0$ so that it exists) to a possible solution for a finite y in the LNF limit obtained by summing an infinite set of diagrams.

C. Relations Between Coefficients in Large- N_f Expansion of β_y and Coefficients in Small- α Expansion of β_α

Expanding each b_ℓ in Eq. (2.3) in terms of its individual terms multiplying different powers of N_f , Eq. (2.7), we have

$$\begin{aligned} \beta_\alpha &= 8\pi a^2 \left[b_1 + \sum_{\ell=2}^{\infty} a^{\ell-1} b_\ell \right] \\ &= 8\pi a^2 \left[b_{1,1} N_f + \sum_{\ell=2}^{\infty} y^{\ell-1} (N_f)^{1-\ell} \sum_{k=1}^{\ell-1} b_{\ell,k} N_f^k \right] \\ &= 8\pi a^2 N_f b_{1,1} \left[1 + \sum_{\ell=2}^{\infty} y^{\ell-1} N_f^{-\ell} \sum_{k=1}^{\ell-1} \check{b}_{\ell,k} N_f^k \right], \end{aligned} \quad (4.19)$$

so that

$$\beta_y = 8\pi b_{1,1} y^2 \left[1 + \sum_{\ell=2}^{\infty} y^{\ell-1} \sum_{k=1}^{\ell-1} \acute{b}_{\ell,k} N_f^{-(\ell-k)} \right]. \quad (4.20)$$

An alternate way to express β_y in the LNF limit is as an expansion in $1/N_f$ around $1/N_f = 0$. Since one is interested here in the large- N_f limit, it will be convenient to define the variable

$$\nu \equiv \frac{1}{N_f}. \quad (4.21)$$

We thus write

$$\beta_y = 8\pi b_{1,1} y^2 \left[1 + \sum_{s=1}^{\infty} \frac{F_s(y)}{N_f^s} \right] = 8\pi b_{1,1} y^2 \left[1 + \sum_{s=1}^{\infty} F_s(y) \nu^s \right]. \quad (4.22)$$

By relating this expansion to the expansion Eq. (4.20), we find that if one expands $F_s(y)$ as a series in powers of y about $y = 0$, the term of lowest degree in y has degree s . We thus write

$$F_s(y) = \sum_{p=s}^{\infty} f_{s,p} y^p. \quad (4.23)$$

Substituting this expansion in Eq. (4.22), we have

$$\beta_y = 8\pi b_{1,1} y^2 \left[1 + \sum_{s=1}^{\infty} N_f^{-s} \sum_{p=s}^{\infty} f_{s,p} y^p \right]. \quad (4.24)$$

Matching the terms involving a given inverse power of N_f in Eqs. (4.22) and (4.20), i.e., the term $f_{s,p} y^p N_f^{-s}$ in Eq. (4.20) and the term $\acute{b}_{\ell,k} y^{\ell-1} N_f^{-(\ell-k)}$, we have $p = \ell - 1$ and $s = \ell - k$, and the identification

$$f_{s,p} = \acute{b}_{p+1,p+1-s} = \frac{3}{4} b_{p+1,p+1-s} \quad (4.25)$$

or equivalently,

$$b_{\ell,k} = b_{1,1} f_{\ell-k,\ell-1} = \frac{4}{3} f_{\ell-k,\ell-1}. \quad (4.26)$$

In these equations, it is understood that the indices ℓ , k , s , and p range over the values in the respective expansions. Although no coefficients with $s = 0$ appear in Eq. (4.24), we formally define $f_{0,0} \equiv (3/4)b_{1,1} = 1$. It will be convenient to define

$$\acute{f}_{s,p} \equiv b_{1,1} f_{s,p} = \frac{4}{3} f_{s,p}. \quad (4.27)$$

Thus, combining these results, we have, explicitly,

$$F_s(y) = \sum_{p=s}^{\infty} f_{s,p} y^p = \frac{3}{4} \sum_{p=s}^{\infty} b_{p+1,p+1-s} y^p \quad (4.28)$$

and

$$b_{\ell} = \sum_{k=1}^{\ell-1} b_{\ell,k} N_f^k = \frac{4}{3} \sum_{k=1}^{\ell-1} f_{\ell-k,\ell-1} N_f^k. \quad (4.29)$$

D. Determination of Coefficient $b_{n,n-1}$ and Application of Criterion for UV Zero

In this section we will use an exact integral representation for $F_1(y)$ from [18] to determine the coefficient of the term in b_n of highest degree in N_f , namely, $b_{n,n-1}$ and, hence to determine the existence or non-existence of a UV zero in β_α at loop order n for large N_f . The integral representation for $F_1(y)$ (in the $\overline{\text{MS}}$ scheme) is [18]

$$F_1(y) = \int_0^{\frac{4y}{3}} dx I_1(x), \quad (4.30)$$

where

$$I_1(x) = \frac{(1+x)(1-2x)(1-2x/3)\Gamma(4-2x)}{b_{1,1}\Gamma(1+x)\Gamma(3-x)[\Gamma(2-x)]^2}, \quad (4.31)$$

Here $\Gamma(x)$ is the Euler gamma function. Some implications of this for a possible nonperturbative zero in the $U(1)$ β function have been discussed in [21].

From an inspection of Eq. (4.31), one sees that the integrand $I_1(x)$ is analytic in the complex x plane in a disk around $x = 0$ of radius $|x| = 5/2$. One can therefore expand this integrand function in a Taylor series about $x = 0$ and integrate term by term. The resultant integral, which is the function $F_1(y)$, is analytic in the complex y plane in a disk around $y = 0$ of radius $|y| = 15/8$. We thus obtain a Taylor series for $F_1(y)$ around $y = 0$ to arbitrarily high order. Now we combine this with the relation (4.25) derived above equating $f_{s,p}$ with $\acute{b}_{p+1,p+1-s}$. We take the $s = 1$ special case of this relation to get

$$\begin{aligned} f_{1,p} &= \acute{b}_{p+1,p} = \frac{3}{4} b_{p+1,p}, \quad i.e., \\ b_{\ell,\ell-1} &= \acute{f}_{1,\ell-1} = \frac{4}{3} f_{1,\ell-1}. \end{aligned} \quad (4.32)$$

Combining this equation expressing $f_{1,p} = (3/4)b_{p+1,p}$ with the result proved above, that the $b_{\ell,\ell-1}$ are invariant under NFI scheme transformations, one proves that the $f_{1,p}$ are invariant under these scheme transformations. Since these coefficients determine $F_1(y)$ via the $s = 1$ special case of the Taylor series expansion (4.23), it follows that $F_1(y)$ is NFI scheme-independent. Although the $F_1(y)$ function is different for a non-Abelian gauge theory (in the latter case, we denote it as $F_1^{(NA)}(\eta)$, where η is defined below in Eq. (6.2)), the steps in the proof of NFI scheme-invariance are the same, so the result holds for both an Abelian and non-Abelian gauge theory. Thus,

$$F_1(y)_{ST} = F_1(y), \quad F_1^{(NA)}(\eta)_{ST} = F_1^{(NA)}(\eta), \quad (4.33)$$

where the subscript ST means the function after the NFI scheme transformation in Eqs. (2.8) and (2.9) has been applied.

Since the Taylor series expansion of $F_1(y)$ yields all $f_{1,p}$ for arbitrarily large p , these determine the $b_{p+1,p}$, i.e.,

$b_{\ell,\ell-1}$, namely the terms in b_ℓ of highest degree in N_f for arbitrarily large loop order ℓ . We list some of the low-order coefficients $f_{1,p}$ in Appendix C for reference. It is readily verified that the results for $b_{\ell,\ell-1}$ obtained via this Taylor series expansion in conjunction with Eq. (4.32) agree with the known results from b_ℓ for loop orders $\ell = 2$ to $\ell = 5$, namely, $b_{2,1} = \dot{f}_{1,1} = 4$, as in Eq. (3.4) above,

$$b_{3,2} = \dot{f}_{1,2} = -\frac{44}{9} = -4.88889, \quad (4.34)$$

$$b_{4,3} = \dot{f}_{1,3} = -\frac{1232}{243} = -5.06996, \quad (4.35)$$

and

$$b_{5,4} = \dot{f}_{1,4} = \frac{856}{243} + \frac{128\zeta(3)}{27} = 9.22127, \quad (4.36)$$

where here and below, we leave factorizations of integers implicit, and numerical values are given to the indicated floating-point accuracy.

Going beyond these known coefficients, we have calculated analytic expressions for the $b_{\ell,\ell-1}$ to higher loop order ℓ . For $\ell = 6, 7, 8$ we obtain

$$b_{6,5} = \dot{f}_{1,5} = \frac{16064}{3645} - \frac{11264\zeta(3)}{1215} + \frac{512\pi^4}{6075} = 1.47275 \quad (4.37)$$

$$\begin{aligned} b_{7,6} &= \dot{f}_{1,6} = \frac{4288}{729} - \frac{78848\zeta(3)}{6561} - \frac{5632\pi^4}{32805} + \frac{4096\zeta(5)}{243} \\ &= -7.80879 \end{aligned} \quad (4.38)$$

and

$$\begin{aligned} b_{8,7} &= \dot{f}_{1,7} = \frac{52480}{6561} + \frac{438272\zeta(3)}{45927} - \frac{22528\pi^4}{98415} \\ &\quad - \frac{180224\zeta(5)}{5103} + \frac{32768\pi^6}{964467} + \frac{32768\zeta(3)^2}{5103} \\ &= 2.49243. \end{aligned} \quad (4.39)$$

In Appendix D we list the analytic expressions that we have calculated for the next two $b_{\ell,\ell-1}$ coefficients, namely those for loop order $\ell = 9$ and $\ell = 10$. These analytic expressions for the $b_{\ell,\ell-1}$ with $6 \leq \ell \leq 10$ are new results here. As is evident, the expressions become rather lengthy as ℓ increases. The numerical values of $b_{\ell,\ell-1}$ are given for loop order ℓ up to 24 in Table III (with $\ell \equiv n$ in the notation). This extends the numerical values given in [18]. (Note that the coefficient $(\beta)_n$ in [18] is equal to $4^{-(n+1)}b_{n+2,n+1}$ in our notation.)

From Eq. (4.32) and Eq. (4.17), it follows that the sign of $b_{n,n-1}$ determines whether, for large N_f , there is a UV zero in the n -loop beta function, $\beta_{\alpha,n\ell}$; this UV zero exists if and only if $\text{sgn}(b_{n,n-1})$ is negative. Thus, the results listed in Table III determine whether the n -loop beta function has a UV zero for large N_f . For example, since $b_{3,2}$ is negative, the three-loop beta function,

$\beta_{\alpha,3\ell}$ has a UV zero for large N_f , and the same is true for the four-loop beta function, $\beta_{\alpha,4\ell}$, since $b_{4,3}$ is also negative. But in contrast, since $b_{5,4}$ and $b_{6,5}$ are positive, the respective five-loop and six-loop beta functions do not have any UV zero at large N_f , etc. In order for the β_α function to show evidence for a (stable) UV zero at large N_f , $\beta_{\alpha,n\ell}$ should exhibit such a zero, i.e., the signs of the $b_{n,n-1}$ coefficients should be negative. Instead, as is evident in Table III, one finds a scattering of both positive and negative values of $b_{n,n-1}$ up to $n = 24$. Thus, up to the $n = 24$ loop level that we have calculated, the n -loop β function of the U(1) theory does not exhibit a stable UV zero for large N_f . There is the possibility that as the loop order n increases beyond some value greater than 24, the $b_{n,n-1}$ will become uniformly negative, so that the corresponding n -loop beta function, $\beta_{\alpha,n\ell}$ will develop a stable UV zero at large N_f . However, one would still have to contend with the result (4.18), which does not yield a finite value of $y_{UV,n\ell}$ as $N_f \rightarrow \infty$ for fixed n . These findings do not give support for a UV zero of the beta function in the LNF limit. However, they do not rigorously preclude such a UV zero obtained by the calculation of a leading infinite set of fermion vacuum polarization insertions on gauge boson propagator lines. If such a UV zero should exist, it would mean that calculations of the n -loop beta function for fixed finite n and large N_f are not sensitive to it. Some insight into this possibility is gained from a comparison of the $b_{n,n-1}$ coefficients with certain coefficients $b_{n,n-1}^{(d)}$ defined in Eq. (4.52) below.

E. Calculations of Higher Coefficients $f_{s,p}$ for $2 \leq s \leq 4$

Although the $F_s(y)$ with $s \geq 2$ have not been calculated exactly, we can determine some of the coefficients $f_{s,p}$ in these $F_s(y)$ by reversing the procedure used in the previous section, using Eq. (4.25) together with the known coefficients b_ℓ . For general s , Eq. (4.25) gives $f_{s,p}$ in terms of $\dot{b}_{p+1,p+1-s}$, so if the b_ℓ have been calculated up to ℓ -loop order, then this relation enables one to determine the $f_{s,p}$ for $s \leq p \leq \ell - 1$. Since the b_ℓ have been computed up to $\ell = n = 5$ loops (in the $\overline{\text{MS}}$ scheme) [10, 11], one can thus determine $f_{s,p}$ for $s \leq p \leq 4$ for $F_s(y)$ with $2 \leq s \leq 4$.

We consider $F_2(y)$ first. Here, Eq. (4.25) gives $f_{2,p} = \dot{b}_{p+1,p-1}$ with $p \geq 2$. The first two of coefficients $f_{2,p}$ are known:

$$f_{2,2} = \dot{b}_{3,1} = -\frac{3}{2}, \quad (4.40)$$

and

$$f_{2,3} = \dot{b}_{4,2} = \frac{190}{9} - \frac{208\zeta(3)}{3} = -62.23150. \quad (4.41)$$

For $f_{2,4}$, we find [44]:

$$\begin{aligned} f_{2,4} &= \acute{b}_{5,3} = \frac{-10879}{54} + \frac{4000\zeta(3)}{9} - \frac{52\pi^4}{45} - 320\zeta(5) \\ &= -(1.1159395 \times 10^2) . \end{aligned} \quad (4.42)$$

We next study $F_3(y)$. Setting $s = 3$ in Eq. (4.25) gives $f_{3,p} = \acute{b}_{p+1,p-2}$ with $p \geq 3$. Here,

$$f_{3,3} = \acute{b}_{4,1} = -\frac{69}{2} = -34.5 \quad (4.43)$$

and

$$\begin{aligned} f_{3,4} &= \acute{b}_{5,2} = -\frac{3731}{6} - 744\zeta(3) + 2040\zeta(5) \\ &= 5.9916895 \times 10^2 . \end{aligned} \quad (4.44)$$

Finally, we consider $F_4(y)$. Setting $s = 4$ in Eq. (4.25) yields $f_{4,p} = \acute{b}_{p+1,p-3}$ with $p \geq 4$. Thus,

$$\begin{aligned} f_{4,4} &= \acute{b}_{5,1} = \frac{4157}{8} + 96\zeta(3) \\ &= 6.3502246 \times 10^2 . \end{aligned} \quad (4.45)$$

F. Role of $F_1(y)$ Regarding a Possible UV Zero of the Beta Function

We address here the question of whether β_y might have a UV zero for large N_f . From Eq. (4.22), one can make several general statements. First, since the LNF limit of Eq. (4.1) is $N_f \rightarrow \infty$ with y held fixed and finite, if the value of y is such that the $F_s(y)$ are finite, then each terms, $F_s(y)/N_f^s$ in the series $\sum_{s=1}^{\infty}$ vanishes, and β_y reduces to just the first term, $\beta_y = 8\pi y^2 b_{1,1} = (32\pi/3)y^2$, which does not vanish for any nonzero y . The question then is whether for some large but finite $N_f \gg 1$, β_y might vanish.

As is evident from Eq. (4.22), to order $O(1/N_f)$, the condition that β_y vanishes is that $1 + F_1(y)/N_f = 0$. This condition is satisfied for large N_f if and only if there is a value of y for which $F_1(y)$ is very large and negative. Now the expansion variable $1/N_f$ in Eq. (4.22) (for fixed y) can be regarded as being formally analogous to the expansion variable a or α in Eq. (2.3) (for fixed N_f).

One may ask whether the necessary (and sufficient) condition that, to order $1/N_f$, β_y can vanish, i.e., that there exists a value of y such that $F_1(y)$ is large and negative, is satisfied. Using the exact result from [18], Holdom observed that this is, indeed, the case [21]. We recall his analysis. The integrand function $I_1(x)$ in Eq. (4.31) has simple poles at

$$x_{pote} = \frac{5}{2} + p \quad \text{for } p = 0, 1, 2, 3, \dots \quad (4.46)$$

These produce divergences in $F_1(y)$. The first of these is a logarithmic divergence at

$$y_{div} = \frac{15}{8} . \quad (4.47)$$

Next, one expands the integrand around this singularity, uses the Taylor-Laurent series for $\Gamma(x)$ for $x = -n + \epsilon$, with $n = 0, 1, 2, \dots$ and $\epsilon \rightarrow 0$ (see Appendix B), and integrates the result term by term. One finds that for y slightly less than y_{div} , the logarithmically divergent part of $F_1(y)$ is

$$F_1(y)_{div} \sim \frac{7}{15\pi^2} \ln |1 - (y/y_{div})| + \text{finite} . \quad (4.48)$$

Since this diverges negatively as $y \nearrow y_{div}$, it follows that there exists a value of y , which we denote y_0 , for which the term $F_1(y)/N_f$ is equal to -1 , so that $\beta_y = 0$ to this order in the $1/N_f$ expansion. This value y_0 is exponentially close to the value y_{div} where $F_1(y)$ has a (logarithmic) divergence;

$$y_0 \simeq \frac{15}{8} - \text{const.} \times e^{-15\pi^2 N_f/7} . \quad (4.49)$$

Obviously, one must treat this result with caution, because of the divergence in $F_1(y)$ at y_{div} . However, it suggests that for large N_f , β_y might have a UV zero. Somewhat analogously to the situation with the nonlinear σ model [4], the existence of this UV zero would be the result of having summed an infinite subset of Feynman diagrams that are dominant in the limit $N_f \rightarrow \infty$. One way to study a possible zero in β_y further entails an analysis of contributions to β_y from terms that are of higher order in $1/N_f$, namely, $\sum_{s=2}^{\infty} F_s(y)/N_f^s$. Since the $F_s(y)$ for $s \geq 2$ have not been calculated exactly, this analysis is, perforce, exploratory. The $F_s(y)$ with $s \geq 2$ are also NFI scheme-dependent, in contrast with $F_1(y)$. It has been suggested that in the interval $I_y = [0, y_{div}]$, $F_2(y)$ might contain a pole at $y = y_{div}$, the $F_s(y)$ with $s \geq 2$ might have higher-order poles, and the effect of these might be to remove the UV zero in β_y in this interval obtained with just the leading $F_1(y)/N_f$ term [21]. That is, for a fixed large value of N_f , as $y \rightarrow y_{div}$, higher-order terms in the expansion (4.22) could dominate over the $F_1(y)/N_f$ term. For fixed y slightly different from y_{div} , provided that the $F_s(y)$ with $s \geq 2$ are finite at this value of y , one can make N_f large enough to reduce the contributions of any finite set of terms $F_s(y)/N_f^s$ with $s \geq 2$ to values smaller than $F_1(y)/N_f$. But this does not necessarily hold for the infinite series $\sum_{s=2}^{\infty} F_s(y)/N_f^s$. In our analysis above, we have used a complementary method, namely to investigate the presence or absence of a UV zero in $\beta_{\alpha, n\ell}$ at large N_f .

We note that a possible UV zero in β_y would be different from, for example, an IR zero in the two-loop beta function of a non-Abelian (NA) gauge theory, $\beta_{\alpha, 2\ell}^{(NA)} = (\alpha^2/(2\pi))(b_1^{(NA)} + b_2^{(NA)}a)$ at $a_{IR, 2\ell} = -b_1^{(NA)}/b_2^{(NA)}$, where $b_1^{(NA)} < 0$ and $b_2^{(NA)} > 0$, because, while the latter occurs at a fixed value of a , here the zero would occur at a fixed value of $y = N_f a$, with a itself having been driven toward zero because of the condition that y be fixed as $N_f \rightarrow \infty$ in the LNF limit. A second difference

can be seen as follows. By either analytically continuing N_f from non-negative integer values to non-negative real values or by keeping N_f fixed at positive integer values and letting N_c become large, one can choose N_f to be very slightly less than the value at which $b_1^{(NA)}$ goes through zero and reverses sign, and hence one can make the value of an IR zero of $\beta_\alpha^{(NA)}$ arbitrarily small. This is not the case with a possible UV zero of β_y calculated to a given order in $1/N_f$. A third difference, closely related to the second, is that, whereas the condition for an IR zero in $\beta_\alpha^{(NA)}$, say at two loops, viz., $b_1^{(NA)} + b_2^{(NA)} a = 0$, does not require $b_2^{(NA)}$ to be extremely large, since $b_1^{(NA)}$ can be quite small, the condition for a zero in β_y at large N_f does generically require $|F_1(y)|$ to be very large.

G. Calculations of the Coefficients $b_{n,n-1}^{(d)}$ and Comparison with the $b_{n,n-1}$

Since the large negative value of $F_1(y)$ as y approaches y_{div} from below is a key feature in the possibility of a zero in β_y to order $1/N_f$, it is interesting to investigate how close the actual calculated values of $f_{1,p}$ and hence, by Eq. (4.26), the $b_{n,n-1}$ with $n = p + 1$ for the U(1) gauge theory are to the values that one would get by calculating a Taylor series expansion (around $y = 0$) of the leading divergent term in Eq. (4.48). The coefficients in this Taylor series expansion are

$$f_{1,p}^{(d)} = \frac{1}{p!} \left. \frac{d^p F_1(y)_{div}}{dy^p} \right|_{y=0} \quad (4.50)$$

(where d stands for div), so, in analogy with Eq. (4.23),

$$F_1(y)_{div} = \sum_{p=1}^{\infty} f_{1,p}^{(d)} y^p. \quad (4.51)$$

From Eq. (4.32), one then has the corresponding coefficients

$$b_{n,n-1}^{(d)} = \frac{4}{3} f_{1,n-1}^{(d)}. \quad (4.52)$$

We list the $b_{n,n-1}^{(d)}$ in Table IV for n up to 24. The $f_{1,p}^{(d)}$, and hence $b_{n,n-1}^{(d)}$ with $n = p + 1$, are all negative, and they decrease in magnitude monotonically and sufficiently with increasing n as to be consistent with a finite limit above unity for large, fixed N_f :

$$\lim_{n \rightarrow \infty} \left(-\frac{4N_f}{3b_{n,n-1}^{(d)}} \right)^{\frac{1}{n-1}} = \lim_{n \rightarrow \infty} \left(-\frac{N_f}{f_{1,n-1}^{(d)}} \right)^{\frac{1}{n-1}}. \quad (4.53)$$

An important conclusion follows from comparing the results for the series expansion coefficients $b_{n,n-1}$ for the full function $F_1(y)$, listed in Table III, with the corresponding coefficients $b_{n,n-1}^{(d)}$ that we have calculated from $F_1(y)_{div}$ and listed in Table IV. Two key differences are

evident, namely that (i) while the $b_{n,n-1}$ exhibit a scattering of positive and negative signs, the $b_{n,n-1}^{(d)}$ are uniformly negative, and (ii) while the $|b_{n,n-1}|$ do not decrease monotonically, the $|b_{n,n-1}^{(d)}|$ do, as discussed above. We conclude from these differences that the series expansion in y for the full function $F_1(y)$, up to order $n = 24$, is not dominated by the effect of the logarithmic divergence at $y = y_{div} = 15/8$.

V. IMPLICATIONS FROM ILLUSTRATIVE TEST FUNCTIONS FOR β_y

In this section we consider several illustrative test functions for β_y that, by construction, for fixed $\nu = 1/N_f$, have a UV zero at a value of y that we denote as y_z (where the subscript stands for “zero”). We explore how the properties of these test functions are manifested in the two types of series expansions studied here, namely the expansion in powers of ν around the point $\nu = 0$ for fixed $y = N_f a$ in Eq. (4.22) and the expansion in powers of a , or equivalently α , around $a = 0$ for fixed large N_f in Eq. (2.3). The choice of these test functions is restricted by some general properties. We recall from our discussion above that the term of lowest degree in the expansion of $F_s(y)$ about $y = 0$ has degree s in y . The factor $[1 + \sum_{s=1}^{\infty} F_s(y) \nu^s]$ in Eq. (4.22) has the properties that (i) it is equal to 1 for arbitrary y if $\nu = 0$, and (ii) it is equal to 1 for arbitrary ν if $y = 0$. We define a function $\Phi(y, \nu)$ as

$$\Phi(y, \nu) \equiv \frac{\beta_y}{4\pi b_{1,1} y^2} = \frac{3\beta_y}{16\pi y^2}. \quad (5.1)$$

The function $\Phi(y, \nu)$ thus satisfies the properties

$$\Phi(y, 0) = 1 \quad (5.2)$$

and

$$\Phi(0, \nu) = 1. \quad (5.3)$$

As noted above, since our purpose in constructing and analyzing these $\Phi(y, \nu)$ test functions is to investigate their implications for a possible UV zero in the actual beta function of a U(1) gauge theory, they are designed to have a zero at $y = y_z$, as approached from below (in an interval connected with small y), i.e.,

$$\lim_{y \nearrow y_z} \Phi(y, \nu) = 0. \quad (5.4)$$

We will study these test functions for y in the interval

$$I_y: \quad 0 \leq y \leq y_z \quad (5.5)$$

and focus on the region near $\nu = 0$, i.e., large N_f . The functions $F_s(y)$ in Eq. (4.22) are then given by

$$F_s(y) = \frac{1}{s!} \left. \frac{d^s \Phi(y, \nu)}{d\nu^s} \right|_{\nu=0}. \quad (5.6)$$

Note that in taking this derivative, it is necessary to extend the definition of ν from (a subset of) \mathbb{Q}_+ to \mathbb{R} ; this extension is to be implicitly understood where necessary. Our analysis here also applies to the non-Abelian case to be discussed in the next section, with the replacement of y by η as given in Eq. (6.2). The question of what analytic form $\Phi(y, \nu)$ takes for $y > y_z$ is beyond the scope of our study here, since we are only interested in the behavior in the interval I_y , i.e., the question of a UV zero that would be reached by the coupling via renormalization-group evolution, starting from an initial small value in the infrared. Although, by design, our $\Phi(y, \nu)$ test functions satisfy the necessary conditions (5.2)-(5.4), and it is hoped that they give insight into the behavior of the true function $\Phi(y, \nu)$, no implication is made that they are fully realistic.

A particularly simple test function, applicable in the interval I_y , is the power-law form

$$\Phi(y, \nu) = [1 - (y/y_z)]^\nu, \quad (5.7)$$

where ν is a positive number. With this test function, we now calculate the resultant functions $F_s(y)$ appearing in the expansion (4.22). Since

$$[1 - (y/y_z)]^\nu = e^{\nu \ln[1 - (y/y_z)]} = 1 + \sum_{s=1}^{\infty} \frac{\nu^s}{s!} \left[\ln \left(1 - (y/y_z) \right) \right]^s, \quad (5.8)$$

it follows that

$$F_s(y) = \frac{1}{s!} \left[\ln \left(1 - (y/y_z) \right) \right]^s. \quad (5.9)$$

If we were to truncate this series with the $s = 1$ term, this would be analogous in form to the dominant negative logarithmic divergence in the actual $F_1(y)$ for the U(1) gauge theory, as given in Eq. (4.48). Since ν is not, in general, an integer, and, indeed, we are interested in the regime where ν is approaching zero, $\Phi(y, \nu)$ generically has a branch point singularity at $y = y_z$. Thus, generically, $F_s(y)$ is analytic about the point $y = 0$ in the complex y plane inside a disk of radius $|y| = y_z$. (If one were to set ν equal to an integer, $F_s(y)$ would be analytic everywhere in the y plane.) We calculate the Taylor series expansion of $F_s(y)$ around $y = 0$ to get the coefficients $f_{s,p}$ in Eq. (4.23). The results obey the requisite condition for the U(1) (and non-Abelian) gauge theory that the Taylor expansion of $F_s(y)$ around $y = 0$ has, as its lowest-degree term in y , a term proportional to y^s . For the case $s = 1$ that yields the leading terms in the large- N_f (small- ν) limit, we calculate the coefficients $f_{1,p}$ in the expansion of $F_1(y)$ to be

$$f_{1,p} = -\frac{1}{p}. \quad (5.10)$$

We have also constructed and studied a family of illustrative $\Phi(y, \nu)$ functions, applicable in the interval I_y , each of which has an essential zero at $y = y_z$, namely,

$$\Phi(y, \nu) = \exp \left[\nu \left(-\frac{1}{(1 - (y/y_z))^k} + 1 \right) \right]. \quad (5.11)$$

Here, in addition to y_z , $\Phi(y, \nu)$ depends on a second parameter, the positive real number k , which determines the nature of the essential zero. In particular, if k is a positive integer, then $\ln[\Phi(y, \nu)]$ has a pole of order k at y_z . For the $\Phi(y, \nu)$ in Eq. (5.11), we calculate

$$F_s(y) = \frac{1}{s!} \left[-\frac{1}{(1 - (y/y_z))^k} + 1 \right]^s. \quad (5.12)$$

From this we compute the $f_{s,p}$ coefficients in Eq. (4.23). The results again obey, as they must, the requirement that $F_s(y)$ has, as its lowest-degree term in y , a term proportional to y^s . For $s = 1$, we calculate

$$f_{1,p} = -\binom{k+p-1}{p} y_z^{-p}. \quad (5.13)$$

where $\binom{m}{n} = m!/[n!(m-n)!]$ is the binomial coefficient. Although we are only concerned here with the behavior in the interval I_y , we note parenthetically that one could recast the test function (5.7) so as to remain real for $y > y_z$ by using $\Phi(y, \nu) = [(1 - (y/y_z)^2)^2]^\nu$, and one could restrict k to be even in (5.11) so that the resultant $\Phi(y, \nu)$ vanishes instead of diverging as y approaches y_z from above. With these definitions, $\Phi(y, \nu)$ would be a bounded (real, positive) function for $y > y_z$, so that y_z would be an infrared zero of β_y for $y > y_z$. Test functions for which y_z would be an ultraviolet zero of β_y for $y > y_z$ can also be envisaged.

An important property of these coefficients $f_{1,p}$ in Eq. (5.10) for the $\Phi(y, \nu)$ function (5.7) and in Eq. (5.13) for the $\Phi(y, \nu)$ function (5.11), is that they are all negative. From Eq. (4.25) or (4.26), it follows that the $b_{n,n-1}$ for $n \geq 2$ corresponding to these $\Phi(y, \nu)$ functions are all negative. Recalling Eq. (4.12) and the condition (4.17), this property of negative $f_{1,p}$ and hence negative $b_{n,n-1}$ coefficients resulting from both of the illustrative $\Phi(y, \nu)$ functions above implies that, regarding the other type of expansion, namely the expansion of β_α in powers of a , the resultant n -loop beta function $\beta_{\alpha, n\ell}$ would always exhibit a UV zero for small ν (i.e., large N_f). As we have discussed and as is evident from Table III, the actual $b_{n,n-1}$ coefficients for the U(1) gauge theory are not uniformly negative, but instead exhibit a scatter of signs up to the maximal loop order $n = 24$ which we have studied. As was noted above, this means that, up to this loop order, these $b_{n,n-1}$ do not exhibit evidence for a UV zero in the respective beta functions at large N_f .

The coefficients $f_{s,p}$ in the small- y expansions of the $F_s(y)$ can also be calculated for higher values of s . For example, for the illustrative $\Phi(y, \nu)$ function in Eq. (5.7) with a power-law zero, we find

$$F_2(y) = \frac{1}{2}y^2 + \frac{1}{2}y^3 + \frac{11}{24}y^4 + \frac{5}{12}y^5 + O(y^6), \quad (5.14)$$

$$F_3(y) = -\frac{1}{6}y^3 - \frac{1}{4}y^4 - \frac{7}{24}y^5 - O(y^6), \quad (5.15)$$

$$F_4(y) = \frac{1}{24}y^4 + \frac{1}{12}y^5 + O(y^6), \quad (5.16)$$

and so forth for higher s . For the $\Phi(y, \nu)$ with $k = 1$ in Eq. (5.11), we compute

$$F_2(y) = \frac{1}{2}y^2 + y^3 + \frac{3}{2}y^4 + 2y^5 + O(y^6), \quad (5.17)$$

$$F_3(y) = -\frac{1}{6}y^3 - \frac{1}{2}y^4 - y^5 - O(y^6), \quad (5.18)$$

$$F_4(y) = \frac{1}{24}y^4 + \frac{1}{6}y^5 + O(y^6), \quad (5.19)$$

and so forth for higher s . We have also calculated these Taylor series expansions of $F_s(y)$ for other values of k . A general property that we find is that for both the $\Phi(y, \nu)$ functions with a power-law zero and an essential zero, the nonzero coefficients $f_{s,p}$ are negative for s odd and positive for even s , i.e.,

$$\text{sgn}(f_{s,p}) = (-1)^s. \quad (5.20)$$

for these functions.

In passing, we add a comment concerning the power-law and essential zeros in the test function $\Phi(y, \nu)$ and hence β_y . Although an essential zero in $\Phi(y, \nu)$ at a given point, here, $y = y_z$, cannot be detected to any order of a series expansion about y_z , since all of the coefficients vanish identically, the zero does manifest itself in the series expansion of $F_s(y)$ about $y = 0$. Indeed, as we have shown, the coefficients $f_{s,p}$ obtained from the expansion of $F_s(y)$ about $y = 0$ share important properties in common, such as (5.20) for both the power-law-zero form (5.7) and the essential-zero form (5.11) of $\Phi(y, \nu)$.

VI. NON-ABELIAN GAUGE THEORY

A. Structure of Beta Function

In this section we discuss the question of a possible UV zero of the β function for a non-Abelian gauge theory with a sufficiently large fermion content that this β function is positive near the origin. We will consider a theory with gauge group G and N_f massless (Dirac) fermions transforming according to a representation R of G . As noted above, this large- N_f non-Abelian gauge theory is IR-free and is similar in this sense to the U(1) gauge theory. Let $T_f \equiv T(R)$, where $T(R)$ is the usual trace invariant defined by [45]

$$\sum_{i,j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij} \mathcal{D}_R(T_b)_{ji} = T(R) \delta_{ab}. \quad (6.1)$$

where a is a group index running from 1 to the order of the group, and $\mathcal{D}_R(T_a)$ is the R -representation (*Darstellung*) of the generator T_a of the Lie algebra of G .

We consider the limit analogous to Eq. (4.1), namely

$$N_f \rightarrow \infty \quad \text{with} \quad \eta(\mu) \equiv T_f N_f a(\mu) = \frac{T_f N_f \alpha(\mu)}{4\pi}, \quad (6.2)$$

where the function $\eta(\mu)$ is a finite function of the Euclidean scale μ in the $N_f \rightarrow \infty$. As noted above, to distinguish the quantities for the non-Abelian gauge theory from the analogous quantities for the Abelian U(1) gauge theory, we will use the superscript (NA). We thus write, for the beta function,

$$\beta_\alpha^{(NA)} = 2\alpha \sum_{\ell=1}^{\infty} b_\ell^{(NA)} \alpha^\ell = 2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell^{(NA)} \alpha^\ell. \quad (6.3)$$

For small coupling α , since $\beta > 0$, as the Euclidean momentum scale decreases toward the infrared, $\alpha(\mu) \rightarrow 0$, i.e., the theory is formally free in the infrared.

Since the coefficient of the one-loop term in $\beta^{(NA)}$ is [23, 45]

$$b_1^{(NA)} = \frac{1}{3}(-11C_A + 4T_f N_f), \quad (6.4)$$

we are interested in the interval

$$N_f > N_{f,b1z}, \quad (6.5)$$

where $N_{f,b1z}$ denotes the value of N_f where $b_1^{(NA)}$ vanishes with sign reversal as a function of N_f , namely,

$$N_{f,b1z} = \frac{11C_A}{4T_f}. \quad (6.6)$$

The two-loop coefficient is [25]

$$b_2^{(NA)} = \frac{1}{3}[-34C_A^2 + 4(5C_A + 3C_f)T_f N_f]. \quad (6.7)$$

This is negative for small N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{17C_A^2}{2(5C_A + 3C_f)T_f}. \quad (6.8)$$

For arbitrary G and R , $N_{f,b2z} < N_{f,b1z}$. Hence, our restriction to the range $N_f > N_{f,b1z}$, for which $b_1^{(NA)} > 0$, implies that $b_2^{(NA)} > 0$ also. Consequently, for arbitrary G and R , this theory has no UV zero at the two-loop level. This is similar to the situation in the U(1) theory.

The one-loop coefficient has the form

$$b_1^{(NA)} = b_{1,0}^{(NA)} + b_{1,1}^{(NA)} T_f N_f, \quad (6.9)$$

where

$$b_{1,0}^{(NA)} = -\frac{11}{3}C_A \quad (6.10)$$

and

$$b_{1,1}^{(NA)} = b_{1,1} = \frac{4}{3}. \quad (6.11)$$

The coefficients $b_\ell^{(NA)}$ for $\ell \geq 2$ have the generic form of polynomials in $(T_f N_f)$ of lowest degree 0 and highest degree $\ell - 1$:

$$b_\ell^{(NA)} = \sum_{k=0}^{\ell-1} b_{\ell,k}^{(NA)} (T_f N_f)^k \quad \text{for } \ell \geq 2. \quad (6.12)$$

Eq. (6.12) is the analogue, for this non-Abelian gauge theory, of the expansion (2.7) for the U(1) gauge theory. Given that the b_ℓ coefficients for the U(1) theory can be derived from those for the non-Abelian theory by the formal replacements $C_A = 0$, $C_f = 1$, and $T_f = 1$, together with replacements of other group invariants that enter at loop level $\ell \geq 4$ [27], it follows that the $b_{\ell,0}$ terms vanish for the U(1) theory.

Analogously to the U(1) theory, the large- N_f dependence of the coefficients $b_\ell^{(NA)}$ in $\beta_\alpha^{(NA)}$ motivates the definition of rescaled coefficients that have finite limits as $N_f \rightarrow \infty$, namely, for $\ell = 1$, $b_1^{(NA)}/(T_f N_f) = b_{1,0}^{(NA)}/(T_f N_f) + b_{1,1}^{(NA)}$ and

$$\check{b}_\ell^{(NA)} \equiv \frac{b_\ell^{(NA)}}{(T_f N_f)^{\ell-1}} \quad \text{for } \ell \geq 2. \quad (6.13)$$

We define

$$\acute{b}_\ell^{(NA)} \equiv \frac{\check{b}_\ell^{(NA)}}{b_{1,1}^{(NA)}} = \frac{3}{4} \check{b}_\ell^{(NA)} \quad (6.14)$$

and

$$\check{i}_{\ell,k}^{(NA)} \equiv \frac{b_{\ell,k}^{(NA)}}{b_{1,1}^{(NA)}} = \frac{3}{4} b_{\ell,k}^{(NA)}. \quad (6.15)$$

Given that, for large N_f , $b_1^{(NA)} \sim (T_f N_f)$ and, for $\ell \geq 2$, $b_\ell \sim (T_f N_f)^{\ell-1}$, one may construct a rescaled beta function $\beta_\eta^{(NA)}$ that is finite in the LNF limit (6.2) by defining

$$\beta_\eta^{(NA)} = T_f N_f \beta_\alpha. \quad (6.16)$$

We have

$$\beta_\alpha^{(NA)} = 8\pi a^2 \left[b_{1,1}^{(NA)} T_f N_f + b_{1,0}^{(NA)} + \sum_{\ell=2}^{\infty} \check{b}_\ell^{(NA)} \eta^{\ell-1} \right]. \quad (6.17)$$

Thus,

$$\beta_\eta^{(NA)} = 8\pi b_{1,1}^{(NA)} \eta^2 \left[1 + \frac{\acute{b}_{1,0}^{(NA)}}{T_f N_f} + \frac{1}{T_f N_f} \sum_{\ell=2}^{\infty} \acute{b}_\ell^{(NA)} \eta^{\ell-1} \right]. \quad (6.18)$$

The n -loop rescaled beta function, $\beta_{\eta,n\ell}^{(NA)}$, is given by Eq. (6.18) with the upper limit on the summation over loop order ℓ set to $\ell = n$ rather than $\ell = \infty$.

B. Possible UV Zero of Beta Function in a Non-Abelian Gauge Theory for Large N_f

The condition that the n -loop beta function vanishes away from the origin $\eta = 0$ is the polynomial equation (of degree $n - 1$ in η)

$$1 + \frac{\acute{b}_{1,0}^{(NA)}}{T_f N_f} + \frac{1}{T_f N_f} \sum_{\ell=2}^n \acute{b}_\ell^{(NA)} \eta^{\ell-1} = 0. \quad (6.19)$$

By an analysis similar to the one give above for the U(1) theory, in the LNF limit (6.2), of the $n - 1$ roots of Eq. (6.19), the relevant one has the approximate form

$$\eta_{UV,n\ell} \sim \left(-\frac{T_f N_f}{\acute{b}_{n,n-1}^{(NA)}} \right)^{\frac{1}{n-1}} = \left(-\frac{b_{1,1}^{(NA)} T_f N_f}{b_{n,n-1}^{(NA)}} \right)^{\frac{1}{n-1}}. \quad (6.20)$$

Since $b_1^{(NA)}$ and (for NFI STs) $b_{n,n-1}^{(NA)}$ are scheme-independent, this solution for $\eta_{UV,n\ell}$ in Eq. (6.20) is also NFI scheme-independent. From Eq. (6.20), it follows that a (NFI scheme-independent) criterion for existence of a UV zero in the beta function of this non-Abelian gauge theory for large N_f is:

As $N_f \rightarrow \infty$, $\beta_{y,n\ell}^{(NA)}$ has a UV zero if and only if $b_{n,n-1}^{(NA)} < 0$. (6.21)

Let us assume that for some n , $b_{n,n-1}^{(NA)} < 0$, so that $\beta_{\alpha,n\ell}^{(NA)}$ does have a UV zero for large N_f . From Eq. (6.20), it follows that, for fixed loop order n and $N_f \rightarrow \infty$,

$$\lim_{N_f \rightarrow \infty} \eta_{UV,n\ell} = \infty. \quad (6.22)$$

This is the analogue, for the non-Abelian gauge theory, of Eq. (4.18) for the U(1) gauge theory.

We can express the rescaled beta function as

$$\beta_\eta^{(NA)} = 8\pi b_{1,1}^{(NA)} \eta^2 \left[1 + \sum_{s=1}^{\infty} \frac{F_s^{(NA)}(\eta)}{(T_f N_f)^s} \right]. \quad (6.23)$$

As discussed above, $F_1^{(NA)}(\eta)$ is NFI scheme-independent, while the $F_s^{(NA)}(\eta)$ with $s \geq 2$ are scheme-dependent. The $F_s^{(NA)}(\eta)$ are expanded as

$$F_s^{(NA)}(\eta) = \sum_{p=s-1}^{\infty} f_{s,p}^{(NA)} \eta^p. \quad (6.24)$$

From the results in [19], the following closed-form expression has been inferred for $F_1(\eta)^{(NA)}$ [21]:

$$F_1^{(NA)}(\eta) = -\frac{11C_A}{4} + \int_0^{4\eta/3} I_1(x) I_2(x) dx, \quad (6.25)$$

where $I_1(x)$ was given above and

$$I_2(x) = C_f + \frac{(20 - 43x + 32x^2 - 14x^3 + 4x^4)C_A}{4(1-x^2)(2x-1)(2x-3)}. \quad (6.26)$$

The function $I_2(x)$ has a simple pole at $|x| = 1$. Consequently, the integral, $F_1^{(NA)}(\eta)$ is analytic about $\eta = 0$ in a disk in the complex η plane of radius $|\eta| = 3/4$. As η approaches the value

$$\eta_{div} = \frac{3}{4} \quad (6.27)$$

from below, $F_1^{(NA)}(\eta)$ diverges through negative values. Thus, as in the U(1) case, this leads to a zero in the expression for $\beta_\eta^{(NA)}$ to leading order in $1/N_f$, i.e., there exists a value of η slightly less than $3/4$ for which $[1 + F_1^{(NA)}(\eta)/N_f] = 0$ for large N_f .

Via Taylor series expansion of the integrand in Eq. (6.25) and integration term-by-term, one can calculate the $f_{s,p}^{(NA)}$ and hence the $b_{\ell,\ell-1}^{(NA)}$. The $b_{\ell,\ell-1}^{(NA)}$ obtained in this way agree with the known results for loop order $\ell = 1$ to $\ell = 4$. For $\ell = 1$,

$$b_{1,1}^{(NA)} = \hat{f}_{1,0}^{(NA)} = b_{1,1} = \frac{4}{3}. \quad (6.28)$$

For $\ell \geq 2$, the $b_{\ell,\ell-1}^{(NA)}$ have the general form

$$b_{\ell,\ell-1}^{(NA)} = b_{\ell,\ell-1} C_f + b_{\ell,\ell-1,C_A} C_A, \quad (6.29)$$

where $b_{\ell,\ell-1}$ is the corresponding coefficient for the U(1) gauge theory. Specifically,

$$b_{2,1}^{(NA)} = \hat{f}_{1,1}^{(NA)} = b_{2,1} C_f + \frac{20}{3} C_A = 4C_f + \frac{20}{3} C_A,$$

$$\begin{aligned} b_{6,5}^{(NA)} = \hat{f}_{1,5}^{(NA)} &= b_{6,5} C_f + \left(-\frac{4832}{1215} - \frac{40448\zeta(3)}{3645} + \frac{512\pi^4}{6075} \right) C_A \\ &= 1.47275C_f - 3.63329C_A \end{aligned} \quad (6.34)$$

and

$$\begin{aligned} b_{7,6}^{(NA)} = \hat{f}_{1,6}^{(NA)} &= b_{7,6} C_f + \left(-\frac{9440}{2187} - \frac{27136\zeta(3)}{6561} - \frac{20224\pi^4}{98415} + \frac{20480\zeta(5)}{729} \right) C_A \\ &= -(7.80879C_f + 0.1746567C_A). \end{aligned} \quad (6.35)$$

For the purpose of our analysis of zeros of the beta function of the general non-Abelian theory in the large- N_f limit, we have calculated the $b_{n,n-1}^{(NA)}$ to higher-loop

$$\begin{aligned} b_{8,7}^{(NA)} = \hat{f}_{1,7}^{(NA)} &= b_{8,7} C_f + \left(-\frac{215680}{45927} - \frac{468992\zeta(3)}{45927} - \frac{54272\pi^4}{688905} - \frac{647168\zeta(5)}{15309} + \frac{163840\pi^6}{2893401} + \frac{163840\zeta(3)^2}{15309} \right) C_A \\ &= 1.42326C_f + 2.49243C_A. \end{aligned} \quad (6.36)$$

In Appendix C and Table V we list our additional results

$$(6.30)$$

$$\begin{aligned} b_{3,2}^{(NA)} = \hat{f}_{1,2}^{(NA)} &= b_{3,2} C_f - \frac{158}{27} C_A \\ &= -\left[\frac{44}{9} C_f + \frac{158}{27} C_A \right] \\ &= -(4.88889C_f + 5.85185C_A) \end{aligned} \quad (6.31)$$

and

$$\begin{aligned} b_{4,3}^{(NA)} = \hat{f}_{1,3}^{(NA)} &= b_{4,3} C_f - \frac{424}{243} C_A \\ &= -\left[\frac{1232}{243} C_f + \frac{424}{243} C_A \right] \\ &= -(5.06996C_f + 1.74486C_A). \end{aligned} \quad (6.32)$$

The coefficients up to $\ell = 7$ are [19]

$$\begin{aligned} b_{5,4}^{(NA)} = \hat{f}_{1,4}^{(NA)} &= b_{5,4} C_f + \left(-\frac{916}{243} + \frac{640}{81} \right) C_A \\ &= 9.22127C_f + 5.72819C_A \end{aligned} \quad (6.33)$$

order. For $\ell = 8$ we find

for the $b_{\ell,\ell-1}^{(NA)}$ with two higher values of $\ell \equiv n$.

In accordance with (6.21), the n -loop beta function, $\beta_{\alpha,n\ell}^{(NA)}$, has a UV zero for large N_f if and only if $b_{n,n-1}^{(NA)} < 0$. We list the $b_{n,n-1}^{(NA)}$ in Table V together with an entry indicating whether or not the n -loop beta function, $\beta_{\alpha,n\ell}^{(NA)}$ has a UV zero. The signs of the $b_{n,n-1}^{(NA)}$ are definite for $b_{2,1}^{(NA)}$ (positive), $b_{3,2}^{(NA)}$ (negative), $b_{4,3}^{(NA)}$ (negative), and $b_{5,4}^{(NA)}$ (positive).

In the case of $b_{6,5}^{(NA)}$ and some $b_{n,n-1}^{(NA)}$ for higher n , the C_f and C_A terms have opposite signs, so that further analysis is necessary. Let us define the ratio

$$r_C \equiv \frac{C_A}{C_f}. \quad (6.37)$$

The ranges of values of r_C for various fermion representations R will be relevant for our analysis and are given in Appendix E. Analytically continuing r_C from \mathbb{Q} to \mathbb{R} , we find that $b_{6,5}^{(NA)}$ is negative (positive) for r_C greater (less) than

$$(r_C)_{6,5} = \frac{2(1255 - 2640\zeta(3) + 24\pi^4)}{5(453 + 1264\zeta(3) - 16\pi^4)} = 0.405348 \quad (6.38)$$

One can reexpress this as a correlated condition on the non-Abelian gauge group, G , and fermion representation, R . For example, consider $G = \text{SU}(N_c)$, and R equal to the fundamental representation. Then (see Appendix E) r_C decreases from $8/3$ at $N_c = 2$ toward 2 as $N_c \rightarrow \infty$, so that the condition that $(r_C)_{6,5}$ be greater than 0.405348 is always satisfied, and hence $\beta_{\alpha,6\ell}^{(NA)}$ has a UV zero for large N_f . All of the other representations considered in Appendix E, including the adjoint, rank-2 symmetric (S_2), and rank-2 antisymmetric (A_2) tensor representations yield r_C values larger than the value in Eq. (6.38), and hence $\beta_{\alpha,6\ell}^{(NA)}$ also has a UV zero for large N_f for all of these representations.

Proceeding to higher-loop values, the coefficient $b_{7,6}^{(NA)}$ is negative-definite, so that $\beta_{\alpha,7\ell}^{(NA)}$ has a UV zero for large N_f . However, the sign of $b_{8,7}^{(NA)}$ is positive, so $\beta_{\alpha,8\ell}^{(NA)}$ does not have a UV zero. The signs of several of the $b_{n,n-1}^{(NA)}$ with $n \geq 9$ depend on r_C . As an example, the sign of $b_{9,8}^{(NA)}$ is negative if and only if $r_C > 2.2970$ (to the indicated floating-point accuracy). Whether or not this condition is satisfied depends on the gauge group and the fermion representation. For definiteness, we again focus on the group $G = \text{SU}(N_c)$. If the fermions are in the fundamental representation, then the condition $r_C > 2.2970$ corresponds to the condition that N_c (analytically continued from positive integer to positive real values) is less than 2.7810. Thus, for this fermion representation and (physical, integer) values of N_c , $b_{9,8}^{(NA)}$ is negative if and only if $N_c = 2$, and for this single value of N_c , $\beta_{\alpha,9\ell}^{(NA)}$ has a UV zero for large N_f . If the theory contains (Dirac) fermions in the adjoint representation, then $r_C =$

1, so $b_{9,8}^{(NA)}$ is positive. Similarly, if the fermions are in the S_2 representation, then r_C is bounded between $8/9$ and 1, so again, $b_{9,8}^{(NA)}$ is positive. If the fermions are in the A_2 representation, then the condition $r_C > 2.2970$ corresponds to the inequality $N_c < 2.965$. Since the A_2 representation is defined only for $N_c \geq 3$, there is no physical N_c that satisfies this inequality. Thus, for the adjoint, S_2 , and A_2 representations, $\beta_{\alpha,9\ell}^{(NA)}$ has no UV zero for large N_f .

At loop order $n = 11$, the condition for $b_{11,10}^{(NA)}$ to be negative is that $r_C < 0.450206$, but this is never satisfied for any of the representations considered here. For these, $b_{11,10}^{(NA)}$ is positive, so $\beta_{\alpha,11\ell}$ does not have any UV zero for large N_f . At loop order $n = 12$, the condition for $b_{12,11}^{(NA)}$ to be negative is that $r_C > 1.3528$. This inequality is always satisfied for the fundamental representation, but is never satisfied for the adjoint, or S_2 representations. For the A_2 representation, this inequality corresponds to the inequality $N_c < 5.28542$. Hence, for the physical, integer values $N_c = 3, 4, 5$, this inequality is satisfied, so that $b_{12,11}^{(NA)} < 0$ and $\beta_{\alpha,11\ell}^{(NA)}$ has a UV zero at large N_f .

Up to loop order $n = 18$, there are two other cases where the sign of $b_{n,n-1}^{(NA)}$ depends on r_C , but in both of these cases, namely $n = 15$ and $n = 18$, the respective inequalities for $b_{n,n-1}^{(NA)}$ to be negative are satisfied for all of the representations that we consider, so that for $n = 13$ up to this highest loop order, $n = 18$, the respective $\beta_{\alpha,n\ell}^{(NA)}$ has a UV zero for large N_f .

It is thus interesting that for the seven cases $n = 12$ through $n = 18$, $b_{n,n-1}^{(NA)}$ is negative for R equal to the fundamental representation. This is to be contrasted with the U(1) theory, in which some of the $b_{n,n-1}$ in this interval of n (specifically, $n = 12, 15, 18$) are positive. However, even in the cases where $b_{n,n-1}^{(NA)} < 0$, so that there is a solution for $\eta_{UV,n\ell}$, it has the property (6.22). In this respect, the situation with finite-loop calculations of a UV zero in the beta function for a non-Abelian gauge theory for large N_f is similar to the situation with the Abelian theory.

As is evident from Eq. (6.25), $F_1^{(NA)}(\eta)$ is equal to $-(11/4)C_A$ at $\eta = 0$. As a function of η , it increases as η increases from zero through negative values, reaching a broad maximum at $\eta \simeq 0.6$ and then decreases again and diverges logarithmically through negative values as η approaches $3/4$ from below. We will focus on the interval

$$I_\eta : \quad 0 \leq \eta \leq \frac{3}{4}. \quad (6.39)$$

in our analysis. The condition that $\beta_\eta^{(NA)}$, calculated to leading order in $1/(T_f N_f)$, vanishes with a UV zero is

$$1 + \frac{F_1^{(NA)}(\eta)}{T_f N_f} = 0. \quad (6.40)$$

Since $F_1^{(NA)}(\eta)$ diverges negatively as η approaches $3/4$ from below, there exists a value of η slightly less than $3/4$

for which the condition (6.40) is satisfied. This suggests that the rescaled beta function $\beta_\eta^{(NA)}$ might have a UV zero. As in the Abelian case, the contributions of higher-order terms $\sum_{s=2}^{\infty} F_s^{(NA)}(\eta)/(T_f N_f)^s$ might be such that the full beta function does not have a UV zero that can be reached via renormalization-group evolution from small couplings in the infrared.

As we did for the U(1) theory, we may compare these results with findings from analyses of possible UV zeros in the n -loop beta function $\beta_{\alpha, n\ell}^{(NA)}$ for fixed large N_f . The criterion (6.21) determines whether such a UV zero exists for large N_f . We have found a scatter of signs in the $b_{n, n-1}^{(NA)}$, as listed in Table V, although the $b_{n, n-1}^{(NA)}$ for $n = 12$ to $n = 18$ are negative for fermions in the fundamental representation. Even for the values of n for which there is a UV zero, the solution obeys the asymptotic limiting relation (6.22), making it difficult to match with a finite value of η for a possible UV zero in $\beta_\eta^{(NA)}$.

VII. CONCLUSIONS

In this paper we have extended our earlier study [2] of a UV zero in the n -loop beta function, $\beta_{\alpha, n\ell}$ in a U(1) gauge theory to large values of N_f and have investigated how the results relate to exact results for the leading correction term $F_1(y)/N_f$, in the (rescaled) beta function β_y obtained by summation of a certain dominant set of Feynman diagrams up to infinitely high loop order in the LNF limit (4.1). Effects of scheme transformations on the $b_{\ell, k}$ and $f_{s, p}$ have been calculated. A general criterion was given for determining whether or not the n -loop β function has a UV zero for large N_f , namely that $b_{n, n-1}$, which is NFI scheme-independent, must be negative. As part of our study, we have presented new analytic and numerical results for the coefficients $b_{n, n-1}$ that enter as leading- N_f terms in the n -loop coefficients b_n in the beta function. The coefficients $b_{n, n-1}$ show a scatter of both positive and negative values and hence do not give evidence for a stable UV zero in the U(1) beta function at large N_f up to the highest-loop order, namely $n = 24$, to which we probed. We derived, and verified the accuracy of, an approximate analytic expression (4.12) for the UV zero of the n -loop beta function and showed that, even if $b_{n, n-1} < 0$ so that the n -loop beta function has a UV zero, the value of $y_{UV, n\ell}$ diverges as $N_f \rightarrow \infty$. By calculating corresponding coefficients $b_{n, n-1}^{(d)}$ arising from the negative logarithmically divergent term in $F_1(y)$ and determining how these differ from the actual $b_{n, n-1}$, we showed that the latter, at least to loop order $n = 24$, are not sensitive to the negatively divergent term in $F_1(y)$. We analyzed various illustrative test functions for β_y incorporating a UV zero and calculated resultant series expansions in $1/N_f$ and in y for these functions, finding that the resultant $b_{n, n-1}$ are uniformly negative, as is also true of the $b_{n, n-1}^{(d)}$.

We have also considered the analogous question of a UV zero in the n -loop beta function in a non-Abelian gauge theory at large N_f . We have shown that for some loop orders n , $\text{sgn}(b_{n, n-1})$, and hence the existence of a UV zero in the n -loop beta function at large N_f , can depend on the fermion representation, R , and have discussed the consequences of the ranges of values of the relevant ratio $r_C \equiv C_A/C_f$ for various representations. In general, our conclusions for the non-Abelian gauge theory are broadly similar to those that we reach for the U(1) theory. It is hoped that these results will be a useful addition to the understanding of the properties of the beta functions of Abelian and non-Abelian gauge theories.

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Appendix A: Effects of Scheme Transformations on $b_{\ell, k}$ and $f_{s, p}$

1. Transformations of $b_{\ell, k}$

In Sect. IIB we gave a general formula for the effect of NFI scheme transformations on the coefficients $b_{\ell, k}$ multiplying N_f^k in the ℓ -loop term, b_ℓ in the beta function of a U(1) gauge theory. Here we list some of these relations explicitly for $3 \leq \ell \leq 6$. In addition to the invariance relation (2.18), we have

$$\begin{aligned} b'_{3,1} &= b_{3,1} + h_{3,2}b_{2,1} + h_{3,1}b_{1,1} \\ &= b_{3,1} + k_1b_{2,1} + (k_1^2 - k_2)b_{1,1} , \end{aligned} \quad (\text{A1})$$

$$b'_{4,2} = b_{4,2} + h_{4,3}b_{3,2} = b_{4,2} + 2k_1b_{3,2} , \quad (\text{A2})$$

$$\begin{aligned} b'_{4,1} &= b_{4,1} + h_{4,3}b_{3,1} + h_{4,2}b_{2,1} + h_{4,1}b_{1,1} \\ &= b_{4,1} + 2k_1b_{3,1} + k_1^2b_{2,1} + 2(-k_1^3 + 2k_1k_2 - k_3)b_{1,1} , \end{aligned} \quad (\text{A3})$$

$$b'_{5,3} = b_{5,3} + h_{5,4}b_{4,3} = b_{5,3} + 3k_1b_{4,3} , \quad (\text{A4})$$

$$\begin{aligned} b'_{5,2} &= b_{5,2} + h_{5,4}b_{4,2} + h_{5,3}b_{3,2} \\ &= b_{5,2} + 3k_1b_{4,2} + (2k_1^2 + k_2)b_{3,2} , \end{aligned} \quad (\text{A5})$$

$$\begin{aligned}
b'_{5,1} &= b_{5,1} + h_{5,4}b_{4,1} + h_{5,3}b_{3,1} + h_{5,2}b_{2,1} + h_{5,1}b_{1,1} & b'_{6,3} &= b_{6,3} + h_{6,5}b_{5,3} + h_{6,4}b_{4,3} \\
&= b_{5,1} + 3k_1b_{4,1} + (2k_1^2 + k_2)b_{3,1} & &= b_{6,3} + 4k_1b_{5,3} + 2(2k_1^2 + k_2)b_{4,3} , \quad (\text{A8}) \\
&+ (-k_1^3 + 3k_1k_2 - k_3)b_{2,1} \\
&+ (4k_1^4 - 11k_1^2k_2 + 6k_1k_3 + 4k_2^2 - 3k_4)b_{1,1} , & b'_{6,2} &= b_{6,2} + h_{6,5}b_{5,2} + h_{6,4}b_{4,2} + h_{6,3}b_{3,2} \\
& & &= b_{6,2} + 4k_1b_{5,2} + 2(2k_1^2 + k_2)b_{4,2} + 4k_1k_2b_{3,2} , \\
& & & (\text{A9})
\end{aligned}$$

$$b'_{6,4} = b_{6,4} + h_{6,4}b_{5,4} = b_{6,4} + 4k_1b_{5,4} , \quad (\text{A7}) \quad \text{and}$$

$$\begin{aligned}
b'_{6,1} &= b_{6,1} + h_{6,5}b_{5,1} + h_{6,4}b_{4,1} + h_{6,3}b_{3,1} + h_{6,2}b_{2,1} + h_{6,1}b_{1,1} \\
&= b_{6,1} + 4k_1b_{5,1} + 2(2k_1^2 + k_2)b_{4,1} + 4k_1k_2b_{3,1} + (2k_1^4 - 6k_1^2k_2 + 4k_1k_3 + 3k_2^2 - 2k_4)b_{2,1} \\
&+ 4(-2k_1^5 + 7k_1^3k_2 - 4k_1^2k_3 - 5k_1k_2^2 + 2k_1k_4 + 3k_2k_3 - k_5)b_{1,1} , \quad (\text{A10})
\end{aligned}$$

and similarly for $\ell \geq 7$. These formulas determine the corresponding transformations of the $f_{s,p}$ in Eqs. (4.23) and (6.24). Similar results hold for a non-Abelian gauge theory.

2. Transformations of $f_{s,p}$ and $F_s(y)$

Here we present calculations of the $f'_{s,p}$ in terms of $f_{s,p}$ for NFI scheme transformations, and the resultant transformation of $F_s(y)$, for $s = 2, 3$. Since for $\ell \geq 3$, the $b'_{\ell,k}$ are not, in general, equal to the $b_{\ell,k}$ for the lower values of k , $1 \leq k \leq \ell - 2$, it follows that the $f_{s,p}$ and $F_s(y)$ with $s \geq 2$ are scheme-dependent even for NFI STs. Using Eqs. (4.25) or equivalently, (4.26) together with our general formulas for the scheme transformation of $b_{\ell,k}$ with $1 \leq k \leq \ell - 2$, Eqs. (2.20) and (2.21), we can calculate how the $f_{s,p}$ and hence $F_s(y)$ with $s \geq 2$ change under an NFI scheme transformation. For the coefficients $f_{2,p}$ that occur in the $s = 2$ special case of Eq. (4.23), we find

$$\begin{aligned}
f'_{2,2} &= f_{2,2} + h_{3,2}f_{1,1} + h_{3,1} \\
&= f_{2,2} + k_1f_{1,1} + (k_1^2 - k_2) , \quad (\text{A11})
\end{aligned}$$

and

$$\begin{aligned}
f'_{2,p} &= f_{2,p} + h_{p+1,p}f_{1,p-1} \\
&= f_{2,p} + (p-1)k_1f_{1,p-1} \quad \text{for } p \geq 3 . \quad (\text{A12})
\end{aligned}$$

Substituting these results into the $s = 2$ special case of Eq. (4.23), we thus derive the effect of an NFI scheme

transformation on $F_2(y)$:

$$\begin{aligned}
F_2(y)_{ST} &= F_2(y) + (h_{32}f_{1,1} + h_{3,1})y^2 + \sum_{p=3}^{\infty} h_{p+1,p}f_{1,p-1}y^p \\
&= F_2(y) + (k_1f_{1,1} + k_1^2 - k_2)y^2 \\
&+ k_1 \sum_{p=3}^{\infty} (p-1)f_{1,p-1}y^p . \quad (\text{A13})
\end{aligned}$$

For $s = 3$ we find

$$\begin{aligned}
f'_{3,3} &= f_{3,3} + h_{4,3}f_{2,2} + h_{4,2}f_{1,1} + h_{4,1} \\
&= f_{3,3} + 2k_1f_{2,2} + k_1^2f_{1,1} + (-2k_1^3 + 4k_1k_2 - 2k_3) \\
& \quad (\text{A14})
\end{aligned}$$

and

$$f'_{3,p} = f_{3,p} + h_{p+1,p}f_{2,p-1} + h_{p+1,p-1}f_{1,p-2} \quad \text{for } p \geq 4 . \quad (\text{A15})$$

Hence,

$$\begin{aligned}
F_3(y)_{ST} &= F_3(y) + (h_{4,3}f_{2,2} + h_{4,2}f_{1,1} + h_{4,1})y^3 \\
&+ \sum_{p=4}^{\infty} (h_{p+1,p}f_{2,p-1} + h_{p+1,p-1}f_{1,p-2})y^p . \quad (\text{A16})
\end{aligned}$$

Similar results follow for $s \geq 4$.

Appendix B: Some Relevant Properties of the Euler Gamma Function

We recall the definition of the Euler gamma function $\Gamma(x)$:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt . \quad (\text{B1})$$

The function $\Gamma(x)$ has simple poles at $x = -n + \epsilon$, with $n = 0, 1, 2, \dots$. The Taylor-Laurent series for $\Gamma(x)$ as x approaches one of these poles is

$$\Gamma(-n + \epsilon) \sim \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \left(-\gamma_E + \sum_{m=1}^n \frac{1}{m} \right) + O(\epsilon) \right] . \quad (\text{B2})$$

where the term $\sum_{m=1}^n \frac{1}{m}$ is absent for $n = 0$ and γ_E is the Euler-Mascheroni constant,

$$\gamma_E = \lim_{k \rightarrow \infty} \left[\left(\sum_{s=1}^k \frac{1}{s} \right) - \ln k \right] = 0.5772\dots \quad (\text{B3})$$

The expansion of $I_1(x)$ about $x = 0$ and hence $F_1(y)$ about $y = 0$ makes use of the Taylor series expansions of the gamma functions in $I_1(x)$ about nonsingular points. These in turn rely upon the basic results (e.g., [46])

$$\Gamma(x + 1) = \sum_{k=0}^{\infty} c_k x^k , \quad (\text{B4})$$

where $c_0 = 1$ and

$$c_{n+1} = (n+1)^{-1} \sum_{m=0}^n (-1)^{m+1} s_{m+1} c_{n-m} \quad (\text{B5})$$

with

$$s_1 = \gamma_E, \quad s_m = \zeta(m) \text{ for } m \geq 2 \quad (\text{B6})$$

and

$$\frac{1}{\Gamma(x+1)} = \sum_{k=0}^{\infty} d_k x^k , \quad (\text{B7})$$

where $d_0 = 1$ and

$$d_{n+1} = (n+1)^{-1} \sum_{m=0}^n (-1)^m c_{m+1} d_{n-m} . \quad (\text{B8})$$

These relations explain the presence of the Riemann $\zeta(s)$ functions in various $f_{s,p}$ and $b_{n,k}$. As noted in the text, the $\zeta(m)$ with even $m = 2r$ are evaluated using the identity

$$\zeta(2r) = \frac{(-1)^{r+1} B_{2r} (2\pi)^{2r}}{2(2r)!} , \quad (\text{B9})$$

where the B_n are the Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} . \quad (\text{B10})$$

Thus, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_{2r+1} = 0$ for $r \geq 1$. Note that $\text{sgn}(B_{2r}) = (-1)^{r-1}$. The relations for $\zeta(2r)$ are then

$$\zeta(2) = \frac{\pi^2}{6} , \quad \zeta(4) = \frac{\pi^4}{90} , \quad \zeta(6) = \frac{\pi^6}{945} , \quad (\text{B11})$$

and so forth for higher values of $2r$.

Appendix C: Taylor Series Expansions of $F_1(y)$ and $F_1^{(NA)}(\eta)$

In this appendix we list the coefficients $f_{s,p}$ and $f_{s,p}^{(NA)}$ in the respective Taylor series expansions of $F_1(y)$ and $F_1^{(NA)}(\eta)$ for the U(1) and non-Abelian gauge theories. For the U(1) theory we have

$$f_{1,1} = 3 , \quad (\text{C1})$$

$$f_{1,2} = -\frac{11}{3} = -3.66667 , \quad (\text{C2})$$

$$f_{1,3} = -\frac{308}{81} = -3.80247 , \quad (\text{C3})$$

$$f_{1,4} = \frac{214}{81} + \frac{32\zeta(3)}{9} = 6.915955 , \quad (\text{C4})$$

$$f_{1,5} = \frac{4016}{1215} - \frac{2816\zeta(3)}{405} + \frac{128\pi^4}{2025} = 1.10456 , \quad (\text{C5})$$

$$\begin{aligned} f_{1,6} &= \frac{1072}{243} - \frac{19712\zeta(3)}{2187} - \frac{1408\pi^4}{10935} + \frac{1024\zeta(5)}{81} \\ &= -5.85659 , \end{aligned} \quad (\text{C6})$$

and so forth for higher $f_{s,p}$ coefficients.

For the non-Abelian gauge theory we have

$$f_{1,0}^{(NA)} = -\frac{11C_A}{4} , \quad (\text{C7})$$

$$f_{1,1}^{(NA)} = 3C_f + 5C_A , \quad (\text{C8})$$

$$f_{1,2}^{(NA)} = -\left[\frac{11C_f}{3} + \frac{79C_A}{18} \right] , \quad (\text{C9})$$

Appendix D: Higher-Loop Coefficients $b_{n,n-1}$ and $b_{n,n-1}^{(NA)}$

$$f_{1,3}^{(NA)} = - \left[\frac{308C_f}{81} + \frac{106C_A}{81} \right], \quad (C10)$$

$$f_{1,4}^{(NA)} = \left(\frac{214}{81} + \frac{32\zeta(3)}{9} \right) C_f + \left(-\frac{229}{81} + \frac{160\zeta(3)}{27} \right) C_A, \quad (C11)$$

and so forth for higher orders.

1. U(1) Gauge Theory

In the text, for the U(1) gauge theory with N_f fermions we have listed the $b_{n,n-1}$ up to $n = 8$. Here we list the higher-loop coefficients for $n = 9, 10$:

$$b_{9,8} = \dot{f}_{1,8} = \frac{214720}{19683} + \frac{257024\zeta(3)}{19683} + \frac{54784\pi^4}{295245} - \frac{315392\zeta(5)}{6561} - \frac{90112\pi^6}{1240029} - \frac{90112\zeta(3)^2}{6561} + \frac{16384\zeta(7)}{243} + \frac{8192\pi^4\zeta(3)}{32805} = 2.35202 \quad (D1)$$

and

$$b_{10,9} = \dot{f}_{1,9} = \frac{2630656}{177147} + \frac{1097728\zeta(3)}{59049} + \frac{2056192\pi^4}{7971615} + \frac{7012352\zeta(5)}{177147} - \frac{1441792\pi^6}{14348907} - \frac{10092544\zeta(3)^2}{531441} - \frac{2883584\zeta(7)}{19683} - \frac{1441792\pi^4\zeta(3)}{2657205} + \frac{229376\pi^8}{13286025} + \frac{1048576\zeta(3)\zeta(5)}{19683} = -1.71453. \quad (D2)$$

2. Non-Abelian Gauge Theory

Similarly, we calculate

$$b_{9,8}^{(NA)} = \dot{f}_{1,8}^{(NA)} = b_{9,8} C_f + \left(-\frac{32992}{6561} - \frac{77312\zeta(3)}{6561} - \frac{58624\pi^4}{295245} - \frac{108544\zeta(5)}{6561} - \frac{323584\pi^6}{3720087} - \frac{323584\zeta(3)^2}{19683} + \frac{81920\zeta(7)}{729} + \frac{8192\pi^4\zeta(3)}{19683} \right) C_A = 2.35202C_f - 1.02394C_A \quad (D3)$$

and

$$b_{10,9}^{(NA)} = \dot{f}_{1,9}^{(NA)} = b_{10,9} C_f + \left(-\frac{921088}{177147} - \frac{2416640\zeta(3)}{177147} - \frac{618496\pi^4}{2657205} - \frac{7503872\zeta(5)}{177147} - \frac{3473408\pi^6}{100442349} - \frac{3473408\zeta(3)^2}{531441} - \frac{10354688\zeta(7)}{59049} - \frac{5177344\zeta(3)\pi^4}{7971615} + \frac{229376\pi^8}{7971615} + \frac{5242880\zeta(3)\zeta(5)}{59049} \right) C_A = -(1.71453C_f + 0.059843C_A). \quad (D4)$$

Appendix E: Ranges of Values of r_C

The ranges of values of the ratio r_C in Eq. (6.37) for various fermion representations R will be important for our analysis of possible UV zeros of the beta function in a non-Abelian gauge theory. For definiteness, we focus on the gauge group $G = \text{SU}(N_c)$; it is straightforward

to deal with other gauge groups. For R equal to the fundamental representation of $\text{SU}(N_c)$, one has

$$r_C = \frac{2N_c^2}{N_c^2 - 1} \quad \text{for } R = \text{fund}. \quad (E1)$$

This has the value $r_C = 8/3$ at $N_c = 2$ and decreases monotonically with increasing N_c , approaching the lim-

iting value of 2 as $N_c \rightarrow \infty$. Setting r_C equal to a given value and solving for N_c yields a quadratic equation for N_c , with a unique physical solution,

$$N_c = \left(\frac{r_C}{r_C - 2} \right)^{1/2}. \quad (\text{E2})$$

We next consider two-index representations. For R equal to the adjoint representation, r_C has the unique value $r_C = 1$. If R is the symmetric rank-2 tensor representation, denoted S_2 , then

$$r_C = \frac{N_c^2}{(N_c + 2)(N_c - 1)} \quad \text{for } R = S_2. \quad (\text{E3})$$

This has the value 1 at $N_c = 2$ and decreases to a minimum of $8/9$ at $N_c = 4$, after which it increases for larger N_c , approaching the limiting value of 1 from below as $N_c \rightarrow \infty$. Regarding the inverse relation, there is no solution if $r_C \notin [8/9, 1)$, and there are two solutions for N_c (as a formal real variable) if $r_C \in [8/9, 1)$. These solutions are

$$N_c = \frac{r_C + \sqrt{r_C(9r_C - 8)}}{2(1 - r_C)} \quad (\text{E4})$$

and

$$N_c = \frac{r_C - \sqrt{r_C(9r_C - 8)}}{2(1 - r_C)} \quad (\text{E5})$$

If R is the antisymmetric rank-2 tensor representation (defined for $N_c \geq 3$), denoted A_2 ,

$$r_C = \frac{N_c^2}{(N_c - 2)(N_c + 1)} \quad \text{for } R = A_2. \quad (\text{E6})$$

This has the value $r_C = 8/3$ for $N_c = 3$ and decreases monotonically with N_c , approaching the limiting value 1 from above as $N_c \rightarrow \infty$.

One can also consider more complicated representations. For example, if R is the symmetric rank-3 tensor representation, S_3 , then

$$r_C = \frac{2N_c^2}{3(N_c + 3)(N_c - 1)} \quad \text{for } R = S_3. \quad (\text{E7})$$

This has the value $r_C = 8/15 = 0.5333$ at $N_c = 2$ and increases with N_c , approaching the value $2/3$ from below as $N_c \rightarrow \infty$. For R equal to the antisymmetric rank-3 tensor representation A_3 (defined for $N_c \geq 4$),

$$r_C = \frac{2N_c^2}{3(N_c - 3)(N_c + 1)} \quad \text{for } R = A_3. \quad (\text{E8})$$

This has the value $32/15 = 2.1333$ for $N_c = 4$ and decreases monotonically with N_c , approaching the limit $2/3$ from above as $N_c \rightarrow \infty$.

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- [39] There have been many studies of scheme transformations applicable in the vicinity of the origin, $\alpha = 0$, designed to optimize the accuracy and stability of perturbative QCD calculations; we do not consider these here, since our focus is on a possible UV zero away from the origin.
- [40] Since our previous related works [2, 3, 34–36] focused on asymptotically free non-Abelian gauge theories, it was convenient to extract an overall minus sign multiplying the beta function. Because we focus here on infrared-free theories, we write the beta function as in Eqs. (2.3) and (6.3).
- [41] One can generalize this to scheme transformations with $f(a')$ functions that have an essential zero at $a' = 0$, as discussed in footnote [25] of the first paper of [3]. The results for b'_ℓ are the same for this case as for a transformation function $f(a')$ that is analytic at $a' = 0$.
- [42] In passing, we note that this result, that b_1 and b_2 have the same sign, so the theory has no two-loop UV zero, applies more generally to a vectorial U(1) gauge theory with N_{f_i} fermions of different charges q_i for $i = 1, \dots, k$, and, even more generally, to a chiral U(1) gauge theory, with left-handed fermions $\psi_{i,L}$, $1 \leq i \leq N_{f,L}$ and charges q_{iL} , and right-handed fermions $\psi_{i,R}$, $1 \leq i \leq N_{f,R}$ and charges q_{iR} . In the chiral case, one requires that $\sum_{i=1}^{N_{f,L}} q_{i,L}^3 - \sum_{j=1}^{N_{f,R}} q_{j,R}^3 = 0$ so that there is no chiral gauge anomaly.
- [43] Here and below, when an expression is given for N_f that formally evaluates to a non-integral real value, it is understood implicitly that one infers an appropriate integral value of N_f from this.
- [44] Our expansions agree with, and extend to higher order, the results in [21] except for the A^4 term in Eq. (5) of [21]; that term was derived from an earlier partial calculation of b_5 in P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **88**, 012001 (2001) which was superseded by the full calculation in [11].
- [45] Our notation for Casimir invariants is $C_2(G) \equiv C_A$ and, for a fermion representation R , $C_2(R) \equiv C_f$ and $T(R) = T_f$; our normalizations are standard, so that, e.g., for $G = \text{SU}(N_c)$, $C_A = N_c$, and, for R equal to the fundamental representation, $C_f = (N_c^2 - 1)/(2N_c)$ and $T_f = 1/2$.
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TABLE I: Values of the UV zero in the β_α function of the U(1) gauge theory with N_f fermions, at n -loop ($n\ell$) order, for $n = 3, 4, 5$, in the $\overline{\text{MS}}$ scheme, denoted $\alpha_{UV,n\ell}$. The symbol $-$ indicates that there is no zero in β for the given order and value of N_f . See text for further details.

N_f	$\alpha_{UV,3\ell}$	$\alpha_{UV,4\ell}$	$\alpha_{UV,5\ell}$
1	10.2720	3.0400	–
2	6.8700	2.4239	–
3	5.3689	2.0776	–
4	4.5017	1.8463	–
5	3.9279	1.67685	2.5570
6	3.5156	1.5455	1.8469
7	3.2027	1.4397	1.6243
8	2.9555	1.3519	1.4851
9	2.7545	1.2776	1.3863
10	2.5871	1.2135	1.3120
20	1.7262	0.8483	–
100	0.7081	0.33265	–
500	0.3038	0.1203	–
10^3	0.2127	0.07678	–
10^4	0.016614	0.016965	–

TABLE II: Values of $\kappa_{n\ell}$ (defined in Eq. (4.16)) as a function of N_f in a U(1) gauge theory for $n = 3$ and $n = 4$ loops. See text for further details.

N_f	$\kappa_{3\ell}$	$\kappa_{4\ell}$
1	1.8580	1.1249
2	1.6248	1.0772
3	1.5105	1.0490
4	1.4404	1.0297
5	1.3920	1.0155
6	1.3562	1.0045
7	1.3284	0.9957
8	1.3060	0.9885
9	1.2875	0.9825
10	1.2719	0.9774
20	1.1883	0.9515
100	1.0812	0.9365
500	1.0356	0.9512
10^3	1.0251	0.9588
10^4	1.0079	0.9786

TABLE III: List of terms in b_n of highest degree (equal to degree $n-1$) in N_f for $2 \leq n \leq 24$, denoted $b_{n,n-1}$ in a U(1) gauge theory. The n -loop beta function in this theory, $\beta_{\alpha,n\ell}$, has a UV zero at large N_f if and only if $b_{n,n-1}$ is negative. Notation $a\text{e-}m$ means $a \times 10^{-m}$. See text for further details.

n	$b_{n,n-1}$
2	4
3	-4.8889
4	-5.0700
5	9.2213
6	1.47275
7	-7.8088
8	2.4924
9	2.3520
10	-1.7145
11	-0.022326
12	0.37396
13	-0.108455
14	-0.025088
15	0.020247
16	-2.2434e-3
17	-1.3959e-3
18	0.49000e-3
19	1.40184e-6
20	-0.32360e-4
21	0.59182e-5
22	0.65388e-6
23	-0.39871e-6
24	0.33916e-7

TABLE IV: List of coefficients $b_{n,n-1}^{(d)}$, as defined in Eq. (4.52) for $2 \leq n \leq 24$ in a U(1) gauge theory. Notation $ae-m$ means $a \times 10^{-m}$. See text for further details.

n	$b_{n,n-1}^{(d)}$
2	-0.033624
3	-0.89663e-2
4	-0.31880e-2
5	-1.2752e-3
6	-0.54409e-3
7	-0.24182e-3
8	-1.10545e-4
9	-0.51588e-4
10	-0.24456e-4
11	-1.1739e-5
12	-0.56917e-5
13	-0.27826e-5
14	-1.3699e-6
15	-0.67842e-6
16	-0.33770e-6
17	-1.6885e-7
18	-0.84757e-7
19	-0.42692e-7
20	-0.21571e-7
21	-1.0929e-8
22	-0.55514e-8
23	-0.28261e-8
24	-1.4417e-9

TABLE V: List of terms in $b_n^{(NA)}$ of highest degree (equal to degree $n - 1$) in N_f for $2 \leq n \leq 18$, denoted $b_{n,n-1}^{(NA)}$, and resultant determination of the existence or non-existence of a UV zero of the n -loop beta function, $\beta_{\alpha,n\ell}$, for large N_f (entries Y and N denote yes and no). For some n , $\text{sgn}(b_{n,n-1}^{(NA)})$ depends on $r_C \equiv C_A/C_f$, as indicated. Notation $ae\text{-}m$ means $a \times 10^{-m}$. See text for further discussion.

n	$b_{n,n-1}^{(NA)}$	$\exists \alpha_{UV,n\ell}$ at large N_f ?
2	$4C_f + 6.6667C_A$	N
3	$-(4.8889C_f + 5.85185C_A)$	Y
4	$-(5.0700C_f + 1.7449C_A)$	Y
5	$9.2213C_f + 5.7282C_A$	N
6	$1.47275C_f - 3.6333C_A$	Y iff $r_C > 0.40535$
7	$-(7.8088C_f + 0.17466C_A)$	N
8	$2.4924C_f + 1.4233C_A$	N
9	$2.3520C_f - 1.0239C_A$	Y iff $r_C > 2.2970$
10	$-(1.7145C_f + 0.059843C_A)$	Y
11	$-0.022326C_f + 0.049590C_A$	Y iff $r_C < 0.450206$
12	$0.37396C_f - 0.27643C_A$	Y iff $r_C > 1.3528$
13	$-(0.108455C_f + 0.21716C_A)$	Y
14	$-(0.025088C_f + 0.25622C_A)$	Y
15	$0.020247C_f - 0.33847C_A$	Y iff $r_C > 0.059818$
16	$-(0.0022434C_f + 0.41598C_A)$	Y
17	$-(0.0013959C_f + 0.51918C_A)$	Y
18	$0.00049000C_f - 0.652215C_A$	Y iff $r_C > 0.75129 \times 10^{-3}$