

Radiative Leptonic $B_c \rightarrow \gamma \ell \bar{\nu}$ Decay in Effective Field Theory beyond Leading Order

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Abstract

We study the radiative leptonic $B_c \rightarrow \gamma \ell \bar{\nu}$ decays in the nonrelativistic QCD effective field theory, and consider a fast-moving photon. As a result the interactions with the heavy quarks can be integrated out, and thus we arrive at a factorization formula for the decay amplitude. We calculate not only the relevant short-distance coefficients at leading order and next-to-leading order in α_s , but also the nonrelativistic corrections at the order $|v|^2$ in our analysis. We find out that the QCD corrections can sizably decrease the branching ratio and thus is of great importance in extracting the long-distance operator matrix elements of B_c . For the phenomenological application, we present our results for the photon energy, lepton energy and lepton-neutrino invariant mass distribution.

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I. INTRODUCTION

The search for new degrees of freedom can proceed under two distinctive directions. At the high energy frontier, new particles have different signatures with the standard model (SM) particles, and measurements of their production may provide definitive evidence on their existence. On the other hand, it is likely that low energy processes will be influenced through loop effects. Rare decays of heavy mesons, with tiny decay rates in the SM, are sensitive to the new degrees of freedom and thus can be exploited as indirect searches of these unknown effects, for a recent review see Ref. [1].

The B_c meson is the unique pseudo-scalar meson that is long lived and composed of two different heavy flavors. Since this hadron is stable against strong interactions, its weak decays provide a rich phenomena for the study of CKM matrix elements, and also a platform to study the effects of weak interactions in a heavy quarkonium system [2, 3]. In the past decades it has received growing attentions since the first observation by the CDF collaboration [4]. This can be particularly witnessed by the recent LHCb measurements of the B_c lifetime [5, 6], the decay widths of $B_c \rightarrow J/\psi\pi$ and $B_c \rightarrow J/\psi\ell\bar{\nu}$ [7, 8], and various other decay modes [9–12]. One may expect that more decay channels of B_c can be measured by the LHCb, ATLAS and CMS experiments [13–15].

On theoretical side, various approaches have been applied to calculate the decay width of B_c decays [16–52], but most of them are phenomenological. Since both constituents of the B_c are heavy and can only be treated nonrelativistically, an effective field theory can be established [53]. Taking the $B_c \rightarrow J/\psi\ell\bar{\nu}$ as the example, one may derive the conjectured non-relativistic QCD (NRQCD) factorization formula for its decay amplitude:

$$\mathcal{A}(B_c \rightarrow J/\psi) \propto C_{ij} \langle 0 | \mathcal{O}_i^{\bar{b}} | \bar{B}_c \rangle \times \langle J/\psi | \mathcal{O}_j^{\bar{c}} | 0 \rangle, \quad (1)$$

where the $\mathcal{O}_{i,j}^{ff'}$ are constructed by low energy operators. The short-distance, or *hard*, contributions at the length scale $1/m_{b,c}$ are encapsulated into the coefficients C_{ij} that can be computed in perturbation theory.

The long-distance, or *soft* part of, matrix elements have to be extracted in a nonperturbative approach, for instance the Lattice QCD simulation, or constrained by much simpler processes for instance the annihilation modes $B_c \rightarrow \ell\bar{\nu}$ and $B_c \rightarrow \gamma\ell\bar{\nu}$. However, the usefulness of the $B_c \rightarrow \ell\bar{\nu}$ is challenged by two aspects. Firstly its decay rate is given by

$$\Gamma(B_c \rightarrow \ell\bar{\nu}_\ell) = \frac{G_F^2}{8\pi} |V_{cb}|^2 f_{B_c}^2 m_{B_c}^3 \frac{m_\ell^2}{m_{B_c}^2} \left(1 - \frac{m_\ell^2}{m_{B_c}^2}\right)^2, \quad (2)$$

in which the suppression factor $m_\ell^2/m_{B_c}^2$ arises from the helicity flip. As a result, the $B_c \rightarrow \mu\bar{\nu}_\mu$ and $B_c \rightarrow e\bar{\nu}_e$ have tiny branching fractions that may be out of the detector capability at the current experimental facilities. Secondly, there is only one physical observable, namely the decay rate, and thus the $B_c \rightarrow \ell\bar{\nu}$ is not capable to uniquely determine all, typically more than one when relativistic corrections are taken into account, long-distance matrix elements (LDMEs).

On the contrary, the $B_c \rightarrow \gamma\ell\bar{\nu}$ can provide a wealth of information [54–58], in terms of a number of observables ranging from the decay probabilities, polarizations to an angular analysis. It is interesting to notice that the counterpart in B sector, $B \rightarrow \gamma\ell\bar{\nu}$, has been widely discussed towards the understanding of the B meson light-cone distribution amplitudes [59–63]. The small branching fraction of $B_c \rightarrow \gamma\ell\bar{\nu}$ can be compensated by the high

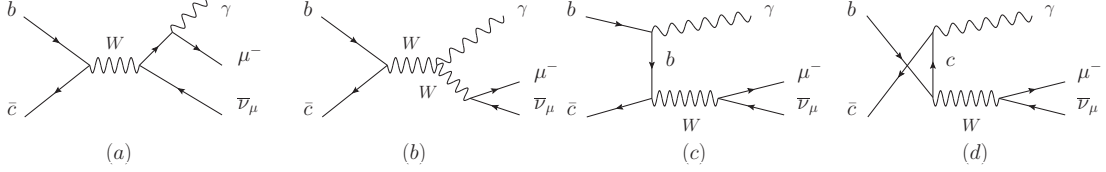


FIG. 1: Leading order Feynman diagrams for the radiative leptonic $B_c \rightarrow \gamma \mu \bar{\nu}_\mu$ decay in the SM. The lepton μ can also be e or τ . The photon emission from a virtual W -boson shown in the second panel is suppressed by $1/m_W^2$ compared to the other contributions.

luminosity at the ongoing hadron colliders and the under-design experimental facilities. The main purpose of this paper is to explore the $B_c \rightarrow \gamma \ell \bar{\nu}$ at next-to-leading order (NLO) in α_s and in $|\mathbf{v}|^2$, which shall catch up the progress in the $B_c \rightarrow \ell \bar{\nu}$ [55, 64]. For the leptonic decay constant, the two-loop calculation is also available in Ref. [65].

The rest of this paper is organized as follows. In Sec. I, we will derive the formulas for various partial decay widths of $B_c \rightarrow \gamma \ell \bar{\nu}$. Sec. III is extensively devoted to the next-to-leading order calculation. We will discuss the phenomenological results in Sec. IV. We summarize our findings and conclude in Sec. V. We relegate the calculation details to the Appendix.

In the SM, leading order (LO) Feynman diagrams for the $B_c \rightarrow \gamma \ell \bar{\nu}$ decay are shown in Fig 1. The photon emission from a virtual W -boson is suppressed by $1/m_W^2$ compared to other contributions, and thus the second diagram in Fig. 1 can be neglected. Integrating out the off-shell W -boson, we arrive at the effective electro-weak Hamiltonian

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cb} \bar{c} \gamma_\mu (1 - \gamma_5) b \bar{\ell} \gamma^\mu (1 - \gamma_5) \nu + h.c., \quad (3)$$

where V_{cb} is the CKM matrix element. The decay amplitude, matrix element of the above

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Hamiltonian between the B_c and $\gamma \ell \bar{\nu}$ state,

$$\mathcal{A} = \langle \gamma \ell^- \bar{\nu} | H_{\text{eff}} | \bar{B}_c \rangle \quad (4)$$

is responsible for the process $B_c \rightarrow \gamma \ell \bar{\nu}$.

A. Differential decay widths

Since there is no strong interaction connection between the leptonic and hadronic part, the decay amplitude can be decomposed into two individual sectors:

$$\mathcal{A} = \frac{G_F}{\sqrt{2}} V_{cb} \left\{ \langle 0 | \bar{c} \gamma_\mu (1 - \gamma_5) b | \bar{B}_c \rangle \times \langle \gamma \ell^- \bar{\nu} | \bar{\ell} \gamma^\mu (1 - \gamma_5) \nu | 0 \rangle \right. \\ \left. + \langle \gamma | \bar{c} \gamma_\mu (1 - \gamma_5) b | \bar{B}_c \rangle \times \langle \bar{u}_l \gamma^\mu (1 - \gamma_5) v_\nu \rangle \right\}, \quad (5)$$

with the matrix elements encoding the hadronic effects:

$$\langle 0 | \bar{c} \gamma_\mu (1 - \gamma_5) b | \bar{B}_c \rangle, \quad \langle \gamma | \bar{c} \gamma_\mu (1 - \gamma_5) b | \bar{B}_c \rangle. \quad (6)$$

The first one defines the B_c decay constant

$$\langle 0 | \bar{c} \gamma_\mu \gamma_5 b | \bar{B}_c(p_{B_c}) \rangle = i f_{B_c} p_{B_c, \mu}, \quad (7)$$

while the $B_c \rightarrow \gamma$ transition is parametrized by two form factors:

$$\langle \gamma(\epsilon, k) | \bar{c} \gamma_\mu b | \bar{B}_c(p_{B_c}) \rangle = -e \frac{V(L^2)}{p_{B_c} \cdot k} \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu} p_{B_c}^\rho k^\sigma, \quad (8)$$

$$\langle \gamma(\epsilon, k) | \bar{c} \gamma_\mu \gamma_5 b | \bar{B}_c(p_{B_c}) \rangle = ie A(L^2) \left(\epsilon_\mu^* - k_\mu \frac{p_{B_c} \cdot \epsilon^*}{p_{B_c} \cdot k} \right) - \frac{ie}{p_{B_c} \cdot k} f_{B_c} p_{B_c, \mu} p_{B_c} \cdot \epsilon^*, \quad (9)$$

with the momentum transfer $L = p_{B_c} - k$. Here and throughout this work we adopt the convention $\epsilon^{0123} = +1$. The above equations are similar with the parameterization of the $B \rightarrow \gamma$ form factors as given in Ref. [66]. The last term in Eq. (9) that is proportional to the B_c decay constant has been added in order to maintain the gauge invariance of the full amplitude [67, 68], and see appendix A for a derivation.

Substituting Eqs. (7), (8), (9) into Eq. (5), we obtain

$$\mathcal{A} = -i \frac{G_F}{\sqrt{2}} V_{cb} e f_{B_c} \bar{u}_\ell \gamma^\mu (1 - \gamma_5) v_\nu \left\{ [1 + a(s_l)] \left(\epsilon_\mu^* - k_\mu \frac{p_{B_c} \cdot \epsilon^*}{p_{B_c} \cdot k} \right) - \frac{iv(s_l)}{p_{B_c} \cdot k} \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu} p_{B_c}^\rho k^\sigma \right\}, \quad (10)$$

where $s_l = L^2$ and terms due to lepton mass corrections have been neglected. Apparently, this expression is gauge invariant. For the sake of simplicity, we have defined two abbreviations in the above ¹

$$a(L^2) \equiv \frac{A(s_l)}{f_{B_c}}, \quad v(s_l) \equiv \frac{V(s_l)}{f_{B_c}}. \quad (11)$$

In terms of the decay constant and form factors, the differential decay width for the $B_c \rightarrow \gamma \ell^- \bar{\nu}_\ell$ is given as

$$\begin{aligned} \frac{d^2\Gamma}{dE_k dE_l} &= \frac{1}{64m_{B_c} \pi^3} |\mathcal{A}|^2 \\ &= \frac{\alpha_{\text{em}} f_{B_c}^2 |V_{cb}|^2 G_F^2 m_{B_c}}{4\pi^2 x_k^2} (1 - x_k) \times \left[a^2 (x_k^2 + 2x_k(x_l - 1) + 2(x_l - 1)^2) \right. \\ &\quad + 2a ((v + 1)x_k^2 + 2(v + 1)x_k(x_l - 1) + 2(x_l - 1)^2) + 2vx_k(x_k + 2x_l - 2) \\ &\quad \left. + v^2 (x_k^2 + 2x_k(x_l - 1) + 2(x_l - 1)^2) + x_k^2 + 2x_k x_l - 2x_k + 2x_l^2 - 4x_l + 2 \right], \end{aligned} \quad (12)$$

¹ One shall distinguish the form factor v from the relative velocity v to be defined in the following.

where $x_k = 2E_k/m_{B_c}$ and $y = 2E_l/m_{B_c}$, and E_k and E_l is the energy of the photon and charged lepton in the B_c rest frame, respectively. One can integrate out the E_l and obtain

$$\frac{d\Gamma}{dE_k} = \frac{\alpha_{\text{em}} f_{B_c}^2 |V_{cb}|^2 G_F^2 m_{B_c}^2 x_k (1-x_k) ((1+a)^2 + v^2)}{12\pi^2}. \quad (13)$$

The differential distributions can also be converted to

$$\begin{aligned} \frac{d^2\Gamma}{ds_l d\cos\theta_l} = & \frac{m_{B_c}^2 - s_l}{32m_{B_c}\pi^2} |V_{cb}|^2 \alpha_{\text{em}} f_{B_c}^2 G_F^2 (1-x_k) \frac{1}{x_k^2} \left[a^2 (x_k^2 + 2x_k(x_l - 1) + 2(x_l - 1)^2) \right. \\ & + 2a ((v+1)x_k^2 + 2(v+1)x_k(x_l - 1) + 2(x_l - 1)^2) + 2vx_k(x_k + 2x_l - 2) \\ & \left. + v^2 (x_k^2 + 2x_k(x_l - 1) + 2(x_l - 1)^2) + x_k^2 + 2x_k x_l - 2x_k + 2x_l^2 - 4x_l + 2 \right], \end{aligned} \quad (14)$$

using the relation:

$$E_k = \frac{m_{B_c}^2 - s_l}{2m_{B_c}}, \quad (15)$$

$$E_l = \frac{1}{4m_{B_c}} [(m_{B_c}^2 + s_l) - (m_{B_c}^2 - s_l) \cos\theta_l]. \quad (16)$$

The θ_l is the polar angle between the lepton ℓ flight direction and the opposite direction of the B_c meson in the rest frame of the $\ell\bar{\nu}_\ell$ pair. Likewise one can integrate out the θ_l

$$\frac{d\Gamma}{ds_l} = \frac{\alpha_{\text{em}} f_{B_c}^2 |V_{cb}|^2 G_F^2 (m_{B_c}^2 - s_l) s_l ((1+a)^2 + v^2)}{24\pi^2 m_{B_c}^3}. \quad (17)$$

B. NRQCD factorization

The factorization properties for the $B_c \rightarrow \gamma\ell\bar{\nu}$ depend on the kinematics of the photon. In this work, we will not study the soft-photon contribution as discussed in B decays [69], and leave it for future work. In the region where the photon is a collinear (fast-moving) object, its interaction with heavy quarks is highly virtual and thus should be encoded in the short distance coefficients. In the NRQCD scheme, we only need retain those color-singlet operator matrix elements that connect the B_c state to the vacuum. To the desired order, one expects the following factorization formula:

$$f_{B_c} = \sqrt{\frac{2}{m_{B_c}}} \left[c_0^f \langle 0 | \chi_c^\dagger \psi_b | \bar{B}_c(\mathbf{p}) \rangle + \frac{c_2^f}{m_{B_c}^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{B}_c(\mathbf{p}) \rangle + \mathcal{O}(v^4) \right], \quad (18)$$

$$V = \sqrt{\frac{2}{m_{B_c}}} \left[\frac{c_0^V}{m_{B_c}} \langle 0 | \chi_c^\dagger \psi_b | \bar{B}_c(\mathbf{p}) \rangle + \frac{c_2^V}{m_{B_c}^3} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{B}_c(\mathbf{p}) \rangle + \mathcal{O}(v^4) \right], \quad (19)$$

$$A = \sqrt{\frac{2}{m_{B_c}}} \left[\frac{c_0^A}{m_{B_c}} \langle 0 | \chi_c^\dagger \psi_b | \bar{B}_c(\mathbf{p}) \rangle + \frac{c_2^A}{m_{B_c}^3} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{B}_c(\mathbf{p}) \rangle + \mathcal{O}(v^4) \right], \quad (20)$$

where v denotes half relative velocity between the charm and bottom quarks in the meson, $c_0^{f,V,A}$ and $c_2^{f,V,A}$ are the dimensionless short-distance coefficients that can be expanded in terms of the strong coupling constant ². We shall calculate the one-loop corrections to the $c_0^{f,V,A}$, but give only the LO results for $c_2^{f,V,A}$ since the latter ones are already power-suppressed. ψ_Q and χ_Q^\dagger represent Pauli spinor fields that annihilate the heavy quark Q and anti-quark \bar{Q} , respectively. Besides, one need note that the state $|H(p)\rangle$ in QCD has the standard normalization: $\langle H(p')|H(p)\rangle = 2E_p(2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')$, while an additional factor $2E_p$ is abandoned in the nonrelativistic normalization where $\langle H(\mathbf{p}')|H(\mathbf{p})\rangle = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')$.

III. NEXT-TO-LEADING ORDER CALCULATION

A. Kinematics

Let p_1 and p_2 represent the momenta for the heavy quark Q and anti-quark \bar{Q}' . Without loss of generality, one may adopt the decomposition:

$$p_1 = \alpha P_{B_c} - q, \quad (21)$$

$$p_2 = \beta P_{B_c} + q, \quad (22)$$

where P_{B_c} is the total momentum of the quark pair. q is a half of the relative momentum between the quark pair with $P_{B_c} \cdot q = 0$. α and β are the energy fraction for Q and \bar{Q}' in the meson, respectively. The explicit expressions for all the momentum in the rest frame of the B_c meson are given by

$$P_{B_c}^\mu = (E_1 + E_2, 0), \quad (23)$$

$$q^\mu = (0, \mathbf{q}), \quad (24)$$

$$p_1^\mu = (E_1, -\mathbf{q}), \quad (25)$$

$$p_2^\mu = (E_2, \mathbf{q}). \quad (26)$$

In the rest frame, the meson momentum becomes purely timelike while the relative momentum is spacelike. One can obtain the relations $\alpha = \sqrt{m_b^2 - q^2}/(\sqrt{m_b^2 - q^2} + \sqrt{m_c^2 - q^2})$ and $\beta = 1 - \alpha$ with the on-shell conditions $E_1 = \sqrt{m_b^2 - q^2}$, $E_2 = \sqrt{m_c^2 - q^2}$, and $q^2 = -\mathbf{q}^2$.

B. Covariant projection method

In the following calculation, we will adopt the covariant spin-projector method, which can be applied to all orders in v .

The Dirac spinors for the B_c system may be written as

$$u_b(p_1, \lambda) = \sqrt{\frac{E_1 + m_b}{2E_1}} \begin{pmatrix} \xi_\lambda \\ \frac{\vec{\sigma} \cdot \vec{p}_1}{E_1 + m_b} \xi_\lambda \end{pmatrix}, \quad (27)$$

² Throughout this paper, we shall use the superscripts (0) and (1) to indicate the LO and NLO contributions in α_s and the subscripts 0 and 2 to denote the LO and NLO contributions in the velocity.

$$v_c(p_2, \lambda) = \sqrt{\frac{E_2 + m_c}{2E_2}} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}_2}{E_2 + m_c} \xi_\lambda \\ \xi_\lambda \end{pmatrix}, \quad (28)$$

where ξ_λ is the two-component Pauli spinors and λ is the polarization parameters. It is straightforward to derive the covariant form of the spin-singlet combinations of spinor bilinears:

$$\begin{aligned} \Pi_0(q) &= -i \sum_{\lambda_1, \lambda_2} u_b(p_1, \lambda_1) \bar{v}_c(p_2, \lambda_2) \langle \frac{1}{2} \lambda_1 \frac{1}{2} \lambda_2 | 00 \rangle \otimes \frac{1_c}{\sqrt{N_c}} \\ &= \frac{i}{4\sqrt{2E_1 E_2} \omega} (\alpha \not{p}_{B_c} - \not{q} + m_b) \frac{\not{p}_{B_c} + E_1 + E_2}{E_1 + E_2} \gamma_5 (\beta \not{p}_{B_c} + \not{q} - m_c) \otimes \frac{1_c}{\sqrt{N_c}}, \end{aligned} \quad (29)$$

with the auxiliary parameter $\omega = \sqrt{E_1 + m_b} \sqrt{E_2 + m_c}$. Here 1_c is the unit matrix in the fundamental representation of the color SU(3) group.

C. Perturbative matching

Due to the simplicity of the final state, one can directly match the QCD currents onto the NRQCD ones. To determine the values of c_0 and c_2 , we follow the spirit that those short-distance coefficients are insensitive to the long-distance hadronic dynamics. As a convenient choice, one can replace the physical B_c^- meson by a free $\bar{c}b$ pair of the quantum number $^1S_0^{[1]}$, so that both the full amplitude, $\mathcal{A}[\bar{c}b(^1S_0^{[1]}) \rightarrow \gamma \ell \bar{\nu}]$, and the NRQCD operator matrix elements can be directly accessed in perturbation theory. The short-distance coefficients c_i can then be solved by equating the QCD amplitude \mathcal{A} and the corresponding NRQCD amplitude, order by order in α_s . For this purpose, we introduce a decay constant and two form factors at the free quark level:

$$\langle 0 | \bar{c} \gamma_\mu \gamma_5 b | \bar{c} b(^1S_0^{[1]}) \rangle = i \mathcal{U} g_{\mu 0}, \quad (30)$$

$$\langle \gamma(\epsilon, k) | \bar{c} \gamma_\mu b | \bar{c} b(^1S_0^{[1]}) \rangle = -e \frac{1}{k \cdot p_{B_c}} \nabla \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu} p_{B_c}^\rho k^\sigma, \quad (31)$$

$$\langle \gamma(\epsilon, k) | \bar{c} \gamma_\mu \gamma_5 b | \bar{c} b(^1S_0^{[1]}) \rangle = ie \mathbb{A} \left(\epsilon_\mu^* - k_\mu \frac{p_{B_c} \cdot \epsilon^*}{p_{B_c} \cdot k} \right) - ie \frac{1}{p_{B_c} \cdot k} \mathcal{U} p_{B_c \mu} p_{B_c} \cdot \epsilon^*. \quad (32)$$

Analogous to (18,19,20), one can write down the matching formula:

$$\mathcal{U} = c_0^f \langle 0 | \chi_c^\dagger \psi_b | \bar{c} b(^1S_0^{[1]}) \rangle + \frac{c_2^f}{(m_b + m_c)^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{c} b(^1S_0^{[1]}) \rangle, \quad (33)$$

$$\mathbb{V} = \frac{1}{m_b + m_c} \left[c_0^V \langle 0 | \chi_c^\dagger \psi_b | \bar{c} b(^1S_0^{[1]}) \rangle + \frac{c_2^V}{(m_b + m_c)^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{c} b(^1S_0^{[1]}) \rangle \right], \quad (34)$$

$$\mathbb{A} = \frac{1}{m_b + m_c} \left[c_0^A \langle 0 | \chi_c^\dagger \psi_b | \bar{c} b(^1S_0^{[1]}) \rangle + \frac{c_2^A}{(m_b + m_c)^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{c} b(^1S_0^{[1]}) \rangle \right], \quad (35)$$

where we have adopted the nonrelativistic normalization.

One can organize the full amplitudes defined in Eqs. (30,31,32) in powers of the relative momentum between \bar{c} and b , denoted by \mathbf{q} . To the desired accuracy, one can truncate the series at $\mathcal{O}(\mathbf{q}^2)$, with the first two Taylor coefficients. We will compute both amplitudes at LO in α_s in subsection III D, and the calculation at NLO in α_s will be conducted in subsection III E.

The NRQCD matrix elements encountered in the above equations are particularly simple at LO in α_s :

$$\begin{aligned}\langle 0|\chi^\dagger\psi|\bar{c}b(^1S_0^{[1]})\rangle^{(0)} &= \sqrt{2N_c}, \\ \langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|\bar{c}b(^1S_0^{[1]})\rangle^{(0)} &= \sqrt{2N_c}\mathbf{q}^2,\end{aligned}\quad (36)$$

where the factor $\sqrt{2N_c}$ is due to the spin and color factors of the normalized $\bar{c}b(^1S_0^{[1]})$ state. The computation of these matrix elements to $\mathcal{O}(\alpha_s)$ will be addressed in subsection III F.

D. Tree-level amplitude

Adopting the above notation, one can easily obtain the tree-level amplitude for the decay constant

$$\begin{aligned}\langle 0|\bar{c}\gamma_\mu\gamma_5 b|\bar{c}b(^1S_0^{[1]})\rangle^{(0)} &= \text{Tr}[\Pi_0(q)\gamma_\mu\gamma_5] \\ &= ip_{B_c}^\mu\sqrt{2N_c}\frac{(E_1+m_b)(E_2+m_c)+q^2}{2\sqrt{E_1E_2(E_1+m_b)(E_2+m_c)}(E_1+E_2)} \\ &= ig_{\mu 0}\sqrt{2N_c}\left(1-\frac{q^2}{8m_{red}^2}\right),\end{aligned}\quad (37)$$

where the q^μ terms have been omitted and

$$m_{red} = \frac{m_b m_c}{m_b + m_c}, \quad (38)$$

is defined as the reduced mass of the $\bar{c}b$ system.

The vector current is similarly evaluated as:

$$\begin{aligned}\langle \gamma|\bar{c}\gamma_\mu b|\bar{c}b(^1S_0^{[1]})\rangle^{(0)} &= \text{Tr}[\Pi_0(q)iee_c\epsilon^{*\nu}\frac{i(\not{k}-\not{p}_2+m_c)}{(k-p_2)^2-m_c^2}\gamma_\mu] + \text{Tr}[\Pi_0(q)\gamma_\mu\frac{i(\not{p}_1-\not{k}+m_b)}{(p_1-k)^2-m_b^2}iee_b\epsilon^{*\nu}] \\ &= -\frac{e\sqrt{2N_c}}{4w\sqrt{E_1E_2}}\left(\frac{e_c}{E_2k\cdot p_{B_c}+Ek\cdot q} + \frac{e_b}{E_1k\cdot p_{B_c}-Ek\cdot q}\right) \\ &\quad \times \left\{E_{bc}\epsilon_{\mu\nu\rho\sigma}\epsilon^{*\nu}k^\rho p_{B_c}^\sigma + E(E_1+E_2+m_b-m_c)\epsilon_{\mu\nu\rho\sigma}\epsilon^{*\nu}k^\rho q^\sigma\right\}.\end{aligned}\quad (39)$$

We have introduced the abbreviation $E = E_1 + E_2$, and $E_{bc} = (E_1 + m_b)(E_2 + m_c) + q^2$. Here $e_c = 2/3$ and $e_b = -1/3$ is the electric charge of the c and b quark, respectively.

One can perform the Taylor expansion of the amplitudes in powers of q^μ :

$$\mathcal{A}(q) = \mathcal{A}(0) + \frac{\partial\mathcal{A}(0)}{\partial q^\mu}\Big|_{q=0} q^\mu + \frac{1}{2!}\frac{\partial^2\mathcal{A}(0)}{\partial q^\mu\partial q^\nu}\Big|_{q=0} q^\mu q^\nu + \dots \quad (40)$$

Those terms linear in q should be dropped since this auxiliary momentum introduced at the quark level has no correspondence at the hadron level. In this paper, the $\mathcal{O}(|q|^2)$ contributions will be retained. In order to simplify the calculation in the covariant derivation, one shall use the following replacement:

$$q^\mu q^\nu \rightarrow \frac{|q|^2}{D-1}(-g^{\mu\nu} + \frac{P_{B_c}^\mu P_{B_c}^\nu}{P_{B_c}^2}). \quad (41)$$

The result for the axial-vector current is a bit lengthy:

$$\begin{aligned} \langle \gamma | \bar{c} \gamma_\mu \gamma_5 b | \bar{c} b ({}^1S_0^{[1]}) \rangle^{(0)} &= -ie \sqrt{2N_c} \frac{1}{4\sqrt{E_1 E_2 (E_1 + m_b)(E_2 + m_c)}} \\ &\times \left\{ \epsilon_\mu^* e_c \frac{k \cdot p_{B_c} E_{bc} + k \cdot q E (E_1 - E_2 + m_b - m_c)}{E_2 k \cdot p_{B_c} + E k \cdot q} \right. \\ &- \epsilon_\mu^* e_b \frac{k \cdot p_{B_c} E_{bc} + k \cdot q E (E_1 - E_2 + m_b - m_c)}{E_1 k \cdot p_{B_c} - E k \cdot q} \\ &+ q_\mu e_c \frac{2(E_1 - E_2 + m_b - m_c)(E_2 \epsilon^* \cdot p_{B_c} + E \epsilon^* \cdot q)}{E_2 k \cdot p_{B_c} + E k \cdot q} \\ &- q_\mu e_b \frac{2(E_1 - E_2 + m_b - m_c)(E_1 \epsilon^* \cdot p_{B_c} - E \epsilon^* \cdot q)}{E_1 k \cdot p_{B_c} - E k \cdot q} \\ &+ p_{B_c \mu} e_c \frac{2E_{bc}(E_2 \epsilon^* \cdot p_{B_c} + E \epsilon^* \cdot q)}{E(E_2 k \cdot p_{B_c} + E k \cdot q)} \\ &- p_{B_c \mu} e_b \frac{2(E_1 E_{bc} \epsilon^* \cdot p_{B_c} + E \epsilon^* \cdot q(E_{bc} + q^2))}{E(E_1 k \cdot p_{B_c} - E k \cdot q)} \\ &- k_\mu e_c \frac{E_{bc} \epsilon^* \cdot p_{B_c} + E \epsilon^* \cdot q(E_1 - E_2 + m_b - m_c)}{E_2 k \cdot p_{B_c} + E k \cdot q} \\ &\left. + k_\mu e_b \frac{E_{bc} \epsilon^* \cdot p_{B_c} + E \epsilon^* \cdot q(E_1 - E_2 + m_b - m_c)}{E_1 k \cdot p_{B_c} - E k \cdot q} \right\}. \quad (42) \end{aligned}$$

In order to extract the \mathcal{A} form factor, we only need to keep the ϵ_μ term which corresponds to Feynman gauge $\epsilon \cdot p_{B_c} = 0$, but we have explicitly checked the gauge invariance up to v^2 order.

The tree-level NRQCD matrix elements for the $\bar{c}b$ have been given in Eq. (36), and thus the above results in Eqs. (37,39,42) lead to the tree-level Wilson coefficients

$$c_0^{f,0} = 1, \quad (43)$$

$$c_2^{f,0} = -\frac{\tilde{z}^4}{8z^2}, \quad (44)$$

$$c_0^{V,0} = -\frac{e_c}{2z} - \frac{e_b}{2}, \quad (45)$$

$$c_2^{V,0} = -\tilde{z}^2 \left(\frac{e_c(3z^2 + 2z + 11)}{48z^3} + \frac{e_b(11z^2 + 2z + 3)}{48z^2} \right), \quad (46)$$

$$c_0^{A,0} = \frac{e_b}{2} - \frac{e_c}{2z}, \quad (47)$$

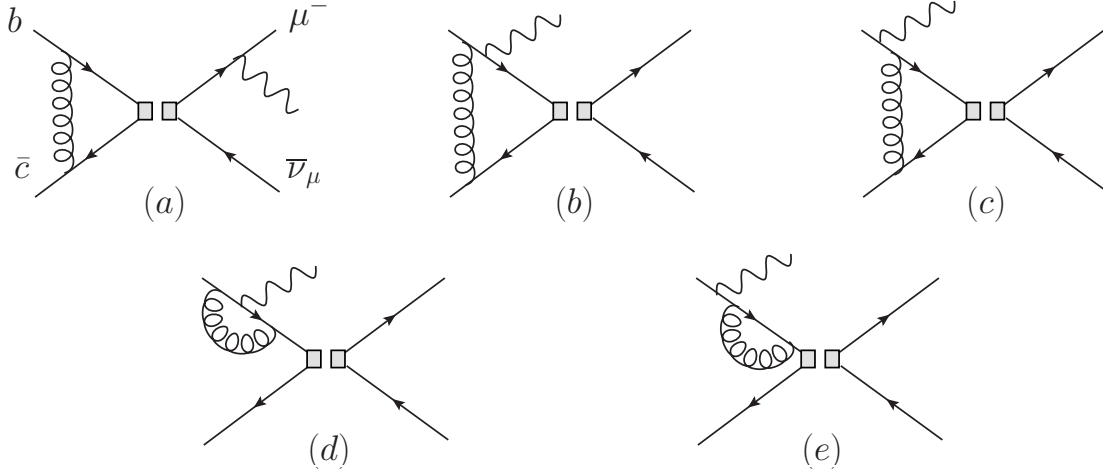


FIG. 2: Typical NLO Feynman diagrams for the radiative leptonic $B_c \rightarrow \gamma \mu \bar{\nu}_\mu$ decay in the SM. The other four diagrams can be easily obtained by interchanging the bottom and anti-charm quarks lines.

$$c_2^{A,0} = -\tilde{z}^2 \left(\frac{e_c[(3z^2 + 2z + 11) + 8z(1-z)m_b/E_k]}{48z^3} - \frac{e_b[(11z^2 + 2z + 3) - 8z(1-z)m_b/E_k]}{48z^2} \right). \quad (48)$$

In the above results, we have defined $z = m_c/m_b$ and $\tilde{z} = 1 + z$. $c_i^{f,0}$ means the LO of Wilson coefficient c_i^f . It is interesting to notice that the Wilson coefficients $c_2^{A,0}$ depends on the energy of the emitted photon, which will induce nontrivial behaviors as will be demonstrated later.

E. NLO amplitudes in QCD

Typical one-loop diagrams for the QCD corrections to the $B_c \rightarrow \gamma \ell \bar{\nu}_\ell$ decay are shown in Fig. 2. In calculating the one-loop amplitudes, we use the dimensional regularization to regulate the ultraviolet (UV) and infrared (IR) divergence.

The diagram (a) in Fig. 2 contributes to the NLO decay constant:

$$\mathcal{U}_{0,a}^1 = \sqrt{2N_c} \frac{C_F \alpha_s}{4\pi} \left[\frac{1}{\hat{\epsilon}_{UV}} + \frac{2}{\hat{\epsilon}_{IR}} + 3 \ln \frac{\mu^2}{m_b^2} - 2 + 2t_1 - \frac{6 \ln z}{z+1} \right], \quad (49)$$

with

$$t_1 = \frac{1}{2|v|} \left(\pi^2 - i\pi \left[\frac{1}{\hat{\epsilon}_{IR}} - \ln \frac{16m_{red}^2|v|^2}{\mu^2} \right] \right), \quad (50)$$

$$v = \frac{q}{2m_{red}}.$$

We have introduced the abbreviation

$$\frac{1}{\hat{\epsilon}_{UV,IR}} = \frac{1}{\epsilon_{UV,IR}} - \gamma_E + \ln 4\pi. \quad (51)$$

The heavy quark field renormalization and mass term are given as

$$\begin{aligned} Z_q^{OS} &= 1 - \frac{C_F \alpha_s}{4\pi} \left[\frac{1}{\hat{\epsilon}_{UV}} + \frac{2}{\hat{\epsilon}_{IR}} + 3 \ln \frac{\mu^2}{m^2} + 4 \right], \\ \delta_m &= -\frac{3m C_F \alpha_s}{4\pi} \left[\frac{1}{\hat{\epsilon}_{UV}} + \ln \frac{\mu^2}{m^2} + \frac{4}{3} \right]. \end{aligned} \quad (52)$$

For the vector current form factor, the sub-diagram in Fig. 2 gives out the corresponding contribution

$$\begin{aligned} \mathbb{V}_b &= \frac{\sqrt{2N_c} e_b C_F \alpha_s}{4\pi m_b} \left[-\frac{1}{\hat{\epsilon}_{IR}} + \frac{4\tilde{z}}{y^2 - \tilde{z}^2} + \frac{\tilde{z}^2 + y^2}{y^2 \tilde{z} - \tilde{z}^3} b_1 - \frac{1}{\tilde{z}} b_2 + \frac{2(y^2 - z(z+1))}{z(y^2 - \tilde{z}^2)} b_3 \right. \\ &\quad \left. - \frac{2y^2}{z(y^2 - \tilde{z}^2)} b_4 - \frac{y^2}{\tilde{z}} c_4 - (1-z)c_3 - (\tilde{z}^2 - y^2)d_1 \right], \\ \mathbb{V}_c &= \frac{\sqrt{2N_c} e_b C_F \alpha_s}{4\pi m_b} \left[-\frac{1}{2\hat{\epsilon}_{UV}} + \frac{y^2 + z^2 + 4z + 3}{\tilde{z}^2 - y^2} + \frac{\tilde{z} + y^2}{\tilde{z}^2 - y^2} b_1 - \frac{\tilde{z}(3y^2 - z^2 + 1)}{2z(\tilde{z}^2 - y^2)} b_4 \right. \\ &\quad \left. + \frac{(2z^2 + 3z - 1)\tilde{z} - y^2(2z + 3)}{2z(y^2 - \tilde{z}^2)} b_3 + (\tilde{z} + y^2 - z^2)c_4 \right], \\ \mathbb{V}_d &= \frac{\sqrt{2N_c} e_b C_F \alpha_s}{4\pi m_b} \left[-\frac{1}{2\hat{\epsilon}_{UV}} + \frac{y^2 - z^2 + 4z + 5}{\tilde{z}^2 - y^2} + \frac{y^2 - z^2 + z + 2}{\tilde{z}^2 - y^2} b_2 \right. \\ &\quad \left. + \frac{y^2 - z^2 + 4z + 5}{2(y^2 - \tilde{z}^2)} b_3 + c_1 \right], \\ \mathbb{V}_e &= \frac{\sqrt{2N_c} e_b C_F \alpha_s}{4\pi m_b} \left[-\frac{y^2 + z^2 + 8z + 7}{2(\tilde{z}^2 - y^2)} + \frac{\tilde{z} - y^2}{2(y^2 - z\tilde{z})} - \frac{\tilde{z}(y^2 - z^2 + 1)}{2(y^2 - z\tilde{z})(y^2 - \tilde{z}^2)} b_2 \right. \\ &\quad \left. + \frac{(z^2 + 6z + 1)\tilde{z}^2 + y^4 - 2y^2(z^2 + 4z + 3)}{2(y^2 - \tilde{z}^2)(y^2 - z\tilde{z})} b_3 \right], \end{aligned} \quad (53)$$

where the auxiliary functions b_i , c_i , and d_i are defined in Appendix B.

The counter-mass terms and wave function renormalization corrections give:

$$\begin{aligned} \mathbb{V}_{CT-m} &= \frac{\sqrt{2N_c} e_b C_F \alpha_s}{4\pi m_b} \left[\frac{3\tilde{z}}{y^2 - \tilde{z}^2} \left(\frac{1}{\hat{\epsilon}_{UV}} + \ln \frac{\mu^2}{m_b^2} + \frac{4}{3} \right) \right], \\ \mathbb{V}_{CT-F} &= \frac{\sqrt{2N_c} e_b C_F \alpha_s}{4\pi m_b} \left[\frac{1}{\hat{\epsilon}_{IR}} + \frac{1}{2\hat{\epsilon}_{UV}} + \frac{3}{2} \ln \frac{\mu^2}{zm_b^2} + 2 \right]. \end{aligned} \quad (54)$$

For the axial-vector current form factor, the sub-diagram has gauge-dependent contributions, however, the summed result is gauge-invariant. We will show the detail in Appendix C.

F. NLO amplitudes in NRQCD

The NRQCD Lagrangian can be derived by integrating out the degrees of freedom of order heavy quark mass [53]:

$$\begin{aligned} \mathcal{L}_{\text{NRQCD}} &= \psi^\dagger \left(iD_t + \frac{\mathbf{D}^2}{2m} \right) \psi + \psi^\dagger \frac{\mathbf{D}^4}{8m^3} \psi + \frac{c_F}{2m} \psi^\dagger \boldsymbol{\sigma} \cdot g_s \mathbf{B} \psi \\ &\quad + \frac{c_D}{8m^2} \psi^\dagger (\mathbf{D} \cdot g_s \mathbf{E} - g_s \mathbf{E} \cdot \mathbf{D}) \psi + \frac{ic_S}{8m^2} \psi^\dagger \boldsymbol{\sigma} \cdot (\mathbf{D} \times g_s \mathbf{E} - g_s \mathbf{E} \times \mathbf{D}) \psi \\ &\quad + (\psi \rightarrow i\sigma^2 \chi^*, A_\mu \rightarrow -A_\mu^T) + \mathcal{L}_{\text{light}}. \end{aligned} \quad (55)$$

The replacement in the last line implies that the corresponding heavy anti-quark bilinear sector can be obtained through the charge conjugation transformation. $\mathcal{L}_{\text{light}}$ represents the Lagrangian for the light quarks and gluons. The coefficients c_D , c_F , and c_S have perturbative expansions in powers of α_s , which can be written as $c_i = 1 + \mathcal{O}(\alpha_s)$.

The matrix element of the $\bar{c}b$ to vacuum at NLO can be written as

$$\langle 0 | \chi^\dagger \psi | \bar{c}b(^1S_0^{[1]}) \rangle^{(1)} = \sqrt{2N_c} \frac{\alpha_s C_F}{2\pi 2|v|} \left(\pi^2 - i\pi \left[\frac{1}{\epsilon_{IR}} - \ln \frac{16m_{red}^2 |v|^2}{\mu^2} \right] \right). \quad (56)$$

This is in agreement with the results in Ref. [70].

G. Determination of c_i : Matching QCD to NRQCD

Up to α_s and v^2 , one can expand the decay constant and form factors as

$$\begin{aligned} \mathfrak{U} &= c_0^{f,0} \langle 0 | \chi_c^\dagger \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)} + c_0^{f,1} \langle 0 | \chi_c^\dagger \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)} + c_0^{f,0} \langle 0 | \chi_c^\dagger \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(1)} \\ &\quad + \frac{c_2^{f,0}}{(m_b + m_c)^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{D} \right)^2 \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)}, \end{aligned} \quad (57)$$

$$\begin{aligned} \mathfrak{V} &= \frac{1}{m_b + m_c} [c_0^{V,0} \langle 0 | \chi_c^\dagger \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)} + c_0^{V,1} \langle 0 | \chi_c^\dagger \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)} \\ &\quad + c_0^{V,0} \langle 0 | \chi_c^\dagger \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(1)} + \frac{c_2^{V,0}}{(m_b + m_c)^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{D} \right)^2 \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)}], \end{aligned} \quad (58)$$

$$\begin{aligned} \mathfrak{A} &= \frac{1}{m_b + m_c} [c_0^{A,0} \langle 0 | \chi_c^\dagger \psi_b^{(0)} | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)} + c_0^{A,1} \langle 0 | \chi_c^\dagger \psi_b^{(0)} | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)} \\ &\quad + c_0^{A,0} \langle 0 | \chi_c^\dagger \psi_b^{(0)} | \bar{c}b(^1S_0^{[1]}) \rangle^{(1)} + \frac{c_2^{A,0}}{(m_b + m_c)^2} \langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{D} \right)^2 \psi_b | \bar{c}b(^1S_0^{[1]}) \rangle^{(0)}]. \end{aligned} \quad (59)$$

Matching the QCD results onto the NRQCD, one can obtain the UV and IR finite short-distance coefficient

$$c_0^{f,1} = -\frac{3C_F\alpha_s}{4\pi} \left(2 + \frac{1-z}{1+z} \ln z \right), \quad (60)$$

$$\begin{aligned} c_0^{V,1} &= \frac{C_F\alpha_s}{4\pi} \left\{ e_b \left[\ln \frac{\mu^2}{zm_b^2} - \frac{\tilde{z}^2 (-3z\tilde{z} + \tilde{z} + 2y^2) + y^4}{2(y^2 - z\tilde{z})(y^2 - \tilde{z}^2)} + \frac{\tilde{z}^3 + y^2(3z-1)}{4(\tilde{z}^3 - y^2\tilde{z})} b_1 + \frac{y^2 - 2z\tilde{z}}{2z(y^2 - z\tilde{z})} b_3 \right. \right. \\ &\quad \left. \left. + \frac{1}{4} \left(\frac{2\tilde{z}}{y^2 - z\tilde{z}} + \frac{2}{\tilde{z} - y} + \frac{2}{\tilde{z} + y} - \frac{4}{\tilde{z}} - 3 \right) b_2 + \frac{-z\tilde{z}^2 + \tilde{z}^2 + 3y^2z - y^2}{2z(y^2 - \tilde{z}^2)} b_4 \right. \right. \\ &\quad \left. \left. + \frac{-\tilde{z} - y^2z + z^3 + z^2}{z\tilde{z}} c_1 + \frac{y^2z - z^3 + 2z + 1}{z} c_2 + (z-1)c_3 + (y^2 - \tilde{z}^2)d_1 \right] \right. \\ &\quad \left. + (e_b \rightarrow \frac{e_c}{z}, z \rightarrow \frac{1}{z}, y \rightarrow \frac{y}{z}) \right\}, \end{aligned} \quad (61)$$

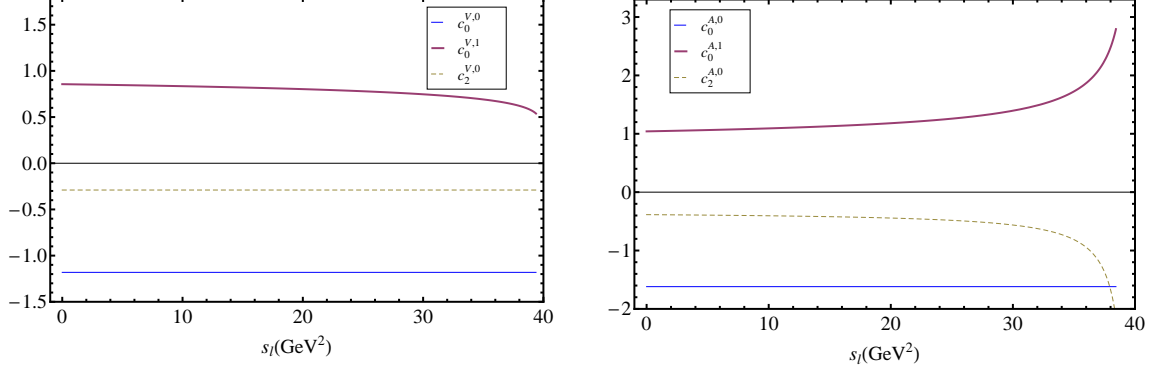


FIG. 3: Dependence of short-distance coefficients $c^{V(A)}$ on the s_l . The solid line denotes the coefficient $c_0^{V(A),0}$, the dotted line is the coefficient $c_2^{V(A),0}$ from relativistic corrections, and the thick curve is the coefficient $c_0^{V(A),2}$ from α_s corrections.

$$\begin{aligned}
c_0^{A,1} = & \frac{C_F \alpha_s}{4\pi} \left\{ e_b \left[-\ln \frac{\mu^2}{zm_b^2} + \frac{1}{2(y^2 - z\tilde{z})(y^2 - \tilde{z}^2)^2} (y^4(z+11)\tilde{z} - y^2(z(5z+34) + 5)\tilde{z}^2 \right. \right. \\
& + (z(z(3z+23) + 5) + 1)\tilde{z}^3 + y^6) + \frac{b_1}{4\tilde{z}(y^2 - \tilde{z}^2)^2} (-2y^2(z-3)\tilde{z}^2 - (z^2 + 14z \\
& - 3)\tilde{z}^3 + y^4(3z-1)) + \frac{b_2}{4\tilde{z}(y^2 - z\tilde{z})(y^2 - \tilde{z}^2)} (y^2(y^2(3z+7) - (2z+3)(3z-1)\tilde{z}) \\
& + (3(z-1)z - 2)\tilde{z}^3) - \frac{b_3}{2z(z\tilde{z} - y^2)(y^2 - \tilde{z}^2)^2} (y^2(13z^2 - 2z + 1)\tilde{z}^2 - 2(3z^3 + z)\tilde{z}^3 \\
& + y^4(y^2 - 8z^2 - 6z + 2)) - \frac{(z-1)^2\tilde{z}^3 + y^4(3z+1) - 2y^2(2z^3 + 5z^2 + 2z - 1)}{2z(y^2 - \tilde{z}^2)^2} b_4 \\
& + \frac{y^2(y^2(-z) + z^2(2z+5) - 3) - (z-1)(z(z+4) - 1)\tilde{z}^2}{z\tilde{z}(\tilde{z}^2 - y^2)} c_1 \\
& - \frac{((z-2)z(z+4) + 1)\tilde{z}^2 + y^2(z(y^2 - 2z(z+2) + 3) + 3)}{z(y - \tilde{z})(\tilde{z} + y)} c_2 \\
& \left. + \frac{(z-1)(-y^2 + z^2 - 1)}{y^2 - \tilde{z}^2} c_3 + (-y^2 + z^2 + 4z - 1)d_1 \right] \\
& \left. - (e_b \rightarrow \frac{e_c}{z}, z \rightarrow \frac{1}{z}, y \rightarrow \frac{y}{z}) \right\}. \tag{62}
\end{aligned}$$

Note that the scale dependent term in the brace of Eqs. (61) and (62) will be cancelled each other, the residual dependence only lies in the strong coupling constant.

IV. PHENOMENOLNICAL RESULTS

The input parameters are adopted as [71]: $m_{B_c} = 6.2756\text{GeV}$; $G_F = 1.16637 \times 10^{-5}\text{GeV}^{-2}$; $\alpha = 1/128$; for the CKM parameters, we adopt $|V_{cb}| = 0.041$. For the heavy quark mass, we adopt $m_b = 4.8\text{GeV}$ and $m_c = 1.5\text{GeV}$ [46]. The B_c -meson lifetime is using the latest measurement by the LHCb Collaboration, i.e. $\tau_{B_c} = 0.50\text{ps}$ [5, 6].

We first present numerical results for the decay constant f_{B_c} :

$$\begin{aligned} c_2^{f,0} &= -\frac{\tilde{z}^4}{8z^2} = -3.8, \\ c_0^{f,1} &= -\frac{3C_F\alpha_s}{4\pi} \left(2 + \frac{1-z}{1+z} \ln z \right) = -0.44 \times \alpha_s. \end{aligned} \quad (63)$$

The strong coupling constant at the Z-boson peak is [71]

$$\alpha_s(m_Z) = 0.1185 \pm 0.0006, \quad (64)$$

which corresponds to

$$\alpha_s(m_b) = 0.218, \quad \alpha_s(m_c) = 0.368. \quad (65)$$

With these values, one can see the α_s corrections can reduce the decay constant by approximately 9.5% – 16.2%.

To estimate the size of $\mathcal{O}(|\mathbf{v}|^2)$ effects, one requests the size of non-perturbative LDMEs, for which we use Buchmüller-Tye (B-T) potential model [72]:

$$\langle 0 | \chi_c^\dagger \psi_b | \bar{B}_c(\mathbf{p}) \rangle = \sqrt{\frac{N_c}{2\pi}} |R_S^{\text{B-T}}(0)| \simeq 0.884 \text{GeV}^{3/2}, \quad (66)$$

$$\langle 0 | \chi_c^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi_b | \bar{B}_c(\mathbf{p}) \rangle \simeq \mathbf{q}^2 \langle 0 | \chi_c^\dagger \psi_b | \bar{B}_c(\mathbf{p}) \rangle. \quad (67)$$

For an estimate of \mathbf{q}^2 , one may make use of the relative velocity. Using the heavy quarks kinetic and potential energy approximation [53], we have

$$|\mathbf{v}| \simeq \alpha_s(2m_{red}|\mathbf{v}|). \quad (68)$$

Choosing $m_b = 4.8 \text{ GeV}$ and $m_c = 1.5 \text{ GeV}$, and using two-loop strong coupling constant, we get

$$|\mathbf{v}|_{J/\psi}^2 \approx 0.267, \quad |\mathbf{v}|_{\Upsilon}^2 \approx 0.108, \quad |\mathbf{v}|_{B_c}^2 \approx 0.186. \quad (69)$$

For a value $\langle \mathbf{v}^2 \rangle_{B_c} \simeq 0.186$, we have

$$\mathbf{q}^2 \simeq 0.9718 \text{GeV}^2. \quad (70)$$

As a result, the decay constant will be further reduced by about 9%.

For the short-distance coefficients for $B_c \rightarrow \gamma$ transition form factors V and A , our results are shown in Fig. 3. The solid line denotes the leading-order coefficient $c_0^{V(A),0}$, the dotted line correspond to the coefficient $c_2^{V(A),0}$ from relativistic corrections, and the thick curve is the coefficient $c_0^{V(A),2}$ from α_s corrections. From these figures, one can see the relativistic corrections give constructive contributions, but the $\mathcal{O}(\alpha_s)$ QCD corrections are destructive and thus have important consequences. Note that the factorization in Eqs. (18,19,20) is valid only for a hard photon, while the soft-photon contribution needs special treatment [69]. Thus a cut-off on the photon energy should be introduced, however we have checked that the cut-off will not affect the results significantly in Tabs. I and II.

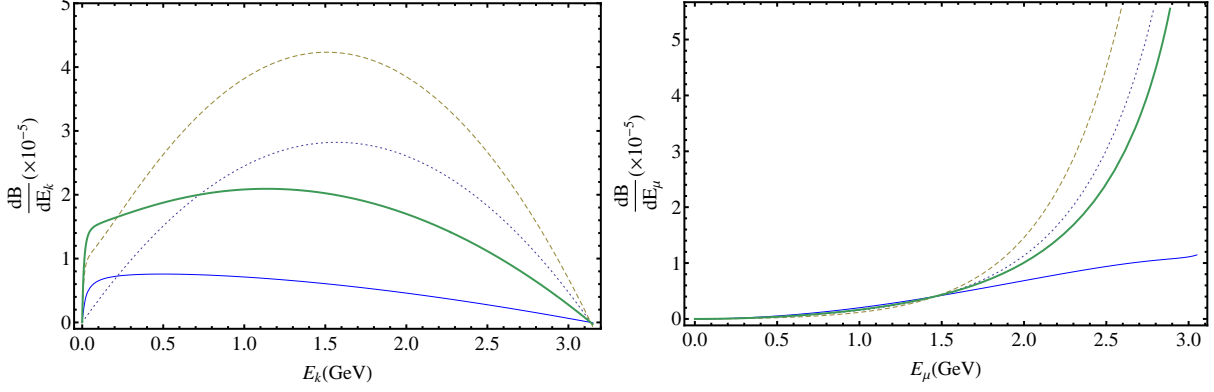


FIG. 4: The dependence of the branching ratio $\mathcal{B}(B_c \rightarrow \gamma\mu\bar{\nu}_\mu)$ on the photon and lepton energy. The dotted line denotes the leading-order result, the dashed line is the result with relativistic corrections, the blue line is the result with QCD corrections, and the thick curve denotes the total results with both the QCD and relativistic corrections.

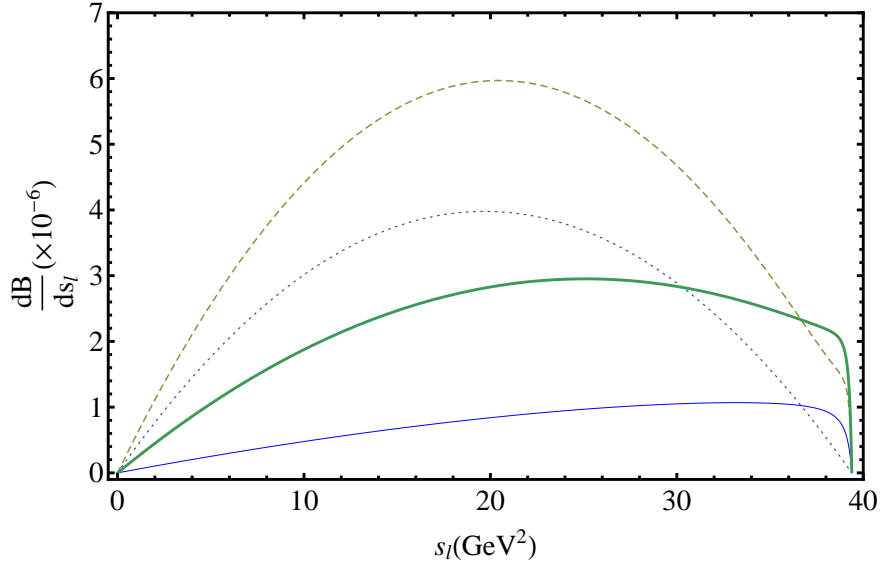


FIG. 5: Similar with Fig. 4 but for the s_l dependence.

With the estimated long-distance matrix elements, results for differential distributions are given in Figs. 4 and 5, where the QCD and relativistic corrections are shown respectively. The integrated branching ratios of $B_c \rightarrow \gamma\ell\bar{\nu}$ and $B_c \rightarrow \ell\bar{\nu}$ are presented in Tabs. I and II. Ignoring the lepton mass, the branching ratio of $B_c \rightarrow \gamma e\bar{\nu}_e$ is identical to that of $B_c \rightarrow \gamma\mu\bar{\nu}_\mu$. The LO results are in agreement with Ref. [54–58] with the same input parameters. From the calculation, one can see that both the QCD and relativistic corrections give destructive contributions to the process $B_c \rightarrow \ell\bar{\nu}$. However, relativistic corrections produce a constructive contribution to the $B_c \rightarrow \gamma\ell\bar{\nu}$. Our results have demonstrated that the QCD and relativistic corrections are mandatory towards a more accurate extraction of the value of LDMEs for B_c system.

TABLE I: Branching ratios of $B_c \rightarrow \gamma \ell \bar{\nu}$ and $B_c \rightarrow \ell \nu$. Here $\tau_{B_c} = 0.50\text{ps}$, and we vary the heavy quark masses with $m_b = 4.8 \pm 0.1\text{GeV}$ and $m_c = 1.5 \mp 0.1\text{GeV}$.

Channels	Tree-level	$ \mathbf{v} ^2$ -corrections	QCD corrections	This work (NLO)
$B_c \rightarrow \tau \bar{\nu}_\tau$	2.90×10^{-2}	-0.54×10^{-2}	$-0.56_{-0.04}^{+0.03} \times 10^{-2}$	$1.80_{-0.04}^{+0.03} \times 10^{-2}$
$B_c \rightarrow \mu \bar{\nu}_\mu$	12.10×10^{-5}	-2.25×10^{-5}	$-2.32_{-0.16}^{+0.14} \times 10^{-5}$	$7.53_{-0.16}^{+0.14} \times 10^{-5}$
$B_c \rightarrow e \bar{\nu}_e$	2.82×10^{-9}	-0.53×10^{-9}	$-0.54_{-0.04}^{+0.03} \times 10^{-9}$	$1.75_{-0.04}^{+0.03} \times 10^{-9}$
$B_c \rightarrow \gamma \mu \bar{\nu}_\mu$	$10.49_{-1.80}^{+2.27} \times 10^{-5}$	$5.46_{-1.07}^{+1.35} \times 10^{-5}$	$-7.68_{+1.54}^{-1.97} \times 10^{-5}$	$8.23_{-1.33}^{+1.65} \times 10^{-5}$

TABLE II: Branching ratios of $B_c \rightarrow \gamma \ell \bar{\nu}$ and $B_c \rightarrow \ell \nu$ compared with other theories or models, including Lattice QCD (LQCD), Light front model (LFM), Constituent quark model (CQM). Here $\tau_{B_c} = 0.50\text{ps}$ is adopted.

	This work	LQCD [74]	LFM [56]	CQM [58]	Ref. [75]	Ref. [55]
$10^2 B(B_c \rightarrow \tau \bar{\nu}_\tau)$	$1.80_{-0.04}^{+0.03}$	2.12	1.52	1.44	1.8	1.6
$10^5 B(B_c \rightarrow \mu \bar{\nu}_\mu)$	$7.53_{-0.16}^{+0.14}$	8.86	6.09	6.2	7.6	5.7
$10^9 B(B_c \rightarrow e \bar{\nu}_e)$	$1.75_{-0.04}^{+0.03}$	2.06	1.41	1.47	1.7	1.5
$10^5 B(B_c \rightarrow \gamma \mu \bar{\nu}_\mu)$	$8.23_{-1.33}^{+1.65}$	–	2.2(5)	4.71	–	4.78

V. SUMMARY

In this work, we have analyzed the radiative leptonic $B_c \rightarrow \gamma \ell \bar{\nu}$ decays in the NRQCD effective field theory. NRQCD factorization ensures the separation of short-distance and long-distance effects of $B_c \rightarrow \gamma \ell \bar{\nu}$ into all order of α_s . Treating the photon as a collinear object whose interactions with the heavy quarks can be integrated out, we arrive at a factorization formula for the decay amplitude.

We have calculated not only the short-distance coefficients at leading order and next-to-leading order in α_s , but also the nonrelativistic corrections at the order $|\mathbf{v}|^2$ in our analysis. We found that the QCD corrections can sizably decrease the branching ratio, which has very important impact on extracting the long-distance operator matrix elements of B_c . For phenomenological applications, we have estimated the long-distance matrix elements, which are further used to explore the photon energy, lepton energy and lepton-neutrino invariant mass distribution. These results can be examined at the LHCb experiment.

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Appendix A: Ward identities for matrix elements

In this section, we will derive the constraints on the $B_c \rightarrow \gamma$ form factors following a Ward identity for the conservation of the electromagnetic current. To be more specific, let us consider the following matrix element:

$$\langle \gamma(k, \epsilon) | (\bar{c}\gamma_\nu\gamma_5 b)(0) | \bar{B}_c \rangle = ie\epsilon^{*\mu} \int d^4x e^{ik \cdot x} \langle 0 | T j_\mu^{\text{e.m.}}(x) (\bar{c}\gamma_\nu\gamma_5 b)(0) | \bar{B}_c \rangle. \quad (\text{A1})$$

In this case, the electromagnetic current includes contributions from heavy quarks $j_\mu^{\text{e.m.}} = e_c \bar{c}\gamma_\mu c + e_b \bar{b}\gamma_\mu b$.

The conservation of the electromagnetic current implies a Ward identity for the matrix element of the time-ordered product in (A1)

$$\begin{aligned} & ik^\mu \int d^4x e^{ik \cdot x} \langle 0 | T j_\mu^{\text{e.m.}}(x) | (\bar{c}\gamma_\nu\gamma_5 b)(0) | \bar{B}_c \rangle \\ &= \int d^3x e^{ik \cdot x} (\langle 0 | j_0^{\text{e.m.}}(x) (\bar{c}\gamma_\nu\gamma_5 b)(0) | \bar{B}_c \rangle \theta(x^0) + \langle 0 | (\bar{c}\gamma_\nu\gamma_5 b)(0) j_0^{\text{e.m.}}(x) | \bar{B}_c \rangle \theta(-x^0)) \Big|_{x^0 \rightarrow -\infty}^{x^0 \rightarrow \infty} \\ &= \int d^3x e^{-i\vec{k} \cdot \vec{x}} (\langle 0 | j_0^{\text{e.m.}}(\vec{x}) (\bar{c}\gamma_\nu\gamma_5 b)(0) | \bar{B}_c \rangle - \langle 0 | (\bar{c}\gamma_\nu\gamma_5 b)(0) j_0^{\text{e.m.}}(\vec{x}) | \bar{B}_c \rangle) \\ &= \int d^3x e^{-i\vec{k} \cdot \vec{x}} \langle f | [j_0^{\text{e.m.}}(\vec{x}), (\bar{c}\gamma_\nu\gamma_5 b)(\vec{0})] | \bar{B}_c \rangle. \end{aligned} \quad (\text{A2})$$

The commutator on the right-hand side is non-vanishing since the operator $\bar{c}\gamma_\nu\gamma_5 b$ carries an electric charge. It can be evaluated as:

$$\begin{aligned} & \int d^3x e^{-i\vec{k} \cdot \vec{x}} \langle 0 | [j_0^{\text{e.m.}}(\vec{x}), (\bar{c}\gamma_\nu\gamma_5 b)(\vec{0})] | \bar{B}(p_{B_c}) \rangle \\ &= \int d^3x e^{-i\vec{k} \cdot \vec{x}} \langle 0 | [e_c c_m^\dagger(\vec{x}) c_m(\vec{x}) + e_b b_m^\dagger(\vec{x}) b_m(\vec{x}), c_n^\dagger(0) (\gamma^0 \gamma_\nu \gamma_5)_{ns} b_s(0)] | \bar{B}(p_{B_c}) \rangle \\ &= (e_c - e_b) \langle 0 | (\bar{c}\gamma_\nu\gamma_5 b)(\vec{0}) | \bar{B}_c(p_{B_c}) \rangle \\ &= i(e_c - e_b) f_{B_c p_{B_c, \nu}}. \end{aligned} \quad (\text{A3})$$

The most general parametrization of the matrix element on the left-hand side without k^μ can be written in terms of five form factors $f_i(k^2, p_{B_c} \cdot k)$

$$\begin{aligned} i \int d^4x e^{ik \cdot x} \langle 0 | T j_\mu^{\text{e.m.}}(x) (\bar{c}\gamma_\nu\gamma_5 b)(0) | \bar{B}_c \rangle &= i[f_1 g_{\mu\nu} + f_2 p_{B_c, \mu} p_{B_c, \nu} + f_3 k_\mu k_\nu \\ &\quad + f_4 k_\mu p_{B_c, \nu} + f_5 p_{B_c, \mu} k_\nu]. \end{aligned} \quad (\text{A4})$$

The Ward identity (A3) implies two constraints on these form factors

$$(p_{B_c} \cdot k) f_2 + k^2 f_4 = (e_c - e_b) f_{B_c}, \quad f_1 + k^2 f_3 + (p_{B_c} \cdot k) f_5 = 0. \quad (\text{A5})$$

For a real photon $k^2 = 0$, these constraints fix uniquely the form factor $f_2(0, p_{B_c} \cdot k)$, and relate $f_1(0, p_{B_c} \cdot k)$ and $f_5(0, p_{B_c} \cdot k)$, which leads to

$$\langle \gamma(\epsilon, k) | \bar{c}\gamma_\mu\gamma_5 b | \bar{B}_c(p_{B_c}) \rangle = ie p_{B_c} \cdot k f_5 \left(\epsilon_\mu^* - k_\mu \frac{p_{B_c} \cdot \epsilon^*}{p_{B_c} \cdot k} \right) - \frac{ie}{p_{B_c} \cdot k} f_{B_c p_{B_c, \mu} p_{B_c} \cdot \epsilon^*}. \quad (\text{A6})$$

This is the same as the result in Eq. (9) as presented in text, with the identification $p_{B_c} \cdot k f_5 = A$.

Appendix B: Passarino-Veltman integrals

The coefficients b_i , c_i and d_i are related to the scalar Passarino-Veltman integrals defined in Ref. [76, 77], and we have split the finite pieces $b_i = B_i^{finite}$, $c_i = C_i^{finite}/m_b^2$ and $d_i = D_i^{finite}/m_b^4$:

$$\begin{aligned}
B_1 &= B_0(0, z^2 m_b^2, z^2 m_b^2), \\
B_2 &= B_0(0, m_b^2, m_b^2), \\
B_3 &= B_0(m_b^2(y^2 - z\tilde{z})/\tilde{z}, 0, m_b^2), \\
B_4 &= B_0(y^2 m_b^2, m_b^2, z^2 m_b^2), \\
C_1 &= C_0(m_b^2, 0, m_b^2(y^2 - z\tilde{z})/\tilde{z}, 0, m_b^2, m_b^2), \\
C_2 &= C_0(\tilde{z}^2 m_b^2, y^2 m_b^2, 0, m_b^2, z^2 m_b^2, m_b^2), \\
C_3 &= C_0(m_b^2, z^2 m_b^2, \tilde{z}^2 m_b^2, m_b^2, 0, z^2 m_b^2), \\
C_4 &= C_0(m_b^2(y^2 - z\tilde{z})/\tilde{z}, m_b^2 y^2, m_b^2 z^2, 0, m_b^2, m_b^2 z^2), \\
D_1 &= D_0(m_b^2, z^2 m_b^2, y^2 m_b^2, 0, \tilde{z}^2 m_b^2, m_b^2(y^2 - z\tilde{z})/\tilde{z}, m_b^2, 0, z^2 m_b^2, m_b^2). \tag{B1}
\end{aligned}$$

Here we give the the results of divergence integrals.

$$\begin{aligned}
B_1 &= \frac{1}{\epsilon_{UV}} + \ln \frac{\mu^2}{z^2 m_b^2}, \\
B_2 &= \frac{1}{\epsilon_{UV}} + \ln \frac{\mu^2}{m_b^2}, \\
B_3 &= \frac{1}{\epsilon_{UV}} + \ln \frac{\mu^2}{m_b^2} - \frac{(y^2 - \tilde{z}^2) \ln(\tilde{z} - \frac{y^2}{\tilde{z}})}{y^2 - z\tilde{z}} + 2, \\
B_4 &= \frac{1}{\epsilon_{UV}} + \ln \frac{\mu^2}{y^2 m_b^2} + 2 + \sum_{i=1}^2 (\gamma_i(y) \ln(\frac{\gamma_i(y) - 1}{\gamma_i(y)}) - \ln(\gamma_i(y) - 1)), \\
C_3 &= -\frac{1}{2z m_b^2} \left(\frac{1}{\epsilon_{IR}} + t_1 + \ln \frac{\mu^2}{m_b^2} - 2 - \frac{2 \ln z}{1+z} \right), \\
D_1 &= \frac{\tilde{z}}{2m_b^4 z (\tilde{z}^2 - y^2)} \left(\frac{1}{\epsilon_{IR}} + t_1 + \ln \frac{\mu^2}{m_b^2} - 2 \ln \frac{\tilde{z}^2 - y^2}{\tilde{z}} + \frac{1}{(y - \tilde{z})(y + \tilde{z})} (2(\tilde{z}^2 - 2y^2 \ln y - (y^2 + z^2 - 1) \ln z - y^2(1 + 2 \ln 2)) + (-g_5 + y^2 + z^2 - 1)g_1 + (g_5 + y^2 - z^2 + 1)g_2 + (-g_5 + y^2 - z^2 + 1)g_3 + (g_5 + y^2 + z^2 - 1)g_4) \right), \tag{B2}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{1,2}(x) &= \frac{\pm\sqrt{(x^2 - z^2 + 1)^2 - 4x^2 + x^2 - z^2 + 1}}{2x^2}, \\
g_1 &= \ln\left(\sqrt{(y^2 - z^2 + 1)^2 - 4y^2} - y^2 - z^2 + 1\right), \\
g_2 &= \ln\left(\sqrt{(y^2 - z^2 + 1)^2 - 4y^2} + y^2 - z^2 + 1\right), \\
g_3 &= \ln\left(-\sqrt{(y^2 - z^2 + 1)^2 - 4y^2} + y^2 - z^2 + 1\right), \\
g_4 &= \ln\left(-\sqrt{(y^2 - z^2 + 1)^2 - 4y^2} - y^2 - z^2 + 1\right), \\
g_5 &= \sqrt{y^4 - 2y^2(z^2 + 1) + (z^2 - 1)^2}. \tag{B3}
\end{aligned}$$

Appendix C: One loop corrections to the axial-vector form factor A

The most general structure of the matrix element of the axial-vector current is parametrized by:

$$\langle\gamma(\epsilon, k)|\bar{c}\gamma_\mu\gamma_5b|[\bar{c}b(^1S_0^{[1]})]\rangle = ie\left(\epsilon_\mu^*\mathbb{A}^\epsilon - k_\mu\frac{p_{B_c}\cdot\epsilon^*}{p_{B_c}\cdot k}\mathbb{A}^k\right) - ie\frac{p_{B_c}\cdot\epsilon^*}{p_{B_c}\cdot k}\mathbb{U}^A p_{B_c\mu}. \tag{C1}$$

This section will be devoted to demonstrate the gauge invariance at the one-loop level in α_s , namely

$$\mathbb{A}^\epsilon = \mathbb{A}^k \equiv \mathbb{A}, \tag{C2}$$

$$\mathbb{U}^A = \mathbb{U}. \tag{C3}$$

The contributions from individual diagrams to \mathbb{A}^ϵ are given as

$$\begin{aligned} \mathbb{A}_b^\epsilon = & \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{1}{\hat{\epsilon}_{IR}} - \frac{4(z-1)\tilde{z}^2}{(y^2 - \tilde{z}^2)^2} + \frac{y^2 \tilde{z}^2 - 2(z-1)\tilde{z}^3 - y^4}{(y^2 - \tilde{z}^2)^2 \tilde{z}} b_1 + \frac{y^2}{(y^2 - \tilde{z}^2)\tilde{z}} b_2 \right. \\ & + \frac{2y^2 \tilde{z} - y^4 + (z^2 - 1)^2}{z(y^2 - \tilde{z}^2)^2} b_3 + \frac{-2y^2 \tilde{z} + (z-1)\tilde{z}^3 + y^4}{z(y^2 - \tilde{z}^2)^2} b_4 \\ & - \frac{2(3z-1)\tilde{z}^2 + y^4 - y^2(z^2 + 4z + 3)}{(\tilde{z}^2 - y^2)\tilde{z}} c_4 - \frac{(z^2 - 1)(y^2 - z^2 + 1)}{(y^2 - \tilde{z}^2)\tilde{z}} c_3 \\ & \left. + (-y^2 + z^2 + 4z - 1) d_1 \right], \end{aligned} \quad (C4)$$

$$\begin{aligned} \mathbb{A}_c^\epsilon = & \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{1}{2\hat{\epsilon}_{UV}} + \frac{\tilde{z} + y^2}{y^2 - \tilde{z}^2} b_1 + \frac{-y^2(2z+3) + 2z^3 + 5z^2 + 2z - 1}{2z(\tilde{z}^2 - y^2)} b_3 \right. \\ & \left. + \frac{y^2 + z^2 + 4z + 3}{y^2 - \tilde{z}^2} + \frac{\tilde{z}(-3y^2 + z^2 - 1)}{2z(y^2 - \tilde{z}^2)} b_4 + (-y^2 + z^2 + z - 1) c_4 \right], \end{aligned} \quad (C5)$$

$$\mathbb{A}_d^\epsilon = \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{1}{2\hat{\epsilon}_{UV}} + \frac{y^2 - z^2 + 1}{y^2 - \tilde{z}^2} + \frac{y^2 - z\tilde{z}}{y^2 - \tilde{z}^2} b_2 + -\frac{y^2 - z^2 + 1}{2(y^2 - \tilde{z}^2)} b_3 - \tilde{z} c_1 \right], \quad (C6)$$

$$\mathbb{A}_e^\epsilon = \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[-\frac{1}{2\hat{\epsilon}_{UV}} + \frac{y^2 - z^2 + 1}{2(y^2 - z\tilde{z})} + \frac{\tilde{z}}{2y^2 - 2z\tilde{z}} b_2 - \frac{y^2 - z^2 + 1}{2(y^2 - z\tilde{z})} b_3 \right], \quad (C7)$$

The mass counter term and wave function renormalization give the contributions:

$$\begin{aligned} \mathbb{A}_{CT-m}^\epsilon &= 0, \\ \mathbb{A}_{CT-F}^\epsilon &= -\mathbb{V}_{CT-F}. \end{aligned} \quad (C8)$$

The contributions from individual diagrams to \mathbb{A}^k are given as

$$\begin{aligned} \mathbb{A}_b^k = & \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[-\frac{\tilde{z}(y^2(-7z^2 + 10z + 1)\tilde{z}^2 + (z-1)\tilde{z}^5 + y^4(3y^2 + 3z^2 - 8z - 11))}{(y^2 - z\tilde{z})(y^3 - y\tilde{z}^2)^2} \right. \\ & + \frac{2y^2(3-2z)\tilde{z}^2 + z\tilde{z}^4 + y^4(-(z+2))}{(y^3 - y\tilde{z}^2)^2} b_1 + \frac{\tilde{z} + y^2}{y^4 - y^2 z\tilde{z}} b_2 - \frac{2\tilde{z}(-y^2 + z^2 + 3)}{(y^2 - \tilde{z}^2)} c_4 \\ & + \frac{-y^2 \tilde{z}^2 (y^2 + z^2 - 4z + 5) + (z^3 - 3z^2 + 5z + 1)\tilde{z}^3 + y^6}{z(-y^2 + z^2 + z)(y^2 - \tilde{z}^2)^2} b_3 \\ & \left. + \frac{y^2(3z-5)\tilde{z}^3 - (z-1)\tilde{z}^5 + y^4(y^2 + z^2 - 1)}{z(y^3 - y\tilde{z}^2)^2} b_4 \right] + \mathbb{A}_b^\epsilon, \end{aligned} \quad (C9)$$

$$\begin{aligned} \mathbb{A}_c^k = & \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{\tilde{z}(y^2(-7z^2 + 10z + 1)\tilde{z}^2 + (z-1)\tilde{z}^5 + y^4(3y^2 + 3z^2 - 8z - 11))}{(y^2 - z\tilde{z})(y^3 - y\tilde{z}^2)^2} \right. \\ & + \frac{2y^2(2z-3)\tilde{z}^2 - z\tilde{z}^4 + y^4(z+2)}{(y^3 - y\tilde{z}^2)^2} b_1 + \frac{\tilde{z} + y^2}{y^4 - y^2 z\tilde{z}} b_2 - \frac{2\tilde{z}(-y^2 + z^2 + 3)}{(y^2 - \tilde{z}^2)} c_4 \\ & - \frac{-y^2 \tilde{z}^2 (y^2 + z^2 - 4z + 5) + (z^3 - 3z^2 + 5z + 1)\tilde{z}^3 + y^6}{z(z\tilde{z} - y^2)(y^2 - \tilde{z}^2)^2} b_3 \\ & \left. - \frac{y^2(3z-5)\tilde{z}^3 - (z-1)\tilde{z}^5 + y^4(y^2 + z^2 - 1)}{z(y^3 - y\tilde{z}^2)^2} b_4 \right] + \mathbb{A}_c^\epsilon, \end{aligned} \quad (C10)$$

$$\begin{aligned}\mathbb{A}_d^k &= \mathbb{A}_d^\epsilon + \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{\tilde{z} (5y^2 - 5z^2 - 6z - 1)}{(y^2 - \tilde{z}^2)(y^2 - z\tilde{z})} + \frac{\tilde{z} (3y^2 - 3z^2 - 4z - 1)}{(y^2 - \tilde{z}^2)(y^2 - z\tilde{z})} b_2 \right. \\ &\quad \left. + \frac{\tilde{z} (-3y^2 + 3z^2 + 4z + 1)}{(y^2 - \tilde{z}^2)(y^2 - z\tilde{z})} b_3 \right],\end{aligned}\quad (C11)$$

$$\begin{aligned}\mathbb{A}_e^k &= \mathbb{A}_e^\epsilon + \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{3\tilde{z}}{y^2 - \tilde{z}^2} \frac{1}{\hat{\epsilon}_{UV}} + \frac{\tilde{z}}{-y^2 + z^2 + z} + \frac{\tilde{z}^2}{(y^2 - \tilde{z}^2)(y^2 - z\tilde{z})} b_2 \right. \\ &\quad \left. + \frac{\tilde{z} (3y^2 - 3z^2 - 4z - 1)}{(y^2 - \tilde{z}^2)(y^2 - z\tilde{z})} b_3 \right].\end{aligned}\quad (C12)$$

Similar, the mass counter-terms and wave function renormalization corrections give:

$$\begin{aligned}\mathbb{A}_{CT-m}^k &= \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi m_b} \left[\frac{3\tilde{z}}{\tilde{z}^2 - y^2} \left(\frac{1}{\hat{\epsilon}_{UV}} + \ln \frac{\mu^2}{m_b^2} + \frac{4}{3} \right) \right], \\ \mathbb{A}_{CT-F}^k &= \mathbb{A}_{CT-F}^\epsilon.\end{aligned}\quad (C13)$$

Adding the above contributions, one may derive the relation $\mathbb{A}^\epsilon = \mathbb{A}^k$, which is guaranteed by gauge invariance. One can obtain the one-loop results for \mathbb{A} by adding up the anti-symmetrical part with $e_b \rightarrow e_c$ and $m_b \leftrightarrow m_c$.

The contributions from individual diagrams to \mathcal{U}^A are given as

$$\begin{aligned}\mathcal{U}_b^A &= \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi} \left[-\frac{2}{\hat{\epsilon}_{IR}} + \frac{y^2 (3z^2 - 6z - 1) \tilde{z}^2 - (z - 1) \tilde{z}^5 + y^4 (-3y^2 + z^2 + 8z + 7)}{y^2 (y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} \right. \\ &\quad + \frac{-3y^2 (z - 1) \tilde{z}^2 + z \tilde{z}^4 - y^4}{y^2 \tilde{z} (y^2 - \tilde{z}^2)} b_1 + \frac{-(y^4 + 3)z + (y^2 - 1)z^3 + (y^2 - 3)z^2 - 1}{y^2 \tilde{z} (y^2 - z\tilde{z})} b_2 \\ &\quad + \frac{4y^2 \tilde{z} + (z^2 - 4z - 1) \tilde{z}^2 - y^4}{z (y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} b_3 + \frac{\tilde{z} (-(z - 1) \tilde{z}^3 + y^4 + 2y^2 (z^2 - z - 2))}{y^2 z (y^2 - \tilde{z}^2)} b_4 \\ &\quad \left. - 2\tilde{z}c_4 + 4zc_3 \right],\end{aligned}\quad (C14)$$

$$\begin{aligned}\mathcal{U}_c^A &= \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi} \left[-\frac{1}{\hat{\epsilon}_{UV}} + \frac{y^2 (-5z^2 + 4z + 1) \tilde{z}^2 + (z - 1) \tilde{z}^5 + y^4 (y^2 + 3z^2 - 2z - 5)}{y^2 (y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} \right. \\ &\quad - \frac{\tilde{z} (z \tilde{z}^2 + y^2 (2 - 3z))}{y^4 - y^2 \tilde{z}^2} b_1 + \frac{\tilde{z}^2}{y^4 - y^2 z \tilde{z}} b_2 + \frac{-4y^2 \tilde{z} + (-z^2 + 4z + 1) \tilde{z}^2 + y^4}{z (y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} b_3 \\ &\quad \left. + \frac{\tilde{z} ((z - 1) \tilde{z}^3 - y^2 (y^2 + 2z^2 - 2z - 4))}{y^2 z (y^2 - \tilde{z}^2)} b_4 + 2\tilde{z}c_4 \right],\end{aligned}\quad (C15)$$

$$\begin{aligned}\mathcal{U}_d^A &= \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi} \left[-\frac{1}{\hat{\epsilon}_{UV}} + \frac{(z^2 + 10z + 1) \tilde{z}^2 + y^4 - 2y^2 (z^2 + 6z + 5)}{(y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} \right. \\ &\quad \left. + \frac{\tilde{z} (-5y^2 + 5z^2 + 6z + 1)}{(y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} b_2 - \frac{(z^2 + 6z + 1) \tilde{z}^2 + y^4 - 2y^2 (z^2 + 4z + 3)}{(y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} b_3 \right],\end{aligned}\quad (C16)$$

$$\begin{aligned}\mathcal{U}_e^A &= \frac{e_b C_F \alpha_s \sqrt{2N_c}}{4\pi} \left[\frac{y^2 - z^2 - 8z - 7}{y^2 - \tilde{z}^2} \frac{1}{\hat{\epsilon}_{UV}} + \frac{\tilde{z}^2 - y^2}{y^2 - z\tilde{z}} + \frac{\tilde{z} (-y^2 + z^2 - 1)}{(y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} b_2 \right. \\ &\quad \left. + \frac{(z^2 + 6z + 1) \tilde{z}^2 + y^4 - 2y^2 (z^2 + 4z + 3)}{(y^2 - \tilde{z}^2) (y^2 - z\tilde{z})} b_3 \right],\end{aligned}\quad (C17)$$

$$\begin{aligned}\mathcal{U}_{CT-m}^A &= -\frac{2}{z}\mathbb{A}_{CT-m}^\epsilon, \\ \mathcal{U}_{CT-F}^A &= -\frac{2}{z}\mathbb{A}_{CT-F}^\epsilon.\end{aligned}\tag{C18}$$

The sum of them is

$$\mathcal{U}_{b-e+CT}^A = -\frac{3e_b C_F \alpha_s \sqrt{2N_c} ((z-1)\ln(z) - 2\tilde{z} + 2/3t_1)}{4\pi\tilde{z}}.\tag{C19}$$

We can get the one-loop result in Eq. 60 after adding up the symmetrical part with $e_b \rightarrow e_c$ and $m_b \leftrightarrow m_c$.

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