

## Lambda Phenomena: The Lambda points of liquid Helium and chiral QCD

---

**Sourendu Gupta\***

*TIFR Mumbai*

*E-mail: [sgupta@tifr.res.in](mailto:sgupta@tifr.res.in)*

**Rishi Sharma**

*TIFR Mumbai*

*E-mail: [rishi@theory.tifr.res.in](mailto:rishi@theory.tifr.res.in)*

The superfluid transition of liquid Helium shares an interesting phenomenon with the chiral limit of QCD: the specific heat,  $c_V$ , is finite at the critical point, but has a cusp. From this follows an interesting mixture of universal and non-universal features at the critical point. Through the CP symmetry of chiral QCD, this has implications for the fourth order baryon number susceptibility,  $\chi_B^4$ , and susceptibilities of higher orders. Investigations of such a scaling will show us whether O(4) scaling is an accurate description of baryon-free QCD when the pion mass is realistic.

*9th International Workshop on Critical Point and Onset of Deconfinement - CPOD2014,*

*17-21 November 2014*

*ZiF (Center of Interdisciplinary Research), University of Bielefeld, Germany*

---

\*Speaker.

## 1. Introduction

In the chiral limit of QCD, obtained by setting the light quark masses to zero, the global symmetry of QCD is chiral  $SU(2) \times SU(2)$ . This is homomorphic to a  $O(4)$  symmetry. The universality conjecture then leads us expect that the critical indices and critical amplitude ratios in chiral QCD should be the same as for  $O(4)$  magnetic systems [1].

In recent times, the chiral condensate of QCD (which is equivalent to the spontaneous magnetization of the magnet), its derivatives with respect to the quark mass (equivalent to the magnetic susceptibility), and their scaling towards the chiral limit, have been studied with somewhat ambiguous results [2].

The scaling of the internal energy and the specific heat of  $O(N)$  symmetric systems is intricate, as is known from the phenomenology of liquid Helium. However, with the extensive lattice QCD computations now available on quark number susceptibilities in QCD at zero baryon density [3], their important role in heavy-ion collisions [4], and their connection with temperature derivatives of the free energy, it is important to initiate the scaling analysis of these quantities. That is the purpose of this talk.

## 2. Scaling and the limits of universality

The thermodynamics of QCD is characterized by a free energy, which is a function of some number of intensive control parameters. These could include the temperature,  $T$ , and the baryon chemical potential,  $\mu$ . If the pion mass were exactly vanishing, then QCD would have a  $O(4)$  global chiral symmetry. Since we are interested in real QCD, where the pion is not massless, an explicit chiral symmetry breaking parameter is needed. This is the quark mass,  $m$ , which plays the same role in QCD as a magnetic field does for the  $O(4)$  magnet.

Near the critical point one can decompose the free energy,  $F(T, m)$ , into the sum of two terms. One of these is a regular part,  $F_r(T, m)$ , and the other is a singular part,  $F_s(T, m)$ .  $F_r(T, m)$  is Taylor expandable around the critical point,  $T = T_c$  and  $m = 0$  with some large radius of convergence. The modern theory of critical phenomena starts from the observation that the most singular part is a scaling function

$$F_s(T, m) = t^{2-\alpha} \Phi(\tau), \quad \text{where} \quad t = \left| 1 - \frac{T}{T_c} \right|, \quad \tau = \frac{t}{(m/M_0)^{1/\Delta}}, \quad (2.1)$$

where we have chosen the scaling variables  $t$  and  $\tau$  to be dimensionless,  $T_c$  is the critical temperature,  $M_0$  is any mass scale which remains finite in the chiral limit, and  $\alpha, \Delta = \beta\delta$  are critical exponents. The function  $F_s(t, \tau)$ , defined so, is universal, in the sense that whether we examine an  $O(4)$  Heisenberg magnet, QCD, a non-linear sigma model of pions, or the Nambu-Jona-Lasinio (NJL) model, the  $F_s(t, \tau)$  we obtain from all of them are the same. As a result, the universal properties of thermal QCD know nothing about QCD, aside from its chiral symmetry. However, the various models differ in  $F_r(T, m)$ , so this is the piece which gives information about the actual degrees of freedom involved in the QCD phase transition.

Even if the magnitude of  $F_s$  is comparable to  $F_r$ , since it is singular, its effect may be enhanced by taking sufficient number of derivatives. For example, the specific heat,  $c_v \propto t^{-\alpha}$ , and hence

Model	Example	$\beta$	$\delta$	$\alpha$	Ref
O( $\infty$ )		1/2	5	-1	[5]
O(4)	chiral QCD	0.380	4.86	-0.2268	[6]
O(3)	?	0.365	4.79	-0.115	[8]
O(2)	liquid He	0.349	4.78	-0.0172	[7]
O(1)	liquid-gas	0.325	4.8	0.11	[8]
MFT		1/2	3	0	

**Table 1:** Critical exponents of O(N) models in three spatial dimensions. The exponent  $\alpha$  is obtained from the other reported exponents using scaling identities. O(1) should be taken to mean the Ising model. MFT stands for mean field theory. There are no known examples of O(3) models, since real ferromagnets have relevant terms which break this symmetry.

diverges at  $T = T_c$ , provided that  $\alpha > 0$ . As one can see from Table 1, this is true of the Ising model. However, for all other  $O(N)$  models  $\alpha < 0$ , and, as a result, the singular contribution to the specific heat exactly vanishes for  $T = T_c$ .

This seems to contradict our knowledge of the specific heat of liquid He, which is in the O(2) universality class and has a cusp at the critical point. The resolution of this puzzle comes from noticing that the peak of  $c_v$  is only finite, and hence is regular. It is the shape which is singular. So the specific heat has to arise through a playoff between the singular and regular parts. In fact, a very precise microgravity experiment has been done over the range  $|T - T_c| \leq 2$  nK [9], and the results fitted to the formula

$$c_v = A_r + t^{-\alpha}(B + Ct^{-\Delta'}), \quad (2.2)$$

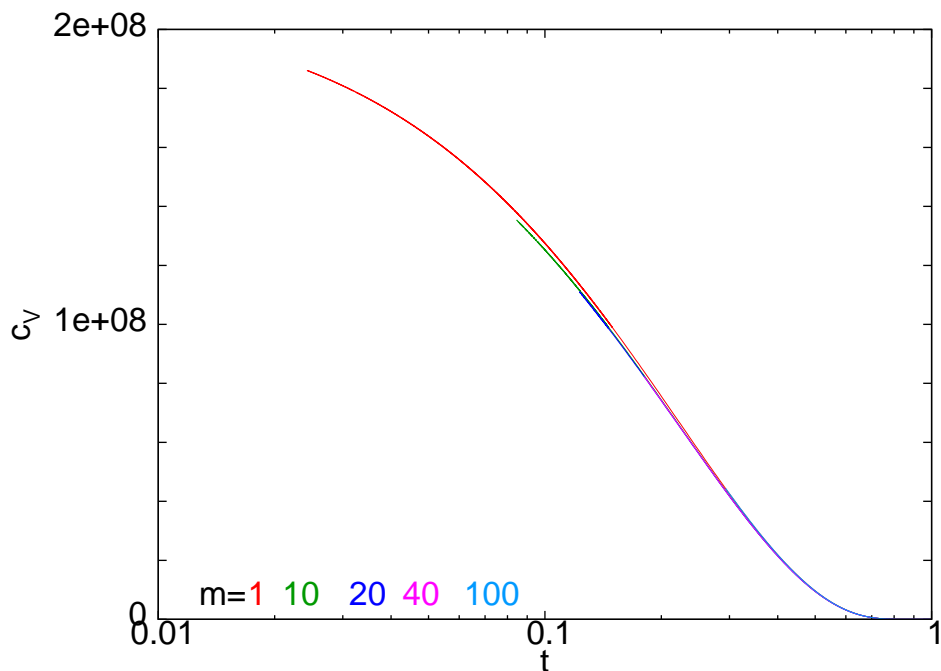
where  $A_r$  comes only from  $F_r$ .  $\Delta'$  is a possible correction-to-scaling exponent.  $A_r$  is positive and  $B$  is constrained to be negative. It can be shown that  $B$  can be negative without violating the thermodynamic consistency criterion that  $c_v > 0$ . A result of the microgravity experiment is that  $\alpha = -0.01285(38)$ . Interestingly this is in disagreement with the careful work of [7].

This mechanism also works for QCD and other O(N) symmetric models. For these we may write

$$c_v(T, m) = A_r + \frac{t^{-\alpha}}{T_c} \Psi(t, \tau). \quad (2.3)$$

$T_c$  and  $A_r$  are non-universal, and change from QCD to various effective theories for it, but the exponent  $\alpha$ , and the regular function  $\Psi$  (which may be written in terms of the scaling function  $\Phi$ , and its derivatives,  $\Phi'$  and  $\Phi''$ , if desired) are universal. So the shape of the specific heat cusp is universal but its height and width must be determined in QCD. Furthermore, these two parameters are good tests of possible effective models, since a bad model of QCD will not reproduce its non-universal properties.

An interesting statement about the scaling of  $c_v$  with mass arises from this. Suppose we succeeded in measuring (on the lattice)  $c_v$  for QCD with various different light quark masses. By plotting the data as a function of  $t$  and scaling  $c_v$  appropriately, can we observe scaling in the form of data collapsing on to an universal curve? Clearly, there are no singularities of the free energy if  $T$  is varied around  $T_c$  at fixed non-vanishing  $m$ . As a result, taking  $t \rightarrow 0$  and  $\tau \rightarrow 0$  simultaneously



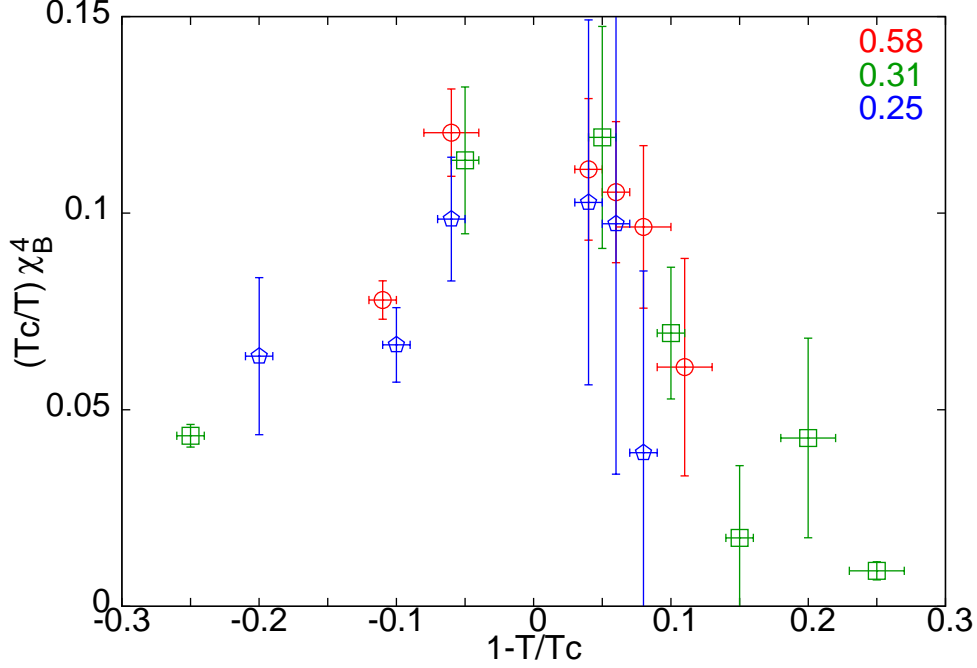
**Figure 1:** Data collapse obtained in an MFT treatment of the NJL model when keeping only the data for  $\tau > 50$ . Tiny violations of scaling are visible; these can be controlled by increasing the cut on  $\tau$ . The value of this cut depends on the choice of  $M_0$  (here it was taken to be  $T_c$ ), and the renormalization scheme, when going beyond MFT. Also, since this value is not universal, it could be different in QCD. Only data for  $T < T_c$  is used in this plot to avoid having to subtract a large regular part, as discussed in the text.

will not reveal scaling. Instead, one must take the limit  $m \rightarrow 0$  first and  $T \rightarrow T_c$  next, which means that one must take  $\tau \rightarrow \infty$  first and then  $t \rightarrow 0$  in order to see data collapse.

One can test this in the NJL model even at tree level, *i.e.*, in the MFT approximation. The high temperature limit of this model contains weakly interacting quarks, so the regular contribution to  $c_v$  actually increases fairly rapidly with temperature. As a result, one may miss the pseudo-critical behaviour in  $c_v$  unless the temperature range is scanned finely to discover a peak sitting over a rising background, or the free quark contribution is subtracted to make the peak stand out over a falling background. This difficulty would also occur in QCD [10], but not in the O(4) Heisenberg magnet.

With this MFT one sees that data collapse is possible when one plots  $c_v$  against  $t$  provided that one selects only  $\tau > 50$ . This is sufficient to ensure that for any finite  $m$  one does not approach  $t = 0$  too closely. Figure 1 shows that one may relax the condition  $\tau \rightarrow \infty$  provided one is willing to tolerate small enough violation of scaling. Since experimental data or Monte Carlo computations come with errors, it should be possible to tune the cutoff on  $\tau$  in order to find the scaling curve within the errors.

We end this section with a remark about the scaling fields. In making use of effective models to study universal properties of QCD, most works make the assumption that the scaling fields of the effective theory ( $T$  and  $m$ ) are identical to those of QCD. Whether or not this assumption is



**Figure 2:** Data collapse obtained for  $\chi_B^4$  using the results of [14], when keeping only the data for  $\tau > 0.04$  with O(4) exponents. The data is plotted as a function of  $1 - T/T_c$  instead of  $t$  because the regular parts on the two sides of  $T_c$  are have not been removed. The colour coding corresponds to the values of  $m_\pi^2/m_\rho^2$  given in the figure legend.

correct can be tested, but, to the best of our knowledge, such tests have not been performed.

### 3. Relevance to the phase diagram of chiral QCD

The phase diagram of chiral QCD can be extended to finite baryon chemical potential,  $\mu$ . Since this scaling field preserves the O(4) symmetry, the critical point of chiral QCD at  $\mu = 0$  gets stretched into a line. The global CP symmetry of QCD implies that  $F(T, m = 0, \mu) = F(T, m = 0, -\mu)$ , so

$$T_c(\mu) = T_c + \frac{1}{2}\kappa\mu^2 + \dots \quad (3.1)$$

The curvature has been studied in lattice QCD for about a decade. Different determinations agree roughly on its value [11].

If one assumes that  $\mu$  enters the scaling function of eq. (2.1) only through the dependence of  $T_c$  on  $\mu$  as given in eq. (3.1), then, as in [12], we can write

$$\left. \frac{\partial}{\partial T} g(t, \tau) \right|_{\mu=0} = - \frac{T_c}{T\kappa} \left. \frac{\partial^2}{\partial \mu^2} g(t, \tau) \right|_{\mu=0}. \quad (3.2)$$

Qualitative evidence for this relation between derivatives was obtained very long back [13]. The

relation above implies a connection between  $c_V$  and the fourth order baryon number susceptibility—

$$\chi_B^4(T, m) = \left. \frac{\partial^4 P(T, m, \mu)}{\partial \mu^4} \right|_{\mu=0} \simeq (\kappa T_c)^2 \left( \frac{T}{T_c} \right)^4 \frac{c_V}{T^3}. \quad (3.3)$$

The last expression for  $\chi_B^4$  comes from retaining only the most singular contribution. This suggests a scaling test of  $\chi_B^4$  similar to that for  $c_V$ .

This test can be performed with the results of [14]. For the treatment of lattice QCD computations we may replace  $m/M_0$  in the definition of  $\tau$  by  $m_\pi^2/m_\rho^2$ , so that  $t$ ,  $\tau$  and  $\chi_B^4$  are all renormalized quantities. In Figure 2 we plot the full measured  $T_c \chi_B^4/T$ . Data collapse should be expected in the region where the contribution from the singular part dominates.

The regular parts at temperatures well above and well below  $T_c$  are expected to be different, since effective theories in these two regions are the hadron gas model ( $T \ll T_c$ ) and the weak coupling expansion of QCD ( $T \gg T_c$ ). We have plotted Figure 2 to show these two branches separately. It seems that in the region  $t \leq 0.1$ , the differences in the regular parts may be neglected within the precision of the data. One particular implication is that the gas model should not work for  $\chi_B^4$  within 10% of  $T_c$ .

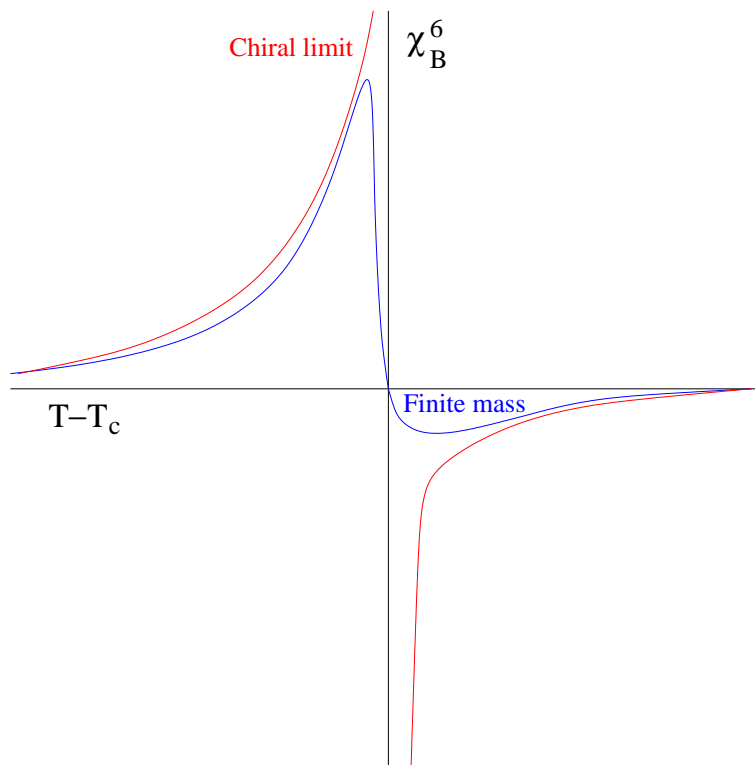
While this gives us a first test (in this sector) of scaling at  $T = 0$  at surprisingly large quark masses, the current errors are large. Improvement in errors would allow us to test scaling better, and also to test the importance of the variation of the regular part of these quantities with approximately 10% change in  $t$ . These requirements set benchmarks for future measurements of  $\chi_B^4$ .

It is clear from eq. (2.1) and the values of  $\alpha$  in Table 1 that derivatives of  $c_V$  with respect to  $T$  would diverge in the vicinity of the critical point. As a result, one should be able to observe scaling of the higher order baryon number susceptibilities. For example, the sixth order quantity,  $\chi_B^6$  would be universal, and is likely to have a shape similar to that shown in Figure 3. It would be interesting to test this in future, when improvement in statistics makes these tests significant.

#### 4. Some remarks

It is interesting to recall that before the modern theory of critical phenomena was developed, the Ehrenfest classification of phase transitions was in vogue. This attempted to define orders of phase transitions according to which derivative of the free energy diverged. In the case of O(N) models one sees very clearly that such a classification runs into trouble. On examining the chiral (magnetic) susceptibilities, one comes to the conclusion that the QCD transition is of second order. However, on examining  $c_V$  one comes to the conclusion that the same transition is of third order, since  $c_V$  does not diverge, but its derivative with respect to  $T$  does. We realize today that the differences are due only to the value of associated critical index.

We conclude by reiterating the importance of scaling tests such as that suggested here. They constitute a new domain of tests of the universality hypothesis in the context of QCD. Not only is this important in its own right, but also serves to put bounds on the region of applicability of models such as the hadron gas model. This model is a mixture of ideal gases and hence contains no singular part, whereas O(4) universality is based entirely on the singularity due to pions in the chiral limit. Since these are mutually exclusive descriptions of the free energy, the success of one rules out the other.



**Figure 3:** In the chiral limit  $\chi_B^6$  diverges with critical exponent  $1 + \alpha$  and is proportional to the temperature derivative of  $c_V$ . At finite  $m_\pi^2/m_\rho^2$  these divergences would be rounded off as shown.

We thank Deepak Dhar for discussions.

## References

- [1] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29** (1984) 338.
- [2] S. Ejiri, F. Karsch, E. Laermann, C. Miao, S. Mukherjee, P. Petreczky, C. Schmidt and W. Soeldner *et al.*, Phys. Rev. D **80** (2009) 094505 [arXiv:0909.5122 [hep-lat]].
- [3] A. Bazavov, H.-T. Ding, P. Hegde, F. Karsch, C. Miao, S. Mukherjee, P. Petreczky and C. Schmidt *et al.*, Phys. Rev. D **88** (2013) 9, 094021 [arXiv:1309.2317 [hep-lat]]; S. Datta, R. V. Gai and S. Gupta, PoS LATTICE **2013** (2014) 202; S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo, Phys. Rev. Lett. **111** (2013) 062005 [arXiv:1305.5161 [hep-lat]]; A. Bazavov *et al.* [HotQCD Collaboration], Phys. Rev. D **86** (2012) 034509 [arXiv:1203.0784 [hep-lat]]; R. V. Gai and S. Gupta, Phys. Rev. D **78** (2008) 114503 [arXiv:0806.2233 [hep-lat]]; R. V. Gai and S. Gupta, Phys. Rev. D **71** (2005) 114014 [hep-lat/0412035].
- [4] R. V. Gai and S. Gupta, Phys. Lett. B **696** (2011) 459 [arXiv:1001.3796 [hep-lat]].
- [5] S. A. Antonenko and A. I. Sokolov, Phys. Rev. E **51** (1995) 1894 [hep-th/9803264].
- [6] J. Engels and T. Mendes, Nucl. Phys. B **572** (2000) 289 [hep-lat/9911028].
- [7] J. Engels, S. Holtmann, T. Mendes and T. Schulze, Phys. Lett. B **492** (2000) 219 [hep-lat/0006023].

- 
- [8] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. **39** (1977) 95.
- [9] J. A. Lipa, D. R. Swanson, J. A. Nissen, T. C. P. Chui and U. E. Israelsson, Phys. Rev. Lett. **76** (1996) 944.
- [10] In fact, after this talk, the HotQCD collaboration has found a peak in  $c_V$  by performing such a subtraction. See P. Hegde, talk given in the workshop “Perspectives and Challenges in Lattice Gauge Theory”, TIFR Mumbai, February 2015.
- [11] O. Kaczmarek, F. Karsch, E. Laermann, C. Miao, S. Mukherjee, P. Petreczky, C. Schmidt and W. Soeldner *et al.*, Phys. Rev. D **83** (2011) 014504 [arXiv:1011.3130 [hep-lat]]; R. Falcone, E. Laermann and M. P. Lombardo, PoS LATTICE **2010** (2010) 183 [arXiv:1012.4694 [hep-lat]]; G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, JHEP **1104** (2011) 001 [arXiv:1102.1356 [hep-lat]]; C. Bonati, M. D’Elia, M. Mariti, M. Mesiti, F. Negro and F. Sanfilippo, Phys. Rev. D **90** (2014) 11, 114025 [arXiv:1410.5758 [hep-lat]]; L. Cosmai, P. Cea and A. Papa, arXiv:1410.2471 [hep-lat].
- [12] B. Friman, F. Karsch, K. Redlich and V. Skokov, Eur. Phys. J. C **71** (2011) 1694 [arXiv:1103.3511 [hep-ph]].
- [13] R. V. Gavai and S. Gupta, Phys. Rev. D **72** (2005) 054006 [hep-lat/0507023].
- [14] S. Gupta, N. Karthik and P. Majumdar, Phys. Rev. D **90** (2014) 3, 034001 [arXiv:1405.2206 [hep-lat]].