Curvilinear integral theorems for monogenic functions in commutative associative algebras

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Abstract. We consider an arbitrary finite-dimensional commutative associative algebra, $\mathbb{A}_n^m$, with unit over the field of complex number with m idempotents. Let $e_1 = 1, e_2, e_3$ be elements of $\mathbb{A}_n^m$ which are linearly independent over the field of real numbers. We consider monogenic (i. e. continuous and differentiable in the sense of Gateaux) functions of the variable $xe_1 + ye_2 + ze_3$, where x, y, z are real. For mentioned monogenic function we prove curvilinear analogues of the Cauchy integral theorem, the Morera theorem and the Cauchy integral formula.

1   Introduction.

The Cauchy integral theorem and Cauchy integral formula for the holomorphic function of the complex variable are a fundamental result of the classical complex analysis. Analogues of these results are also an important tool in commutative algebras of dimensional more that 2.

In the paper of E. R. Lorch [1] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the curvilinear integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. E. K. Blum [2] withdrew a convexity condition of a domain in the mentioned results from [1].

Let us note that *a priori* the differentiability of a function in the sense of Gateaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Therefore, we consider a

monogenic functions defined as a continuous and differentiable in the sense of Gateaux. Also we assume that a monogenic function is given in a domain of three-dimensional subspace of an arbitrary commutative associative algebra with unit over the field of complex numbers. In this situation the results established in the papers [1, 2] is not applicable for a mentioned monogenic function, because it deals with an integration along a curve on which the function is not given, generally speaking.

In the papers [3, 4, 5] for monogenic function the curvilinear analogues of the Cauchy integral theorem, the Cauchy integral formula and the Morera theorem are obtained in special finite-dimensional commutative associative algebras.

In this paper we generalize results of the papers [3, 4, 5] for an arbitrary commutative associative algebra over the field of complex numbers.

Let us note that some analogues of the curvilinear Cauchy integral theorem and the Cauchy integral formula for another classes of functions in special commutative algebras are established in the papers [6, 7, 8, 9, 10].

2   The algebra $\mathbb{A}_n^m$.

Let $\mathbb{N}$ be the set of natural numbers. We fix the numbers $m, n \in \mathbb{N}$ such that $m \leq n$. Let $\mathbb{A}_n^m$ be an arbitrary commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan [11, pp. 33 – 34] proved that in the algebra $\mathbb{A}_n^m$ there exist a basis $\{I_k\}_{k=1}^n$ satisfies the following multiplication rules:

1. $\forall \ r, s \in [1, m] \cap \mathbb{N} :$ $\qquad I_r I_s = \begin{cases} 0 & \text{if} \ \ r \neq s, \\ I_r & \text{if} \ \ r = s; \end{cases}$

2. $\forall \ r, s \in [m + 1, n] \cap \mathbb{N} :$ $\qquad I_r I_s = \displaystyle\sum_{k=\max\{r,s\}+1}^{n} \Upsilon_{r,k}^s I_k \, ;$

3. $\forall \ s \in [m + 1, n] \cap \mathbb{N} \ \ \exists! \ u_s \in [1, m] \cap \mathbb{N} \ \ \forall \ r \in [1, m] \cap \mathbb{N} :$

$$I_r I_s = \begin{cases} 0 \ \ \text{if} \ \ r \neq u_s \, , \\ I_s \ \ \text{if} \ \ r = u_s \, . \end{cases} \tag{1}$$

Furthermore, the structure constants $\Upsilon_{r,k}^s \in \mathbb{C}$ satisfy the associativity conditions:

(A 1). $(I_r I_s) I_p = I_r (I_s I_p) \quad \forall \ r, s, p \in [m+1, n] \cap \mathbb{N}$;

(A 2). $(I_u I_s) I_p = I_u (I_s I_p) \quad \forall \ u \in [1, m] \cap \mathbb{N} \ \forall \ s, p \in [m+1, n] \cap \mathbb{N}$.

Obviously, the first m basis vectors $\{I_u\}_{u=1}^m$ are the idempotents and, respectively, form the semi-simple subalgebra. Also the vectors $\{I_r\}_{r=m+1}^n$ form the nilpotent subalgebra of algebra $\mathbb{A}_n^m$. The unit of $\mathbb{A}_n^m$ is the element $1 = \sum_{u=1}^m I_u$. Therefore, we will write that the algebra $\mathbb{A}_n^m$ is a semi-direct sum of the m-dimensional semi-simple subalgebra S and $(n-m)$-dimensional nilpotent subalgebra N, i. e.

$$\mathbb{A}_n^m = S \oplus_s N.$$

In the cases where $\mathbb{A}_n^m$ has some specific properties, the following propositions are true.

Proposition 1 [15]. *If there exists the unique $u_0 \in [1, m] \cap \mathbb{N}$ such that $I_{u_0} I_s = I_s$ for all $s = m + 1, \ldots, n$, then the associativity condition* (A 2) *is satisfied.*

Thus, under the conditions of Proposition 1, the associativity condition (A 1) is only required. It means that the nilpotent subalgebra of $\mathbb{A}_n^m$ with the basis $\{I_r\}_{r=m+1}^n$ can be an arbitrary commutative associative nilpotent algebra of dimension $n - m$. We note that such nilpotent algebras are fully described for the dimensions $1, 2, 3$ in the paper [12], and some four-dimensional nilpotent algebras can be found in the papers [13], [14].

Proposition 2 [15]. *If all u_r are different in the multiplication rule 3, then $I_s I_p = 0$ for all $s, p = m + 1, \ldots, n$.*

Thus, under the conditions of Proposition 2, the multiplication table of the nilpotent subalgebra of $\mathbb{A}_n^m$ with the basis $\{I_r\}_{r=m+1}^n$ consists only of zeros, and all associativity conditions are satisfied.

The algebra $\mathbb{A}_n^m$ contains m maximal ideals

$$\mathcal{I}_u := \left\{ \sum_{k=1,\, k \neq u}^n \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}, \quad u = 1, 2, \ldots, m,$$

the intersection of which is the radical

$$\mathcal{R} := \left\{ \sum_{k=m+1}^n \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}.$$

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We define m linear functionals $f_u : \mathbb{A}_n^m \to \mathbb{C}$ by put

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall\, \omega \in \mathcal{I}_u\,, \quad u = 1, 2, \ldots, m.$$

Since the kernels of functionals f_u are, respectively, the maximal ideals $\mathcal{I}_u$, then these functionals are also continuous and multiplicative (see [16, p. 147]).

3   Monogenic functions.

We consider the vectors $e_1 = 1, e_2, e_3$ in $\mathbb{A}_n^m$ which are linearly independent over the field of real number $\mathbb{R}$. It means that the equality

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Let the vectors $e_1 = 1, e_2, e_3$ have the following decompositions with respect to the basis $\{I_k\}_{k=1}^n$:

$$e_1 = 1, \quad e_2 = \sum_{k=1}^n a_k I_k\,, \quad e_3 = \sum_{k=1}^n b_k I_k\,, \tag{2}$$

where $a_k, b_k \in \mathbb{C}$.

Let $\zeta := x e_1 + y e_2 + z e_3$, where $x, y, z \in \mathbb{R}$. It is also obvious that $\xi_u := f_u(\zeta) = x + y a_u + z b_u, \; u = 1, 2, \ldots, m$. Let $E_3 := \{\zeta = x e_1 + y e_2 + z e_3 : \; x, y, z \in \mathbb{R}\}$ be the linear span of vectors e_1, e_2, e_3 over the field of real numbers $\mathbb{R}$. We note that in the further investigations, it is essential assumption: $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$, where $f_u(E_3)$ is the image of E_3 under the mapping f_u. Obviously, it holds if and only if for every fixed $u = 1, 2, \ldots, m$ at least one of the numbers a_u or b_u belongs to $\mathbb{C} \setminus \mathbb{R}$.

With a set $Q \subset \mathbb{R}^3$ we associate the set $Q_\zeta := \{\zeta = x e_1 + y e_2 + z e_3 : (x, y, z) \in Q\}$ in E_3. We also note that the topological properties of a set Q_ζ in E_3 understood as a corresponding topological properties of a set Q in $\mathbb{R}^3$. For example, a homotopicity of a curve $\gamma_\zeta \subset E_3$ to the zero means a homotopicity of $\gamma \subset \mathbb{R}^3$ to the zero, etc.

Let Ω be a domain in $\mathbb{R}^3$.

A continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ is *monogenic* in Ω_ζ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i. e. if

for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_n^m$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3. \tag{3}$$

$\Phi'(\zeta)$ is the *Gateaux derivative* of the function Φ in the point ζ.

Consider the decomposition of a function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ with respect to the basis $\{I_k\}_{k=1}^n$:

$$\Phi(\zeta) = \sum_{k=1}^n U_k(x, y, z)\, I_k. \tag{4}$$

In the case where the functions $U_k : \Omega \to \mathbb{C}$ are $\mathbb{R}$-differentiable in Ω, i. e. for every $(x, y, z) \in \Omega$,

$$U_k(x+\Delta x, y+\Delta y, z+\Delta z) - U_k(x, y, z) = \frac{\partial U_k}{\partial x}\, \Delta x + \frac{\partial U_k}{\partial y}\, \Delta y + \frac{\partial U_k}{\partial z}\, \Delta z +$$

$$+ o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\right), \qquad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \to 0,$$

the function Φ is monogenic in the domain Ω_ζ if and only if the following Cauchy – Riemann conditions are satisfied in Ω_ζ:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x}\, e_2, \quad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\, e_3. \tag{5}$$

Expansion of the resolvent is of the form

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^m \frac{1}{t - \xi_u}\, I_u + \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{Q_{k,s}}{(t - \xi_{u_s})^k}\, I_s \tag{6}$$

$$\forall t \in \mathbb{C} : t \neq \xi_u, \quad u = 1, 2, \ldots, m,$$

where $Q_{k,s}$ are determined by the following recurrence relations:

$$Q_{2,s} := T_s, \quad Q_{k,s} = \sum_{r=k+m-2}^{s-1} Q_{k-1,r}\, B_{r,s}, \quad k = 3, 4, \ldots, s - m + 1. \tag{7}$$

with

$$T_s := ya_s + zb_s, \quad B_{r,s} := \sum_{k=m+1}^{s-1} T_k \Upsilon_{r,s}^k, \quad s = m + 2, \ldots, n,$$

and natural numbers u_s are defined in the rule 3 of the multiplication table of the algebra $\mathbb{A}_n^m$.

From the relations (6) follows that the points $(x, y, z) \in \mathbb{R}^3$ corresponding to the noninvertible elements $\zeta \in \mathbb{A}_n^m$ form the straight lines

$$L_u : \quad \begin{cases} x + y \operatorname{Re} a_u + z \operatorname{Re} b_u = 0, \\ \quad\quad y \operatorname{Im} a_u + z \operatorname{Im} b_u = 0 \end{cases}$$

in the three-dimensional space $\mathbb{R}^3$.

Denote by $D_u \subset \mathbb{C}$ the image of Ω_ζ under the mapping f_u, $u = 1, 2, \ldots, m$. A constructive description of all monogenic functions in the algebra $\mathbb{A}_n^m$ by means of holomorphic functions of the complex variable are obtained in the paper [15]. Namely, it is proved the theorem:

Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Then any monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ can be expressed in the form

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1}\, dt + \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int\limits_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1}\, dt,$$

(8)

where F_u is the certain holomorphic function in a domain D_u; G_s is the certain holomorphic function in a domain D_{u_s}; Γ_q is a closed Jordan rectifiable curve lying in the domain D_q surround a point ξ_q and containing no points ξ_ℓ, $\ell, q = 1, 2, \ldots, m, \ell \neq q$.

4 Cauchy integral theorem for a curvilinear integral.

Let γ be a Jordan rectifiable curve in $\mathbb{R}^3$. For a continuous function $\Psi : \gamma_\zeta \to \mathbb{A}_n^m$ of the form

$$\Psi(\zeta) = \sum_{k=1}^{n} U_k(x, y, z)\, I_k + i \sum_{k=1}^{n} V_k(x, y, z)\, I_k, \tag{9}$$

where $(x, y, z) \in \gamma$ and $U_k : \gamma \to \mathbb{R}$, $V_k : \gamma \to \mathbb{R}$, we define an integral along a Jordan rectifiable curve γ_ζ by the equality:

$$\int\limits_{\gamma_\zeta} \Psi(\zeta)d\zeta := \sum_{k=1}^n I_k \int\limits_\gamma U_k(x, y, z)dx + \sum_{k=1}^n e_2 I_k \int\limits_\gamma U_k(x, y, z)dy +$$

$$+ \sum_{k=1}^n e_3 I_k \int\limits_\gamma U_k(x, y, z)dz + i \sum_{k=1}^n I_k \int\limits_\gamma V_k(x, y, z)dx +$$

$$+ i \sum_{k=1}^n e_2 I_k \int\limits_\gamma V_k(x, y, z)dy + i \sum_{k=1}^n e_3 I_k \int\limits_\gamma V_k(x, y, z)dz,$$

where $d\zeta := dx + e_2 dy + e_3 dz$.

Also we define a surface integral. Let Σ be a piece-smooth surface in $\mathbb{R}^3$. For a continuous function $\Psi : \Sigma_\zeta \to \mathbb{A}_n^m$ of the form (9), where $(x, y, z) \in \Sigma$ and $U_k : \Sigma \to \mathbb{R}$, $V_k : \Sigma \to \mathbb{R}$, we define a surface integral on Σ_ζ with the differential form $dxdy$, by the equality

$$\int\limits_{\Sigma_\zeta} \Psi(\zeta)dxdy := \sum_{k=1}^n I_k \int\limits_\Sigma U_k(x, y, z)dxdy + i \sum_{k=1}^n I_k \int\limits_\Sigma V_k(x, y, z)dxdy.$$

A similarly defined the integrals with the forms $dydz$ and $dzdx$.

If a function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ is continuous together with partial derivatives of the first order in a domain Ω_ζ, and Σ is a piece-smooth surface in Ω, and the edge γ of surface Σ is a rectifiable Jordan curve, then the following analogue of the Stokes formula is true:

$$\int\limits_{\gamma_\zeta} \Psi(\zeta)d\zeta = \int\limits_{\Sigma_\zeta} \left(\frac{\partial\Psi}{\partial x}e_2 - \frac{\partial\Psi}{\partial y} \right) dxdy + \left(\frac{\partial\Psi}{\partial y}e_3 - \frac{\partial\Psi}{\partial z}e_2 \right) dydz +$$

$$+ \left(\frac{\partial\Psi}{\partial z} - \frac{\partial\Psi}{\partial x}e_3 \right) dzdx. \tag{10}$$

Now, the next theorem is a result of the formula (10) and the equalities (5).

Theorem 1. *Suppose that $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ is a monogenic function in a domain Ω_ζ, and Σ is a piece-smooth surface in Ω, and*

the edge γ of surface Σ is a rectifiable Jordan curve. Then

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)d\zeta = 0. \tag{11}$$

In the case where a domain Ω is convex, then by the usual way (see, e. g., [17]) the equality (11) can be prove for an arbitrary closed Jordan rectifiable curve γ_ζ.

In the case where a domain Ω is an arbitrary, then similarly to the proof of Theorem 3.2 [2] we can prove the following

Theorem 2. *Let $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ be a monogenic function in a domain Ω_ζ. Then for every closed Jordan rectifiable curve γ homotopic to a point in Ω, the equality (11) is true.*

5   The Morera theorem.

To prove the analogue of Morera theorem in the algebra $\mathbb{A}_n^m$, we introduce auxiliary notions and prove some auxiliary statements.

Let us consider the algebra $\mathbb{A}_n^m(\mathbb{R})$ with the basis $\{I_k, iI_k\}_{k=1}^n$ over the field $\mathbb{R}$ which is isomorphic to the algebra $\mathbb{A}_n^m$ over the field $\mathbb{C}$. In the algebra $\mathbb{A}_n^m(\mathbb{R})$ there exist another basis $\{e_k\}_{k=1}^{2n}$, where the vectors e_1, e_2, e_3 are the same as in the Section 3.

For the element $a := \sum\limits_{k=1}^{2n} a_k e_k, \ a_k \in \mathbb{R}$ we define the Euclidian norm

$$\|a\| := \sqrt{\sum_{k=1}^{2n} a_k^2}\,.$$

Accordingly, $\|\zeta\| = \sqrt{x^2 + y^2 + z^2}$ and $\|e_1\| = \|e_2\| = \|e_3\| = 1$.

Using the Theorem on equivalents of norms, for the element $b := \sum\limits_{k=1}^n (b_{1k} + ib_{2k})I_k, \ b_{1k}, b_{2k} \in \mathbb{R}$ we have the following inequalities

$$|b_{1k} + ib_{2k}| \le \sqrt{\sum_{k=1}^{2n} \left(b_{1k}^2 + b_{2k}^2\right)} \le c\|b\|, \tag{12}$$

where c is a positive constant does not depend on b.

Lemma 1. *If γ is a closed Jordan rectifiable curve in $\mathbb{R}^3$ and function $\Psi : \gamma_\zeta \to \mathbb{A}_n^m$ is continuous, then*

$$\left\| \int\limits_{\gamma_\zeta} \Psi(\zeta)\, d\zeta \right\| \leq c \int\limits_{\gamma_\zeta} \|\Psi(\zeta)\| \|d\zeta\|, \qquad (13)$$

where c is a positive absolutely constant.

Proof. Using the representation of function Ψ in the form (9) for $(x, y, z) \in \gamma$, we obtain

$$\left\| \int\limits_{\gamma_\zeta} \Psi(\zeta) d\zeta \right\| \leq \sum_{k=1}^{n} \|I_k\| \int\limits_{\gamma} \left| U_k(x, y, z) + i V_k(x, y, z) \right| dx +$$

$$+ \sum_{k=1}^{n} \|e_2 I_k\| \int\limits_{\gamma} \left| U_k(x, y, z) + i V_k(x, y, z) \right| dy +$$

$$+ \sum_{k=1}^{n} \|e_3 I_k\| \int\limits_{\gamma} \left| U_k(x, y, z) + i V_k(x, y, z) \right| dz.$$

Now, taking into account the inequality (12) for $b = \Psi(\zeta)$ and the inequalities $\|e_s I_k\| \leq c_s$, $s = 1, 2, 3$, where c_s are positive absolutely constants, we obtain the relation (13). The lemma is proved.

Using Lemma 1, for functions taking values in the algebra $\mathbb{A}_n^m$, the following Morera theorem can be established in the usual way.

Theorem 3. *If a function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ is continuous in a domain Ω_ζ and satisfies the equality*

$$\int\limits_{\partial \triangle_\zeta} \Phi(\zeta) d\zeta = 0 \qquad (14)$$

for every triangle $\triangle_\zeta$ such that closure $\overline{\triangle_\zeta} \subset \Omega_\zeta$, then the function Φ is monogenic in the domain Ω_ζ.

6 Cauchy integral formula for a curvilinear integral.

Let $\zeta_0 := x_0 e_1 + y_0 e_2 + z_0 e_3$ be a point in a domain $\Omega_\zeta \subset E_3$. In a neighborhood of ζ_0 contained in Ω_ζ let us take a circle $C_\zeta(\zeta_0, \varepsilon)$ of radius ε with the center at the point ζ_0. By $C_u(\xi_u^{(0)}, \varepsilon) \subset \mathbb{C}$ we denote the image of $C_\zeta(\zeta_0, \varepsilon)$ under the mapping f_u, $u = 1, 2, \ldots, m$. We assume that the circle $C_\zeta(\zeta_0, \varepsilon)$ embraces the set $\{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^{m} L_u\}$. It means that the curve $C_u(\xi_u^{(0)}, \varepsilon)$ bounds some domain D'_u and $f_u(\zeta_0) = \xi_u^{(0)} \in D'_u$, $u = 1, 2, \ldots, m$.

We say that the curve $\gamma_\zeta \subset \Omega_\zeta$ embraces once the set $\{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^{m} L_u\}$, if there exists a circle $C_\zeta(\zeta_0, \varepsilon)$ which embraces the mentioned set and is homotopic to γ_ζ in the domain $\Omega_\zeta \setminus \{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^{m} L_u\}$.

Since the function ζ^{-1} is continuous on the curve $C_\zeta(0, \varepsilon)$, then there exist the integral

$$\lambda := \int_{C_\zeta(0,\varepsilon)} \zeta^{-1} d\zeta. \tag{15}$$

The following theorem is an analogue of Cauchy integral theorem for monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$.

Theorem 4. *Suppose that a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Suppose also that $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ is a monogenic function in Ω_ζ. Then for every point $\zeta_0 \in \Omega_\zeta$ the following equality is true:*

$$\lambda \, \Phi(\zeta_0) = \int_{\gamma_\zeta} \Phi(\zeta) \, (\zeta - \zeta_0)^{-1} \, d\zeta, \tag{16}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ, that embraces once the set $\{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^{m} L_u\}$.

Proof. Inasmuch as γ_ζ is homotopic to $C_\zeta(\zeta_0, \varepsilon)$ in the domain

$\Omega_\zeta \setminus \{\zeta - \zeta_0 : (x, y, z) \in \bigcup\limits_{u=1}^{m} L_u\}$, it follows from Theorem 2 that

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta = \int\limits_{C_\zeta(\zeta_0,\varepsilon)} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta. \qquad (17)$$

Taking into account the equality (17) we represent the integral on the right-hand side of equality (16) as the sum of the following two integrals:

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta = \int\limits_{C_\zeta(\zeta_0,\varepsilon)} (\Phi(\zeta) - \Phi(\zeta_0))\,(\zeta - \zeta_0)^{-1}\,d\zeta +$$

$$+\Phi(\zeta_0) \int\limits_{C_\zeta(\zeta_0,\varepsilon)} (\zeta - \zeta_0)^{-1}\,d\zeta =: J_1 + J_2.$$

Let us note that from the relation (17) follows that if there exist the integral in the equality (15) then it does not depend on ε. As a consequence of the equalities (15), (17), we have the following relation

$$J_2 = \Phi(\zeta_0) \int\limits_{C_\zeta(0,\varepsilon)} \tau^{-1} d\tau = \lambda\,\Phi(\zeta_0), \qquad (18)$$

where $\tau := \zeta - \zeta_0$.

The integrand in the integral J_1 is bounded by a constant which does not depend on ε: when $\varepsilon \to 0$ the integrand tends to $\Phi'(\zeta_0)$ (see Lemma 5 [15]). Therefore, using the Lemma 1 the integral J_1 tends to zero as $\varepsilon \to 0$. The theorem is proved.

Below, it will be shown that the constant λ is an invertible element in $\mathbb{A}_n^m$.

7   A constant λ.

In some special algebras (see [3, 4, 5]) the Cauchy integral formula (16) has the form

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \int\limits_{\gamma_\zeta} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta, \qquad (19)$$

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i. e.
$$\lambda = 2\pi i. \tag{20}$$

In this Section we indicate a set of algebras $\mathbb{A}_n^m$ for which (20) holds. In this a way we first consider some auxiliary statements.

As a consequence of the expansion (6), we obtain the following equality:
$$\zeta^{-1} = \sum_{k=1}^{n} \widetilde{A}_k \, I_k \tag{21}$$

with the coefficients $\widetilde{A}_k$ determined by the following relations:
$$\widetilde{A}_u = \frac{1}{\xi_u}, \quad u = 1, 2, \ldots, m,$$
$$\widetilde{A}_s = \sum_{k=2}^{s-m+1} \frac{\widetilde{Q}_{k,s}}{\xi_{u_s}^k}, \quad s = m+1, m+2, \ldots, n, \tag{22}$$

where $\widetilde{Q}_{k,s}$ are determined by the following recurrence relations:
$$\widetilde{Q}_{2,s} := -T_s, \quad \widetilde{Q}_{k,s} = -\sum_{r=k+m-2}^{s-1} \widetilde{Q}_{k-1,r} \, B_{r,s}, \quad k = 3, 4, \ldots, s-m+1. \tag{23}$$

where T_s and $B_{r,s}$ are the same as in the equalities (7), and natural numbers u_s defined in the rule 3 of the multiplication table of the algebra $\mathbb{A}_n^m$.

Taking into account the equality (21) and the relation
$$d\zeta = dx e_1 + dy e_2 + dz e_3 = \sum_{u=1}^{m} \left(dx + dy \, a_u + dz \, b_u \right) I_u +$$
$$+ \sum_{r=m+1}^{n} \left(dy \, a_r + dz \, b_r \right) I_r = \sum_{u=1}^{m} d\xi_u \, I_u + \sum_{r=m+1}^{n} dT_r \, I_r \, ,$$

we have the following equality
$$\zeta^{-1} d\zeta = \sum_{u=1}^{m} \widetilde{A}_u \, d\xi_u \, I_u + \sum_{r=m+1}^{n} \widetilde{A}_{u_r} \, dT_r \, I_r +$$
$$+ \sum_{s=m+1}^{n} \widetilde{A}_s \, d\xi_{u_s} \, I_s + \sum_{s=m+1}^{n} \sum_{r=m+1}^{n} \widetilde{A}_s \, dT_r \, I_s I_r =: \sum_{k=1}^{n} \sigma_k \, I_k \, . \tag{24}$$

Now, taking into account the denotation (24) and the equality (22), we calculate:

$$\int\limits_{C_\zeta(0,R)} \sum_{u=1}^{m} \sigma_u \, I_u = \sum_{u=1}^{m} I_u \int\limits_{C_u(\xi_u,R)} \frac{d\xi_u}{\xi_u} = 2\pi i \sum_{u=1}^{m} I_u = 2\pi i.$$

Therefore,

$$\lambda = 2\pi i + \sum_{k=m+1}^{n} I_k \int\limits_{C_\zeta(0,R)} \sigma_k \,. \qquad (25)$$

We note that form the relations (25), (21), and (22) that λ is an invertible element.

Thus, the equality (20) holds if and only if

$$\int\limits_{C_\zeta(0,R)} \sigma_k = 0 \qquad \forall\, k = m+1, \ldots, n. \qquad (26)$$

But, for satisfying the equality (26) the differential form σ_k must be a total differential of some function. We note that the property of being a total differential is invariant under admissible transformations of coordinates [18, Theorem 2, p. 328]. In our situation, if we show that σ_k is a total differential of some function depend of the variables $\frac{T_{m+1}}{\xi}, \ldots, \frac{T_k}{\xi}$, then it means that σ_k is a total differential of some function depending on x, y, z.

7.1

In this subsection we indicate a set of algebras in which the vectors (2) chosen arbitrarily and the equality (20) holds. We remind that an arbitrary commutative associative algebra, $\mathbb{A}_n^m$, with unit over the field of complex number $\mathbb{C}$ can be represented as $\mathbb{A}_n^m = S \oplus_s N$, where S is m-dimensional semi-simple subalgebra and N is $(n-m)$-dimensional nilpotent subalgebra (see Section 2).

Theorem 5. *If $\mathbb{A}_n^m \equiv S$, then the equality (20) holds.*

The proof immediately follows from the conditions $\sigma_k \equiv 0$ for $k = m+1, \ldots, n$ and (25). This theorem is obtained in the paper [5].

Theorem 6. *If $\mathbb{A}_n^m = S \oplus_s N$ and N is a zero nilpotent subalgebra, then the equality* (20) *holds.*

Proof. From the condition of theorem follows that in the relations (22) all $B_{k,p} = 0$. Therefore, (22) takes the form

$$\widetilde{A}_k = -\frac{T_k}{\xi_{u_k}^2}, \qquad k = m+1,\ldots,n. \tag{27}$$

Since $I_s I_r = 0$ for $r, s = m+1,\ldots,n$, then form the denotation (24) and the identity (27), we obtain

$$\sigma_k = \frac{dT_k}{\xi_{u_k}} + \widetilde{A}_k \, d\xi_{u_k} = \frac{dT_k}{\xi_{u_k}} - \frac{T_k}{\xi_{u_k}^2} \, d\xi_{u_k} = d\left(\frac{T_k}{\xi_{u_k}}\right) =: d\tau_k, \quad k = m+1,\ldots,n.$$

Under the transformation $(x, y, z) \to \tau_k$ the circle $C_\zeta(0, R)$ maps into a closed smooth curve $\widetilde{C}$ (Jordan or not) and the singularity $\xi_{u_k} = 0$ maps on $\tau_k = \infty$. Consequently, in an interior of the curve $\widetilde{C}$ does not exist singular points. By the Cauchy theorem in complex plane [18, p. 90], we have:

$$\int\limits_{C_\zeta(0,R)} \sigma_k = \int\limits_{\widetilde{C}} d\tau_k = 0.$$

So, the equality (20) is a consequence of the last relation and (25). The theorem is proved.

The Theorem 6 implies the formula (19) for monogenic functions in the three-dimensional algebra $\mathbb{A}_2$ which investigated in the paper [19].

Further we consider the case where N in non-zero nilpotent subalgebra. For this goal we establish an explicitly form of $\sigma_{m+1}, \sigma_{m+2}, \sigma_{m+3}$ and σ_{m+4}.

From the relation (24) follows the equalities

$$\sigma_{m+1} = \frac{dT_{m+1}}{\xi_{u_{m+1}}} + \widetilde{A}_{m+1} \, d\xi_{u_{m+1}},$$

$$\sigma_k = \frac{dT_k}{\xi_{u_k}} + \widetilde{A}_k \, d\xi_{u_k} + \sum_{r,s=m+1}^{k-1} \widetilde{A}_r \, dT_s \Upsilon_{r,k}^s, \quad k = m+2,\ldots,n. \tag{28}$$

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Now, the equalities (22) and (23) implies the following equalities:

$$\widetilde{A}_{m+1} = -\frac{T_{m+1}}{\xi_{u_{m+1}}^2}, \quad \widetilde{A}_{m+2} = -\frac{T_{m+2}}{\xi_{u_{m+2}}^2} + \frac{T_{m+1}^2}{\xi_{u_{m+2}}^3}\Upsilon_{m+1,m+2}^{m+1},$$

$$\widetilde{A}_{m+3} = -\frac{T_{m+3}}{\xi_{u_{m+3}}^2} + \frac{T_{m+1}^2}{\xi_{u_{m+3}}^3}\Upsilon_{m+1,m+3}^{m+1} + 2\frac{T_{m+1}T_{m+2}}{\xi_{u_{m+3}}^3}\Upsilon_{m+2,m+3}^{m+1}-$$

$$-\frac{T_{m+1}^3}{\xi_{u_{m+3}}^4}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1} + \frac{T_{m+2}^2}{\xi_{u_{m+3}}^3}\Upsilon_{m+2,m+3}^{m+2} - \frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+3}}^4}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1},$$

$$\widetilde{A}_{m+4} = -\frac{T_{m+4}}{\xi_{u_{m+4}}^2} + \frac{T_{m+1}^2}{\xi_{u_{m+4}}^3}\Upsilon_{m+1,m+4}^{m+1} + 2\frac{T_{m+1}T_{m+3}}{\xi_{u_{m+4}}^3}\Upsilon_{m+3,m+4}^{m+1}+$$

$$+2\frac{T_{m+1}T_{m+2}}{\xi_{u_{m+4}}^3}\Upsilon_{m+2,m+4}^{m+1} + 2\frac{T_{m+2}T_{m+3}}{\xi_{u_{m+4}}^3}\Upsilon_{m+3,m+4}^{m+2} + \frac{T_{m+2}^2}{\xi_{u_{m+4}}^3}\Upsilon_{m+2,m+4}^{m+2}-$$

$$-\frac{T_{m+1}^3}{\xi_{u_{m+4}}^4}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+4}^{m+1} - \frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+4}^{m+2}-$$

$$-\frac{T_{m+1}^2 T_{m+3}}{\xi_{u_{m+4}}^4}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+2} + \frac{T_{m+3}^2}{\xi_{u_{m+4}}^3}\Upsilon_{m+3,m+4}^{m+3} - \frac{T_{m+1}^3}{\xi_{u_{m+4}}^4}\Upsilon_{m+1,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+1}-$$

$$-\frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4}\Upsilon_{m+1,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+2} - \frac{T_{m+1}^2 T_{m+3}}{\xi_{u_{m+4}}^4}\Upsilon_{m+1,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+3}-$$

$$-2\frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+1} - 2\frac{T_{m+1}T_{m+2}^2}{\xi_{u_{m+4}}^4}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+2}-$$

$$-2\frac{T_{m+1}T_{m+2}T_{m+3}}{\xi_{u_{m+4}}^4}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+3} + \frac{T_{m+1}^4}{\xi_{u_{m+4}}^5}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+1}+$$

$$+\frac{T_{m+1}^3 T_{m+2}}{\xi_{u_{m+4}}^5}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+2} + \frac{T_{m+1}^3 T_{m+3}}{\xi_{u_{m+4}}^5}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+3}-$$

$$-\frac{T_{m+1}T_{m+2}^2}{\xi_{u_{m+4}}^4}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+3,m+4}^{m+1} - \frac{T_{m+2}^3}{\xi_{u_{m+4}}^4}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+3,m+4}^{m+2}-$$

$$-\frac{T_{m+2}^2 T_{m+3}}{\xi_{u_{m+4}}^4}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+3,m+4}^{m+3} + \frac{T_{m+1}^3 T_{m+2}}{\xi_{u_{m+4}}^5}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+1}+$$

$$+\frac{T_{m+1}^2 T_{m+2}^2}{\xi_{u_{m+4}}^5}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+2}+\frac{T_{m+1}^3 T_{m+2}T_{m+3}}{\xi_{u_{m+4}}^5}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+3}\,.$$

Finally, a consequence of the previous equalities and the relations (28) is the following differential representation of $\sigma_{m+1},\sigma_{m+2},\sigma_{m+3}$ and σ_{m+4}:

$$\sigma_{m+1}=d\left(\frac{T_{m+1}}{\xi_{u_{m+1}}}\right),\quad \sigma_{m+2}=d\left(\frac{T_{m+2}}{\xi_{u_{m+2}}}-\frac{1}{2}\Upsilon_{m+1,m+2}^{m+1}\frac{T_{m+1}^2}{\xi_{u_{m+2}}^2}\right),$$

$$\tag{29}$$

$$\sigma_{m+3}=d\left(\frac{T_{m+3}}{\xi_{u_{m+3}}}-\frac{1}{2}\Upsilon_{m+1,m+3}^{m+1}\frac{T_{m+1}^2}{\xi_{u_{m+3}}^2}-\Upsilon_{m+2,m+3}^{m+1}\frac{T_{m+1}T_{m+2}}{\xi_{u_{m+3}}^2}-\right.$$

$$\left.-\frac{1}{2}\Upsilon_{m+2,m+3}^{m+2}\frac{T_{m+2}^2}{\xi_{u_{m+3}}^2}+\frac{1}{3}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\frac{T_{m+1}^3}{\xi_{u_{m+3}}^3}\right)+$$

$$+\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+2}\sigma_{m+3}^{(1)}\,,\tag{30}$$

$$\sigma_{m+4}=d\left(\frac{T_{m+4}}{\xi_{u_{m+4}}}-\frac{1}{2}\Upsilon_{m+1,m+4}^{m+1}\frac{T_{m+1}^2}{\xi_{u_{m+4}}^2}-\Upsilon_{m+3,m+4}^{m+1}\frac{T_{m+1}T_{m+3}}{\xi_{u_{m+4}}^2}-\right.$$

$$-\Upsilon_{m+2,m+4}^{m+1}\frac{T_{m+1}T_{m+2}}{\xi_{u_{m+4}}^2}-\frac{1}{2}\Upsilon_{m+2,m+4}^{m+2}\frac{T_{m+2}^2}{\xi_{u_{m+4}}^2}-\Upsilon_{m+3,m+4}^{m+2}\frac{T_{m+2}T_{m+3}}{\xi_{u_{m+3}}^2}+$$

$$+\frac{1}{3}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+4}^{m+1}\frac{T_{m+1}^3}{\xi_{u_{m+4}}^3}-\frac{1}{2}\Upsilon_{m+3,m+4}^{m+3}\frac{T_{m+3}^2}{\xi_{u_{m+4}}^2}+\frac{1}{3}\Upsilon_{m+1,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+1}\frac{T_{m+1}^3}{\xi_{u_{m+4}}^3}-$$

$$\left.-\frac{1}{4}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+1}\frac{T_{m+1}^4}{\xi_{u_{m+4}}^4}+\frac{1}{3}\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+3,m+4}^{m+2}\frac{T_{m+2}^3}{\xi_{u_{m+4}}^3}\right)+$$

$$+\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+4}^{m+2}\sigma_{m+4}^{(1,2)}+\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+2}\sigma_{m+4}^{(2,2)}+\Upsilon_{m+1,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+2}\sigma_{m+4}^{(3,2)}+$$

$$+\Upsilon_{m+3,m+4}^{m+3}\Upsilon_{m+1,m+3}^{m+1}\sigma_{m+4}^{(4,3)}+\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+1}\sigma_{m+4}^{(5,1)}+\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+2}\sigma_{m+4}^{(6,2)}+$$

$$+\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+3}\sigma_{m+4}^{(7,3)}-\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+2}\sigma_{m+4}^{(8,2)}-$$

$$-\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+2,m+3}^{m+1}\Upsilon_{m+3,m+4}^{m+3}\sigma_{m+4}^{(9,3)}+\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+3,m+4}^{m+1}\sigma_{m+4}^{(10,1)}+$$

$$+\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+3,m+4}^{m+3}\,\sigma_{m+4}^{(11,3)}-\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+1}\,\sigma_{m+4}^{(12,1)}-$$

$$-\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+2}\,\sigma_{m+4}^{(13,2)}-\Upsilon_{m+2,m+3}^{m+2}\Upsilon_{m+1,m+2}^{m+1}\Upsilon_{m+3,m+4}^{m+3}\,\sigma_{m+4}^{(14,3)}\,,\tag{31}$$

where

$$\sigma_{m+3}^{(1)}:=\frac{T_{m+1}^2}{\xi_{u_{m+3}}^3}\left(dT_{m+2}-\frac{T_{m+2}}{\xi_{u_{m+3}}}\,d\xi_{u_{m+3}}\right),\tag{32}$$

and $\sigma_{m+4}^{(\ell,r)}$, $\ell=1,2,\ldots,14$ are determined by the following relations:

$$\sigma_{m+4}^{(\ell,r)}:=\begin{cases}\dfrac{T_{m+1}^2}{\xi_{u_{m+4}}^3}\,g(r) & \text{for} & \ell=1,2,3,4,\\[2ex]\dfrac{2\,T_{m+1}T_{m+2}}{\xi_{u_{m+4}}^3}\,g(r) & \text{for} & \ell=5,6,7,\\[2ex]\dfrac{T_{m+1}^3}{\xi_{u_{m+4}}^4}\,g(r) & \text{for} & \ell=8,9,\\[2ex]\dfrac{T_{m+2}^2}{\xi_{u_{m+4}}^3}\,g(r) & \text{for} & \ell=10,11,\\[2ex]\dfrac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4}\,g(r) & \text{for} & \ell=12,13,14,\end{cases}\tag{33}$$

where $g(r):=dT_{m+r}-\dfrac{T_{m+r}}{\xi_{u_{m+4}}}\,d\xi_{u_{m+4}}$.

Theorem 7. *If $\mathbb{A}_n^m=S\oplus_s N$ and $\dim_{\mathbb{C}}N\leq 3$, then the equality* (20) *holds.*

Proof. From the equality (29) for σ_{m+1}, we have

$$\sigma_{m+1}=d\left(\frac{T_{m+1}}{\xi_{u_{m+1}}}\right)=:d\tau_{m+1}\,.$$

Now, the identity $\int_{C_\zeta(0,R)}\sigma_{m+1}=0$ proved as in Theorem 6.

Consider σ_{m+2} from the equality (29), which is a total differential of the certain function depending on the variables $\frac{T_{m+1}}{\xi_{u_{m+2}}},\frac{T_{m+2}}{\xi_{u_{m+2}}}$. Under the transformation $(x,y,z)\to\left(\frac{T_{m+1}}{\xi_{u_{m+2}}},\frac{T_{m+2}}{\xi_{u_{m+2}}}\right)$ the circle $C_\zeta(0,R)$ maps into a closed smooth curve $\widetilde{\widetilde{C}}$ (Jordan or not) and the singularity $\xi_{u_{m+2}}=0$ maps on ∞. Consequently, in an interior of the curve $\widetilde{\widetilde{C}}$ does not exist singular points. Then by the Cauchy theorem in

the space $\mathbb{C}^2$ [18, p. 334], we have:

$$\int\limits_{C_\zeta(0,R)} \sigma_{m+2}(x,y,z) = \int\limits_{\widetilde{\widehat{C}}} \sigma_{m+2}\left(\frac{T_{m+1}}{\xi_{u_{m+2}}}, \frac{T_{m+2}}{\xi_{u_{m+2}}}\right) = 0.$$

Finally, we prove the equality (26) for $k = m + 3$. In the paper [12] is described all commutative associative nilpotent algebras over the field $\mathbb{C}$ of dimensional $1, 2, 3$. From results of the paper [12] (Table 1) immediately follows that for all mentioned algebras the relation $\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+2,m+3} = 0$ is always satisfied. Therefore, the equality (30) implies that under the conditions of theorem σ_{m+3} is always a total differential of the certain function depending on the variables $\frac{T_{m+1}}{\xi_{u_{m+3}}}, \frac{T_{m+2}}{\xi_{u_{m+3}}}, \frac{T_{m+3}}{\xi_{u_{m+3}}}$.

Now as before, under the transformation

$$(x, y, z) \to \left(\frac{T_{m+1}}{\xi_{u_{m+3}}}, \frac{T_{m+2}}{\xi_{u_{m+3}}}, \frac{T_{m+3}}{\xi_{u_{m+3}}}\right)$$

the circle $C_\zeta(0, R)$ maps into a closed smooth curve $\widehat{C}$ (Jordan or not) and the singularity $\xi_{u_{m+3}} = 0$ maps on ∞. Hence, in an interior of the curve $\widehat{C}$ does not exist singular points. Then by the Cauchy theorem in the space $\mathbb{C}^3$ [18, p. 334], we have:

$$\int\limits_{C_\zeta(0,R)} \sigma_{m+3}(x,y,z) = \int\limits_{\widehat{C}} \sigma_{m+3}\left(\frac{T_{m+1}}{\xi_{u_{m+3}}}, \frac{T_{m+2}}{\xi_{u_{m+3}}}, \frac{T_{m+3}}{\xi_{u_{m+3}}}\right) = 0.$$

So, the equality (20) is a consequence of the last relation and (25). The theorem is proved.

Let us note that from the Theorem 7 follows the formula (19) for monogenic functions in the three-dimensional algebra $\mathbb{A}_3$ (see [3]) and in the three-dimensional algebra $\mathbb{A}_2$ which considered in the paper [19].

Theorem 8. *Let $\mathbb{A}^m_n = S \oplus_s N$ and $\dim_{\mathbb{C}} N = 4$. Then the*

equality (20) *holds if the following relations satisfied*

$$\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+2,m+3} = \Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+2,m+4} = \Upsilon^{m+1}_{m+1,m+3}\Upsilon^{m+2}_{m+3,m+4} =$$

$$= \Upsilon^{m+3}_{m+3,m+4}\Upsilon^{m+1}_{m+1,m+3} = \Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+1}_{m+3,m+4} = \Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+2}_{m+3,m+4} =$$

$$= \Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+3}_{m+3,m+4} = \Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+2}_{m+3,m+4} =$$

$$= \Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+3}_{m+3,m+4} = \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+3,m+4} = \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+3}_{m+3,m+4} =$$

$$= \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+1}_{m+3,m+4} = \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+3,m+4} =$$

$$= \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+3}_{m+3,m+4} = 0. \tag{34}$$

Proof. From the equalities (30) and (31) it obvious that under conditions (34) expressions for σ_{m+3} and σ_{m+4} are total differentials. Further proof is similar to proof of the Theorem 7. The theorem is proved.

Further we consider some examples of algebras, which satisfy the relations (34).

Examples.

- Consider the algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3, \ I_2 I_4 = I_5$$

and other products are zeros (for nilpotent subalgebra see [14], Table 21, algebra $\mathcal{J}_{69}$ and [13], page 590, algebra $A_{1,4}$).

- Consider the algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3$$

and other products are zeros (for nilpotent subalgebra see [13], page 590, algebra $A_{1,2} \oplus A_{0,1}^2$).

- The algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3, \ I_4^2 = I_5$$

and other products are zeros (for nilpotent subalgebra see [13], page 590, algebra $A_{1,2} \oplus A_{1,2}$).

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- The algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3, \quad I_2 I_3 = I_4$$

and other products are zeros (for nilpotent subalgebra see [14], Table 21, algebra $\mathcal{J}_{71}$).

Now we consider an example of algebra, which does not satisfy the relations (34). Moreover, we choose the vectors e_1, e_2, e_3 of the form (2) such that the equality (20) is not true.

Example.

Consider the algebra $\mathbb{A}_5$ with the basis $\{1, \rho, \rho^2, \rho^3, \rho^4\}$, where $\rho^5 = 0$ (see [4] and [10], par. 11). Here $n = 5$, $m = 1$. It is obvious that $\Upsilon_{2,3}^2 \Upsilon_{3,4}^3 = 1$ and the relations (34) are not true. Consider the vectors:

$$e_1 = 1, \quad e_2 = i + \rho^2 + \rho^4, \quad e_3 = (1 - i)\rho + \left(\frac{1}{4} - \frac{3}{4}i\right)\rho^3,$$

which are linearly independent over $\mathbb{R}$ and satisfy the equality

$$e_1^2 + e_2^2 + e_3^2 = 0.$$

Let $\zeta = xe_1 + ye_2 + ze_3$. In the algebra $\mathbb{A}_5$ for given ζ, we have

$$\xi_{u_2} = \xi_{u_3} = \xi_{u_4} = \xi_{u_5} = x + iy =: \xi.$$

The inverse element ζ^{-1} is of the form (21), where

$$\widetilde{A}_0 = \frac{1}{\xi}, \quad \widetilde{A}_1 = \frac{z(i-1)}{\xi^2}, \quad \widetilde{A}_2 = -\frac{y}{\xi^2} + \frac{z^2(1-i)^2}{\xi^3},$$

$$\widetilde{A}_3 = \frac{1}{4}\frac{z(3i-1)}{\xi^2} + \frac{2yz(1-i)}{\xi^3} - \frac{z^3(1-i)^3}{\xi^4},$$

$$\widetilde{A}_4 = -\frac{y}{\xi^2} + \frac{y^2 + \frac{1}{2}z^2(1-i)(1-3i)}{\xi^3} - \frac{3yz^2(1-i)^2}{\xi^4} + \frac{z^4(1-i)^4}{\xi^5}.$$

Let us set

$$C_\zeta(0, R) := \{\zeta = xe_1 + ye_2 \in E_3 : x^2 + y^2 = R^2\}. \qquad (35)$$

On the circle of integration (35), we obtain:

$$\widetilde{A}_0 = \frac{1}{\xi}, \quad \widetilde{A}_1 = \widetilde{A}_3 = 0, \quad \widetilde{A}_2 = -\frac{y}{\xi^2}, \quad \widetilde{A}_4 = -\frac{y}{\xi^2} + \frac{y^2}{\xi^3}. \quad (36)$$

As a consequence of the equations (28), (36) on the circle (35) we obtain the following expression

$$\sigma_5 = \left(\frac{1}{\xi} - \frac{y}{\xi^2} \right) dy + \left(-\frac{y}{\xi^2} + \frac{y^2}{\xi^3} \right) d\xi.$$

It is easy to calculate that

$$\int_{C_\zeta(0,R)} \sigma_5 = \frac{\pi i}{2}$$

and

$$\int_{C_\zeta(0,R)} \sigma_1 = \int_{|\xi|=R} \frac{d\xi}{\xi} = 2\pi i, \qquad \int_{C_\zeta(0,R)} \sigma_k = 0, \qquad k = 2, 3, 4.$$

Hence, in this example

$$\lambda = \int_{C_\zeta(0,R)} \zeta^{-1} d\zeta = 2\pi i + \frac{\pi i}{2} \rho^4.$$

7.2

In this subsection we indicate sufficient conditions on a choose of the vectors (2) for which the equality (20) is true. Let the algebra $\mathbb{A}_n^m$ be represented as $\mathbb{A}_n^m = S \oplus_s N$. Let us note that the condition $\zeta \in E_3 \subset S$ means that in the decomposition (2) $a_k = b_k = 0$ for all $k = m + 1, \ldots, n$.

Theorem 9. *If $\mathbb{A}_n^m = S \oplus_s N$ and $\zeta \in E_3 \subset S$, then the equality* (20) *holds.*

Proof. Since $\zeta \in S$, then $T_k = 0$ for $k = m + 1, \ldots, n$ (see denotation (7)). From (23), (22) follows that $\widetilde{A}_k = 0$, and now from (28) follows that $\sigma_k = 0$ for $k = m + 1, \ldots, n$. The equality (20) is a consequence of the equality $\sigma_k = 0$ and the relation (25). The theorem is proved.

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Let us note that by essentially the Theorem 9 generalizes the Theorem 3 of the paper [20].

Now we consider a case where $\zeta \notin S$. If $\mathbb{A}_n^m = S \oplus_s N$ and $\dim_{\mathbb{C}} N \leq 3$, then by Theorem 7 the equality (20) holds for any $\zeta \in E_3$.

Theorem 10. *Let $\mathbb{A}_n^m = S \oplus_s N$ and $\dim_{\mathbb{C}} N = 4$. Then the equality (20) holds if the following two conditions satisfied:*

1. *$a_{m+1} = b_{m+1} = 0$;*

2. *at least one of the relations $a_{m+2} = b_{m+2} = 0$ or $a_{m+3} = b_{m+3} = 0$ are true.*

Proof. It follows from the condition of theorem that $T_{m+1} = 0$ and at least one of the equalities $T_{m+2} = 0$ or $T_{m+3} = 0$ are true. To prove (20) it is need to prove the equality (25) for $k = m+1, \ldots, m+4$. The equality (25) is proved in Theorem 7 for $k = m+1, m+2$. Under the condition $T_{m+1} = 0$ from (32), we have $\sigma_{m+3}^{(1)} = 0$. Since now σ_{m+3} is a total differential, then similar to proof of Theorem 7 can be proved the equality (25) for $k = m+3$.

Moreover, under the conditions of theorem from the denotation (33) follows the equalities $\sigma_{m+4}^{(\ell,r)} = 0$ for all $\ell = 1, \ldots, 14$. Therefore, σ_{m+4} is a total differential, then similar to proof of Theorem 7 can be proved the equality (25) for $k = m+4$.
The theorem is proved.

References

[1] E. R. Lorch, *The theory of analytic function in normed abelin vector rings*, Trans. Amer. Math. Soc., **54** (1943), 414 − 425.

[2] E. K. Blum, *A theory of analytic functions in banach algebras*, Trans. Amer. Math. Soc., **78** (1955), 343 − 370.

[3] V. S. Shpakivskyi and S. A. Plaksa, *Integral theorems and a Cauchy formula in a commutative three-dimensional harmonic algebra*, Bulletin Soc. Sci. Lettr. Lódź, **60** (2010), 47 − 54.

[4] S. A. Plaksa and V. S. Shpakivskyi, *Monogenic functions in a finite-dimensional algebra with unit and radical of maximal dimensionality*, J. Algerian Math. Soc., **1** (2014), 1 − 13.

[5] S. A. Plaksa and R. P. Pukhtaievych, *Constructive description of monogenic functions in n-dimensional semi-simple algebra*, An. Şt. Univ. Ovidius Constanţa, **22** (2014), no. 1, 221 – 235.

[6] V. Gončarov, *Sur l'intégrale de Cauchy dans le domaine hypercomplexe*, Bulletin de l'Académie des Sciences de l'URSS. Classe des sciences mathématiques et na, **10** (1932), 1405 – 1424.

[7] P. W. Ketchum, *Analytic functions of hypercomplex variables*, Trans. Amer. Math. Soc., **30** (1928), no. 4, 641 – 667.

[8] P. W. Ketchum, *A complete solution of Laplace's equation by an infinite hypervariable*, Amer. J. Math., **51** (1929), 179 – 188.

[9] M. N. Roşculeţ, *O teorie a funcţiilor de o variabilă hipercomplexă în spaţiul cu trei dimensiuni*, Studii şi Cercetări Matematice, **5**, nr. 3–4 (1954), 361 – 401.

[10] M. N. Roşculeţ, *Algebre liniare asociative şi comutative şi fincţii monogene ataşate lor*, Studii şi Cercetări Matematice, **6**, nr. 1–2 (1955), 135 – 173.

[11] E. Cartan, *Les groupes bilinéares et les systèmes de nombres complexes*, Annales de la faculté des sciences de Toulouse, **12** (1898), no. 1, 1–64.

[12] D. Burde and W. de Graaf, *Classification of Novicov algebras*, Applicable Algebra in Engineering, Communication and Computing, **24**(2013), no. 1, 1–15.

[13] D. Burde and A. Fialowski, *JacobiJordan algebras*, Linear Algebra Appl., **459** (2014), 586 – 594.

[14] M. E. Martin, *Four-dimensional Jordan algebras*, Int. J. Math. Game Theory Algebra **20** (4) (2013) 41 – 59.

[15] V. S. Shpakivskyi, *Constructive description of monogenic functions in a finite-dimensional commutative associative algebra*, submitted to J. Math. Anal. Appl.

[16] E. Hille and R. S. Phillips, Functional analysis and semi-groups [Russian translation], Inostr. Lit., Moscow (1962).

[17] I. I. Privalov, Introduction to the Theory of Functions of a Complex Variable, GITTL, Moscow, 1977. (Russian)

[18] B. V. Shabat, Introduction to complex analysis, Part 2, Nauka, Moskow (1976). (Russian)

[19] S. A. Plaksa and R. P. Pukhtaevich, *Constructive description of monogenic functions in a three-dimensional harmonic algebra with one-dimensional radical*, Ukr. Math. J., **65** (2013), no. 5, 740–751.

[20] V. S. Shpakivskyi and T. S. Kuzmenko, *Integral theorems for the quaternionic G-monogenic mappings*, submitted to An. Şt. Univ. Ovidius Constanţa.

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