Pole mass of the W boson at two-loop order in the pure $\overline{\text{MS}}$ scheme

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I provide a calculation at full two-loop order of the complex pole squared mass of the W boson in the Standard Model in the pure $\overline{\text{MS}}$ renormalization scheme, with Goldstone boson mass effects resummed. This approach is an alternative to earlier ones that use on-shell or hybrid renormalization schemes. The renormalization scale dependence of the real and imaginary parts of the resulting pole mass are studied. Both deviate by about ± 4 MeV from their median values as the renormalization scale is varied from 50 GeV to 200 GeV, but the theory error is likely larger. A surprising feature of this scheme is that the 2-loop QCD correction has a larger scale-dependence, but a smaller magnitude, than the 2-loop non-QCD correction, unless the renormalization scale is chosen very far from the top-quark mass.

Contents

I. INTRODUCTION

The discovery [1, 2] of the 125 GeV Higgs boson h at the Large Hadron Collider (LHC) has completed the minimal Standard Model of electroweak symmetry breaking. Since the LHC has also not discovered any superpartners or other new fundamental particles, it is now more motivated than ever to perform precision analyses of the masses and interactions of the known particles of the completed theory. This paper concerns the complex pole mass [3–9] of the W boson,

$$s_{\text{pole}}^{W} \equiv M_W^2 - i\Gamma_W M_W \tag{1.1}$$

calculated at 2-loop order.

There have already been many studies [11–37] that calculate contributions to the physical W boson mass, including all 2-loop order contributions and some QCD-enhanced effects at 3- and 4-loop order besides. (These are reviewed in refs. [35, 36], for example.) Indeed, the accuracy of the most advanced of these calculations exceeds that of the present paper when it comes to predicting the W-boson mass in terms of other measured quantities. However, the existing calculations have been done in on-shell or hybrid $\overline{\text{MS}}$/on-shell schemes, or use expansions in small squared mass ratios, as in the case of ref. [25, 26]. In this paper, I will provide a calculation that does not employ mass ratio expansions and uses a "pure" $\overline{\text{MS}}$ scheme, which means that the complete set of input parameters consists of only the renormalized running $\overline{\text{MS}}$ quantities

$$v,\ g,\ g',\ \lambda,\ y_t,\ g_3 \tag{1.2}$$

at a given renormalization scale Q. Here, $v(Q)$ is defined to be the minimum of the radiatively corrected effective potential in Landau gauge, which is now known to full 2-loop order [38] with 3-loop contributions at leading order in g_3 and y_t [39], with Goldstone boson mass contributions resummed [40, 41]. This allows v to be traded for the Higgs squared mass parameter $m^2(Q)$. The normalizations of v, m^2 and λ are such that the Higgs potential is

$$V = m^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \tag{1.3}$$

and $\langle \Phi \rangle = v/\sqrt{2}$, with a canonically normalized Higgs doublet field Φ.

In principle, the input parameters should also include the other quark and lepton Yukawa couplings, but these make only a very small difference in the present paper, as discussed below. In the pure $\overline{\text{MS}}$ scheme approach, all of the complex pole masses and other observables, such as the Fermi decays constant, are outputs, to be computed in terms of the quantities in eq. (1.2). In practice, global fits to data may be used to obtain the relationship. In this paper, the input parameters of eq. (1.2) are all understood to be in the full non-decoupled (6-quark) Standard Model theory. Note that if the renormalization scale Q is chosen between M_W and M_t, the largest logarithms encountered in calculations of the physical masses of W, Z, h, t will be at most $\ln(M_t^2/M_W^2) \approx 1.5$.

It has been argued that the experimental vector boson masses $M_{V,\text{exp}}$ as measured at colliders are related to the complex pole mass quantities by, approximately [42], [5, 6]:

$$M_{V,\text{exp}}^2 = M_V^2 + \Gamma_V^2. \tag{1.4}$$

Numerically, this amounts to $M_{W,\text{exp}} \approx M_W + 27$ MeV in the case of the W boson, assuming the Standard Model prediction for the width. Here, M_V^2 is the the real part of the complex

pole of the propagator, while $M_{V,\text{exp}}^2$ corresponds to what is sometimes called the "on-shell" mass. In the following, I will refer to M_W rather than $M_{W,\text{exp}}$. The current experimental value [43] is $M_{W,\text{exp}} = 80.385 \pm 0.015$ GeV.

At the present time, the pure $\overline{\text{MS}}$ scheme is not quite competitive in numerical accuracy with the on-shell or hybrid schemes for the W-boson mass calculation (although it is for the Higgs boson mass, which has been obtained to 2-loop order with the leading 3-loop corrections [44, 45]). However, as the technology for loop calculations improves, it is quite possible that this will change. As a matter of opinion, I find the modular approach of the pure $\overline{\text{MS}}$ scheme to be conceptually simpler, and it can be easily extended to include contributions from new particles beyond the Standard Model, and the methods used can even be applied to other vector bosons (such as a W') in different theories. In any case, there is hopefully some value in being able to compare different schemes for the Standard Model observables, given their importance.

II. W BOSON COMPLEX POLE MASS AT 2-LOOP ORDER

In this section, I describe the calculation of the W-boson complex pole mass. The calculation reported here is restricted to Landau gauge, because only in that gauge has the effective potential been evaluated to full 2-loop order with leading 3-loop corrections, and this is necessary to obtain the relationship between the Higgs vacuum expectation value (VEV) and the Lagrangian squared mass parameter, used implicitly in the calculation below. However, the complex pole mass [3–9] is a physical observable. It is therefore independent of the gauge fixing parameters [10], as well as renormalization group invariant.

In order to obtain the W-boson complex pole mass, one first obtains, in terms of bare parameters in the regulated theory in $d = 4 - 2\epsilon$ dimensions, the transverse self-energy function

$$\Pi(s) = \frac{1}{16\pi^2}\Pi^{(1)}(s) + \frac{1}{(16\pi^2)^2}\Pi^{(2)}(s). \tag{2.1}$$

This is obtained by constructing the W-boson self-energy function $\Pi_{\mu\nu}^{WW}(s)$ from the sum of all 1-particle-irreducible 2-point Feynman diagrams, and then contracting with $(\eta^{\mu\nu} - p^\mu p^\nu/p^2)/(d-1)$, where p^μ is the external momentum and $s = -p^2$, using a metric with Euclidean or $(-,+,+,+)$ signature. Factors of $1/(16\pi^2)^\ell$ are used to signify the loop order ℓ. Rather than including counterterm diagrams separately, it is more convenient and efficient to do the calculation in terms of the bare quantities: the VEV v_B and the bare Higgs squared mass parameter m_B^2, and the couplings g_B, g_B', and λ_B, y_{tB}, g_{3B}, and then rewrite the results in terms of the $\overline{\text{MS}}$ quantities.

The finite, renormalization-group invariant, and gauge-fixing invariant complex pole

squared mass can be written at 2-loop order:

$$s_{\text{pole}}^W = W_B + \frac{1}{16\pi^2}\Pi^{(1)}(W_B) + \frac{1}{(16\pi^2)^2}\left[\Pi^{(2)}(W_B) + \Pi^{(1)\prime}(W_B)\Pi^{(1)}(W_B)\right], \quad (2.2)$$

where $W_B = g_B^2 v_B^2/4$. The bare quantities are then eliminated in favor of the $\overline{\text{MS}}$ renormalized parameters using:

$$v_B^2 = \mu^{-2\epsilon}v^2\left[1 + \frac{1}{16\pi^2}\frac{c_{1,1}^\phi}{\epsilon} + \frac{1}{(16\pi^2)^2}\left(\frac{c_{2,2}^\phi}{\epsilon^2} + \frac{c_{2,1}^\phi}{\epsilon}\right) + \dots\right], \quad (2.3)$$

$$g_B = \mu^\epsilon\left[g + \frac{1}{16\pi^2}\frac{c_{1,1}^g}{\epsilon} + \frac{1}{(16\pi^2)^2}\left(\frac{c_{2,2}^g}{\epsilon^2} + \frac{c_{2,1}^g}{\epsilon}\right) + \dots\right], \quad (2.4)$$

$$g_B' = \mu^\epsilon\left[g' + \frac{1}{16\pi^2}\frac{c_{1,1}^{g'}}{\epsilon} + \dots\right], \quad (2.5)$$

$$\lambda_B = \mu^{2\epsilon}\left[\lambda + \frac{1}{16\pi^2}\frac{c_{1,1}^\lambda}{\epsilon} + \dots\right], \quad (2.6)$$

$$m_B^2 = m^2 + \frac{1}{16\pi^2}\frac{c_{1,1}^{m^2}}{\epsilon} + \dots, \quad (2.7)$$

$$y_{tB} = \mu^\epsilon\left[y_t + \frac{1}{16\pi^2}\frac{c_{1,1}^{y_t}}{\epsilon} + \dots\right], \quad (2.8)$$

$$g_{3B} = \mu^\epsilon\left[g_3 + \dots\right], \quad (2.9)$$

to obtain s_{pole}^W in terms of the renormalized parameters. Here μ is the dimensional regularization scale. The $\overline{\text{MS}}$ renormalization scale Q is related to it by

$$Q^2 = 4\pi e^{-\gamma_E}\mu^2, \quad (2.10)$$

where γ_E is the Euler-Mascheroni constant. The counterterm coefficients were listed, in exactly the same conventions as in this paper, in ref. [44], except for:

$$c_{2,1}^g = \frac{35}{24}g^5 + 3g^3g_3^2 + \frac{3}{8}g^3g'^2 - \frac{3}{8}g^3y_t^2, \quad (2.11)$$

$$c_{2,2}^g = \frac{361}{96}g^5. \quad (2.12)$$

All of these counterterm coefficients can be obtained from the 2-loop beta functions and scalar anomalous dimension found in refs. [47–50], [38]; see for example the discussion surrounding eqs. (4.5)-(4.14) of ref. [39].

The procedure for the rest of the calculation is quite similar to that in ref. [44], to which the reader is therefore referred for some more details, in a (perhaps futile) attempt to avoid triggering the arXiv's self-plagiarism detector. The Tarasov algorithm [51] is used to reduce

the 2-loop integrals to a basis set. The program `TARCER` [52] that is often used for this purpose was apparently unable to handle a few of the necessary reductions in a finite time, so I wrote a new Mathematica program `RedTint` implementing the Tarasov algorithm. (This program will be publicly released soon.) After expansion in $\epsilon = (4 - d)/2$, the Tarasov basis integrals were then written in terms of a set of basis integrals defined and described in detail in refs. [53, 54]. The 1-loop basis integrals are:

$$A(x), \ \ B(x, y) \tag{2.13}$$

and the 2-loop basis integral list is

$$I(x, y, z), \ \ S(x, y, z), \ \ T(x, y, z), \ \ \overline{T}(0, x, y), \ \ U(x, y, z, u), \ \ M(x, y, z, u, v). \tag{2.14}$$

The arguments $x, y, \ldots$ are squared masses, and $B, S, T, \overline{T}, U, M$ also each have an implicit dependence on the external momentum invariant $s = -p^2$, while $A, B, I, S, T, \overline{T}, U$ have an implicit dependence on the renormalization scale Q. The computer program `TSIL` [54] can then be used for the efficient numerical evaluation of these basis integrals. `TSIL` uses Runge-Kutta integration of differential equations similar to that suggested in ref. [55], and also includes relevant analytical results found in refs. [53, 56–62].

After writing bare quantities in terms of $\overline{\text{MS}}$ quantities and expanding in ϵ, the tree-level squared-mass arguments of the basis integrals used in the final result are:

$$W = g^2 v^2 / 4, \tag{2.15}$$
$$Z = (g^2 + g'^2) v^2 / 4, \tag{2.16}$$
$$t = y_t^2 v^2 / 2, \tag{2.17}$$
$$h = 2\lambda v^2 \tag{2.18}$$

and 0 for photons and gluons. As in [44], the Goldstone boson squared masses are eliminated by using the condition for the minimization of the effective potential after resummation,

$$m^2 + \lambda v^2 = \frac{1}{16\pi^2} \left\{ 2N_c y_t^2 A(t) - 3\lambda A(h) - \frac{g^2}{2} [3A(W) + 2W] \right.$$
$$\left. - \frac{g^2 + g'^2}{4} [3A(Z) + 2Z] \right\} + \ldots, \tag{2.19}$$

as explained in section 4 of ref. [40] (see also [41] and [63]). The same relation is used to eliminate m^2 from the tree-level Higgs boson squared mass, which appears as h rather than $H = m^2 + 3\lambda v^2$. In a future 3-loop calculation of the W (or Z) pole mass, the 2-loop version of eq. (2.19) should be used; this can be found in eqs. (4.18)-(4.20) of ref. [40].

The 2-loop W boson squared pole mass is thus obtained, after finally taking $\epsilon \to 0$, as:

$$s^W_{\text{pole}} = M_W^2 - i\Gamma_W M_W = W + \frac{1}{16\pi^2}\Delta_W^{(1)} + \frac{1}{(16\pi^2)^2}\left[\Delta_W^{(2),\text{QCD}} + \Delta_W^{(2),\text{non}-\text{QCD}}\right], \quad (2.20)$$

where the right-hand side is a function of $v, g, g', \lambda, y_t, g_3, Q$, with all propagator masses expressed as W, Z, h, t, or 0. The list of 1-loop basis integrals used is

$$
\begin{aligned}
I^{(1)} = \big\{ &A(h),\ A(t),\ A(W),\ A(Z),\ B(0,0),\ B(0,h),\ B(0,t), \\
&B(0,Z),\ B(h,t),\ B(h,W),\ B(t,Z),\ B(W,Z)\big\},
\end{aligned}
\quad (2.21)
$$

while the list of necessary 2-loop basis integrals is:

$$
\begin{aligned}
I^{(2)} = \big\{ &I(0,0,h),\ I(0,0,t),\ I(0,0,W),\ I(0,0,Z),\ I(0,h,W),\ I(0,h,Z),\ I(0,t,W), \\
&I(0,W,Z),\ I(h,h,h),\ I(h,t,t),\ I(h,W,W),\ I(h,Z,Z),\ I(t,t,Z),\ I(W,W,Z), \\
&S(h,h,W),\ S(h,W,Z),\ S(t,t,W),\ S(W,Z,Z),\ T(h,0,0),\ T(h,0,t), \\
&T(h,0,W),\ T(h,W,Z),\ T(t,0,0),\ T(t,0,h),\ T(t,0,Z),\ T(W,0,0), \\
&T(Z,0,0),\ T(Z,0,t),\ T(Z,0,W),\ T(Z,h,W),\ \overline{T}(0,h,W),\ \overline{T}(0,W,Z), \\
&U(0,t,0,W),\ U(0,t,h,t),\ U(0,t,t,Z),\ U(h,W,0,0),\ U(h,W,0,t), \\
&U(h,W,h,W),\ U(h,W,W,Z),\ U(W,0,t,t),\ U(W,h,h,h),\ U(W,h,t,t), \\
&U(W,h,W,W),\ U(W,h,Z,Z),\ U(W,Z,0,0),\ U(W,Z,h,Z),\ U(W,Z,t,t), \\
&U(W,Z,W,W),\ U(Z,W,0,0),\ U(Z,W,0,t),\ U(Z,W,h,W),\ U(Z,W,W,Z), \\
&M(0,0,0,0,0),\ M(0,0,0,0,Z),\ M(0,0,0,W,0),\ M(0,0,t,t,0),\ M(0,0,t,t,Z), \\
&M(0,0,t,W,0),\ M(0,t,W,0,t),\ M(0,W,0,Z,0),\ M(0,W,t,h,t), \\
&M(0,W,t,Z,t),\ M(0,W,W,0,W),\ M(0,W,W,h,W),\ M(0,W,W,Z,W), \\
&M(0,Z,t,W,0),\ M(h,h,W,W,h),\ M(h,W,W,h,W),\ M(h,W,W,Z,W), \\
&M(h,Z,W,W,Z),\ M(W,W,Z,Z,h),\ M(W,Z,Z,W,W)\big\}.
\end{aligned}
\quad (2.22)
$$

In each of the B, S, T, $\overline{T}$, U, and M integrals, the external momentum invariant is the tree-level squared mass, $s = W$.

The 1-loop contribution to the pole mass is:

$$
\begin{aligned}
\Delta_W^{(1)} = g^2 \bigg\{ &N_c|V_{tb}|^2 f(b,t,W) + \left[N_c(n_Q - |V_{tb}|^2) + n_L\right] f(0,0,W) + \left(\frac{1}{4} - \frac{h}{12W}\right)A(h) \\
&+ \left(\frac{4W}{Z} + \frac{h+Z}{12W} - 3\right)A(W) + \left(\frac{2W}{Z} - \frac{2}{3} - \frac{Z}{12W}\right)A(Z)
\end{aligned}
$$

$$+\left(\frac{4W^2}{Z} + \frac{17W - 4Z}{3} - \frac{Z^2}{12W}\right)B(W,Z) + \left(\frac{h}{3} - \frac{h^2}{12W} - W\right)B(h,W)$$
$$-\frac{4W^2}{Z} + \frac{64W}{9} + \frac{h+Z}{6}\Bigg\}, \tag{2.23}$$

where

$$N_c = n_Q = n_L = 3 \tag{2.24}$$

are the numbers of colors, quark doublets, and lepton doublets in the Standard Model, respectively, and the fermion-loop function is

$$f(x,y,s) = \frac{1}{6s}\Big\{\big[(x-y)^2 + s(x+y) - 2s^2\big]B(x,y) + (x-y-2s)A(x)$$
$$+(y-x-2s)A(y)\Big\} + (s - 3x - 3y)/9, \tag{2.25}$$

and the bottom quark mass and $|V_{tb}|^2$ dependence has been included. The lighter quark and lepton masses can also be restored in the obvious way, by changing the 0 arguments of the function f in eq. (2.23) and introducing additional Cabibbo-Kobayashi-Maskawa (CKM) mixing factors. Fortunately, however, the difference made by non-zero masses of $b, \tau, c, \ldots$ and the presence of CKM mixing (assuming CKM unitarity and $V_{tb} = 0.99914$ [43]) is less than about 1 MeV in both M_W and Γ_W for 50 GeV $< Q <$ 200 GeV, and is much less for Q in the middle of that range, so those effects will be neglected for simplicity below.

Note that 1-loop contributions involving $B(0,0)$, $B(0,Z)$ and $B(0,h)$ cancel, when the 0 arguments correspond to Goldstone bosons and unphysical modes of the vector bosons in Landau gauge. This and similar cancellations in the 2-loop order part (mentioned below) are useful checks, as non-cancellation of such terms would have implied imaginary parts of the complex pole squared mass that do not correspond to any real decay mode of the W boson.

The 2-loop QCD contribution is also simple enough to be written on a few lines in terms of the basis functions:

$$\Delta_W^{(2),\text{QCD}} = g_3^2 g^2\left(\frac{N_c^2 - 1}{24}\right)\Big[-4(t-W)^2(2+t/W)M(0,0,t,t,0)$$
$$+8(t-2W)(1+t/W)T(t,0,0) - (10t + 8W)B(0,t)^2$$
$$-(36t/W + 56 + 16W/t)A(t)B(0,t) + (30t^2/W + 42t - 12W)B(0,t)$$
$$-(40/W + 24/t)A(t)^2 + (30t/W + 84)A(t) - 39W + 17t/2$$
$$-(n_Q - 1)W\big\{31 + 12B(0,0) + 8WM(0,0,0,0,0)\big\}\Big]. \tag{2.26}$$

The remaining, non-QCD, 2-loop contributions, are much more complicated, involving a large number of terms. The form of the result is[†]

$$\Delta_W^{(2),\text{non}-\text{QCD}} = \sum_i c_i^{(2)} I_i^{(2)} + \sum_{j \leq k} c_{j,k}^{(1,1)} I_j^{(1)} I_k^{(1)} + \sum_j c_j^{(1)} I_j^{(1)} + c^{(0)}. \qquad (2.27)$$

The coefficients $c_i^{(2)}$ and $c_{j,k}^{(1,1)}$ and $c_j^{(1)}$ and $c^{(0)}$ are given in electronic form in an ancillary file `coefficients.txt` provided with the arXiv source for this article. These coefficients are written exclusively in terms of the quantities W, Z, t, h, v^2 [by using eqs. (2.15)-(2.18) to eliminate g, g', y_t, and λ], as well as the fixed parameters N_c, n_Q, and n_L. The latter can each be set equal to 3 in the Standard Model, but are kept general for checking purposes, and to tag the fermion loop contributions.

It should be noted that the coefficients in the expression of the pole mass in terms of the basis integrals are not unique. This is because different basis integrals are related by special identities that hold when the squared mass arguments are not generic. These identities include eqs. (A.15)-(A.21) of ref. [44], and eqs. (A.14), (A.15), and (A.17)-(A.20) in ref. [64].

When setting $s \to W$ in eq. (2.20), one encounters singular behavior in individual terms, associated with photon lines attached to a W boson propagator. In general, such potentially singular terms should cancel in the complex pole mass [24]. They are dealt with here by using expansions such as[‡]

$$\begin{aligned} B(0, W) &= 1 - A(W)/W + (s - W)[1 + A(W)/W - \overline{\ln}(W - s)]/W \\ &\quad + (s - W)^2[-1 - A(W)/W + \overline{\ln}(W - s)]/W^2 + \mathcal{O}(s - W)^3 \end{aligned} \qquad (2.28)$$

with $\overline{\ln}(x) \equiv \ln(x/Q^2)$. Similar expansions of 2-loop basis functions that have thresholds or pseudo-thresholds at $s = W$ are carried out using the differential equations listed in section IV of ref. [53], using methods similar to those found in [46]. After doing so, all pole and logarithmic singularities in $s - W$ that are found in individual Feynman diagrams cancel in the total eq. (2.20), an important check.

Several other helpful checks were performed on the calculation. First, single and double poles in ϵ cancel in s_{pole}^W. This cancellation relies on agreement between the counter-terms $c_{\ell,n}^X$ (for $X = v, g, g', \lambda, y_t, g_3$) as extracted from the β functions and Higgs scalar anomalous dimension in the literature, and the coefficients of divergent parts of the loop integrations performed here. Second, I checked that logarithms of $G = m^2 + \lambda^2 v^2$ cancel. This is required for the absence of spurious imaginary parts that could occur when the renormalization scale is chosen so that $G < 0$, and spurious divergences that could occur for $G = 0$. Third, I

[†] Of the 78 coefficients $c_{j,k}^{(1,1)}$ for products of 1-loop integrals, 42 vanish.
[‡] Eq. (2.28) is used to eliminate $B(0, W)$ everywhere, explaining its absence in eqs. (2.21) and (2.23).

checked the absence of spurious imaginary parts of s_{pole}^W; note that Γ_W must be identically 0 in the case $n_Q = 1$, $n_L = 0$, because in the Standard Model the W boson can only decay to lighter fermion doublets. This checks cancellations between diagrams with Goldstone boson propagators and the corresponding Landau gauge vector propagator parts with poles at 0 squared mass. Fourth, I checked that in each of the formal[§] limits that the quantities W, Z, t, h, $4W - h$, $4Z - h$, $4t - Z$, $t - W$, $t + W - Z$, or $t + W - h$ vanish, the whole expression for s_{pole}^W is finite and well-behaved, even though many of the individual 2-loop coefficients in eqs. (2.23), (2.26), and (2.27) are singular in one or more of those limits. This again reflects non-trivial relations between different basis integrals when squared mass arguments are not generic. Finally, the result for s_{pole}^W was analytically checked to be renormalization group invariant through terms of 2-loop order. In principle, this should be equivalent to the check of cancellation of $1/\epsilon$ poles, but in practice it tests many intermediate steps of the calculation. This check is written as:

$$0 = Q\frac{d}{dQ}s_{\text{pole}}^W = \left[Q\frac{\partial}{\partial Q} - \gamma_\phi v\frac{\partial}{\partial v} + \sum_X \beta_X\frac{\partial}{\partial X}\right]s_{\text{pole}}^W, \tag{2.29}$$

where $X = \{g, g', \lambda, y_t, g_3\}$, and γ_ϕ is the anomalous dimension of the Higgs field. It uses the derivatives of basis integrals with respect to the implicit argument Q, given in eqs. (4.7)-(4.13) of ref. [53], and derivatives of the 1-loop basis integrals with respect to squared mass arguments, given for example in eqs. (A.5) and (A.6) of ref. [44]. It also uses the beta functions and scalar anomalous dimension given in refs. [47–50], [38]. A corresponding numerical check of renormalization scale invariance is performed in the next section.

III.   NUMERICAL RESULTS

The numerical computation of s_{pole}^W given by eqs. (2.20)-(2.27) is accomplished using the program `TSIL` [54]. This requires only 13 calls of the function `TSIL_Evaluate` (which uses Runge-Kutta solution of coupled differential equations to obtain multiple basis integral functions simultaneously) as well as relatively fast evaluations of the integrals for which analytic formulas in terms of polylogarithms are known and incorporated in `TSIL`.

For purposes of illustration, consider a benchmark set of input data:

$$v(M_t) = 246.647 \text{ GeV}, \tag{3.1}$$
$$g(M_t) = 0.647550, \tag{3.2}$$
$$g'(M_t) = 0.358521, \tag{3.3}$$

[§] None of these limits are close to being realized in the real world.

$$\lambda(M_t) = 0.12597, \tag{3.4}$$

$$y_t(M_t) = 0.93690, \tag{3.5}$$

$$g_3(M_t) = 1.1666, \tag{3.6}$$

where $Q = M_t = 173.34$ GeV is the input renormalization scale. The top Yukawa coupling and strong coupling constant were taken from ref. [65] version 4, and the electroweak gauge couplings were taken from ref. [36]. The VEV $v(M_t)$, which should minimize the radiatively corrected effective potential in the scheme used here, has also been chosen to approximately reproduce the experimental value of the Z boson physical mass, using a separate calculation (similar to the present one, and also in the pure $\overline{\text{MS}}$ scheme) that I plan to report on soon. The Higgs self-coupling λ was simultaneously chosen so as to also obtain a Higgs pole mass of $M_h = 125.09$ GeV, using the calculation of [44] as implemented in the program SMH [45], at an optimal renormalization scale $Q = 160$ GeV. However, in the absence of a true global fit to available data, it is important to emphasize that the benchmark parameters chosen here should be viewed as illustrative, rather than as a prediction of M_W.

The results for the renormalization scale dependences of M_W and Γ_W obtained from $s_{\text{pole}}^W = M_W^2 - i\Gamma_W M_W$, in various approximations, are shown in Figures 3.1 and 3.2. To make the graphs, the input parameters $v, g, g', \lambda, y_t, g_3$ are run, using 3-loop beta functions [66, 67], from the input scale M_t to the scale Q on the horizontal axis, and s_{pole}^W is re-computed at that scale. In the idealized case, M_W and Γ_W would be independent of Q if computed to sufficiently high order in perturbation theory.

In Figure 3.1, the (green) dotted line is the tree-level result W, which shows a severe scale dependence, due to the running of g and v. This is still large, but reduced, in the 1-loop result, given by the (red) short-dashed line. The majority of the remaining scale dependence is eliminated by including the QCD part of the 2-loop result from eq. (2.26) as shown in the (blue) long-dashed line. The (black) solid line shows the full 2-loop result. Note that despite the large scale dependence of the 2-loop QCD correction, it is actually smaller than the 2-loop non-QCD correction in magnitude except for $Q \lesssim 85$ GeV, where the effect of $\overline{\ln}(t)$ starts to become large. The 2-loop non-QCD correction is of order 40 MeV, but is seen to have a quite mild scale dependence.

In Figure 3.2 the (red) short-dashed line shows the running of Γ_W computed at 1-loop order. Adding in the 2-loop QCD contribution, as shown by the (blue) long-dashed line, is a significant effect, but does not eliminate the scale dependence, which is mostly due to the electroweak 1-loop renormalization group running of g and v. However, including the 2-loop non-QCD corrections to s_{pole}^W greatly ameliorates the scale dependence, as it captures and compensates for most of the effect of running of g and v.

For the range 50 GeV $< Q <$ 200 GeV, the deviations of M_W and Γ_W from their median values are both about ± 4 MeV. For M_W, this is shown in close-up as the solid line in Figure
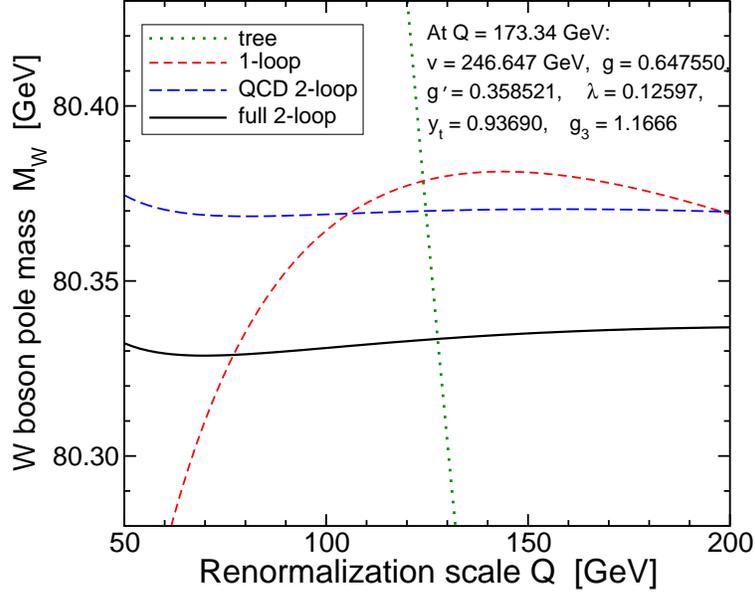
FIG. 3.1: The mass M_W of the W boson, obtained from the complex pole squared mass $s_{\text{pole}}^W = M_W^2 - i\Gamma_W M_W$, as a function of the renormalization scale Q at which s_{pole}^W is computed, in various approximations. The (green) dotted line is the tree-level result W, the (red) short-dashed line is the 1-loop result, the (blue) long-dashed line is the result from the 1-loop and 2-loop QCD contribution, and the (black) solid line is the full 2-loop order result. The input parameters $v, g, g', \lambda, y_t, g_3$ are obtained at the scale Q by 3-loop renormalization group running, starting from eqs. (3.1)-(3.6). Note that the usual Breit-Wigner mass $M_{W,\text{exp}}$ is about 27 MeV larger than M_W.
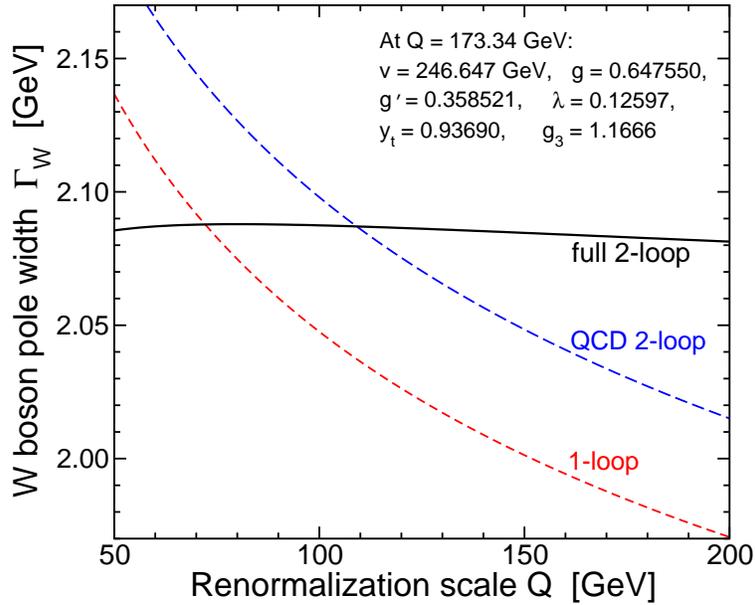

FIG. 3.2: The width Γ_W of the W boson, obtained from the complex pole squared mass $s_{\text{pole}}^W = M_W^2 - i\Gamma_W M_W$, as in Figure 3.1. The (red) short-dashed line is the 1-loop result, the (blue) long-dashed line is the result from the 1-loop and 2-loop QCD contribution, and the (black) solid line is the full 2-loop order result.
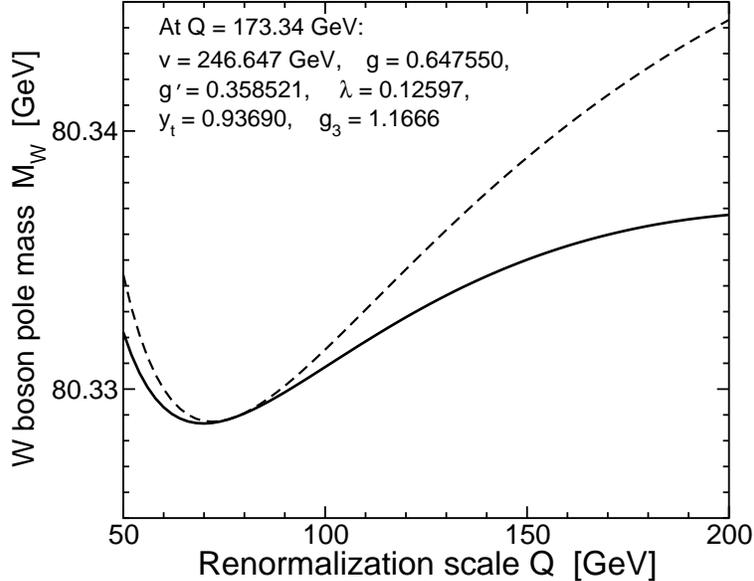
FIG. 3.3: Close-up of the scale dependence of the mass M_W of the W boson, obtained from the complex pole squared mass $s^W_{\text{pole}} = M_W^2 - i\Gamma_W M_W$, as in Figure 3.1. The solid line is the full 2-loop order result, while the dashed line is the same, but after expanding the $\overline{\text{MS}}$ mass t (in the 1-loop part only) about $T = (173.34 \text{ GeV})^2$ to first order, using eq. (3.7).

3.3. While this gives some lower bound on the remaining theory error (not counting the parametric errors in the inputs $v, g, g', \lambda, y_t, g_3$), it is always questionable to assume a direct relationship between scale dependence and theory error. For another handle on the theory error, consider the following exercise. In the top/bottom 1-loop contribution, the running top mass t is used in propagators in the pure $\overline{\text{MS}}$ scheme. However, once the result has been obtained, one can expand t about any other value, for example the top-quark pole mass T. Doing so for the 1-loop contribution only is sensible, since t only appears in propagators, not vertex couplings, in the 1-loop order W boson self-energy. The relevant expansion is:

$$f(0,t,W) = f(0,T,W) + (t-T)\big[A(T) - 2W + (T+W)B(0,T)\big]/2W + \mathcal{O}(t-T)^2. \quad (3.7)$$

If this expansion is extended to, say, 4th order in $t - T$, then the results are easily checked to be nearly indistinguishable from the original $f(0,t,W)$ without expansion. However, terminating the expansion at linear order in $t - T$, as in eq. (3.7), can be considered an alternative consistent 2-loop order result, if $t - T$ is treated as formally of 1-loop order. This version of M_W is shown as the dashed line in Figure 3.3. It clearly has a worse scale dependence, particularly at larger Q, where $T - t$ becomes large. This suggests that the ± 4 MeV scale dependence of the original (solid line) pure $\overline{\text{MS}}$ calculation may be at least partly a fortunate accident. The two curves agree near $Q = 77$ GeV, where the running top-quark mass t equals the physical mass T.

IV. OUTLOOK

In this paper I have reported the results for the complex pole mass of the W boson in the Standard Model in the pure $\overline{\text{MS}}$ scheme, with the vacuum expectation value, defined as the minimum of the Landau gauge effective potential, taken as one of the input parameters. The organization of input and output parameters is quite different from previous works that use the on-shell scheme or hybrid $\overline{\text{MS}}$/on-shell schemes. The state-of-the art computations in these schemes, see respectively e.g. [31] and [36] and references therein, probably both attain a better theory error than the pure $\overline{\text{MS}}$ scheme, for now. Moreover, a direct comparison of numerical results will need at least the corresponding results for the Z boson, which I hope to report on soon. Both results will then be incorporated into a publicly available computer code together with the Higgs boson mass code from [44, 45].

Refs. [25, 26] and the very recent ref. [37] (which appeared as the present paper was being finished) also used the pure $\overline{\text{MS}}$ scheme to compute the complex pole mass of the W boson. However, attempts at direct comparison are complicated† by the fact that these papers used a different definition of the VEV, namely $v_{\text{tree}}^2 = -m^2/\lambda$, rather than v that minimizes the full radiatively corrected effective potential as made here (and, for example, refs. [44] and‡ [36]). The choice of using v_{tree} requires including non-trivial tadpole diagrams, unlike the choice of expanding around v where the sum of Higgs tadpole diagrams (including the tree-level tadpole) simply vanishes. This means that already at 1-loop order, the expressions appear different. Compared to $\Delta_W^{(1)}/(16\pi^2)$ in eq. (2.23) of the present paper, the sum of the bosonic contributions in eq. (B.2) of ref. [25] and the fermionic contributions in (B.2) of ref. [26] differs by:

$$\frac{g^2}{16\pi^2 h}\left[-2N_c t A(t) + \frac{3}{4}hA(h) + 3WA(W) + 2W^2 + \frac{3}{2}ZA(Z) + Z^2\right]. \qquad (4.1)$$

This is simply because the tree-level terms are also different, namely $g^2 v^2/4$ in the present paper and $g^2 v_{\text{tree}}^2/4$ in refs. [25, 26, 37]. To 1-loop order accuracy, the two expressions for the pole mass can easily be checked to be the same, by using eq. (2.19) above, but establishing the connection at 2-loop order would require a somewhat non-trivial re-expansion using the 2-loop relation between v_{tree}^2 and v^2.

Note that, in general, expanding around v_{tree} rather than v has the effect of making

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† Also, refs. [25, 26] use expansions in $1/4 - \sin^2\theta_W$ and Z/h and Z/t (in the notation of the present paper), which further increases the difficulty in making a direct comparison.

‡ However, Ref. [36] uses Feynman gauge instead of Landau gauge, so the VEV referred to in that paper will also not be the same thing as v in the present paper. Note that using v requires choosing a gauge-fixing prescription; choosing Landau gauge has the advantage that the effective potential is much simpler.

the perturbative expansion parameter be $\dfrac{N_c y_t^4}{16\pi^2\lambda}$, rather than the usual $\dfrac{N_c y_t^2}{16\pi^2}$, for the terms leading in the top mass. This can be seen in the presence of the first term in eq. (4.1); in contrast, there is no $g^2 t A(t)/h$ term in eq. (2.23). As mentioned above as one of the checks, at two-loop order there is also no behavior like t^3/h^2 or t^2/h (or any other pole singularity in h, or W or Z) in $\Delta_W^{(2),\text{non}-\text{QCD}}$ in eq. (2.27). As another example, see the discussion surrounding eqs. (4.34)-(4.40) in ref. [40], where the terms of order $\left(\dfrac{y_t^4}{16\pi^2\lambda}\right)^\ell$ in the relation between v_{tree} and v are explicitly identified for loop orders $\ell = 1, 2, 3$ in the limit $y_t^2 \gg \lambda$ in the case $g = g' = 0$. Not surprisingly, expanding around the radiatively corrected VEV leads to faster convergence than expanding around the tree-level VEV, at least formally, although both expansions should converge given enough loop orders, since $\dfrac{N_c y_t^4}{16\pi^2\lambda}$ is still numerically small.

It would clearly be useful to include the 3-loop contributions to W and Z complex pole masses in the pure $\overline{\text{MS}}$ scheme, so that theory errors can be made unambiguously much smaller than all relevant experimental errors. Here it should be remarked that it is not at all obvious that the parametrically QCD-enhanced contributions at 3-loop order will be the largest, especially considering that this was not the case at 2-loop order. A possible scenario is that the QCD-enhanced contributions will have the largest renormalization scale dependence, but not the largest magnitude, since this is what happened at 2-loop order. It seems feasible to eventually include all 3-loop contributions to s_{pole}^W in the pure $\overline{\text{MS}}$ scheme, although to do so without using mass expansions or approximations may require developing new methods for treating 3-loop self-energy contributions.

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