

Torsional response of relativistic fermions in $2 + 1$ dimensions

Manuel Valle

Departamento de Física Teórica, Universidad del País Vasco UPV/EHU, Apartado 644, 48080 Bilbao, Spain

E-mail: manuel.valle@ehu.eus

ABSTRACT: We consider the equilibrium partition function of an ideal gas of Dirac fermions minimally coupled to torsion in $2 + 1$ dimensions. We show that the energy-momentum tensor reproduces the Hall viscosity and other parity violating terms of first order in the torsion. We also consider the modifications of the constitutive relations, and classify the corresponding susceptibilities. An entropy current consistent with zero production of entropy in equilibrium is constructed.

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1 Introduction

The description of transport phenomena and dynamical response in the framework of effective field theory has experienced great progress in the last few years. This is due, to a large extent, to a better understanding of the role played by the quantum anomalies which underlie new macroscopic parity-violating effects at low energy [1]. In this respect, the construction of the thermal partition function as a derivative expansion of a time-independent background has turned out to be an important tool to get information about the non-dissipative part of the constitutive relations of hydrodynamics, without using an entropy approach [2, 3]. In relativistic systems the first order of such expansion usually has a relatively simple structure connected with anomalies [4–6].

For non-relativistic systems there is also considerable interest in establishing the precise connection between the partition function and Hall transport [7–12]. While in these systems the use of torsional Newton-Cartan geometry appears as natural, the role of torsion in a relativistic setting is less clear [13, 14]. Originally the torsion-dependent effective action of massive Dirac fermions at zero temperature and density has been examined in great detail in refs. [15, 16] with the focus on the renormalization effects on the Hall viscosity. Here we are interested in the application of the methods of [2, 3] to the thermal partition function to linear order in the torsion.

The main purpose of this paper is to compute, to linear order in the torsion, the stress tensor and charge current that follow from the equilibrium partition function for a Dirac field minimally coupled to torsion. A first result of our computation is that the relationship between the Hall viscosity and the spin density, $\tilde{\eta} = \langle \ell \rangle / 2 = -\langle \bar{\Psi} \Psi \rangle / 4$, naturally appears as an equilibrium susceptibility relating the stress tensor Θ^{ij} with certain parity-violating combination of components of the torsion tensor. Other results are the explicit construction

of an entropy current consistent with zero production of entropy at equilibrium, and the derivation of new susceptibilities relating the charge current and the pressure with torsion-dependent vector and scalar data.

The organization of the paper is as follows. In section 2 we present some details about the notation, and the partition function we use in the subsequent computations. The form of the stress tensor at thermal equilibrium is given in section 3 in terms of combinations of scalar, vector, and tensor background data depending on torsion. The modification in the Landau frame of the constitutive relations in the presence of torsion is presented in section 4, as well as an expression for the entropy current compatible with zero entropy production. We conclude in section 5 with a short summary of our results.

2 Partition function of a Dirac fermion coupled to torsion

We begin with the action for a Dirac field,

$$S = \int d^3x \det e_\mu^a \left[-\frac{1}{2} \bar{\Psi} \gamma^\mu \vec{\nabla}_\mu \Psi + \frac{1}{2} \bar{\Psi} \overleftarrow{\nabla}_\mu \gamma^\mu \Psi + m \bar{\Psi} \Psi \right], \quad \gamma^\mu(x) = e_\mu^a(x) \gamma^a, \quad (2.1)$$

in the most general static background with torsion

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu = -e^{2\sigma(\mathbf{x})} (dt + a_i(\mathbf{x}) dx^i)^2 + g_{ij}(\mathbf{x}) dx^i dx^j, \\ \mathcal{A} &= A_0 dt + \mathcal{A}_i dx^i, \\ T^a &= \frac{1}{2} T_{\mu\nu}{}^a dx^\mu \wedge dx^\nu. \end{aligned} \quad (2.2)$$

This geometry with torsion has been also considered in other studies of the torsion response [17]. Note that we are using the prescription of minimal coupling, where the action depends on the torsion only through the covariant derivative. The notation of ref. [18] has been adopted, where the Dirac adjoint is defined as $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$. For the Dirac matrices γ^a we choose the representation $\{\gamma^0, \gamma^1, \gamma^2\} = \{-i\sigma_3, \sigma_2, -\sigma_1\}$. The torsion 2-forms T^a are specified in terms of the frame field e_μ^a and the torsion tensor by $T_{\mu\nu}{}^a = T_{\mu\nu}{}^\sigma e_\sigma^a$. The covariant derivatives

$$\begin{aligned} \vec{\nabla}_\mu \Psi &= (\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} - iA_\mu) \Psi, & \gamma_{ab} &\equiv \frac{1}{2} [\gamma_a, \gamma_b] \\ \bar{\Psi} \overleftarrow{\nabla}_\mu &= \bar{\Psi} (\overleftarrow{\partial}_\mu - \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} + iA_\mu), \end{aligned} \quad (2.3)$$

are written in terms of the spin connection $\omega_\mu{}^{ab}$, which is specified by the 1-forms $\omega^{ab} = \omega_\mu{}^{ab} dx^\mu$ appearing in the Cartan structure equation

$$de^a + \omega^a{}_b \wedge e^b = T^a. \quad (2.4)$$

By defining the contortion tensor, antisymmetric in the last two indices, as

$$K_{\mu\nu\rho} = -\frac{1}{2} (T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu}), \quad (2.5)$$

it turns out that the spin connection can be expressed in terms of the Christoffel symbol $\Gamma_{\mu\nu}^\sigma(g)$ and the contortion tensor as follows

$$\begin{aligned}\omega_\mu^{ab} &= -e^{b\nu}\partial_\mu e_\nu^a + e^{b\nu}(\Gamma_{\mu\nu}^\sigma(g) - K_{\mu\nu}^\sigma) e_\sigma^a \\ &= \omega_\mu^{ab}(e) + K_\mu^{ab}.\end{aligned}\tag{2.6}$$

(The square brackets denote antisymmetrization of indices $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.) This formula contains the affine connection $\Gamma_{\mu\nu}^\sigma = \Gamma_{\mu\nu}^\sigma(g) - K_{\mu\nu}^\sigma$ in the presence of torsion, which has an antisymmetric part given by

$$\Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma = T_{\mu\nu}^\sigma.\tag{2.7}$$

The part of the spin connection in the absence of torsion $\omega_\mu^{ab}(e)$, which is called the Levi-Civita connection, is uniquely determined by the frame field. It is obtained from (2.6) by setting $K_{\mu\nu}^\sigma = 0$.

In order to vary the metric and torsion variables independently we will assume, according with ref. [19], that the torsion tensor with the last coordinate upper index $T_{\mu\nu}^\sigma$ and the metric components $G_{\mu\nu}$ are independent of each other. This will be important soon. Note that in ref. [14] the contortion is considered as an independent variable, instead of the torsion.

With the previous relations and

$$\begin{aligned}\{\gamma_c, \gamma_{ab}\} &= -2\bar{\epsilon}_{cab}, & \bar{\epsilon}_{012} &= 1 = -\bar{\epsilon}^{012}, \\ \epsilon^{\mu\nu\rho} &= \det e_\sigma^d \bar{\epsilon}^{abc} e_a^\mu e_b^\nu e_c^\rho,\end{aligned}\tag{2.8}$$

we can see the effect of the torsion on the three-dimensional Dirac action. The substitution of (2.6) and (2.7) into (2.1) reveals that S may be viewed as the action of a Dirac field in a torsion-free background, but with a modified mass. The new mass \tilde{m} , that now depends on the torsion, is obtained by the replacement

$$m \rightarrow \tilde{m} = m + \delta m = m - \frac{1}{8}\epsilon^{\mu\nu\rho} T_{\mu\nu}^\lambda G_{\lambda\rho}.\tag{2.9}$$

It is remarkable that a generic non-minimal coupling of the fermion to torsion may be included simply by replacing δm by $\xi\delta m$, where ξ is a free parameter [16].

In this way, the partition function at zeroth derivative order in the variables $(\mathcal{A}_\mu, G_{\mu\nu}, \tilde{m})$ is written as

$$W^0 = \int d^2x \sqrt{g_2} \frac{e^\sigma}{T_0} \mathcal{P}(T, \mu, \tilde{m}), \quad T = T_0 e^{-\sigma}, \quad \mu = A_0 e^{-\sigma},\tag{2.10}$$

where T_0^{-1} is the period of the imaginary time, and the function \mathcal{P} is the pressure in terms of the temperature and chemical potential. Therefore, we expect that the functional

$$W[G_{\mu\nu}, T_{\mu\nu}^\lambda] = -\frac{1}{8} \int d^2x \sqrt{g_2} \frac{e^\sigma}{T_0} \langle \bar{\Psi} \Psi \rangle \epsilon^{\mu\nu\rho} T_{\mu\nu}^\lambda G_{\lambda\rho},\tag{2.11}$$

contains all the information about the static linear response to lowest order in the derivative expansion of the torsion. Here we have used the fact that the spin density is related to

the pressure by $\partial\mathcal{P}/\partial m = \langle \bar{\Psi}\Psi \rangle$, which follows from the form of the Dirac Lagrangian, eq. (2.1).

Although no specific form of \mathcal{P} is required in the following computations, we write down the expressions of the pressure and the spin density at non-zero temperature and density in the free-field case,

$$\mathcal{P}(T, \mu, m) = -\frac{T^3}{2\pi} \left[\text{Li}_3 \left(-\exp\left(\frac{\mu - |m|}{T}\right) \right) + \text{Li}_3 \left(-\exp\left(\frac{-\mu - |m|}{T}\right) \right) \right] - \frac{|m|T^2}{2\pi} \left[\text{Li}_2 \left(-\exp\left(\frac{\mu - |m|}{T}\right) \right) + \text{Li}_2 \left(-\exp\left(\frac{-\mu - |m|}{T}\right) \right) \right], \quad (2.12)$$

$$\langle \bar{\Psi}\Psi \rangle = -\frac{mT}{2\pi} \left[\ln \left(1 + \exp\left(\frac{\mu - |m|}{T}\right) \right) + \ln \left(1 + \exp\left(\frac{-\mu - |m|}{T}\right) \right) \right], \quad (2.13)$$

where $\text{Li}_n(x)$ is the polylogarithm function.

3 The equilibrium stress tensor at linear order in the torsion

We can now find the energy-momentum tensor $\Theta_{\mu\nu}$ which follows from the partition function by differentiation with respect to σ, a_j and g^{ij} , with the understanding that $T_{\mu\nu}{}^\lambda$ is independent of the metric. From eq. (2.11), the variational formula for Θ_{00} leads to

$$\Theta_{00} = -\frac{T_0 e^\sigma}{\sqrt{g_2}} \frac{\delta W}{\delta \sigma} = -e^{2\sigma} \left(\frac{\partial \langle \bar{\Psi}\Psi \rangle}{\partial \sigma} + \langle \bar{\Psi}\Psi \rangle \right) \delta m - e^{2\sigma} \langle \bar{\Psi}\Psi \rangle \frac{\partial \delta m}{\partial \sigma}. \quad (3.1)$$

The most obvious way to obtain $\partial \langle \bar{\Psi}\Psi \rangle / \partial \sigma$ is to consider the energy density rewritten in the form

$$\begin{aligned} \varepsilon &= -\mathcal{P} + T \frac{\partial \mathcal{P}}{\partial T} + \mu \frac{\partial \mathcal{P}}{\partial \mu} \\ &= -\mathcal{P} - \frac{\partial \mathcal{P}}{\partial \sigma}. \end{aligned} \quad (3.2)$$

Then differentiating with respect to m yields the relationship

$$\frac{\partial \varepsilon}{\partial m} = -\langle \bar{\Psi}\Psi \rangle - \frac{\partial \langle \bar{\Psi}\Psi \rangle}{\partial \sigma}, \quad (3.3)$$

and eq. (3.1) becomes

$$\begin{aligned} \Theta_{00} &= e^{2\sigma} \frac{\partial \varepsilon}{\partial m} \delta m - e^{2\sigma} \langle \bar{\Psi}\Psi \rangle \frac{\partial \delta m}{\partial \sigma} \\ &= e^{2\sigma} \frac{\partial \varepsilon}{\partial m} \delta m + \langle \bar{\Psi}\Psi \rangle \left(\frac{e^\sigma}{2} \epsilon^{ij} (T_{0ij} - a_j T_{0i0}) - e^{2\sigma} \delta m \right). \end{aligned} \quad (3.4)$$

In the computation of the last derivative eq. (2.9) has been used, while keeping $T_{\mu\nu}{}^\lambda$ fixed. Here $\epsilon^{12} = 1/\sqrt{g_2}$. Note the invariance of this result under time reparametrization, $t \rightarrow t + \phi(\mathbf{x})$, $\mathbf{x} \rightarrow \mathbf{x}$. In the light of (3.4), it is convenient to introduce another pseudo-scalar

quantity (besides δm) constructed from the torsion as $\Xi \equiv -u^\mu T_{\mu\nu\rho} \epsilon^{\nu\rho\lambda} u_\lambda$. This quantity, evaluated for the equilibrium fluid velocity, $u_K^\mu = \delta_0^\mu e^{-\sigma}$, becomes $\Xi = -e^{-\sigma} \epsilon^{ij} \mathfrak{T}_{0ij}$, where $\mathfrak{T}_{0ij} = T_{0ij} - a_j T_{0i0}$. Thus the change of Θ_{00} induced by the torsion reads

$$\Theta_{00} = e^{2\sigma} \frac{\partial \varepsilon}{\partial m} \delta m + e^{2\sigma} \langle \bar{\Psi} \Psi \rangle \left(-\frac{\Xi}{2} - \delta m \right) \Big|_{\text{eq}}. \quad (3.5)$$

The other components of the energy-momentum tensor are easily computed:

$$\begin{aligned} \Theta_0^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g_2}} \frac{\delta W}{\delta a_i} \\ &= \langle \bar{\Psi} \Psi \rangle \left(\frac{e^\sigma}{4} \epsilon^{ij} (T_j - a_j T_0) - \frac{e^{-\sigma}}{2} \epsilon^{ij} T_{0j0} \right), \end{aligned} \quad (3.6)$$

where T_μ is the torsion vector defined by $T_\mu = T_{\mu\nu}{}^\nu$. This expression suggests the introduction of two pseudo-vectors orthogonal to u^μ given by

$$\begin{aligned} \tilde{X}^\mu &= -\epsilon^{\mu\nu\rho} u_\nu u^\lambda T_{\lambda\rho\sigma} u^\sigma, \\ \tilde{W}^\mu &= -\epsilon^{\mu\nu\rho} u_\nu T_\rho. \end{aligned} \quad (3.7)$$

In equilibrium these quantities evaluate to

$$\begin{aligned} \tilde{X}^i &= e^{-2\sigma} \epsilon^{ij} T_{0j0}, \\ \tilde{W}^i &= \epsilon^{ij} (T_j - a_j T_0). \end{aligned} \quad (3.8)$$

For the equilibrium stress tensor the functional differentiation of W yields

$$\begin{aligned} \Theta^{ij} &= -\frac{2T_0 e^{-\sigma}}{\sqrt{g_2}} g^{ik} g^{jm} \frac{\delta W}{\delta g^{km}} \\ &= -\frac{e^{-\sigma}}{4} \langle \bar{\Psi} \Psi \rangle (\epsilon^{ik} \mathfrak{T}_{0km} g^{mj} + \epsilon^{jk} \mathfrak{T}_{0km} g^{mi}). \end{aligned} \quad (3.9)$$

Remarkably, this expression may be expressed in terms of the pseudo-tensor given by

$$\tilde{\sigma}^{\mu\nu} = -\frac{1}{2} (\epsilon^{\mu\lambda\rho} u_\lambda \sigma_\rho{}^\nu + \epsilon^{\nu\lambda\rho} u_\lambda \sigma_\rho{}^\mu), \quad (3.10)$$

where $\sigma^{\mu\nu}$ is the shear tensor,

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - G_{\alpha\beta} \nabla_\rho u^\rho), \quad \Delta^{\mu\alpha} = G^{\mu\alpha} + u^\mu u^\alpha. \quad (3.11)$$

While in the absence of torsion the equilibrium values of $\tilde{\sigma}^{\mu\nu}$ and $\sigma^{\mu\nu}$ vanish, it turns out that when torsion is present, the covariant derivative of the equilibrium velocity $u^\mu = \delta_0^\mu e^{-\sigma}$ acquires a non-zero contribution proportional to the torsion, which now is included in the affine connection. Correspondingly, the shear tensor at equilibrium is given by

$$\begin{aligned} \sigma^{\mu\nu} &= \Delta^{\mu\alpha} \Delta^{\nu\beta} \left[(K_{\alpha\beta}{}^\lambda + K_{\beta\alpha}{}^\lambda) u_\lambda + G_{\alpha\beta} K_{\nu\lambda}{}^\nu u^\lambda \right] \\ &= \Delta^{\mu\alpha} \Delta^{\nu\beta} \left[-(T_{\lambda\alpha\beta} + T_{\lambda\beta\alpha}) u^\lambda + G_{\alpha\beta} T_{\lambda\nu}{}^\nu u^\lambda \right], \quad u^\lambda = \delta_0^\lambda e^{-\sigma}. \end{aligned} \quad (3.12)$$

Hence, apart from $\tilde{\sigma}^{ij}$ no more pseudo-tensors are needed to express the equilibrium value of Θ^{ij} . Thus eq. (3.9) may be rewritten in the form

$$\Theta^{ij} = \frac{1}{4} \langle \bar{\Psi} \Psi \rangle (\tilde{\sigma}^{ij} - \Xi g^{ij}) \Big|_{\text{eq}}. \quad (3.13)$$

In a general configuration eq. (3.12) is not longer satisfied, and in addition to $\tilde{\sigma}^{\mu\nu}$, it is convenient to define another pseudo-tensor $\tilde{t}_{(2)}^{\mu\nu}$ related to torsion given by

$$\tilde{t}_{(2)}^{\mu\nu} = -\frac{1}{2} (\epsilon^{\mu\lambda\rho} u_\lambda t_{(2)\rho}^\nu + \epsilon^{\nu\lambda\rho} u_\lambda t_{(2)\rho}^\mu), \quad (3.14)$$

where $t_{(2)}^{\mu\nu} = -\Delta^{\mu\alpha} \Delta^{\nu\beta} (T_{\lambda\alpha\beta} + T_{\lambda\beta\alpha} - G_{\alpha\beta} T_\lambda) u^\lambda$. This will be important when we consider a possible generalization to non minimal coupling in the last section, because then $\tilde{t}_{(2)}^{\mu\nu}$ will appear in the constitutive relations.

It is also possible to derive these results directly from the energy-momentum tensor obtained from the action (2.1), without using the partition function (2.11). With the assumption that the spin connection does not depend on the frame field, the functional derivative of the action with respect to e_a^ρ produces the non-symmetric tensor

$$T'^{\mu\nu} = -\frac{1}{e} e_a^\mu \frac{\delta S}{\delta e_a^\rho} g^{\rho\nu} = \frac{1}{2} \bar{\Psi} (\gamma^\mu \overleftrightarrow{\nabla}^\nu - \overleftarrow{\nabla}^\nu \gamma^\mu) \Psi. \quad (3.15)$$

The symmetry may be restored by adding the tensor $\Delta T^{\mu\nu}$ given by

$$\Delta T^{\mu\nu} = \frac{1}{8} (\nabla_\lambda + T_\lambda) (\bar{\Psi} \{ \gamma^{\lambda\mu}, \gamma^\nu \} \Psi), \quad (3.16)$$

where the covariant derivative is performed with the affine connection including the contortion, and T_λ denotes the torsion vector $T_\lambda \equiv T_{\mu\rho}^\rho$. Using the equations of motion this gives the usual form of the stress tensor

$$\Theta^{\mu\nu} = T'^{\mu\nu} + \Delta T^{\mu\nu} = \frac{1}{4} \bar{\Psi} (\gamma^\mu \overleftrightarrow{\nabla}^\nu - \overleftarrow{\nabla}^\nu \gamma^\mu + (\mu \leftrightarrow \nu)) \Psi. \quad (3.17)$$

To obtain the dependence of $\langle \Theta^{\mu\nu} \rangle$ with the torsion, we note that the contortion tensor in the covariant derivatives always multiplies the term $\langle \bar{\Psi} \Psi \rangle$. A second kind of contribution comes from the fact that the fermion field satisfies a Dirac equation with a torsion-dependent mass \tilde{m} . As a consequence the corresponding perfect fluid constitutive relation

$$\langle \Theta^{\mu\nu} \rangle = \varepsilon(T, \mu, \tilde{m}) u^\mu u^\nu + \mathcal{P}(T, \mu, \tilde{m}) \Delta^{\mu\nu}, \quad (3.18)$$

contributes to the expectation values of Θ_{00} and Θ^{ij} with terms proportional to $\partial\varepsilon/\partial m$ and $\partial\mathcal{P}/\partial m$ respectively. Then the explicit evaluation of K_μ^{ab} in the spin connection shows that the equilibrium values may be written as the combinations

$$\langle \Theta_{00} \rangle = \langle \bar{\Psi} \Psi \rangle \left(\frac{e^\sigma}{2} \epsilon^{ij} \mathfrak{T}_{0ij} - e^{2\sigma} \delta m \right) + e^{2\sigma} \frac{\partial\varepsilon}{\partial m} \delta m, \quad (3.19)$$

$$\langle \Theta^{ij} \rangle = -\langle \bar{\Psi} \Psi \rangle \left(\frac{e^{-\sigma}}{4} (\epsilon^{ik} \mathfrak{T}_{0km} g^{mj} + \epsilon^{jk} \mathfrak{T}_{0km} g^{mi}) + \delta m g^{ij} \right) + \frac{\partial\mathcal{P}}{\partial m} \delta m g^{ij}. \quad (3.20)$$

These are in perfect agreement with (3.4) and (3.9). The result for $\langle \Theta_0^j \rangle$ coincides with (3.6), and does not include contributions of the second type because they are not generated by the perfect fluid constitutive relation.

3.1 Kubo formula for Hall viscosity from a contact term

So far we have been working at the level of the partition function, but it is instructive to consider the response to a general time-dependent applied torsion in the framework of linear response theory. With this in view, we now consider eq. (3.20) as the static limit of a Kubo formula for the stress tensor response to an externally applied time-dependent torsion (not necessarily uniform). In order to obtain such a formula, we assume that the only non-zero components of the torsion tensor are $T_{0jk} = -T_{j0k}$, and we also consider that the metric is Minkowskian. Thus the change of the Hamiltonian to linear order in the torsion becomes

$$\begin{aligned} H_1 &= - \int d^2x \bar{\Psi} \Psi \delta m \\ &= - \frac{1}{4} \int d^2x \bar{\Psi} \Psi \epsilon^{ij} T_{0ij}. \end{aligned} \quad (3.21)$$

The induced change of the expectation value of Θ_{ij} is obtained from linear response theory in the form

$$\begin{aligned} \delta \langle \Theta_{ij}(\mathbf{x}, t) \rangle &= \int d^2y \left\langle \frac{\delta \Theta_{ij}(\mathbf{x}, t)}{\delta T_{0km}(\mathbf{y}, t)} \right\rangle_0 T_{0km}(\mathbf{y}, t) \\ &\quad + \frac{i}{4} \int_{-\infty}^t d\bar{t} e^{-\eta(t-\bar{t})} \int d^2y \langle [\Theta_{ij}(\mathbf{x}, t), \bar{\Psi} \Psi(\mathbf{y}, \bar{t})] \rangle_0 \epsilon^{km} T_{0km}(\mathbf{y}, \bar{t}), \end{aligned} \quad (3.22)$$

where $\eta \rightarrow 0^+$, and the subindex 0 means an equilibrium average with respect to the unperturbed Hamiltonian. The first term (contact term) has its origin in the explicit dependence of Θ with the torsion, and corresponds exactly to the first summand of eq. (3.20),

$$\left\langle \frac{\delta \Theta_{ij}(\mathbf{x}, t)}{\delta T_{0km}(\mathbf{y}, t)} \right\rangle_0 = -\frac{1}{4} \langle \bar{\Psi} \Psi \rangle_0 \left(\epsilon^{ik} \delta^{mj} + \epsilon^{jk} \delta^{mi} + \epsilon^{km} \delta^{ij} \right) \delta(\mathbf{x} - \mathbf{y}). \quad (3.23)$$

The second term involves the retarded stress tensor-spin density correlator defined by

$$\mathcal{Y}_{ij}(\mathbf{x}, t) \equiv -\frac{i}{2} \lim_{\eta \rightarrow 0^+} \theta(t) \langle [\Theta_{ij}(\mathbf{x}, t), \bar{\Psi} \Psi(\mathbf{0}, 0)] \rangle_0 e^{-\eta t}. \quad (3.24)$$

We can express eq. (3.22) in terms of a response function to the torsion $\chi_{ijkm}(\mathbf{q}, \omega)$, which in the Fourier domain relates $\delta \langle \theta_{ij} \rangle = \chi_{ijkm} T_{0km}$. Therefore eq. (3.22) leads to the following Kubo-type formula

$$\chi_{ijkm}(\mathbf{q}, \omega) = -\frac{1}{4} \langle \bar{\Psi} \Psi \rangle_0 \left(\epsilon^{ik} \delta^{mj} + \epsilon^{jk} \delta^{mi} \right) - \frac{1}{2} \left(Y_{ij}(\mathbf{q}, \omega) + \frac{1}{2} \langle \bar{\Psi} \Psi \rangle_0 \delta_{ij} \right) \epsilon^{km}, \quad (3.25)$$

where

$$Y_{ij}(\mathbf{q}, \omega) = \int_0^\infty dt e^{i(\omega+i0^+)t} \int d^2x e^{-i\mathbf{q}\cdot\mathbf{x}} \langle [\Theta_{ij}(\mathbf{x}, t), \bar{\Psi} \Psi(\mathbf{0}, 0)] \rangle_0. \quad (3.26)$$

The response function $Y_{ij}(\mathbf{q}, \omega)$ is similar to the (integrated) stress-strain form of the response function in ref. [20], where the role of the spin density is played by the antisymmetric part of the strain generators $J_{\alpha\beta}$.

We see that the Kubo formula (3.25) is consistent with the static result of eq. (3.20) only if the ordered limit $\lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} Y_{ij}(\mathbf{q}, \omega)$ satisfies

$$\lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} Y_{ij}(\mathbf{q}, \omega) = \lim_{q \rightarrow 0} Y_{ij}(\mathbf{q}, \omega = 0) = -\frac{\delta_{ij}}{2} \frac{\partial \mathcal{P}}{\partial m}. \quad (3.27)$$

This requirement can be expressed as a thermodynamic sum rule valid for $q \rightarrow 0$,

$$\lim_{q \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\text{Im} Y_{ij}(\mathbf{q}, \omega)}{\omega - i0^+} = -\frac{\delta_{ij}}{2} \frac{\partial \mathcal{P}}{\partial m}, \quad (3.28)$$

which is similar to other sum rules giving different susceptibilities. It follows that the static response to T_{0km} is determined by a part of the contact term, namely

$$\lim_{q \rightarrow 0} \chi_{ij km}(\mathbf{q}, \omega = 0) = \tilde{\eta} \left(\epsilon^{ik} \delta^{mj} + \epsilon^{jk} \delta^{mi} \right), \quad (3.29)$$

so that

$$\tilde{\eta} = \frac{1}{4} \lim_{q \rightarrow 0} \delta_{ij} \epsilon_{km} \chi_{ij km}(\mathbf{q}, \omega = 0). \quad (3.30)$$

The coefficient $\tilde{\eta} = -\langle \bar{\Psi} \Psi \rangle_0 / 4$ will be identified below with the Hall viscosity.

4 Torsion-dependent constitutive relations

We now consider the corrections to the constitutive relations imposed by the partition function (2.11). The argument essentially follows ref. [2]. As these corrections involve the expectation value of the particle current $J^\mu = i \bar{\Psi} \gamma^\mu \Psi$, we also need the modification induced by the torsion on the equilibrium current. This is simply given by

$$\delta J_0 = -\frac{T_0 e^\sigma}{\sqrt{g_2}} \frac{\delta W}{\delta A_0} = -e^\sigma \frac{\partial n}{\partial m} \delta m, \quad \delta J^i = 0. \quad (4.1)$$

This may be immediately obtained from the relation between the particle density and the pressure, $n = e^\sigma \partial \mathcal{P} / \partial A_0$, which implies

$$\frac{\partial \langle \bar{\Psi} \Psi \rangle}{\partial A_0} = e^{-\sigma} \frac{\partial n}{\partial m}. \quad (4.2)$$

4.1 Stress tensor and charge current

The most general parity violating modification of the currents that includes torsion must be written in terms of the non-zero quantities in equilibrium $\delta m, \Xi, \tilde{W}^\mu, \tilde{X}^\mu$ and $\tilde{\eta}^{\mu\nu}$. Then the non-dissipative parts of the constitutive relations in the Landau frame take the form

$$\begin{aligned} T^{\mu\nu} &= \varepsilon u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} - (\tilde{\chi}_\Gamma \delta m + \tilde{\chi}_\Xi \Xi) \Delta^{\mu\nu} - \tilde{\eta} \tilde{\sigma}^{\mu\nu}, \\ J^\mu &= n u^\mu + \tilde{\chi}_W \tilde{W}^\mu + \tilde{\chi}_X \tilde{X}^\mu, \end{aligned} \quad (4.3)$$

which involve four new transport coefficients $\tilde{\chi}_\Gamma, \tilde{\chi}_\Xi, \tilde{\chi}_W, \tilde{\chi}_X$ and the Hall viscosity $\tilde{\eta}$.¹ To find them, we specialize to equilibrium by using $u^\mu = \delta_0^\mu e^{-\sigma}$ to zero order in the torsion,

¹ The term $(\tilde{\chi}_\Gamma \delta m + \tilde{\chi}_\Xi \Xi) \Delta^{\mu\nu}$ would be replaced by $(\tilde{\chi}_B B + \tilde{\chi}_\Omega \Omega + \tilde{\chi}_\Gamma \delta m + \tilde{\chi}_\Xi \Xi) \Delta^{\mu\nu}$ in the complete constitutive relation, where $B = \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho}$ and the vorticity is defined by $\Omega = \epsilon^{\mu\nu\rho} u_\mu \partial_\nu u_\rho$ rather than $\epsilon^{\mu\nu\rho} u_\mu \nabla_\nu u_\rho$. Then the effect of the additional term related to the antisymmetric part of the affine connection in the presence of torsion, $\Delta \Omega = -\epsilon^{\mu\nu\rho} u_\mu T_{\nu\rho} u^\lambda$, is considered separately in the coefficients $\tilde{\chi}_\Gamma$ and $\tilde{\chi}_\Xi$.

and $\Delta_{00} = \tilde{\sigma}_{00} = \tilde{X}_0 = \tilde{W}_0 = 0$. Because the torsion-induced corrections can be expressed in terms of changes in the fluid variables $(\delta T, \delta\mu, \delta u^i)$ according to

$$\begin{aligned}
\delta T_{00} &= e^{-2\sigma} \left(\frac{\partial\varepsilon}{\partial T} \delta T + \frac{\partial\varepsilon}{\partial\mu} \delta\mu \right), \\
\delta J_0 &= -e^{-\sigma} \left(\frac{\partial n}{\partial T} \delta T + \frac{\partial n}{\partial\mu} \delta\mu \right), \\
\delta T_0^i &= -e^\sigma (\varepsilon + \mathcal{P}) \delta u^i, \\
\delta J^i &= n \delta u^i + \tilde{\chi}_W \tilde{W}^i + \tilde{\chi}_X \tilde{X}^i, \\
\delta T^{ij} &= \left(\frac{\partial\mathcal{P}}{\partial T} \delta T + \frac{\partial\mathcal{P}}{\partial\mu} \delta\mu - \tilde{\chi}_\Upsilon \delta m - \tilde{\chi}_\Xi \Xi \right) g^{ij} - \tilde{\eta} \tilde{\sigma}^{ij},
\end{aligned} \tag{4.4}$$

we can equate these first three equations to (3.5), (4.1) and (3.6). This gives

$$\begin{aligned}
\left(\frac{\partial\varepsilon}{\partial T} \frac{\partial n}{\partial\mu} - \frac{\partial\varepsilon}{\partial\mu} \frac{\partial n}{\partial T} \right) \delta T &= \left[\frac{\partial n}{\partial\mu} \left(\frac{\partial\varepsilon}{\partial m} - \langle \bar{\Psi} \Psi \rangle \right) - \frac{\partial\varepsilon}{\partial\mu} \frac{\partial n}{\partial m} \right] \delta m - \frac{\langle \bar{\Psi} \Psi \rangle}{2} \frac{\partial n}{\partial\mu} \Xi, \\
\left(\frac{\partial\varepsilon}{\partial T} \frac{\partial n}{\partial\mu} - \frac{\partial\varepsilon}{\partial\mu} \frac{\partial n}{\partial T} \right) \delta\mu &= \left[\frac{\partial n}{\partial T} \left(-\frac{\partial\varepsilon}{\partial m} + \langle \bar{\Psi} \Psi \rangle \right) + \frac{\partial\varepsilon}{\partial T} \frac{\partial n}{\partial m} \right] \delta m + \frac{\langle \bar{\Psi} \Psi \rangle}{2} \frac{\partial n}{\partial T} \Xi, \\
\delta u^i &= \frac{\langle \bar{\Psi} \Psi \rangle}{\varepsilon + \mathcal{P}} \left(\frac{\tilde{X}^i}{2} - \frac{\tilde{W}^i}{4} \right).
\end{aligned} \tag{4.5}$$

We now determine the transport coefficients appearing in the constitutive relations by equating the last two equations in (4.4) with (4.1) and (3.13). With eqs. (4.5) and the thermodynamical derivatives

$$\begin{aligned}
\frac{\partial\mathcal{P}}{\partial\varepsilon} &= \left(\frac{\partial\mathcal{P}}{\partial T} \frac{\partial n}{\partial\mu} - \frac{\partial\mathcal{P}}{\partial\mu} \frac{\partial n}{\partial T} \right) / \left(\frac{\partial\varepsilon}{\partial T} \frac{\partial n}{\partial\mu} - \frac{\partial\varepsilon}{\partial\mu} \frac{\partial n}{\partial T} \right), \\
\frac{\partial\mathcal{P}}{\partial n} &= \left(-\frac{\partial\mathcal{P}}{\partial T} \frac{\partial\varepsilon}{\partial\mu} + \frac{\partial\mathcal{P}}{\partial\mu} \frac{\partial\varepsilon}{\partial T} \right) / \left(\frac{\partial\varepsilon}{\partial T} \frac{\partial n}{\partial\mu} - \frac{\partial\varepsilon}{\partial\mu} \frac{\partial n}{\partial T} \right),
\end{aligned} \tag{4.6}$$

we are left with

$$\begin{aligned}
\tilde{\chi}_\Upsilon &= \langle \bar{\Psi} \Psi \rangle \left(1 - \frac{\partial\mathcal{P}}{\partial\varepsilon} \right), \\
\tilde{\chi}_\Xi &= -\langle \bar{\Psi} \Psi \rangle \left(\frac{1}{2} \frac{\partial\mathcal{P}}{\partial\varepsilon} + \frac{1}{4} \right), \\
\tilde{\eta} &= -\frac{\langle \bar{\Psi} \Psi \rangle}{4}, \\
\tilde{\chi}_W &= \frac{1}{4} \langle \bar{\Psi} \Psi \rangle \frac{n}{\varepsilon + \mathcal{P}}, \\
\tilde{\chi}_X &= -\frac{1}{2} \langle \bar{\Psi} \Psi \rangle \frac{n}{\varepsilon + \mathcal{P}}.
\end{aligned} \tag{4.7}$$

These expressions, which uniquely determine the new transport coefficients in term of the angular momentum density, are one of the main results of this paper.

4.2 The entropy current

Finally we consider the contribution to the entropy current which arises from the partition function in eq. (2.11),

$$W = \int d^2x \sqrt{g_2} \frac{e^\sigma}{T_0} \frac{\partial \mathcal{P}(T, \mu, m)}{\partial m} \delta m, \quad \delta m = -\frac{1}{8} \epsilon^{\mu\nu\rho} T_{\mu\nu\rho}. \quad (4.8)$$

If s denotes the entropy density that follows from the thermodynamic potential at zero derivative order, $s = \partial \mathcal{P} / \partial T$, the standard formula for the entropy yields the contribution

$$S = \frac{\partial(T_0 W)}{\partial T_0} = \int d^2x \sqrt{g_2} \frac{\partial^2 \mathcal{P}}{\partial m \partial T} \delta m = \int d^2x \sqrt{g_2} \frac{\partial s}{\partial m} \delta m. \quad (4.9)$$

Therefore a condition that must be fulfilled by the correction to the entropy current δJ_S^μ is

$$\int d^2x \sqrt{g_2} e^\sigma \delta J_S^0 = \int d^2x \sqrt{g_2} \frac{\partial s}{\partial m} \delta m. \quad (4.10)$$

Let us consider the following tentative expression for the entropy current containing the dependence on the torsion:

$$J_S^\mu = s u^\mu - \frac{\mu}{T} \left(\tilde{\chi}_W \tilde{W}^\mu + \tilde{\chi}_X \tilde{X}^\mu \right) + (b_\Xi \Xi + b_\Upsilon \delta m) u^\mu + d_W \tilde{W}^\mu + d_X \tilde{X}^\mu. \quad (4.11)$$

It follows then from this form of the entropy current that the four coefficients b_Ξ , b_Υ , d_W and d_X are determined, in a unique way, by the condition

$$\begin{aligned} \delta J_S^0 &= s \delta u^0 + e^{-\sigma} \left(\frac{\partial s}{\partial T} \delta T + \frac{\partial s}{\partial \mu} \delta \mu \right) \\ &\quad - \frac{\mu}{T} \left(\tilde{\chi}_W \tilde{W}^\mu + \tilde{\chi}_X \tilde{X}^\mu \right) + (b_\Xi \Xi + b_\Upsilon \delta m) u^\mu + d_W \tilde{W}^\mu + d_X \tilde{X}^\mu \\ &= e^{-\sigma} \frac{\partial s}{\partial m} \delta m. \end{aligned} \quad (4.12)$$

Then one finds that

$$\begin{aligned} b_\Xi &= \frac{\langle \bar{\Psi} \Psi \rangle}{2T}, \\ b_\Upsilon &= \frac{2 \langle \bar{\Psi} \Psi \rangle}{T}, \\ d_W &= \frac{\langle \bar{\Psi} \Psi \rangle}{4T}, \\ d_X &= -\frac{\langle \bar{\Psi} \Psi \rangle}{2T}. \end{aligned} \quad (4.13)$$

Note that in order to obtain these results it is necessary to use the values of δT and $\delta \mu$ given by eqs. (4.5), as well as the expressions for $\delta u^0 = -a_i \delta u^i$, \tilde{W}^0 , \tilde{X}^0 , and the thermodynamical identities

$$\frac{\partial s}{\partial \varepsilon} = \frac{1}{T}, \quad \frac{\partial s}{\partial n} = -\frac{\mu}{T}, \quad \varepsilon + \mathcal{P} = Ts + \mu n. \quad (4.14)$$

Thus one has

$$\frac{\partial s}{\partial m} = \frac{1}{T} \frac{\partial \varepsilon}{\partial m} - \frac{\mu}{T} \frac{\partial n}{\partial m}. \quad (4.15)$$

It is remarkable that the results in eqs. (4.13) imply that the spatial part of δJ_S^μ is identically zero at equilibrium:

$$\begin{aligned} \delta J_S^i &= s \delta u^i + \left(-\frac{\mu}{T} \tilde{\chi}_W + d_W \right) \tilde{W}^i + \left(-\frac{\mu}{T} \tilde{\chi}_X + d_X \right) \tilde{X}^i \\ &= s \delta u^i + \frac{s \langle \tilde{\Psi} \Psi \rangle}{\varepsilon + \mathcal{P}} \left(\frac{\tilde{W}^i}{4} - \frac{\tilde{X}^i}{2} \right) \\ &= 0. \end{aligned} \quad (4.16)$$

Thus the condition of no entropy production is satisfied since the divergence of δJ_S^μ is necessarily zero under these conditions.²

5 Conclusion

In this paper, we have studied the modifications of the stress tensor of an ideal gas of Dirac fermions coupled minimally to torsion. We have concentrated on the part coming from the equilibrium thermal partition function to linear order in the torsion, which in $2 + 1$ dimensions may be easily expressed in terms of an effective mass shift. Differentiating with respect to the metric we have obtained different types of background data linear in the torsion, including two pseudo-scalars, two pseudo-vectors and a pseudo-tensor field. We have also found an expression for the entropy current consistent with zero entropy production at equilibrium. It turns out that the non-dissipative part of the constitutive relations involve five susceptibilities, four of them expressing the modifications of the stress tensor and the current in response to the applied torsion. Because of the way the torsion affects the covariant derivative of the fluid velocity, the remaining susceptibility may be written as the proportionality factor that expresses the equilibrium response to the parity odd shear tensor. Therefore it would be identified as the Hall viscosity.

Throughout this paper we have assumed that the fermions are minimally coupled to torsion. Considering non-minimal couplings introduces an additional parameter on which some of response functions may depend. Specifically, one has to make the replacement $\delta m \rightarrow \xi \delta m$ for arbitrary ξ , therefore leading at first sight to $\tilde{\eta} \rightarrow \xi \tilde{\eta}$. This has been used to question the existence of a connection between Hall viscosity and torsion response [14]. However, the analysis sketched above ignores the fact, pointed out in section 3, that there are actually two independent tensors, given by eqs. (3.10) and (3.14), which coincide at equilibrium (see eq. (3.12)). Therefore, both tensors must be used in the modified constitutive relation

$$\delta T_{\mu\nu} = -\tilde{\eta} \tilde{\sigma}_{\mu\nu} - \tilde{\eta}_2 \tilde{t}_{(2)}^{\mu\nu}. \quad (5.1)$$

²If torsion is present, the covariant differentiation is defined with respect to the non-symmetric connection $\Gamma_{\mu\nu}^\sigma(g) - K_{\mu\nu}^\sigma$. However, it is possible to define a modified divergence (see e.g., ref. [19]) according to $\tilde{\nabla}_\mu \equiv \nabla_\mu + T_{\mu\nu}{}^\nu$, which for a vector field V^μ reduces to the torsion-free result $\tilde{\nabla}_\mu V^\mu = (\sqrt{-g})^{-1} \partial_\mu (\sqrt{-g} V^\mu)$. Thus the conservation of the entropy current found above must correspond to $\tilde{\nabla}_\mu J_S^\mu = 0$.

Now, as the covariant derivative in the presence of torsion involves the affine connection $\Gamma_{\mu\nu}^{\sigma} = \Gamma_{\mu\nu}^{\sigma}(g) - K_{\mu\nu}^{\sigma}$, rather than $\Gamma_{\mu\nu}^{\sigma}(g) - \xi K_{\mu\nu}^{\sigma}$, the parity odd shear tensor $\tilde{\sigma}^{\mu\nu}$ in the last term of eq. (4.3) remains unchanged even if $\xi \neq 1$, and the same applies to $\tilde{\eta}$ which cannot depend on ξ . Therefore the torsional response in the presence of non-minimal coupling requires that

$$\tilde{\eta}_2 = (1 - \xi) \frac{\langle \bar{\Psi} \Psi \rangle}{4}. \quad (5.2)$$

In this way the Hall viscosity remains intact, and in the absence of torsion is the only non-dissipative term in the constitutive relation.

Finally, it would be desirable to combine the results from holographic models with those presented here, but it does not appear to be an easy task. The main reason is that there is no unique way to produce Hall viscosity and angular momentum density in holography [4, 21–23]. For example, the holographic model considered in ref. [4] gives rise to nonzero angular momentum density but vanishing Hall viscosity, while for the model considered in ref. [22] the ratio between these quantities is compatible with the universal value 1/2 from field theory. In this respect, it may be interesting to note that for a gapped system, such as a free massive Dirac fermion, the spin density of eq. (2.13) is nonzero at $\mu = 0$, while the angular momentum density $\langle \ell \rangle$ from certain holographic models [4] vanishes at $\mu = 0$. Thus the relation $\langle \ell \rangle = \frac{1}{2} \partial \mathcal{P} / \partial m$ that determines many features of the results in this paper does not seem to be satisfied in those models. Further studies are required to understand all these points.

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