Lawvere-Tierney sheaves, factorization systems, sections and j-essential monomorphisms in a topos

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Abstract

Let j be a Lawvere-Tierney topology (a topology, for short) on an arbitrary topos $\mathcal{E}$, B an object of $\mathcal{E}$, and $j_B = j \times 1_B$ the induced topology on the slice topos $\mathcal{E}/B$. In this manuscript, we analyze some properties of the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$ which have deal with topology. Then for a left cancelable class $\mathcal{M}$ of all j-dense monomorphisms in a topos $\mathcal{E}$, we achieve some necessary and sufficient conditions for that $(\mathcal{M}, \mathcal{M}^{\perp})$ is a factorization system in $\mathcal{E}$, which is related to the factorization systems in slice topoi $\mathcal{E}/B$, where B ranges over the class of objects of $\mathcal{E}$. Among other things, we prove that an arrow $f : X \to B$ in $\mathcal{E}$ is a j_B-sheaf whenever the graph of f, is a section in $\mathcal{E}/B$ as well as the object of sections $S(f)$ of f, is a j-sheaf in $\mathcal{E}$. Furthermore, we introduce a class of monomorphisms in $\mathcal{E}$, which we call them j-essential. Some equivalent forms of those and some of their properties are presented. Also, we prove that any presheaf in a presheaf topos has a maximal essential extension. Finally, some similarities and differences of the obtained result are discussed if we put a (productive) weak topology j, studied by some authors, instead of a topology.

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1 Introduction and background

A Lawvere-Tierney topology is a logical connective for modal logic. Recently, applications of Lawvere-Tierney topologies in broad topics such as measure theory [7] and quantum Physics [14, 15] are observed. In the spacial case, considerable work has been presented that is dedicated to the study of (weak) Lawvere-Tierney topology on a presheaf topos on a small category and especially on a monoid, see [6, 5]. It is clear that Lawvere-Tierney sheaves in a topos are exactly injective objects (of course, with respect to dense monomorphisms, not to merely monomorphisms) which are separated too. Injectivity with respect to a class $\mathcal{M}$ of morphisms in a slice category $\mathcal{C}/B$ (which its objects are $\mathcal{C}$-arrows with codomain B) has been studied in extensive form, for example we refer the reader to [1, 3]. From this perspective, in this paper we will establish some categorical characterizations of injectives in slice topoi to sheaves. The object of sections $S(f)$ of f is a notion which in [3] it is related to injective objects in a slice category. This object is very useful in synthetic differential geometry (or SDG, for short) (for details, see [11]). For example, considering D as infinitesimals, for any micro-linear object M we have:

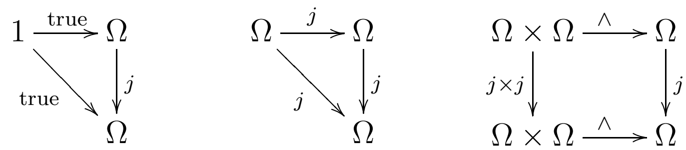
- Let τ be the tangent bundle on M, i.e., $\tau : M^D \to M$, which is defined by $\tau(t) = t(0)$. Then $S(\tau)$ is all vector fields on M.

- Consider $\eta : M^{D\times D} \to M$ which assigns to any micro-square Q of $M^{D\times D}$, the element $Q(0,0)$. Then, $S(\eta)$ is all distributions of dimension 2 on M.

Throughout this paper, $\mathcal{E}$ is a (elementary) topos, two objects $0, 1$ are the initial and terminal objects and the object Ω together with the arrow $1 \overset{\text{true}}{\rightarrowtail} \Omega$ is the subobject classifier of $\mathcal{E}$. Also, the arrow $\wedge : \Omega \times \Omega \to \Omega$ is the meet operation on Ω. Now, we express some basic concepts from [12] which will be needed in sequel.

Definition 1.1. A *Lawvere-Tierney topology* on $\mathcal{E}$ is a map $j : \Omega \to \Omega$ in $\mathcal{E}$ satisfies the following properties

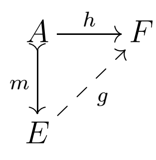
(a) $j \circ \text{true} = \text{true}$; (b) $j \circ j = j$; (c) $j \circ \wedge = \wedge \circ (j \times j)$;



Form now on, we say briefly to a Lawvere-Tierney topology on $\mathcal{E}$, a *topology* on $\mathcal{E}$.

Recall [12] that topologies on $\mathcal{E}$ are in one to one correspondence with universal closure operators. For a topology j on $\mathcal{E}$, considering $\overline{(\,\cdot\,)}$ as the universal closure operator corresponding to j, a monomorphism $k : A \rightarrowtail C$ in $\mathcal{E}$ is called j*-dense* whenever $\overline{A} = C$, as two subobjects of C. Also, we say that k is j*-closed* if we have $\overline{A} = A$, again as subobjects of C.

Definition 1.2. For a topology j on $\mathcal{E}$, an object F of $\mathcal{E}$ is called *a* j*-sheaf* whenever for any j-dense monomorphism $m : A \rightarrowtail E$, one can uniquely extend any arrow $h : A \to F$ to a map g on all of E,



(1)

We say that F is j*-separated* if the arrow g exists in (1), it is unique.

We will denote the full subcategories of $\mathcal{E}$ consisting of j-sheaves and j-separated objects as $\mathbf{Sh}_j(\mathcal{E})$ and $\mathbf{Sep}_j(\mathcal{E})$, respectively.

We now briefly describe the contents of other sections. We start in Section 2, to study basic properties of the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$, for any object B of $\mathcal{E}$, along with the unique map $!_B : B \to 1$. Afterwards, we would like to achieve, for a left cancelable class $\mathcal{M}$ of all j-dense monomorphisms in a topos $\mathcal{E}$, some necessary and sufficient conditions for that $(\mathcal{M}, \mathcal{M}^{\perp})$ to be a factorization system in $\mathcal{E}$, which is related to the factorization systems in slice topoi $\mathcal{E}/B$. In section 3, among other things, we prove that an arrow $f : X \to B$ in $\mathcal{E}$ is a j_B-sheaf whenever the graph of f, is a section in $\mathcal{E}/B$ as well as the object of sections $S(f)$ of f, is a j-sheaf in $\mathcal{E}$. In section 4, we introduce a class of monomorphisms in an elementary topos $\mathcal{E}$, which we call them 'j-essential monomorphisms'. We present some equivalent forms of these and some of their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension. It is shown that the functor Π_B reflects j-essential extensions. It is seen that some of these results hold for a *(productive) weak topology* j, studied in [10], instead of a topology as well.

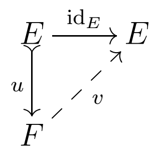
2 Pullback functors, left cancelable dense monomorphisms and factorization systems

The purpose of this section is to present some basic properties of the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$, for any object B of $\mathcal{E}$, along with the unique map $!_B : B \to 1$. Afterwards, for a left cancelable class $\mathcal{M}$ of all j-dense monomorphisms in a topos $\mathcal{E}$ we achieve some necessary and sufficient conditions for that $(\mathcal{M}, \mathcal{M}^{\perp})$ to be a factorization system in $\mathcal{E}$, which is related to the factorization systems in slice topoi $\mathcal{E}/B$.

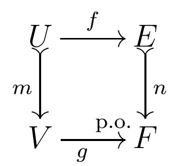
To begin with, the following lemma characterizes sheaves in a topos $\mathcal{E}$.

Lemma 2.1. *Let j be a topology on $\mathcal{E}$. Then an object E of $\mathcal{E}$ is j-sheaf iff E is j-unique absolute retract; that is, any j-dense monomorphism $u : E \rightarrowtail F$, has a unique retraction $v : F \to E$.*

Proof. *Necessity.* Since E is a j-sheaf, for any j-dense monomorphism $u : E \rightarrowtail F$, corresponding to the identity map $\mathrm{id}_E : E \to E$ there exists a unique map $v : F \rightarrowtail E$ such that the following diagram commutes.

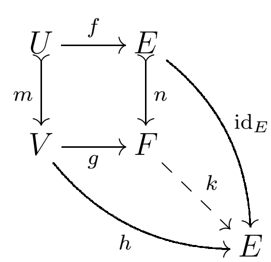


Sufficiency. For each j-dense monomorphism $m : U \rightarrowtail V$ and any map $f : U \to E$, we construct the following pushout diagram in $\mathcal{E}$.



(2)

Since in any topos pushouts transfer j-dense monomorphisms (see [9]), so, in (2), n is j-dense and hence by assumption, there exists a unique retraction $p : F \to E$ such that $pn = \mathrm{id}_E$. Now, for the the arrow $pg : V \to E$ we have $pgm = pnf = \mathrm{id}_E f = f$. To prove that $pg : V \to E$ with this property is unique, let $h : V \to E$ be an arrow in $\mathcal{E}$ in such a way that $hm = f$. Then, in the pushout diagram (2), according to the maps $h : V \to E$ and $\mathrm{id}_E : E \to E$, there exists a unique map $k : F \to E$ such that $kn = \mathrm{id}_E$ and $kg = h$.



Now, k is a retraction of j-dense monomorphism n, so by hypothesis we get $p = k$. Consequently, $pg = kg = h$. $\square$

For an object B of $\mathcal{E}$, we consider the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$ along with the unique map $!_B : B \to 1$, which assigns to any A of $\mathcal{E}$, the second projection $\Pi_B(A) = \pi_B^A : A \times B \to B$ and to any $f : A \to C$, the arrow $f \times \mathrm{id}_B : A \times B \to C \times B$ in $\mathcal{E}$ such that $\pi_B^C(f \times \mathrm{id}_B) = \pi_B^A$. Recall [12] that the object π_B^Ω together with the arrow

$$\mathrm{true} \times \mathrm{id}_B : \mathrm{id}_B \longrightarrow \pi_B^\Omega$$

is the subobject classifier of the slice topos $\mathcal{E}/B$. Also, in a similar vein, we can observe that the meet operation $\wedge_B$ on π_B^Ω is the arrow $\wedge \times 1_B$ in $\mathcal{E}$ such that $\pi_B^\Omega(\wedge \times 1_B) = \pi_B^{\Omega\times\Omega}$,

$$\begin{array}{ccc} \Omega \times \Omega \times B & \xrightarrow{\wedge\times 1_B} & \Omega \times B \\ & {\scriptstyle \pi_B^{\Omega\times\Omega}} \searrow & \downarrow {\scriptstyle \pi_B^\Omega} \\ & & B. \end{array}$$

Now, by Definition 1.1, we easily get the following lemma.

Lemma 2.2. *Let B be any object in a topos $\mathcal{E}$. Then any topology $k : \pi_B^\Omega \to \pi_B^\Omega$ on $\mathcal{E}/B$ is a pair (l, π_B^Ω), for some arrow $l : \Omega \times B \to \Omega$ in $\mathcal{E}$ satisfies the following conditions (as arrows in $\mathcal{E}$)*
(1) $l \circ (l, \pi_B^\Omega) = l$;
(2) $l \circ (\mathrm{true} \times 1_B) = \mathrm{true} \circ !_B$;
(3) $l \circ \wedge_B = \wedge \circ (l \circ (\pi_1, \pi_3), l \circ (\pi_2, \pi_3))$, *where π_i is the i-th projection on $\Omega \times \Omega \times B$, for $i = 1, 2, 3$.*

By Lemma 2.2, for each topology j on $\mathcal{E}$, considering $l = j \circ \pi_\Omega^B$, it is easily seen that $j \times 1_B = (l, \pi_B^\Omega)$ is a topology on $\mathcal{E}/B$ which we denote it by j_B. In this case j_B is called the *induced topology* on $\mathcal{E}/B$ by j.

One can simply see that if an arrow k is a monomorphism in $\mathcal{E}/B$, then k as an arrow in $\mathcal{E}$, is too. Also, for each monomorphism $k : f \rightarrowtail g$ in $\mathcal{E}/B$, where $f : X \to B$ and $g : Y \to B$ in $\mathcal{E}$, we can observe

$$\widetilde{f} \overset{\widetilde{k}}{\rightarrowtail} g = (\overline{X} \xrightarrow{g\overline{k}} B) \overset{\overline{k}}{\rightarrowtail} g, \tag{3}$$

where $\overline{(\cdot)}$ and $\widetilde{(\cdot)}$ are the universal closure operators corresponding to j and j_B on topoi $\mathcal{E}$ and $\mathcal{E}/B$, respectively, in which whole and the middle squares of the following diagram are pullbacks in $\mathcal{E}$,

$$\begin{array}{cccccc} \overline{X} & & & \longrightarrow & & 1 \\ \downarrow {\scriptstyle \overline{k}} & & X & \longrightarrow & 1 & \downarrow {\scriptstyle \mathrm{true}} \\ & & \downarrow {\scriptstyle k} & & \downarrow {\scriptstyle \mathrm{true}} & \\ Y & = & Y & \xrightarrow[\mathrm{char}(k)]{} & \Omega & \xrightarrow{j} \Omega \end{array} \tag{4}$$

(for more details, see [12]). One can construct $\widetilde{k}$ in $\mathcal{E}/B$, similar to the above diagram.

Here, we proceed to improve [2, Vol. III, Proposition 9.2.5] as follows:

Lemma 2.3. *Let j be a topology in a topos $\mathcal{E}$. For every object B of $\mathcal{E}$, the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$ preserves and reflects: denseness (closeness) and j-separated objects (j-sheaves).*

Proof. Let j be a topology on $\mathcal{E}$ and B an object of $\mathcal{E}$. Preserving dense (closed) monomorphisms and sheaves (separated objects) in $\mathcal{E}$ by the pullback functor Π_B, is standard and may be found in [2, Vol. III, Proposition 9.2.5]. To prove the rest of lemma, here we just show that Π_B reflects dense (closed) monomorphisms. To verify this claim, let $g : A \to C$ be an arrow in $\mathcal{E}$ for which $\Pi_B(g)$ is a j_B-dense (j_B-closed) monomorphism. We show that g is j-dense (j-closed) monomorphism. As $\Pi_B(g) = g \times \mathrm{id}_B$ being monomorphism in $\mathcal{E}/B$, the arrow g is monomorphism in $\mathcal{E}$ as well. For, let f, h in $\mathcal{E}$ be two arrows such that $gf = gh$, we will have

$$\begin{aligned} gf = gh &\implies (g \times \mathrm{id}_B)(f \times \mathrm{id}_B) = (g \times \mathrm{id}_B)(h \times \mathrm{id}_B) \\ &\implies f \times \mathrm{id}_B = h \times \mathrm{id}_B \quad (g \times \mathrm{id}_B \text{ is a monomorphism}) \\ &\implies f = h. \end{aligned}$$

Considering $\overline{(\,\cdot\,)}$ and $\widetilde{(\,\cdot\,)}$ as the universal closure operators corresponding to j and j_B, respectively. We get

$$\begin{aligned} \widetilde{\Pi_B(g)} &= \widetilde{g \times \mathrm{id}_B} \\ &= \overline{g \times \mathrm{id}_B} \qquad\qquad \text{(by} \quad (3)) \\ &= \overline{g} \times \mathrm{id}_B, \end{aligned}$$

where the last equality is true since we have $g \times \mathrm{id}_B = (\pi^B_C)^{-1}(g)$, and because of stability of universal closure operators under pullbacks we get $\overline{(\pi^B_C)^{-1}(g)} = (\pi^B_C)^{-1}(\overline{g})$. The above equalities imply that if $\Pi_B(g)$ is j_B-dense (j_B-closed) monomorphism in $\mathcal{E}/B$, then g is j-dense (j-closed) monomorphism in $\mathcal{E}$. $\square$

For any topology j on a topos $\mathcal{E}$, consider $\mathcal{M}$ as the class of all j-dense monomorphisms in $\mathcal{E}$. Also, we denote by $\mathcal{M}^{\perp}$ the class of all arrows $g : C \to D$ in $\mathcal{E}$ such that for any $f : A \to E$ in $\mathcal{M}$ and every commutative square as in

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle w} & \downarrow{\scriptstyle g} \\ E & \xrightarrow[v]{} & D \end{array} \tag{5}$$

there exists a unique arrow $w : E \to C$ in (5) such that the resulting triangles are commutative. In this case, we say that g is *right orthogonal* to f. Moreover, we say that the pair $(\mathcal{M}, \mathcal{M}^{\perp})$ forms a *factorization system* in $\mathcal{E}$ if any arrow f in $\mathcal{E}$ factors as $f = me$, where $m \in \mathcal{M}$ and $e \in \mathcal{M}^{\perp}$ (for more information, see [1]).

Lemma 2.4. *Let j be a topology on a topos $\mathcal{E}$. Then for each object B of $\mathcal{E}$, we have $\mathcal{M}_B^{\perp} \subseteq \mathcal{M}^{\perp}$, where $\mathcal{M}_B$ is the class of all j_B-dense monomorphisms in $\mathcal{E}/B$.*

Proof. By Lemma 2.3 we get $\mathcal{M}_B \subseteq \mathcal{M}$. To reach the conclusion, let $h : f \to g$ be an arrow in $\mathcal{M}_B^{\perp}$, where $f : D \to B$ and $g : E \to B$ are arrows in $\mathcal{E}$. Now, consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & D \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[v]{} & E \end{array} \tag{6}$$

where $m : A \to C$ is in $\mathcal{M}$. Since by Lemma 2.3 the arrow $m : fu \to gv$ in $\mathcal{E}/B$ belongs to $\mathcal{M}_B$ and $h \in \mathcal{M}_B^{\perp}$, there exists a unique arrow $w : gv \to f$ in $\mathcal{E}/B$ such that the following diagram commutes

$$\begin{array}{ccc} fu & \xrightarrow{u} & f \\ {\scriptstyle m}\downarrow & \nearrow{\scriptstyle w} & \downarrow{\scriptstyle h} \\ gv & \xrightarrow[v]{} & g \end{array} \tag{7}$$

The arrow $w : C \to D$ (as an arrow in $\mathcal{E}$) which commutes the resulting triangulares, is unique in the diagram (6). To prove this, let $k : C \to D$ be an arrow in $\mathcal{E}$ such that $km = u$ and $hk = v$. Now, we have $fk = (gh)k = gv$, so $k : gv \to f$ is an arrow in $\mathcal{E}/B$ making all triangles in (7) commutative. Thus, $k = w$ and the proof is complete. □

Definition 2.5. Let j be a topology on a topos $\mathcal{E}$. We say that $\mathcal{E}$ *has enough j-sheaves* if for every object A of $\mathcal{E}$ there is a j-dense monomorphism $A \rightarrowtail F$ where F is a j-sheaf.

Following [1] a class $\mathcal{M}$ of morphisms in $\mathcal{E}$ is a *left cancelable class* if $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$. In the following, we summarize the relation between left cancelable j-dense monomorphisms and factorization systems in a topos $\mathcal{E}$ and its slices.

Theorem 2.6. *Let j be a topology on a topos $\mathcal{E}$. Assume that for any object B of $\mathcal{E}$, the class $\mathcal{M}_B$ of all j_B-dense monomorphisms in $\mathcal{E}/B$ be left cancelable. Then the following are equivalent:*
(i) *for any object B of $\mathcal{E}$, $(\mathcal{M}_B, \mathcal{M}_B^{\perp})$ is a factorization system in $\mathcal{E}/B$;*
(ii) *for any object B of $\mathcal{E}$, $\mathcal{E}/B$ has enough j_B-sheaves;*
(iii) *for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B-separated;*
(iv) *for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B-sheaf;*
(v) *any object of $\mathcal{E}$ is j-sheaf;*
(vi) *any object of $\mathcal{E}$ is j-separated;*
(vii) *$\mathcal{E}$ has enough j-sheaves;*
(viii) *$(\mathcal{M}, \mathcal{M}^{\perp})$ is a factorization system in $\mathcal{E}$.*

Proof. That any j-sheaf is j-separated in $\mathcal{E}$ yields that (v) $\Longrightarrow$ (vi) holds.

(vi) $\Longrightarrow$ (v). That any object of $\mathcal{E}$ is j-separated it follows that $\mathbf{Sep}_j(\mathcal{E})$ is the topos $\mathcal{E}$ and then, every j-separated object is a j-sheaf as in [8, Theorem 2.1].

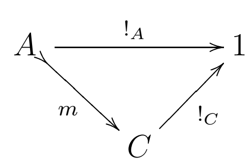
(iii) $\Longrightarrow$ (vi). Setting $B = 1$, then any object of $\mathcal{E}$ is j-separated.

(vi) $\Longrightarrow$ (iii). The claim follows immediately from the fact that for any object B of $\mathcal{E}$,

$$\mathbf{Sep}_{j_B}(\mathcal{E}/B) \cong \mathbf{Sep}_j(\mathcal{E})/B.$$

(see also [9]).

(viii) $\Longrightarrow$ (vii). By (viii), for any object A of $\mathcal{E}$, the unique arrow $!_A : A \to 1$ factors as



where $!_C \in \mathcal{M}_1^{\perp} = \mathcal{M}^{\perp}$ and $m \in \mathcal{M}_1 = \mathcal{M}$. We remark that it is easy to check that for any object B of $\mathcal{E}$, j_B-sheaves in $\mathcal{E}/B$ are exactly the class of all objects of $\mathcal{E}/B$ which belong to $\mathcal{M}_B^{\perp}$. Since $!_C$ is an object in $\mathcal{E}/1 = \mathcal{E}$ which is in $\mathcal{M}_1^{\perp}$, so $!_C$ is a j_1-sheaf, or equivalently, C is a j-sheaf.

(vii) $\Longrightarrow$ (viii). Consider an arrow $f : A \to B$ in $\mathcal{E}$. By using (vii), there exists a j-dense monomorphism $\iota : A \rightarrowtail F$, where F is a j-sheaf in $\mathcal{E}$. Now, we factor f as the composite arrow $A \stackrel{(\iota,f)}{\longrightarrow} F \times B \stackrel{\pi_B^F}{\longrightarrow} B$. Since $\pi_F^B(\iota, f) = \iota \in \mathcal{M}$ and $\mathcal{M}$ is a left cancelable class, so $(\iota, f) \in \mathcal{M}$. Also, F being j-sheaf, by Lemma 2.3 we have π_B^F is a j_B-sheaf in $\mathcal{E}/B$. By Lemma 2.4 we have $\pi_B^F \in \mathcal{M}_B^{\perp} \subseteq \mathcal{M}^{\perp}$, as required.

(vi) $\Longrightarrow$ (vii). First of all we know that any j-separated object of $\mathcal{E}$ can be embedded into a j-sheaf (see, e.g. [12, Proposition V.3.4]). Let A be an object of $\mathcal{E}$. Then, by assumption A is j-separated, and there exists an embedding $A \stackrel{\iota}{\rightarrowtail} F$, where F is a j-sheaf. Now, take the closure of A in F. Since $\overline{A}$ is closed in F, by [12, Lemma V.2.4], it is a j-sheaf. Since A is j-dense in $\overline{A}$ we get the result.

(vii) $\Longrightarrow$ (vi). By assumption for any object A of $\mathcal{E}$, there is a j-dense monomorphism $A \rightarrowtail F$ in $\mathcal{E}$, where F is a j-sheaf. Since any subobject of a j-sheaf is j-separated so A is j-separated.

For any object B of $\mathcal{E}$, setting $\mathcal{E}/B$ instead of $\mathcal{E}$ in (v), (vi), (vii) and (viii), we drive (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv). □

In the following, we will introduce two main classes of dense monomorphisms in a topos $\mathcal{E}$.

Remark 2.7. By diagram (4), one can easily obtain:
(i) Let $j = \mathrm{id}_{\Omega}$ be the trivial topology on $\mathcal{E}$. Then j-dense monomorphisms are only the identity maps. Therefore, any object of $\mathcal{E}$ is a j-sheaf. Also, j-closed monomorphisms are exactly all monomorphisms.
(ii) Let j be the topology $\mathrm{true} \circ !_{\Omega}$ on $\mathcal{E}$, that is, the characteristic map of id_{Ω}. Then, j-dense monomorphisms are exactly all monomorphisms. Furthermore, j-closed monomorphisms are just the identity maps.

Recall [1] that $(Mono, Mono^{\Box})$ is a weak factorization system in any topos $\mathcal{E}$, where $Mono$ is the class of all monomorphisms in $\mathcal{E}$. By Remark 2.7(ii), the class $Mono$ is the class of all j-dense monomorphisms with respect to the topology $j = \mathrm{true} \circ !_\Omega$ on $\mathcal{E}$. Since the class $Mono$ is left cancelable, so we can obtain a special case of Theorem 2.6 as follows. (Notice that by Lemma 2.3 for the topology $j = \mathrm{true} \circ !_\Omega$ and any object B of $\mathcal{E}$, the class $Mono_B$ will be all monomorphisms in $\mathcal{E}/B$.)

Corollary 2.8. *For the topology $j = \mathrm{true} \circ !_\Omega$ on a topos $\mathcal{E}$, the following are equivalent:*
(i) *for any object B of $\mathcal{E}$, $(Mono_B, Mono_B^\perp)$ is a factorization system in $\mathcal{E}/B$;*
(ii) *for any object B of $\mathcal{E}$, $\mathcal{E}/B$ has enough j_B-sheaves;*
(iii) *for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B-sheaf;*
(iv) *for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B-separated;*
(v) *any object of $\mathcal{E}$ is j-sheaf;*
(vi) *any object of $\mathcal{E}$ is j-separated;*
(vii) *$\mathcal{E}$ has enough j-sheaves;*
(viii) *$(Mono, Mono^\perp)$ is a factorization system in $\mathcal{E}$.*

3 Sheaves and sections of an arrow

In this section, among other things, we investigate a relationship between sheaves and sections of an arrow in a topos $\mathcal{E}$. We start to remind [3] that for any object B of $\mathcal{E}$, the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$ has a right adjoint $S : \mathcal{E}/B \to \mathcal{E}$ as for any $f : X \to B$ we have the following pullback

$$\begin{array}{ccc} S(f) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle i_B} \\ X^B & \xrightarrow[f^B]{} & B^B \end{array} \qquad (8)$$

where i_B is the transpose of $\mathrm{id}_B : 1 \times B \cong B \to B$ and f^B is the transpose of the composition arrow $X^B \times B \xrightarrow{ev_X} X \xrightarrow{f} B$ by the exponential adjunction $(-) \times B \dashv (-)^B$; that is, $ev_B(i_B \times \mathrm{id}_B) = \mathrm{id}_B$ and $ev_B(f^B \times \mathrm{id}_B) = f ev_X$, where the natural transformation $ev : (-)^B \times B \to (-)$ is the counit of the exponential adjunction. In fact, in the Mitchell-Bénabou language, we can write

$$S(f) = \{h \mid (\forall c \in B)\ f \circ (h(c)) = c\}.$$

This means that we can call $S(f)$ *the object of sections* of f.

Since any retract of an object in a topos (or in an arbitrary category) is an equalizer, so the topos $\mathbf{Sh}_j(\mathcal{E})$ is closed under retracts. Furthermore, as $\Pi_B \dashv S$, by Lemma 2.3 we have that the pullback functor Π_B preserves dense monomorphisms, so S preserves sheaves (for details, see [9, Corollary 4.3.12]). (Roughly, for any object $B \in \mathcal{E}$ and any adjoint $F \dashv G : \mathcal{E} \to \mathcal{E}/B$ one can easily checked that the functor G preserves sheaves whenever F preserves dense monomorphisms.)

In the following theorem we will find a relationship between sheaves in $\mathcal{E}/B$ and the object of sections of an arrow.

Theorem 3.1. *Let j be a topology on a topos $\mathcal{E}$ and $f : X \to B$ be an object of $\mathcal{E}/B$. Then, f is a j_B-sheaf in $\mathcal{E}/B$, whenever the graph of f which stands for the monomorphism $(\mathrm{id}_X, f) : f \rightarrowtail \pi_B^X$ in $\mathcal{E}/B$, is a section as well as $S(f)$ is a j-sheaf in $\mathcal{E}$.*

Proof. We recall that in [3] it was proved if (id_X, f) is a section in $\mathcal{E}/B$, then f is a retract of $\pi_B^{S(f)}$ in $\mathcal{E}/B$. As $S(f)$ is a j-sheaf, by Lemma 2.3, $\pi_B^{S(f)}$ is a j_B-sheaf in $\mathcal{E}/B$. But $\mathbf{Sh}_{j_B}(\mathcal{E}/B)$ being closed under retracts, therefore f is a j_B-sheaf in $\mathcal{E}/B$. $\square$

To the converse of Theorem 3.1, that the section functor S preserves sheaves it yields that if $f : X \to B$ be a j_B-sheaf in $\mathcal{E}/B$, then $S(f)$ is a j-sheaf in $\mathcal{E}$. Also, by Remark 2.7(ii), for $j = \mathrm{true} \circ !_\Omega$, the monomorphism $(\mathrm{id}_X, f) : f \rightarrowtail \pi_B^X$ is j_B-dense in $\mathcal{E}/B$ and then for a j_B-sheaf $f : X \to B$, it will be a section in $\mathcal{E}/B$.

In the rest of this section, for a small category $\mathcal{C}$ we restrict our attention to obtain a version of Theorem 3.1 for injective presheaves in trivial slices of the presheaf topos $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$ which is close to the version over j-sheaves for the topology $j = \mathrm{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$. (See Proposition 3.5 below.) Note that the topology $j = \mathrm{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$ is associated to the *chaotic or indiscrete Grothendieck topology* on $\mathcal{C}$.
Recall [12] that in the presheaf topos $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$, the exponential

object G^F is defined in each stage C of $\mathcal{C}$ as $G^F(C) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(Y(C) \times F, G)$, where Y is the Yoneda embedding, that is

$$Y : \mathcal{C} \to \widehat{\mathcal{C}}; \quad Y(C) = \mathrm{Hom}_{\mathcal{C}}(-, C).$$

Now, for an arrow $\alpha : G \to F$ consider the arrows $i_F : 1 \to F^F$ and $\alpha^F : G^F \to F^F$ in $\widehat{\mathcal{C}}$ as the transposes of $\mathrm{id}_F : 1 \times F \cong F \to F$ and $\alpha \circ ev_G : G^F \times F \to F$, respectively, by the exponential adjunction. We can observe

$$\forall C \in \mathcal{C}, \quad (i_F)_C : 1(C) = \{*\} \longrightarrow F^F(C); \quad (i_F)_C(*) = \pi_F^{Y(C)}. \tag{9}$$

Also, for any two objects C, D of $\mathcal{C}$, any γ in $G^F(C)$ and any (k, y) in $Y(C)(D) \times F(D)$ we have

$$(\alpha_C^F(\gamma))_D(k, y) = \alpha_D(\gamma_D(k, y)). \tag{10}$$

Remind that a presheaf G has a (unique) global section which means that in each stage C of $\mathcal{C}$ there is a (unique) element $\theta_C \in G(C)$ in such a way that for any arrow $k : D \to C$ in $\mathcal{C}$ we have

$$G(k)(\theta_C) = \theta_D. \tag{11}$$

Here, we find a special case that the exponential object and the object of sections in $\widehat{\mathcal{C}}$ are exactly similar to **Sets**. First, we express some lemma required to achieve the goal.

Lemma 3.2. *Let j be the topology* $\mathrm{true} \circ !_\Omega$ *on $\widehat{\mathcal{C}}$. Then, the following assertions hold:*
(i) *For any j-sheaf G in $\widehat{\mathcal{C}}$, G has a unique global section. More generally, any injective presheaf G of $\widehat{\mathcal{C}}$ has a global section.*
(ii) *For any family $\{G_\lambda\}_{\lambda \in \Lambda}$ in $\widehat{\mathcal{C}}$, the presheaf $G = \prod_{\lambda \in \Lambda} G_\lambda$ is a j-sheaf (injective) in $\widehat{\mathcal{C}}$ iff for all $\lambda \in \Lambda$, G_λ is a j-sheaf (injective) in $\widehat{\mathcal{C}}$.*

Proof. (i) Let G be a j-sheaf in $\widehat{\mathcal{C}}$ and consider the coproduct object $G \sqcup 1$ in $\widehat{\mathcal{C}}$. By Remark 2.7(ii), there exists a unique natural transformation $\eta : G \sqcup 1 \to G$ in $\widehat{\mathcal{C}}$ such that the following diagram commutes (if G being injective, the arrow η is not necessarily unique)

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id}_G} & G \\ {\scriptstyle\iota}\downarrow & \nearrow_{\eta} & \\ G \sqcup 1 & & \end{array}$$

where $\iota : G \to G \sqcup 1$ is the injection arrow. Now, we will denote $\eta_C(*)$ by an element θ_C in $G(C)$ in each stage C of $\mathcal{C}$. Since $\eta : G \sqcup 1 \to G$ is natural, so for any arrow $k : D \to C$ in $\mathcal{C}$ the following square commutes

$$\begin{array}{ccc} (G \sqcup 1)(D) & \xrightarrow{\eta_D} & G(D) \\ {\scriptstyle (G\sqcup 1)(k)}\uparrow & & \uparrow{\scriptstyle G(k)} \\ (G \sqcup 1)(C) & \xrightarrow{\eta_C} & G(C) \end{array}$$

Then, we have

$$\begin{aligned} G(k)(\theta_C) &= G(k)(\eta_C(*)) \\ &= \eta_D((G \sqcup 1)(k)(*)) \\ &= \eta_D(1(k)(*)) = \theta_D. \end{aligned}$$

This is the required result.

(ii) *Necessity.* Let G be a j-sheaf (injective) in $\widehat{\mathcal{C}}$. For any $\lambda, \mu \in \Lambda$, we define $\alpha^{\lambda\mu} : G_\lambda \to G_\mu$ such that in each stage C of $\mathcal{C}$ and for each $x \in G_\lambda(C)$, we have $\alpha^{\lambda\mu}_C(x) = \theta^\mu_C$, where θ^μ_C is the μ-th component of θ_C corresponding to G in (i). Now, we will show that for any $\lambda, \mu \in \Lambda$, $\alpha^{\lambda\mu}$ is a natural transformation in $\widehat{\mathcal{C}}$, that is for any arrow $k : D \to C$ in $\mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} G_\lambda(D) & \xrightarrow{\alpha^{\lambda\mu}_D} & G_\mu(D) \\ {\scriptstyle G_\lambda(k)}\uparrow & & \uparrow{\scriptstyle G_\mu(k)} \\ G_\lambda(C) & \xrightarrow{\alpha^{\lambda\mu}_C} & G_\mu(C) \end{array}$$

For, consider an element $x \in G_\lambda(C)$ we get

$$\begin{aligned} G_\mu(k)(\alpha^{\lambda\mu}_C(x)) &= G_\mu(k)(\theta^\mu_C) \\ &= \theta^\mu_D \qquad\qquad \text{(by (11))} \\ &= \alpha^{\lambda\mu}_D(G_\lambda(k)(x)). \end{aligned}$$

Now, for any $\lambda \in \Lambda$, consider the family $\{\gamma_\mu : G_\lambda \to G_\mu\}_{\mu\in\Lambda}$ in $\widehat{\mathcal{C}}$ such that for each $\lambda \neq \mu \in \Lambda$ we have $\gamma_\mu = \alpha^{\lambda\mu}$ and $\gamma_\lambda = \mathrm{id}_{G_\lambda}$. Since G

is the product $\prod_{\lambda\in\Lambda} G_\lambda$, so there is a unique natural transformation $\gamma : G_\lambda \to G$ such that $p_\mu\gamma = \gamma_\mu$ and $p_\lambda\gamma = \mathrm{id}_{G_\lambda}$, for all $\lambda, \mu \in \Lambda$ and the projections p_λ. Thus, for any $\lambda \in \Lambda, G_\lambda$ is a retract of the j-sheaf (injective) G and then, G_λ is a j-sheaf (injective).

Sufficiency. By the universal property of the product presheaf G, the unique arrow in the definition of a sheaf is easily follows. $\square$

We recall [12] that in each stage C of $\mathcal{C}$ the object $\Omega(C)$ of $\widehat{\mathcal{C}}$ is the set of all sieves on C. Also, the arrow $\mathrm{true}_C : 1(C) = \{*\} \to \Omega(C)$ assigns to $*$, the *maximal sieve* $t(C)$ of $\Omega(C)$, that is all arrows with codomain C of $\mathcal{C}$.

Remark 3.3. Note that the topology $j = \mathrm{true}\circ !_\Omega$ on $\widehat{\mathcal{C}}$ is the unique topology on $\widehat{\mathcal{C}}$ that satisfies Lemma 3.2. To show this, for a j-sheaf G of $\widehat{\mathcal{C}}$, consider the injection $\iota : G \to G \sqcup 1$ in $\widehat{\mathcal{C}}$. In each stage C of $\mathcal{C}$ we have $\mathrm{char}(\iota)_C(*) = \emptyset$. Now, let j be a topology on $\widehat{\mathcal{C}}$. If ι is j-dense monomorphism, then in each stage C of $\mathcal{C}$ we have $j_C(\emptyset) = t(C)$. Now, for any sieve $S \in \Omega(C)$ by Definition 1.1 we get

$$\begin{aligned} t(C) &= j_C(\emptyset) = j_C(\emptyset \cap S) \\ &= j_C(\emptyset) \cap j_C(S) = t(C) \cap j_C(S) = j_C(S). \end{aligned}$$

Thus, j_C is the constant function on $t(C)$, as required.

Let F be the constant presheaf on a set A. One can easily checked that the exponential adjunction $(-) \times F \dashv (-)^F$ is determined by, for any presheaf G in $\widehat{\mathcal{C}}$, the exponential presheaf G^F assigns to any object C of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathbf{Sets}}(A, G(C))$ and to any arrow $f : C \to D$ of $\mathcal{C}$, the function

$$G^F(f) : \mathrm{Hom}_{\mathbf{Sets}}(A, G(D)) \longrightarrow \mathrm{Hom}_{\mathbf{Sets}}(A, G(C))$$

given by $G^F(f)(g) = G(f) \circ g$. As any function $f : A \to G(C)$ can be considered as a sequence $(x_a)_{a\in A} \in \prod_A G(C)$, it yields that one has

$$\forall C \in \mathcal{C}, \quad G^F(C) \cong \prod_A G(C). \tag{12}$$

By (12), (9) and (10), it is convenient to see that for each arrow $\alpha : G \to F$ in $\widehat{\mathcal{C}}$ in which F stands for the constant presheaf on a set A, we get

$$\forall C \in \mathcal{C}, \quad S(\alpha)(C) \cong \prod_{a \in A} \alpha_C^{-1}(a). \tag{13}$$

Now, we will extract a special case of Theorem 3.1 in $\widehat{\mathcal{C}}$. First, let $\alpha : G \to F$ be an arrow in $\widehat{\mathcal{C}}$ in which F is the constant presheaf on a set A. For each element a of A, consider the subpresheaf H_a of G such that $H_a(C) = \alpha_C^{-1}(a)$, for any object C of $\mathcal{C}$. Since limits in $\widehat{\mathcal{C}}$ are constructed pointwise, so (13) shows that $S(\alpha) \cong \prod_{a \in A} H_a$.

Proposition 3.4. *Let j be the topology* $\mathrm{true} \circ !_\Omega$ *on $\widehat{\mathcal{C}}$ and $\alpha : G \to F$ an arrow in $\widehat{\mathcal{C}}$, where F is the constant presheaf on a set A. Then, α is a j_F-sheaf in $\widehat{\mathcal{C}}/F$ iff the monomorphism* $(\mathrm{id}_G, \alpha) : \alpha \rightarrowtail \pi_F^G$ *is a section in $\widehat{\mathcal{C}}/F$ as well as for any $a \in A$, the subpresheaf H_a of G is a j-sheaf in $\widehat{\mathcal{C}}$.*

Proof. We deduce the result by Theorem 3.1, Lemma 3.2(ii) and (13). □

Since in topoi regular monomorphisms are exactly monomorphisms, so by [3, Theorem 1.2], Lemma 3.2(ii) and (13), the following now gives which we are interested in.

Proposition 3.5. *Let $\alpha : G \to F$ be an arrow in $\widehat{\mathcal{C}}$, where F is the constant presheaf on a set A. Then, α is injective in $\widehat{\mathcal{C}}/F$ iff the monomorphism* $(\mathrm{id}_G, \alpha) : \alpha \rightarrowtail \pi_F^G$ *is a section in $\widehat{\mathcal{C}}/F$ as well as for any $a \in A$, the subpresheaf H_a of G is injective.*

In the case when $\mathcal{C}$ is a monoid, we obtain

Example 3.6. Let M be a monoid and M-**Sets** the topos of all (right) representations of a fixed monoid M. Since M is a small category with just one object, for two M-sets X, B we have $X^B = \mathrm{Hom}_{M-\mathbf{Sets}}(M \times B, X)$, where $M \times B$ has the componentwise action. Hence, by (9) and (10), for any equivariant map $f : X \to B$, in the diagram (8) we observe

$$i_B(*) = \pi_B^M : M \times B \to B, \tag{14}$$

and

$$\forall h \in X^B,\ \forall (m,b) \in M \times B, \quad (f^B(h))(m,b) = fh(m,b). \tag{15}$$

Note that one writes any equivariant map $h : M \times B \to X$ in X^B as a sequence $((x_{m,b})_{b\in B})_{m\in M}$, consisting of elements $x_{m,b} = h(m, b)$ of X, for any $(m, b) \in M \times B$. Also, h being equivariant map means that

$$\forall n, m \in M, \forall b \in B, \quad x_{mn,bn} = x_{m,b}n.$$

Hence, we obtain that X^B is equal to

$$\{((x_{m,b})_b)_m \in \prod_{m\in M}\prod_{b\in B} X \ \mid \forall n, m \in M, \forall b \in B, x_{mn,bn} = x_{m,b}n\}. \quad (16)$$

Now, by (8), (14) and (15) we have

$$\begin{aligned} S(f) &= \{((x_{m,b})_b)_m \in X^B \ \mid f^B(((x_{m,b})_b)_m) = \pi^M_B = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \ \mid ((f(x_{m,b}))_b)_m = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \ \mid \forall m \in M, \forall b \in B, x_{m,b} \in f^{-1}(b)\}, \end{aligned}$$

Hence, by (16) we interpret a simple form of underlying set of the M-set $S(f)$ in the topos M-**Sets** as follows

$$\{((x_{m,b})_b)_m \in \prod_{m\in M}\prod_{b\in B} f^{-1}(b) \mid \forall n, m \in M, \forall b \in B, x_{mn,bn} = x_{m,b}n\}.$$

If B has the trivial action $\cdot$, that is $\cdot = \pi_1 : B \times M \to B$ the first projection, then by (12) and (13) we can obtain $X^B \cong \prod_B X$ and $S(f) \cong \prod_{b\in B} f^{-1}(b)$.

Furthermore, recall [12] that for a group G and two G-sets X, B, we have

$$X^B = \{h : B \to X| \ \ h \text{ is a function}\} \cong \prod_B X \quad (17)$$

as two sets. According to the action on X^B, under the isomorphism (17), the action on $\prod_B X$ is given by $(x_b)_{b\in B} \cdot g = (x_{bg^{-1}} \cdot g)_{b\in B}$, for any $g \in G$ and $(x_b)_{b\in B} \in \prod_B X$. Also, by (17) for any equivariant map $f : X \to B$ in G-**Sets**, in a similar way to (13), we have $S(f) \cong \prod_{b\in B} f^{-1}(b)$.

4 j-essential extensions in a topos

This section is devoted to introduce a class of monomorphisms in an elementary topos, which we call these 'j-essential monomorphisms'.

We present some equivalent forms of these and some their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension.

Remind that a monomorphism $\iota : A \rightarrowtail B$ is called *essential* whenever for each arrow $g : B \to C$ such that $g\iota$ is a monomorphism, then g is a monomorphism also. Now, we define a j-essential monomorphism in a topos $\mathcal{E}$ as follows.

Definition 4.1. For a topology j on $\mathcal{E}$, a monomorphism $\iota : A \rightarrowtail B$ is called *j-essential* whenever it is j-dense as well as essential. In this case, we say that B is a *j-essential extension* of A and we write $A \subseteq_j B$.

We shall say an arrow $f : A \to B$ in $\mathcal{E}$ is *j-dense* whenever the subobject $f(A)$, which is the image of f, is j-dense in B. In this way, any epimorphism in $\mathcal{E}$ becomes j-dense. (For the definition of image of an arrow in a topos, see [12].)

The following gives some equivalent definitions of j-essential monomorphisms in a topos $\mathcal{E}$.

Lemma 4.2. *Let j be a topology on $\mathcal{E}$ and $\iota : A \rightarrowtail B$ a j-dense monomorphism. Then, the following are equivalent:*
(i) *for any $g : B \to C$, g is a monomorphism whenever $g\iota$ is a monomorphism;*
(ii) *for any $g : B \to C$, g is a j-dense monomorphism whenever $g\iota$ is a j-dense monomorphism;*
(iii) *for any $g : B \to C$, g is a monomorphism whenever $g\iota$ is a j-dense monomorphism.*

Proof. (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (ii) are proved by [9, A.4.5.11(iii)].
(ii) $\Longrightarrow$ (iii) is clear.
(ii) $\Longrightarrow$ (i). Consider an arrow $g : B \to C$ for which $g\iota$ is monomorphism. We show that g is monomorphism also. Assume that $B \overset{k}{\twoheadrightarrow} g(B) \overset{m}{\rightarrowtail} C$ is the image factorization of the arrow g. Since $g\iota = m(k\iota)$ and $g\iota$ is monomorphism, it follows that the arrow $k\iota$ is a monomor-

phism. Meanwhile, we get

$$\begin{array}{rcll} g(B) & = & k(B) & \text{(as } k \text{ is epic)} \\ & = & k(\overline{A}) & \text{(as } \iota \text{ is dense)} \\ & \subseteq & \overline{k(A)} & \\ & \subseteq & g(B). & \end{array}$$

Therefore, $g(B) = \overline{k(A)} = \overline{k\iota(A)}$. It follows that the compound monomorphism $k\iota : A \rightarrowtail g(B)$ is dense monomorphism and by (ii), k is also. That k is monomorphism and so isomorphism, yields that g is monomorphism. □

We point out that the proof of (ii) $\Longrightarrow$ (i) of Lemma 4.2 shows that any composite $k\iota$, for an epic k and a dense monomorphism ι, is dense.

The follwing shows that j-essential monomorphisms in $\mathcal{E}$ are closed under composition.

Proposition 4.3. *Let j be a topology on $\mathcal{E}$. For two subobjects $A \overset{\iota}{\rightarrowtail} A' \overset{\iota'}{\rightarrowtail} B$ in $\mathcal{E}$, then $A \subseteq_j B$ iff $A \subseteq_j A'$ and $A' \subseteq_j B$.*

Proof. By [9, 13, A.4.5.11(iii)], one has $\iota'\iota$ is j-dense iff ι' and ι are j-dense.

Necessity. First, by Lemma 4.2(i), we show that $A \subseteq_j A'$. To do so, consider an arrow $f' : A' \to C$ for which $f'\iota$ is a monomorphism. Now, by [12, Corollary IV. 10. 3], the object C can be embedded into an injective object D as in $C \overset{\nu}{\rightarrowtail} D$ and hence there is an arrow $\widetilde{f'} : B \to D$ such that $\widetilde{f'}\iota' = \nu f'$. Since $A \subseteq_j B$ and $\widetilde{f'}\iota'\iota = \nu f'\iota$ is a monomorphism, we deduce that $\widetilde{f'}$ is a monomorphism. As $\widetilde{f'}\iota' = \nu f'$ it follows that f' is a monomorphism.

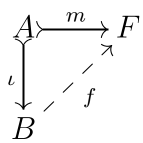
To prove $A' \subseteq_j B$, choose an arrow $f : B \to C$ for which $f\iota'$ is a monomorphism. Then, $f\iota'\iota$ is also a monomorphism. Now $A \subseteq_j B$ implies that f is a monomorphism, as required.

Sufficiency. Let $f : B \to C$ be an arrow in $\mathcal{E}$ such that $f\iota'\iota$ is a monomorphism. Since $A \subseteq_j A'$ and $(f\iota')\iota = f\iota'\iota$ is a monomorphism, it concludes that $f\iota'$ is a monomorphism. Using $A' \subseteq_j B$, we achieve that f is a monomorphism and hence $A \subseteq_j B$. □

In the following, we achieve another property of j-essential monomorphisms in $\mathcal{E}$.

Lemma 4.4. *Let j be a topology on $\mathcal{E}$. If $A \subseteq_j B$ and A is embedded in a j-sheaf F, then B also is embedded in F.*

Proof. Let $\iota : A \rightarrowtail B$ be a j-essential monomorphism and $m : A \rightarrowtail F$ an arbitrary embedding. Since F is a j-sheaf, there exists a unique morphism $f : B \to F$ making the diagram below commutative;



As $A \subseteq_j B$ being j-essential, f is an embedding, as required. $\square$

By Remark 2.7(ii), essential monomorphisms in a topos $\mathcal{E}$ are exactly j-essential monomorphisms in $\mathcal{E}$ with respect to the topology $j = \mathrm{true} \circ !_\Omega$ on $\mathcal{E}$.

Now, we would like to prove that any presheaf in $\widehat{\mathcal{C}}$ has a maximal essential extension.

Theorem 4.5. *Any presheaf in $\widehat{\mathcal{C}}$ has a maximal essential extension.*

Proof. Let F be a presheaf in $\widehat{\mathcal{C}}$ and G an injective presheaf into which F can be embedded. By Lemma 4.4, we can assume that both F and all its essential extensions are subpresheaves of G. Consider $\sum$ as the set of all essential extensions of F which is a poset under subpresheaf inclusion $\subseteq$. Since the arrow id_F is an essential extension of F, it follows that $\sum$ is non-empty. If

$$\ldots \subseteq F_i \subseteq \ldots,$$

$i \in I$, is a chain in $\sum$, then the subpresheaf H of G given by $H(C) = \bigcup_{i \in I} F_i(C)$ for any object C in $\mathcal{C}$ is an upper bound of this chain. Now we show that H lies in $\sum$, i.e., H is an essential extension of F. To achieve this, let $\alpha : H \to K$ be an arrow in $\widehat{\mathcal{C}}$ such that the restriction arrow $\alpha|_F$ is a monomorphism. We prove that α is a monomorphism. To verify this claim, we show that for any $C \in \widehat{\mathcal{C}}$, the function $\alpha_C : \bigcup_{i \in I} F_i(C) \to K(C)$ is one to one. Take $a, b \in \bigcup_{i \in I} F_i(C)$, $a \neq b$. Then there is a $j \in I$ such that $a, b \in F_j(C)$. Denote $\alpha|_{F_j}$ by α_j. Since F_j

is an essential extension of F and $\alpha_j|_F = \alpha|_F$, it implies that α_j is a monomorphism. Now

$$\alpha_C(a) = (\alpha_j)_C(a) \neq (\alpha_j)_C(b) = \alpha_C(b).$$

Therefore, α is a monomorphism. Thus, $H \in \sum$. Now it follows from Zorn's Lemma that there is a maximal element M in $\sum$. Then, M is a maximal essential extension of F. $\square$

It is straightforward to see that any essential extension of B can be embedded in any injective extension of B.

For a topology j on a topos $\mathcal{E}$, by a *j-injective object* we mean an injective object with respect to the class of all j-dense monomorphisms in $\mathcal{E}$.

The following shows that the j-injective presheaves (j-sheaves) in $\widehat{\mathcal{C}}$ have no proper j-essential extension.

Proposition 4.6. *Let j be a topology on $\widehat{\mathcal{C}}$ and F a j-injective presheaf (j-sheaf) in $\widehat{\mathcal{C}}$. Then, F has no proper j-essential extension.*

Proof. Suppose that G is a proper j-essential extension of F and so F is a j-dense subpresheaf of G and $F \neq G$. Thus there is an object C of $\mathcal{C}$ such that $G(C) \not\subset F(C)$ and then, an $a \in G(C)$ such that $a \notin F(C)$. Since F is j-injective (j-sheaf) implies that there is an arrow $\alpha : G \to F$ for which $\alpha|_F = \mathrm{id}_F$. That $a \notin F(C)$ and $\alpha_C(a) \in F(C)$ follows that $a \neq \alpha_C(a)$. But $\alpha_C(\alpha_C(a)) = \alpha_C(a)$. Then, α_C and so α is not a monomorphism although $\alpha|_F = \mathrm{id}_F$ is. This shows that G is not a proper j-essential extension of F and it is a contradiction. $\square$

The following shows that the pullback functor Π_B reflects j-essential extensions.

Proposition 4.7. *Let j be a topology in a topos $\mathcal{E}$. For every object $B \in \mathcal{E}$, the pullback functor $\Pi_B : \mathcal{E} \to \mathcal{E}/B$ reflects j-essential monomorphisms.*

Proof. Let $f : A \to C$ be an arrow in $\mathcal{E}$ such that $\Pi_B(f)$ is a j_B-essential monomorphism. We show that f is a j-essential monomorphism. By Lemma 2.3, f is a j-dense monomorphism in $\mathcal{E}$. Let $g : C \to D$ be an arrow in $\mathcal{E}$ such that gf is a monomorphism.

We show that g is too. Since gf is a monomorphism, the arrow $(g \times \mathrm{id}_B)\Pi_B(f) = (gf) \times \mathrm{id}_B$ is also a monomorphism. As $\Pi_B(f)$ is j_B-essential, so $g \times \mathrm{id}_B$ is a monomorphism. Then g is a monomorphism. This is the required result. $\square$

Recall [10] that a *weak topology* on a topos $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ such that:

(i) $j \circ \mathrm{true} = \mathrm{true}$;

(ii) $j \circ \wedge \leq \wedge \circ (j \times j)$, in which $\leq$ stands for the internal order on Ω. Meanwhile, a weak topology j on $\mathcal{E}$ is said to be *productive* if $j \circ \wedge = \wedge \circ (j \times j)$.

In what follows, we review the whole paper for a weak topology j on a topos $\mathcal{E}$ instead of a topology.

Remark 4.8. Similar to [9, A.4.5.11(ii)], one can easily check that for a weak topology j on $\mathcal{E}$ pushouts also preserve dense monomorphisms. Hence, we can obtain a version of Lemma 2.1 for a weak topology j on $\mathcal{E}$ as well. One can observe that completely analogous assertions to Lemmas 4.2, 4.4 and 2.3, Proposition 4.6 and Theorem 3.1, hold for a weak topology j on $\mathcal{E}$. But, by [10], in the proof of Theorem 2.6, the part (vi) $\Longrightarrow$ (vii) is true for a productive weak topology j on $\mathcal{E}$. The rest parts of this proof satisfies for weak topologies.
Recall [10] that, for a weak topology j on $\mathcal{E}$, it is convenient to see that if the composite subobject mn is dense then so are m and n. In contrast with topologies [9, A.4.5.11(iii)], the converse is not necessarily true. Hence, the sufficiency part of Proposition 4.3 does not necessarily hold for a weak topology j on a topos $\mathcal{E}$. The necessity part of this proposition satisfies for a weak topology j as well.

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