THE HASSE PRINCIPLE FOR BILINEAR SYMMETRIC FORMS OVER A RING OF INTEGERS OF A GLOBAL FUNCTION FIELD

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Abstract. Let C be a smooth projective curve defined over the finite field $\mathbb{F}_q$ (q is odd) and let $K = \mathbb{F}_q(C)$ be its function field. Removing one closed point $C^{\mathrm{af}} = C - \{\infty\}$ results in an integral domain $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[C^{\mathrm{af}}]$ of K, over which we consider a non-degenerate bilinear and symmetric form f with orthogonal group $\underline{\mathbf{O}}_V$. We show that the set $\mathrm{Cl}_\infty(\underline{\mathbf{O}}_V)$ of $\mathcal{O}_{\{\infty\}}$-isomorphism classes in the genus of f of rank $n > 2$, is bijective as a pointed set to the abelian groups $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \cong$ Pic $(C^{\mathrm{af}})/2$, i.e. is an invariant of C^{af}. We then deduce that any such f of rank $n > 2$ admits the local-global Hasse principal if and only if $|\text{Pic }(C^{\mathrm{af}})|$ is odd. For rank 2 this principle holds if the integral closure of $\mathcal{O}_{\{\infty\}}$ in the splitting field of $\underline{\mathbf{O}}_V \otimes_{\mathcal{O}_{\{\infty\}}} K$ is a UFD.

1. Introduction

Let C be a smooth, projective, geometrically connected curve defined over the finite field $\mathbb{F}_q$ with q odd, and let $K = \mathbb{F}_q(C)$ be its function field. For any prime $\mathfrak{p}$ of K, let $v_\mathfrak{p}$ be the induced discrete valuation on K. We remove one closed point ∞ from C, resulting in an affine curve C^{af}, and consider the following ring of $\{\infty\}$-integers of K:

$$\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[C^{\mathrm{af}}] := \{a \in K \mid v_\mathfrak{p}(a) \geq 0 \quad \forall \mathfrak{p} \neq \infty\}.$$

Throughout the paper, an *integral form* on $V \cong \mathcal{O}^n_{\{\infty\}}$ refers to a bilinear and symmetric map $f : V \times V \to \mathcal{O}_{\{\infty\}}$. It will be called *unimodular* if it is non-degenerate at any closed point of C^{af}, which is equivalent in this case to $\det(f) \in \mathbb{F}_q^\times$. Two integral forms f and g on V are $\mathcal{O}_{\{\infty\}}$*-isomorphic* if there exists $Q \in \mathbf{GL}(V)$ such that $f(u,v) = g(Qu, Qv)$ for all $u, v \in V$.

The standard approach to classifying bilinear forms over a global field such as K, basically relies on the Hasse-Minkowski principle which states that this classification, expressed by the first Galois cohomology set $H^1(K, \mathbf{O}_V)$ where $\mathbf{O}_V$ stands for the orthogonal group of f, is equivalent to that obtained locally everywhere, namely, by $\prod_\mathfrak{p} H^1(\hat{K}_\mathfrak{p}, (\mathbf{O}_V)_\mathfrak{p})$ where $\hat{K}_\mathfrak{p}$ is the complete localization of K at a prime $\mathfrak{p}$ and $(\mathbf{O}_V)_\mathfrak{p}$ is the geometric fiber of $\mathbf{O}_V$ there. However, if one considers the classification of integral forms, then this local-global principle fails, leading to the notion of a *genus* of a form. In this paper, we aim to describe geometrically the violation of this principle. We express the classification of integral unimodular forms from the same genus via $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V)$ (Proposition 4.2), where $\underline{\mathbf{SO}}_V$ is the special orthogonal group scheme of f defined over $\operatorname{Spec} \mathcal{O}_{\{\infty\}}$, and then show that this set is bijective as a pointed set for ranks $n > 2$ to the abelian group $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2)$, i.e. it is an invariant of C^{af} (Proposition 4.4). Furthermore, by proving that the Brauer group of the affine curve C^{af} is trivial (Lemma 3.3), we conclude that $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \cong \text{Pic}\,(C^{\text{af}})/2$ (Corollary 3.4).

This description leads us to assert the validity of the Hasse principle for unimodular integral forms of rank $n > 2$ if and only if $|\text{Pic}\,(C^{\text{af}})|$ is odd. For $n = 2$, the Hasse principle holds if the integral closure of $\mathcal{O}_{\{\infty\}}$ in the splitting field of the generic fiber of $\underline{\mathbf{O}}_V$ is a UFD (Theorem 4.5). This result can be considered as a generalization of Theorem 3.1 in [Ger1] in which the elementary case of $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[t]$ is treated (Example 4.6). Its proof was initially based on the reduction by Harder, of the unimodular theory over $\mathbb{F}_q[t]$, to the theory of spaces over $\mathbb{F}_q$ (see [Ger2, Theorem 7.13]).

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2. Classification over rings of integers

The geometrically connected projective curve C remains geometrically connected after removing the closed point ∞, resulting in C^{af}. In order to classify integral forms, we shall refer to the fundamental group $\pi_1(C^{\text{af}}, a)$ of C^{af} w.r.t. some geometric base point a, as defined by Grothendieck in [SGA1, V,§4 and §7]. Up to isomorphism, this group (as a topological group) does not depend on the choice of the base point (see [Mil, Ch.I, Remark 5.1]). Therefore, where one is only concerned with the group-theoretic structure of $\pi_1(C^{\text{af}}, a)$, we may omit the base-point and write just $\pi_1(C^{\text{af}})$.

For any prime $\mathfrak{p}$ of K, let $\mathcal{O}_\mathfrak{p}$ be the discrete valuation ring of K w.r.t. to $v_\mathfrak{p}$ and let $K_\mathfrak{p}$ be its fraction field. Let $\hat{K}_\mathfrak{p}$ be the completion of $K_\mathfrak{p}$ and let $\hat{\mathcal{O}}_\mathfrak{p}$ be its ring of integers. Let $k_\mathfrak{p} = \hat{\mathcal{O}}_\mathfrak{p}/\mathfrak{p}$ be the corresponding (finite) residue field. Let $\hat{K}_\mathfrak{p}^{\text{ur}}$ be the maximal unramified extension of $\hat{K}_\mathfrak{p}$ and let $\hat{\mathcal{O}}_\mathfrak{p}^{\text{sh}}$ be its ring of integers. Given a smooth group scheme $\underline{G}_\mathfrak{p}$ defined over $\text{Spec}\,\hat{\mathcal{O}}_\mathfrak{p}$, the set $H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p})$ is bijective to the Galois cohomology set $H^1(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p})$, while the Galois group taken under consideration is $\text{Gal}(\hat{\mathcal{O}}_\mathfrak{p}^{\text{sh}}/\hat{\mathcal{O}}_\mathfrak{p}) = \text{Gal}(\overline{k}_\mathfrak{p}/k_\mathfrak{p})$ where $\overline{k}_\mathfrak{p}$ stands for the algebraic closure of $k_\mathfrak{p}$. For a smooth group scheme $\underline{G}$ defined over $\mathcal{O}_{\{\infty\}}$, by writing $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{G}) \cong H^1_{\text{flat}}(\mathcal{O}_{\{\infty\}}, \underline{G})$ we shall refer to the action of the aforementioned (total) fundamental group $\pi_1(C^{\text{af}})$ (see [SGA4, VIII Corollaire 2.3] for the étale, flat and Galois cohomology sets bijections in the smooth case).

3. Integral schemes and étale cohomology

Let $\underline{G}$ be an affine, flat and smooth group scheme defined over $\mathrm{Spec}\,\mathcal{O}_{\{\infty\}}$ with generic fiber G. For any prime $\mathfrak{p}$ of K, the localization $(\mathcal{O}_{\{\infty\}})_\mathfrak{p}$ is a base change of $\mathcal{O}_\mathfrak{p}$. Thus the bijection $\mathrm{Spec}\,(\mathcal{O}_{\{\infty\}})_\mathfrak{p} \to \mathrm{Spec}\,\mathcal{O}_\mathfrak{p}$ is faithfully flat (see [Liu, Theorem 3.16]) and so $\underline{G}$, extended to be defined over $\mathrm{Spec}\,\hat{\mathcal{O}}_\mathfrak{p}$ and denoted by $\underline{G}_\mathfrak{p}$, is also smooth. Under these settings, we shall refer to the adelic group $\underline{G}(\mathbb{A})$ and to its subgroup over the ring of $\{\infty\}$-integral adèles $\mathbb{A}_\infty := \hat{K}_\infty \times \prod_{\mathfrak{p}\neq\infty} \hat{\mathcal{O}}_\mathfrak{p}$.

Definition 1. The *class set* of $\underline{G}$ is the set of double cosets $\mathrm{Cl}_\infty(\underline{G}) := \underline{G}(\mathbb{A}_\infty)\backslash\underline{G}(\mathbb{A})/G(K)$. It is finite (cf. [Con1, Thm 1.3.1]). Its cardinality, denoted by $h_\infty(\underline{G})$, is called the *class number* of $\underline{G}$.

Theorem 3.1. (Ye. Nisnevich, [Nis, Theorem I.3.5]). There is an exact sequence of pointed sets:

$$1 \to \mathrm{Cl}_\infty(\underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{G}) \to H^1(K, G) \times \prod_{\mathfrak{p}\neq\infty} H^1(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p}).$$

Lemma 3.2. *Suppose $\underline{G}$ (being affine, flat and smooth defined over $\mathrm{Spec}\,\mathcal{O}_{\{\infty\}}$) is connected, and that G is almost simple, simply connected and $\hat{K}_\infty$-isotropic. Then $H^1_{\acute{e}t}(\mathcal{O}_{\{\infty\}}, \underline{G}) = 0$.*

Proof. At any prime $\mathfrak{p}$, as $\hat{\mathcal{O}}_\mathfrak{p}$ is Henselian, we have $H^i(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p}) \cong H^i(k_\mathfrak{p}, \overline{G}_\mathfrak{p})$ for $i \geq 0$ where $\overline{G}_\mathfrak{p} := \underline{G}_\mathfrak{p} \otimes_{\mathrm{Spec}\,\hat{\mathcal{O}}_\mathfrak{p}} k_\mathfrak{p}$ (see Remark 3.11(a) in [Mil, Ch. III, §3]). The right set for $i = 1$ is trivial by Lang's Theorem (see [Ser, Ch.VI, Prop. 5]). Furthermore, $H^1(K, G)$ is trivial in the simply connected case due to Harder's result (see [Hard, Satz A]), and so Nisnevich's sequence from Theorem 3.1 obtained for $\underline{G}$, shows that $\mathrm{Cl}_\infty(\underline{G})$ is bijective to $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{G})$. But as G is almost simple, simply connected and $\hat{K}_\infty$-isotropic, it admits the strong approximation property w.r.t. $S = \{\infty\}$ (see [Pra, Theorem A]), which means that $\mathrm{Cl}_\infty(\underline{G})$ is trivial, and the assertion follows. □

Lemma 3.3. $\mathrm{Br}(\mathcal{O}_{\{\infty\}}) = 1$.

Proof. As C^{af} is smooth, $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbb{G}}_m) = \mathrm{Br}(\mathcal{O}_{\{\infty\}})$ classifying Azumaya $\mathcal{O}_{\{\infty\}}$-algebras (see [Mil, §2]). Let $[D]$ be a class of central simple algebras in $\mathrm{Br}(K)$. At any prime $\mathfrak{p}$, $[D]$ is associated by the residue map $r_\mathfrak{p}$ with an extension of $k_\mathfrak{p}$, representing thus a class in $H^1(k_\mathfrak{p}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(k_\mathfrak{p}, \mathbb{Q}/\mathbb{Z})$ (the Galois action is trivial). The latter term is isomorphic to $\mathbb{Q}/\mathbb{Z}$, as the absolute Galois group of any finite field $k_\mathfrak{p}$ is isomorphic to $\hat{\mathbb{Z}}$. The ramification map $a := \oplus_\mathfrak{p} r_\mathfrak{p}$ yields then the exact sequence from Class Field Theory (see Theorem 6.5.1 in [GS]):

$$1 \to \mathrm{Br}(K) \xrightarrow{a=\oplus_\mathfrak{p} r_\mathfrak{p}} \bigoplus_\mathfrak{p} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_\mathfrak{p} \mathrm{Cor}_\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \to 1 \tag{3.1}$$

in which the corestriction map $\mathrm{Cor}_\mathfrak{p}$ for any $\mathfrak{p}$ is an isomorphism induced by the Hasse-invariant $\mathrm{Br}(\hat{K}_\mathfrak{p}) \cong \mathbb{Q}/\mathbb{Z}$ (cf. [GS, Proposition 6.3.9]). On the other hand, as all residue fields of K are finite thus perfect, and C^{af} is a one-dimensional regular scheme over $\mathbb{F}_q$, it admits due to Grothendieck the following exact sequence (see [Gro, Proposition 2.1] and [Mil, Example 2.22, case (a)]):

$$1 \to \mathrm{Br}(\mathbb{F}_q[C^{\mathrm{af}}] = \mathcal{O}_{\{\infty\}}) \to \mathrm{Br}(\mathbb{F}_q(C^{\mathrm{af}}) = K) \xrightarrow{\oplus_{\mathfrak{p}\neq\infty} r_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}\neq\infty} \mathbb{Q}/\mathbb{Z}, \tag{3.2}$$

which means that $\mathrm{Br}(\mathcal{O}_{\{\infty\}})$ is the subgroup of $\mathrm{Br}(K)$ of classes that vanish under $r_{\mathfrak{p}}$ at any $\mathfrak{p} \neq \infty$. Thus omitting these $r_{\mathfrak{p}}, \mathfrak{p} \neq \infty$ in sequence (3.1), results in $\mathrm{Br}(\mathcal{O}_{\{\infty\}}) = \ker[\mathbb{Q}/\mathbb{Z} \xrightarrow{\mathrm{Cor}_\infty} \mathbb{Q}/\mathbb{Z}]=1$. □

Corollary 3.4. *There is an isomorphism of abelian groups: Pic* $(C^{af})/2 \cong H^2_{ét}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2)$.

Proof. Étale cohomology applied on the Kummer's exact sequence defined over $\mathrm{Spec}\,\mathcal{O}_{\{\infty\}}$:

$$1 \to \underline{\mu}_2 \to \underline{\mathbb{G}}_m \to \underline{\mathbb{G}}_m \to 1$$

gives rise to the following long exact sequence:

$$\mathrm{Pic}\ (C^{\mathrm{af}}) \xrightarrow{\phi:[\mathcal{L}]\mapsto[2\mathcal{L}]} \mathrm{Pic}\ (C^{\mathrm{af}}) \to H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \to \mathrm{Br}(\mathcal{O}_{\{\infty\}}) \overset{(3.3)}{=} 1$$

in which $\mathrm{Pic}\ (C^{\mathrm{af}})/2 = \mathrm{coker}(\phi) \cong H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2)$. □

Definition 2. Let S be a scheme and G an S-group. G is an *S-torus of rank r* if it is locally isomorphic in the fpqc-topology to $\underline{\mathbb{G}}_m^r$ (cf. [SGA3, Exp. IX, Def. 1.3]).

Lemma 3.5. *Let $h : S' \to S$ be a finite surjective morphism of integral schemes, where S is Noetherian, normal and one dimensional. Then $\underline{N} := R^{(1)}_{S'/S}(\underline{\mathbb{G}}_m)$ is an S-torus iff h is étale.*

Proof. Under the Lemma's hypothesis on h, it is locally free (cf. [Sza, Lemma 5.2.4]). Hence the induced Weil restriction functor $\underline{R} := R_{S'/S}(\underline{\mathbb{G}}_m)$ exists (cf. [BLR, §7.6 Theorem 4]), and so $\underline{N}$, being equal to $\ker[\underline{R} \xrightarrow{\det} \underline{\mathbb{G}}_m]$ (see Ex.¶9(c), p.148 [Bou]), is well defined.

S' is integral thus connected. So given that h is étale over the normal scheme S, it admits a Galois closure $S'' \to S$ that factors through it (see [Sza, Proposition 5.3.9] and [BS, Theorem 4']). Thus the associated fundamental group Γ admits a subgroup $\Gamma_0 := \mathrm{Aut}(S''|S') \subset \Gamma$ consisting of automorphisms of S'' that fix S', such that: $S' \otimes_S S'' \cong (S'')^{|\Gamma/\Gamma_0|}$ (see [Sza, Proposition 5.3.8]). Therefore: $\underline{R} \otimes_S S'' \cong \mathbb{G}_{m,S''}^{|\Gamma/\Gamma_0|}$ and so:

$$\underline{N} \otimes_S S'' \cong \ker\left[\underline{\mathbb{G}}_{m,S''}^{|\Gamma/\Gamma_0|} \xrightarrow{\det} \underline{\mathbb{G}}_{m,S''}\right] \cong \underline{\mathbb{G}}_{m,S''}^{|\Gamma/\Gamma_0|-1},$$

i.e. $\underline{N}$ is an S-torus.

Conversely, if h ramifies at some prime, then $\underline{N}$ is not reductive there (see [Vos, §10.5]). So given that h is locally free thus flat, it must be étale as well. □

4. The Hasse principle and the class group of the orthogonal group

Let $\mathcal{X}$ be the scheme of invertible symmetric $n \times n$-matrices with entries in $\mathcal{O}_{\{\infty\}}$. It is a $\mathrm{Spec}\, \mathcal{O}_{\{\infty\}}$-scheme, and its points correspond to (non-degenerate) n-dimensional integral forms, on which $\underline{\mathbf{GL}}_n$ defined over $\mathrm{Spec}\, \mathcal{O}_{\{\infty\}}$ acts by

$$\forall g \in \underline{\mathbf{GL}}_n,\ F \in \mathcal{X}\ :\ g * F = g^t F g.$$

Let f be an integral unimodular form represented by $F \in \mathcal{X}$. Then the *orthogonal group* $\underline{\mathbf{O}}_V$ associated to (V, f) is the stabilizer of F. Since this action is defined over $\mathcal{O}_{\{\infty\}}$, it is an affine scheme defined over $\mathrm{Spec}\, \mathcal{O}_{\{\infty\}}$. Its generic fiber is $\mathbf{O}_V := \underline{\mathbf{O}}_V \otimes_{\mathrm{Spec}\, \mathcal{O}_{\{\infty\}}} K$. As 2 is a unit in $\mathcal{O}_{\{\infty\}}$, the *special orthogonal group* $\underline{\mathbf{SO}}_V$ is $\ker[\det : \underline{\mathbf{O}}_V \to \underline{\mu}_2]$, where $\underline{\mu}_2 := \mathrm{Spec}\, \mathcal{O}_{\{\infty\}}[x]/(x^2 - 1)$ (see Definition 1.6 and Corollary 2.5 in [Con2]). For the same reason (2 is a unit in $\mathcal{O}_{\{\infty\}}$), $\underline{\mathbf{O}}_V$ is smooth regardless of the parity of n, and $\underline{\mathbf{SO}}_V$ is a smooth closed subgroup with connected fibers ([Con2, Theorem 1.7]). If n is even, then $\underline{\mathbf{O}}_V$ is a semidirect product of $\underline{\mathbf{SO}}_V$ and $\underline{\mu}_2$ (see Corollary 2.5 and Remark 2.6 in [Con2]), and it is a direct product of these if n is odd (cf. [Con2, Proposition 3.4]).

Definition 3. Two integral forms share the same *genus* if they are isomorphic over K and over $\hat{\mathcal{O}}_\mathfrak{p}$ for all primes $\mathfrak{p}$. We denote by $\mathrm{gen}(f)$ the set of all integral forms of the same genus as f.

Definition 4. Given an integral form f, let $c(f)$ denote the number of $\mathcal{O}_{\{\infty\}}$-isomorphism classes in $\mathrm{gen}(f)$. We say that the *Hasse principle* holds for f if $c(f) = 1$.

Platonov and Rapinchuk have shown in [PR, Prop. 8.4] – in the number field case – that $c(f)$ equals the class number of its orthogonal group. In the following, we shall sketch briefly their proof, this time in the function field case:

We consider the above $\mathcal{O}_{\{\infty\}}$-scheme $\underline{\mathbf{GL}}_n$ (in which $\underline{\mathbf{O}}_V$ is embedded), its subgroup $\underline{\mathbf{SL}}_n$ and their extensions defined over $\hat{\mathcal{O}}_\mathfrak{p}$ at any prime $\mathfrak{p}$ (see Section 3) while referring to their adelic groups. Any element of $\underline{\mathbf{O}}_V(\mathbb{A})$ can be put in $\underline{\mathbf{SL}}_n(\mathbb{A})$ by multiplying by a suitable element of $\underline{\mathbf{GL}}_n(\mathbb{A}_\infty)$. Since the K-group $\mathbf{SL}_n$ is split, simple and simply connected, it admits the strong approximation property whence $\underline{\mathbf{SL}}_n(\mathbb{A}) = \underline{\mathbf{SL}}_n(\mathbb{A}_\infty)\mathbf{SL}_n(K)$ (cf. [Pra, Theorem A]). It follows that $\underline{\mathbf{O}}_V(\mathbb{A}) \subseteq \underline{\mathbf{GL}}_n(\mathbb{A}_\infty)\mathbf{GL}_n(K)$. Now according to the Stabilizer Formula [PR, Theorem 8.2], $c(f)$ is equal to the number of double cosets $\underline{\mathbf{O}}_V(\mathbb{A}_\infty) \cdot x \cdot \mathbf{O}_V(K)$ in the principal coset $\underline{\mathbf{GL}}_n(\mathbb{A}_\infty)\mathbf{GL}_n(K)$ which is $h_\infty(\underline{\mathbf{O}}_V)$.

Corollary 4.1. *$c(f) = h_\infty(\underline{\mathbf{O}}_V)$.*

Proposition 4.2. *There is a bijection of finite pointed sets:* $\mathrm{Cl}_\infty(\underline{\boldsymbol{O}}_V) \cong H^1_{\acute{e}t}(\mathcal{O}_{\{\infty\}}, \underline{\boldsymbol{SO}}_V)$.

Proof. Being affine, flat and smooth, $\underline{\mathbf{O}}_V$ admits by the Nisnevich's Theorem 3.1 the exact sequence of pointed sets:

$$1 \to \mathrm{Cl}_\infty(\underline{\mathbf{O}}_V) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{O}}_V) \to H^1(K, \mathbf{O}_V) \times \prod_{\mathfrak{p}\neq\infty} H^1(\hat{\mathcal{O}}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p}). \tag{4.1}$$

Let $W(*)$ denote the Witt ring for the ring $*$. By Witt's Theorem, two forms are isomorphic if and only if they belong to the same Witt class and have the same rank (see [MH, Cor. 3.3]), whence $H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p})$ injects into $W(\hat{\mathcal{O}}_\mathfrak{p})$ and $H^1(\hat{K}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p})$ into $W(\hat{K}_\mathfrak{p})$. Since $\hat{K}_\mathfrak{p}$ is complete, $W(\hat{\mathcal{O}}_\mathfrak{p}) = W(k_\mathfrak{p})$ injects into $W(\hat{K}_\mathfrak{p})$ and we obtain the following commutative diagram:

$$\begin{array}{ccc} H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p}) & \longrightarrow & H^1(\hat{K}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p}) \\ \big\downarrow & & \big\downarrow \\ W(\hat{\mathcal{O}}_\mathfrak{p}) & \hookrightarrow & W(\hat{K}_\mathfrak{p}) \end{array}$$

which shows that $H^1(\hat{\mathcal{O}}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p})$ embeds into $H^1(\hat{K}_\mathfrak{p}, (\underline{\mathbf{O}}_V)_\mathfrak{p})$ for any $\mathfrak{p}$. Then due to Corollary 3.6 in [Nis], sequence (4.1) simplifies to:

$$1 \to \mathrm{Cl}_\infty(\underline{\mathbf{O}}_V) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{O}}_V) \to H^1(K, \mathbf{O}_V). \tag{4.2}$$

As C^{af} is assumed to be smooth, $\operatorname{Spec} \mathcal{O}_{\{\infty\}} = \operatorname{Spec} \mathbb{F}_q[C^{\text{af}}]$ is a normal scheme, i.e. it is integrally closed locally everywhere. In this case any finite étale covering of C^{af} arises by the normalization of $\operatorname{Spec} \mathcal{O}_{\{\infty\}}$ in some separable unramified extension of K (see [Len, Theorem 6.13]). Thus any non-trivial 1-cocycle in $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2)$ remains non-trivial after tensoring with K, i.e. $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2)$ is embedded in its generic fiber. Hence, $\underline{\mu}_2$ being a direct or semidirect summand in $\underline{\mathbf{O}}_V$ (see at the beginning of this section), can be canceled in sequence (4.2), leading to:

$$\mathrm{Cl}_\infty(\underline{\mathbf{O}}_V) \cong \ker[H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) \to H^1(K, \mathbf{SO}_V)].$$

The situation for $\underline{\mathbf{SO}}_V$ is simpler: having (smooth) connected fibers, $H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, (\underline{\mathbf{SO}}_V)_\mathfrak{p})$ vanishes for all primes $\mathfrak{p}$ (by Lang's Lemma), thus not only admitting again due to Corollary 3.6 in [Nis] the exact sequence:

$$1 \to \mathrm{Cl}_\infty(\underline{\mathbf{SO}}_V) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) \xrightarrow{\varphi} H^1(K, \mathbf{SO}_V), \tag{4.3}$$

which shows that $\mathrm{Cl}_\infty(\underline{\mathbf{O}}_V) = \mathrm{Cl}_\infty(\underline{\mathbf{SO}}_V)$, but can be simplfied even more to (cf. [Nis, I.3.5.2] and [Gon1, Theorem 3.4]):

$$1 \to \mathrm{Cl}_\infty(\underline{\mathbf{SO}}_V) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) \to Ш^1_{\{\infty\}}(K, \mathbf{SO}_V) \to 1$$

in which the right term is the first Tate-Shafarevich group w.r.t. $\{\infty\}$, namely:

$$Ш^1_{\{\infty\}}(K,\mathbf{SO}_V) := \ker\left[H^1(K,\mathbf{SO}_V) \to \prod_{\mathfrak{p}\neq\infty} H^1(\hat{K}_{\mathfrak{p}},(\mathbf{SO}_V)_{\mathfrak{p}})\right].$$

The pointed set $H^1(K,\mathbf{SO}_V)$ properly (i.e. by det $=1$ isomorphisms) classifies K-forms isomorphic to f over some finite Galois extensions of K, therefore sharing all the same rank and discriminant. So according to the Hasse-Minkowsky principle (cf. [Lam, VI.3.1]), these forms are classified via their Hasse-invariant locally everywhere. But as the base point f is unimodular, representatives of any class in $H^1(K,\mathbf{SO}_V)$ are $\mathcal{O}_{\{\infty\}}$-regular as well, thus their local Hasse-invariants belong to $\mathrm{Br}(\mathcal{O}_{\{\infty\}})$, being trivial by Lemma 3.3. This means that $\mathrm{Cl}_\infty(\underline{\mathbf{SO}}_V)$ surjects on $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}},\underline{\mathbf{SO}}_V)$. On the other hand, $\mathrm{Cl}_\infty(\underline{\mathbf{SO}}_V)$ is bijective to the first Nisnevich's cohomology set $H^1_{\text{Nis}}(\mathcal{O}_{\{\infty\}},\underline{\mathbf{SO}}_V)$ (cf. [Nis, I. Theorem 2.8] and [Mor, 4.1]), classifying $\underline{\mathbf{SO}}_V$-torsors in the Nisnevich's topology. But Nisnevich's covers are étale, so it is a subset of $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}},\underline{\mathbf{SO}}_V)$, and the assertion follows. □

In particular, Proposition 4.2 plus Corollary 4.1 yield:

Corollary 4.3. *The Hasse principle holds for an integral unimodular form iff* $H^1_{\acute{e}t}(\mathcal{O}_{\{\infty\}},\underline{\mathbf{SO}}_V)=0$.

For rank $n>2$, we consider the following construction (see in [Bas, §2]):
Let $\mathbf{C}(f)$ be the Clifford algebra associated to f. It is a $\mathbb{Z}_2$-graded algebra. The linear map $v\mapsto -v$ on V extends to an algebra automorphism $\alpha:\mathbf{C}(f)\to\mathbf{C}(f)$ (acting as the identity on the even part and negation on the odd part). The *Clifford group* associated to (V,f) is

$$\mathbf{CL}(f) := \{u\in\mathbf{C}(f)^\times \ : \ \alpha(u)vu^{-1}\in V \ \forall v\in V\}.$$

The identity map on V (viewed as its inclusion in the opposite algebra of $\mathbf{C}(f)$) extends to an anti-automorphism of $\mathbf{C}(f)$ which we denote by t. The composition $\alpha\circ t$ mapping $v\mapsto\bar v$ gives rise to the norm $N:\mathbf{CL}(f)\to\mathcal{O}^\times_{\{\infty\}}=\mathbb{F}_q^\times : v\mapsto v\bar v$ (for $v\in V$ it is just $N(v)=-v^2=-f(v,v)$). We define $\underline{\mathbf{Pin}}_V(\mathcal{O}_{\{\infty\}}):=\ker(N)$. This group admits an underlying group scheme over $\operatorname{Spec}\mathcal{O}_{\{\infty\}}$ which we denote by $\underline{\mathbf{Pin}}_V$. The homomorphism $\pi:\underline{\mathbf{Pin}}_V\to\underline{\mathbf{O}}_V$ sending v to the isometry stabilizing it, is a double covering, yielding the following short exact sequence of $\mathcal{O}_{\{\infty\}}$-group schemes:

$$1\to\underline{\mu}_2\to\underline{\mathbf{Spin}}_V\xrightarrow{\pi}\underline{\mathbf{SO}}_V\to 1 \qquad (4.4)$$

where $\underline{\mathbf{Spin}}_V:=\pi^{-1}(\underline{\mathbf{SO}}_V)\subset\underline{\mathbf{Pin}}_V$.

Proposition 4.4. *Let (V,f) be an integral unimodular space of rank $n>2$. Then $Cl_\infty(\underline{\mathbf{O}}_V)$ is bijective as a pointed set to $H^2_{\acute{e}t}(\mathcal{O}_{\{\infty\}},\underline{\mu}_2)$ (being isomorphic to Pic $(C^{af})/2$).*

Proof. The schemes in sequence (4.4) are smooth, whence étale cohomology yields the exact sequence of pointed sets:

$$H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{Spin}}_V) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) \xrightarrow{\delta} H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \tag{4.5}$$

in which since $\mathcal{O}_{\{\infty\}}$ is of Douai type – see Definition 5.2 and Example 5.4(iii) in [Gon2] – and as $\underline{\mathbf{SO}}_V = \underline{\mathbf{Spin}}_V^{\text{ad}}$ while: $Z(\underline{\mathbf{Spin}}_V) = \underline{\mu}_2$, δ is surjective. Furthermore, $\underline{\mathbf{Spin}}_V$ is affine, smooth and connected, and its generic fiber is simple, simply connected. As $\det(f) \in \mathbb{F}_q^\times$, the generic fiber of (V, f) admits a regular model over $\hat{\mathcal{O}}_\infty$ (see [OMe, 92:1]). Its reduction at ∞ remains regular of dimension $n \geq 3$ over the finite field k_∞ thus isotropic ([OMe, 62:1b]). Then its lift back to $\hat{\mathcal{O}}_\infty$ is again isotropic due to Hensel's Lemma (see [EKM, III. Lemma 19.4]), as well as $\mathbf{Spin}_V$ over $\hat{K}_\infty$. Hence $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{Spin}}_V)$ is trivial by Lemma 3.2, which means due to the exactness that $\ker(\delta) = \{0\}$. This does not imply yet that δ is injective, since $\underline{\mathbf{SO}}_V$ is non-commutative for $n > 2$, whence $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V)$ has no reason to be a group. In order to deduce the injectivity of δ, we consider the following diagram induced by some non-trivial $\underline{\mathbf{SO}}_V$-torsor P, as described in [Gir, Cha. IV, Proposition 4.3.4]:

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) & \xrightarrow{\delta} & H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \\ \cong \downarrow \theta_P & & \cong \downarrow r \\ H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, {}^P\underline{\mathbf{SO}}_V) & \xrightarrow{\delta'} & H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \end{array}$$

in which the map δ' is the one obtained by applying étale cohomology on the short exact sequence (4.4) while replacing $\underline{\mathbf{SO}}_V$ by the twisted group scheme ${}^P\underline{\mathbf{SO}}_V = \underline{\mathbf{SO}}_{({}^PV)}$, θ_P is the induced twisting bijection, and r is the translation by $-\delta(P)$. According to [Gir, Cha. IV, Proposition 4.3.4(i),(ii)] this diagram is commutative and there is a bijection:

$$\left\{x \in H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) : \delta(x) = \delta(P)\right\} \cong \ker(\delta).$$

But as shown above, in our case $\ker(\delta)$ is trivial, implying that δ is injective and eventually is a bijection. Due to Proposition 4.2, $\text{Cl}_\infty(\underline{\mathbf{O}}_V)$ is bijective as a pointed set to $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V)$, and therefore is bijective as a pointed set to $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2)$ as well. The rest is Corollary 3.4. □

Theorem 4.5. *Let f be a unimodular form of rank n defined over $\mathbb{F}_q[C^{af}]$.*
The Hasse principle holds for f:
for $n = 2$ if the integral closure of $\mathbb{F}_q[C^{af}]$ in the splitting field of $\mathbf{SO}_V$ is a UFD, and:
for $n > 2$ if and only if $|Pic\ (C^{af})|$ is odd.

Proof. For rank 2, $\underline{\mathbf{SO}}_V$ is a one dimensional norm $\mathcal{O}_{\{\infty\}}$-torus. This derives from being one dimensional and smooth, and from the connectivity of the fibers (see at the beginning of this section). According to Lemma 3.5, such one dimensional norm $\mathcal{O}_{\{\infty\}}$-tori arise from quadratic étale extensions, hence are being classified by $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathbb{Z}/2\mathbb{Z})$. If $\mathcal{O}_{\{\infty\}}$ is a UFD, the Kummer sequence defined over $\operatorname{Spec}\mathcal{O}_{\{\infty\}}$ implies that ($2 \in \mathcal{O}^\times_{\{\infty\}}$, hence the scheme $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\underline{\mu}_2$ over $\operatorname{Spec}\mathcal{O}_{\{\infty\}}$):

$$H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{O}^\times_{\{\infty\}}/(\mathcal{O}^\times_{\{\infty\}})^2 = \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \cong H^1_{\text{ét}}(\mathbb{F}_q, \mathbb{Z}/2\mathbb{Z}).$$

This means that given that $\mathcal{O}_{\{\infty\}}$ is a UFD, any quadratic extension of it, producing a one dimensional norm $\mathcal{O}_{\{\infty\}}$-torus, arises from a quadratic extension of $\mathbb{F}_q$ (recall that $\text{char}(K) \neq 2$, hence the quadratic Artin-Schreier extensions are not to be considered here). Now if $\underline{\mathbf{SO}}_V$ splits over $\mathcal{O}_{\{\infty\}}$, then $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V) = H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbb{G}}_m) = \text{Pic}\,(C^{\text{af}}) = 0$ as $\mathcal{O}_{\{\infty\}}$ is a UFD. Otherwise, it fits into an exact sequence of $\mathcal{O}_{\{\infty\}}$-tori:

$$1 \to \underline{\mathbf{SO}}_V \to \underline{R} := R_{\mathcal{O}'_{\{\infty\}}/\mathcal{O}_{\{\infty\}}}(\underline{\mathbb{G}}_m) \to \underline{\mathbb{G}}_m \to 1 \tag{4.6}$$

in which $\mathcal{O}'_{\{\infty\}}$ is assumed to be a UFD. As $\overline{\mathbf{SO}}_V := \underline{\mathbf{SO}}_V \otimes_{\operatorname{Spec}\mathcal{O}_{\{\infty\}}} \mathbb{F}_q$ is connected, by Lang's Lemma $H^1(\mathbb{F}_q, \overline{\mathbf{SO}}_V) \cong \mathbb{F}_q^\times/\text{Nr}(\mathbb{F}_{q^2}^\times) = 1$. But $\mathbb{F}_{q^2}^\times \subseteq \mathcal{O}'^\times_{\{\infty\}} = \underline{R}(\mathcal{O}_{\{\infty\}})$, which means that $\underline{R}(\mathcal{O}_{\{\infty\}}) \to \mathbb{F}_q^\times$ is surjective. Moreover, by Shapiro's Lemma $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{R}) \cong H^1_{\text{ét}}(\mathcal{O}'_{\{\infty\}}, \underline{\mathbb{G}}_{m,\mathcal{O}'_{\{\infty\}}}) =$ Pic $(\mathcal{O}'_{\{\infty\}})$ being trivial by the assumption. Thus applying étale cohomology on sequence (4.6) implies that $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_V)$ vanishes as well, and the assertion follows from Corollary 4.3.

For higher ranks, this is just Proposition 4.4 plus Corollary 3.4. □

Example 4.6. Let C be of genus zero having a $\mathbb{F}_q$-rational point which we assign as ∞. Then $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[C^{\text{af}}]$ is a UFD, as well as any scalar extension of it (see [Sam, Theorem 5.1]), whose generic fiber may be the splitting field if $n = 2$ of $\mathbf{SO}_V$ (see in the proof of Theorem 4.5). Therefore the Hasse principle holds for any unimodular form defined over it of any rank. So Theorem 4.5 is a generalization of Theorem 3.1 in [Ger1] in which the elementary case of $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[t]$ is treated.

Remark 4.7. The unimodularity condition is essential (though not necessary) for the validity of the Hasse principle even if $\mathcal{O}_{\{\infty\}}$ is a UFD. It is necessary for the Clifford algebra construction if $n > 2$, but is also required for rank 2.

For example, let C be the projective line over $\mathbb{F}_q$ and $\infty = (1/x)$. Then $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[x]$ is a UFD. Let f and g be the $\mathcal{O}_{\{\infty\}}$-forms represented by $F = \text{diag}((1-x^2)^2, 1)$ and $G = \text{diag}((1-x)^2, (1+x)^2)$, respectively. Let

$$Q = \begin{pmatrix} \frac{1}{1+x} & 0 \\ 0 & 1+x \end{pmatrix} \in \mathbf{GL}_2(\hat{\mathcal{O}}_\mathfrak{p}) \quad \forall \mathfrak{p} \neq (1+x)$$

and

$$P = \begin{pmatrix} 0 & \frac{1}{1-x} \\ 1-x & 0 \end{pmatrix} \in \mathbf{GL}_2(\hat{\mathcal{O}}_\mathfrak{p}) \quad \forall \mathfrak{p} \neq (1-x).$$

Then $Q^tFQ = P^tFP = G$. This shows that f and g belong to the same genus. But they are not, however, isomorphic over $\mathcal{O}_{\{\infty\}}$, since mapping the eigenvalue $(1-x^2)^2$ in F to $(1-x)^2$ or $(1+x)^2$ in G can be done only by dividing by a non-constant element, which is not allowed in $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[x]$.

Example 4.8. Let C be an elliptic $\mathbb{F}_q$-curve and suppose ∞ is $\mathbb{F}_q$-rational (such one must exist). The restriction of C to C^{af} gives rise to an exact sequence (see [Hart, Cha.II, Prop.6.5(c)]):

$$0 \to \mathbb{Z} \to \text{Pic }(C) \to \text{Pic }(C^{\text{af}}) \to 0$$

in which the first map $1 \mapsto 1 \cdot \{\infty\}$ is injective because the degree of a curve's divisor is well defined. As we assumed ∞ is $\mathbb{F}_q$-rational, this sequence splits as abelian groups. The degree map on Pic (C) yields another exact sequence which again splits as abelian groups:

$$0 \to \text{Pic }^0(C) \to \text{Pic }(C) \to \mathbb{Z} \to 0.$$

We get an isomorphism of summands Pic $^0(C) \cong$ Pic (C^{af}). Together with another isomorphism of abelian groups: $C(\mathbb{F}_q) \cong \text{Pic }^0(C); P \mapsto [P] - [\infty]$ we may deduce that:

$$C(\mathbb{F}_q) \cong \text{Pic }(C^{\text{af}}). \tag{4.7}$$

Hence according to Theorem 4.5, any unimodular form f of rank ≥ 3 defined over $\text{Spec}\,\mathcal{O}_{\{\infty\}}$ admits the Hasse principle if and only if there is no element of order 2 in $C(\mathbb{F}_q)$. For example, suppose $q > 3$ and $\infty = (0:1:0) \in C(\mathbb{F}_q)$ is removed, so the remaining (non-singular) affine curve C^{af} is given in affine coordinates by the Weierstrass form

$$y^2 = x^3 + ax + b \ \text{ for some } a, b \in \mathbb{F}_q.$$

Then f admits the Hasse principle if and only if C^{af} does not have any $\mathbb{F}_q$-point on the x-axis.

Corollary 4.9. *Let C be an elliptic $\mathbb{F}_q$-curve and suppose $\infty \in C(\mathbb{F}_q)$. Then for any integral unimodular form f of any rank $n > 2$ one has $c(f) = |C(\mathbb{F}_q)/2|$.*

Lemma 4.10. *Let C be an elliptic $\mathbb{F}_q$-curve. Suppose that $-1 \in (\mathbb{F}_q^\times)^2$ and $\infty \in C(\mathbb{F}_q)$. Then $c(1_2) = |C(\mathbb{F}_q)|$.*

Proof. The orthogonal group scheme over $\text{Spec}\,\mathcal{O}_{\{\infty\}}$ of 1_2 is $\underline{\mathbf{O}}_2$. Consider the exact sequence of smooth $\mathcal{O}_{\{\infty\}}$-schemes (recall that char(K) is odd):

$$1 \to \underline{\mathbf{SO}}_2 \to \underline{\mathbf{O}}_2 \to \underline{\mu}_2 \to 1.$$

As $-1 \in (\mathbb{F}_q^\times)^2$, the one dimensional torus $\underline{\mathbf{SO}}_2$ is split, and so $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_2) = \text{Pic }(C^{\text{af}})$. Then due to isomorphism (4.7), $C(\mathbb{F}_q) \cong H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_2)$, classifying according to Proposition 4.2 the integral forms in gen(1_2). According to Hilbert 90 Theorem, this set is also equal to

$\ker[H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_2) \to H^1(K, \mathbf{SO}_2)]$, so the geometric interpretation of this result is that non-trivial principal $\underline{\mathbf{SO}}_2$-bundles, which are in this case non-trivial line bundles of C^{af}, become trivial while tensoring with K. This causes the failure of the Hasse principle. □

Remark 4.11. Lemma 4.10 with isomorphism (4.7) show that the UFD condition for $\mathcal{O}_{\{\infty\}}$ is essential for the validity of the Hasse principle in case of rank 2, even for unimodular forms as 1_2. Moreover, even if $\mathcal{O}_{\{\infty\}}$ is a UFD, it is still essential to assume for $n = 2$ that the integer closure of $\mathcal{O}_{\{\infty\}}$ in the splitting field of $\mathbf{SO}_V$ is a UFD as well.

For example, the elliptic curve $C := \{ZY^2 = X^3 - XZ^2 - Z^3\}$ defined over $\mathbb{F}_3$ (in which -1 is not a square) has a single $\mathbb{F}_3$-rational point $(0 : 1 : 0)$. Suppose we choose it to be ∞. Then $C^{\text{af}} = \{y^2 = x^3 - x - 1\}$ and $\mathcal{O}_{\{\infty\}} = \mathbb{F}_3(C^{\text{af}})$ is a UFD (by (4.7)). But the extension $\mathcal{O}'_{\{\infty\}}$ by $i = \sqrt{-1}$, being the integer closure of $\mathcal{O}_{\{\infty\}}$ in the splitting field of $\mathbf{SO}_2$, gives rise to more rational points of C like $(-1 : i : 1)$. Thus as Pic $(\mathcal{O}_{\{\infty\}}) = 0$, étale cohomology applied on sequence (4.6) implies the bijection of the non-trivial sets: $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mathbf{SO}}_2) \cong \text{Pic}\ (\mathcal{O}'_{\{\infty\}})$ (see in the proof of Theorem 4.5), which shows according to Corollary 4.3 that the Hasse principle fails for the $\mathcal{O}_{\{\infty\}}$-form 1_2.

Example 4.12. Let $C = \{Y^2Z = X^3 + XZ^2\}$ defined over $\mathbb{F}_5$. Then:

$$C(\mathbb{F}_5) = \{(0:0:1), (1:0:2), (1:0:3), (0:1:0)\} \cong \mathbb{Z}/4\mathbb{Z}.$$

Removing $\infty = (0 : 1 : 0)$, we get the affine elliptic curve:

$$C^{\text{af}} = \{y^2 = x^3 + x\} \text{ with: } \mathcal{O}_{\{\infty\}} = \mathbb{F}_5[x, y]/(y^2 - x^3 - x).$$

According to Lemma 4.10, we have four $\mathcal{O}_{\{\infty\}}$-isomorphism classes in $\text{gen}(1_2)$. The key obstruction here for finding explicit integral forms from the same genus of 1_2 which are not $\mathcal{O}_{\{\infty\}}$-isomorphic to it, is using the fact that $\mathcal{O}_{\{\infty\}}$ is not a UFD in such a way that there exist distinct isomorphisms to 1_2, defined over integer rings at distinct places.

Explicitly, the affine supports of the points in $C(\mathbb{F}_5)$ are:

$$\{(0, 0), (1/2, 0) = (3, 0), (1/3, 0) = (2, 0), \infty\}.$$

Each one of the first three points corresponds to an intersection between (y) and another prime:

$$(0, 0) \leftrightarrow (y) \cap (x),\ (3, 0) \leftrightarrow (y) \cap (x - 3),\ (2, 0) \leftrightarrow (y) \cap (x - 2)$$

while ∞ is associated to the all affine curve C^{af}. For any $t \in \{1, x, x + 2, x - 2\}$, the matrix

$$P_{(t)} = \begin{pmatrix} 1 + 3t & 2 - t \\ 3 + t & 1 + 3t \end{pmatrix}$$

with $\det(P_{(t)}) = 2t$ is invertible in $C^{\rm af} - \{t\}$ (in all $C^{\rm af}$ if $t = 1$), giving rise to the integral form represented by

$$G_t = P_{(t)}^t P_{(t)} = \begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix}.$$

On the other hand, using the relation (which is due to the fact that $-1 \in (\mathbb{F}_5^\times)^2$):

$$y^2 = x(x+2)(x-2)$$

one may define the matrices:

$$Q_{(x)} = \frac{y}{(x-2)(x+2)} \begin{pmatrix} x-1 & x+1 \\ -(x+1) & x-1 \end{pmatrix}, \ Q_{(x-2)} = \frac{y}{x(x+2)} \begin{pmatrix} x-1 & x+3 \\ x+3 & -(x-1) \end{pmatrix}$$

$$Q_{(x+2)} = \frac{y}{x(x-2)} \begin{pmatrix} x+1 & x+2 \\ x+2 & -(x+1) \end{pmatrix}$$

satisfying each $Q_{(t)}^t Q_{(t)} = G_t$ as well, and being invertible at the remaining place (t).

We get four non-equivalent 1-cocycles, since $\sqrt{\det(G_{t_1})/\det(G_{t_2})} = t_1/t_2$ is not invertible over $\mathcal{O}_{\{\infty\}}$ for any $t_1 \neq t_2$. The generic fibers of these cocycles are trivial, since both $P_{(t)}$ and $Q_{(t)}$ are well defined over K for any t, and the transition maps $Q_{(t)}^t Q_{(t)} \cdot (P_{(t)}^t P_{(t)})^{-1}$ are trivial. The four corresponding non-isomorphic integral forms in $\mathrm{gen}(1_2)$ are those represented by $\{G_t\}$.

For $n > 2$, however, any unimodular form f defined over this domain $\mathbb{F}_5[C^{\rm af}]$ will admit according to Corollary 4.9 only $|C(\mathbb{F}_q)/2| = 2$ classes in the genus of f.

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