

# Pion decay constants in a strong magnetic field

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## Abstract

Decay constants of the charged and neutral pions in magnetic field are considered in the framework of the effective quark-antiquark lagrangian respecting Gell-Mann–Oakes–Renner (GOR) relations at zero field. The  $\sqrt{\frac{e_q B}{\sigma}}$  dependence is found in strong fields  $e_q B \gg \sigma$  for the neutral pion, while the charged pion constant decreases as  $\sqrt{\frac{\sigma}{e_q B}}$ .

## 1 Introduction

Pion decay constants are basic quantities in the chiral effective theory [1, 2] and are present in the fundamental GOR relations [3], moreover they play an important role of the order parameter, vanishing in the phase of restored chiral symmetry.

The behavior of pion decay constants (pdc) in magnetic field (m.f.) allows to probe the most fundamental properties of the QCD vacuum and hadrons and therefore together with the behavior of chiral condensate was a hot topic in the theoretical community, see [4, 5] for discussion and references.

Specifically, in the case of pdc the analysis was done in the framework of the chiral perturbation theory (ChPT) in [6, 7, 8, 9]. It was argued in [7], that the parameter of the ChPT is  $\xi = \frac{eB}{(4\pi f_\pi)^2}$ ,  $\xi < 1$  and it was found in

[6, 7, 8, 9], that  $f_\pi(eB)$  behaves in the lowest order as

$$\frac{f_\pi^2(eB)}{f_\pi^2(0)} = 1 + \frac{2eB \ln 2}{(4\pi f_\pi(0))^2} + \dots \quad (1)$$

In this analysis only pionic degrees of freedom contribute and the pion constituents, quarks, do not participate. At the same time it is clear, that quark should play an important role for  $eB \gtrsim \sigma$ , where  $\sigma = 0.18 \text{ GeV}^2$  is the string tension, and therefore the result (1), neglecting the pion quark structure, should be modified for  $eB \gtrsim \sigma$ , and possibly also for  $eB \gtrsim m_\pi^2$ .

Therefore it is of interest to study the pdc in m.f. in the approach of [4, 5], where the explicit results were obtained for the charged and neutral pion masses [5] and chiral condensate [4] as a function of m.f.

It was found there, that in the case of  $\pi^0$  the mass is strongly decreasing with  $eB$  (in contrast to much slower decrease in [6, 7]), while for  $\pi^+$  the mass is increasing (in agreement with lattice data [10]). Moreover, in [4] the chiral condensate was found to grow linearly with  $eB$  in good quantitative agreement with lattice data [11], which contradict much smaller slope of ChPT [7].

It is therefore possible, that the results of ChPT are valid in a smaller region, moreover they need modification for charged pions, since as was found in [5], GOR relations are violated for charged pions in m.f., while they are valid for neutral pions, in agreement with [6], [7].

It is a purpose of the present paper to proceed in the framework of the quark-antiquark formalism of [4, 5] to find the m.f. dependence of pdc both for neutral and charged pions. In the next section the general formalism is shortly discussed and the resulting expression for pdc is given. Section 3 is devoted to the neutral pions and section 4 to the charged pions, while section 5 contains summary and prospectives.

## 2 General formalism

The effective chiral quark-antiquark Lagrangian was derived and studied in [12, 13, 14, 15] without m.f.,

$$L_{ECL} = N_c \text{tr} \log[(\hat{\partial} + m_f)\hat{1} + M\hat{U}], \quad (2)$$

where the octet of Nambu-Goldstone (NG) mesons are given by

$$\hat{U} = \exp(i\hat{\phi}\gamma_5), \quad \hat{\phi} = \phi_a t_a, \quad a = 1, \dots, 8. \quad (3)$$

Here  $M = M(x, y)$  is the scalar confinement interaction defined via the vacuum average of field correlators  $\langle tr F_{\mu\nu}(x)\phi F_{\mu\nu}(y)\phi \rangle$ ,  $\phi(x, y) = P \exp ig \int_y^x A_\mu dz_\mu$ .

Expansion of (2) to the quadratic in  $\hat{\phi}$  terms, produces the GOR relations and one obtains definitions of quark condensate  $\langle \bar{q}q \rangle$  and pdc [12, 13], e.g.

$$f_\pi^2 = N_c M(0) \sum_{n=0}^{\infty} \frac{|\psi_n(0)|^2}{m_n^3}, \quad (4)$$

where [13, 14]

$$M(0) = \frac{2\sigma\lambda}{\sqrt{\pi}}(1 + O(\sigma\lambda^2)) \approx 0.15 \text{ GeV} \quad (5)$$

and  $M_n, \psi_n$  are eigenvalues and eigenfunctions of the  $q\bar{q}$  Hamiltonian without chiral degrees of freedom. The corresponding masses have been computed in [12, 13, 14]

$$m_0 = 0.4 \text{ GeV}, \quad m_1 = 1.35 \text{ GeV}, \quad m_2 = 1.85 \text{ GeV} \quad (6)$$

and resulting value of  $f_\pi$  is [12, 13].

$$f_\pi = 96 \text{ MeV} \frac{M(0)}{(150 \text{ MeV})}, \quad -\frac{\langle \bar{q}q \rangle}{n_f} = (217 \text{ MeV})^3. \quad (7)$$

Now one can include m.f. as in [4, 5], which is done using  $\hat{\partial} \rightarrow \hat{D} = \hat{\partial} - ie_q A_\mu^{(e)} \gamma_\mu$ , where  $\mathbf{A}^{(e)} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ .

As a result one obtains the following form of pdc

$$f_\pi^2 = N_c M^2(0) \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{|\psi_{n,i}^{(+)}(0)|^2}{(M_{n,i}^{(+-)})^3} + \frac{|\psi_{n,i}^{(-)}(0)|^2}{(M_{n,i}^{(-+)})^3} \right). \quad (8)$$

Here  $(+-)$  and  $(-+)$  refer to the spin projections of the quark and anti-quark respectively, and  $n, i$  denote the quantum numbers  $n_\perp, n_3$  and  $u, d$  of the  $q\bar{q}$  motion in m.f.

### 3 The case of the neutral pion

We now turn to the  $q\bar{q}$  Hamiltonian in m.f. defining  $M_{n,i}$  and  $\psi_{n,i}$  in (8). In the case of the neutral pion the corresponding expression without chiral

degrees of freedom was derived before in [16, 17] and studied for the case of pion in [5],

$$M_n = \bar{M}_n^{(0)} + \Delta M_{\text{coul}} + \Delta M_{SE} + \Delta M_{ss}. \quad (9)$$

The form of  $\bar{M}_n$  (prior to stationary point insertions  $\omega_i \rightarrow \omega_i^{(0)}(eB)$ ) is

$$\bar{M}_n = \varepsilon_{n_\perp, n_z} + \frac{m_1^2 + \omega_1^2 - e_q \mathbf{B} \sigma_1}{2\omega_1} + \frac{m_2^2 + \omega_2^2 + e_q \mathbf{B} \sigma_2}{2\omega_2}, \quad (10)$$

where

$$\varepsilon_{n_\perp, n_z} = \frac{1}{2\tilde{\omega}} \left[ \sqrt{e^2 B^2 + \frac{4\sigma\tilde{\omega}}{\gamma}} (2n_\perp + 1) + \sqrt{\frac{4\sigma\tilde{\omega}}{\gamma}} \left( n_z + \frac{1}{2} \right) \right] + \frac{\gamma\sigma}{2}. \quad (11)$$

We start with the  $B = 0$  case and write  $\bar{M}_n$ , Eq. (10), for  $m_q = 0, \omega_1 = \omega_2, e_q = e$ .

$$\bar{M}_n = \omega + \frac{3}{2} \sqrt{\frac{2\sigma\omega}{\gamma}} + \frac{\gamma\sigma}{2}. \quad (12)$$

Minimizing in  $\omega, \gamma$  one obtains

$$\bar{M}_n(\omega_0, \gamma_0) = 4\omega_0, \quad \omega_0 = \frac{\sqrt{3\sigma}}{2} = 0.367 \text{ GeV}, \quad (13)$$

and  $(\Delta M_{\text{coul}} + \Delta M_{se})$  cancel approximately 1/2 of  $\bar{M}_n$  ([4,5]), so that the final value of the mass in (8) (without  $(\Delta M_{ss})$  for  $B = 0$  is  $m_0^{(+-)} = m_0^{(-+)} = 2\omega_0$ . Consider now the case of small  $eB \ll \sigma$ . In this case to the lowest order in  $\frac{eB}{\sigma}$  one obtains

$$\frac{|\psi^{(+-)}(0)|^2}{(M^{(+-)})^3} + \frac{|\psi^{(-+)}(0)|^2}{(M^{(-+)})^3} = 2 \left( \frac{\sigma}{2\pi} \right)^{3/2} \frac{1}{(2\omega_0)^3} + O \left( \left( \frac{eB}{\sigma} \right)^2 \right). \quad (14)$$

For  $|\psi(0)|^2$  one has

$$|\psi(0)|^2 = \frac{1}{\pi^{3/2} r_\perp^2 r_3}, \quad \frac{1}{r_\perp^2} = \frac{1}{2} \sqrt{(eB)^2 + \sigma^2 c}, \quad \frac{1}{r_3} = \left( \frac{\sigma^2 c}{4} \right)^{1/4}, \quad (15)$$

where  $c = \frac{4\tilde{\omega}}{\gamma\sigma}$ , and for  $eB = 0, C(eB = 0) = 1$ .

At large  $eB$ ,  $eB \gg \sigma$  from (9-11) one obtains

$$M_0^{(+-)} = 2\omega_0^{(+-)} = 3^{-1/4}\sqrt{\sigma}, \quad M_0^{(-+)} = 2\sqrt{2eB} \quad (16)$$

and for  $|\phi|^2$  one has from (15)

$$|\psi_{n_\perp=0, n_3}^{(+-)}(0)|^2 \cong \frac{\sqrt{\sigma}\sqrt{e_q^2 B^2 + \sigma^2}}{(2\pi)^{3/2}} \quad (17)$$

$$|\psi_{n_\perp=0, n_3}^{(-+)}(0)|^2 = \left(\frac{\sigma}{2\pi}\right)^{3/2} (c_{-+})^{3/4} \sqrt{1 + \left(\frac{e_q B}{\sigma}\right)^2 \frac{1}{c_{-+}}}, \quad (18)$$

where  $c_{-+}(B) = \left(1 + \frac{8e_q B}{\sigma}\right)^{2/3}$  (cf [4, 5]).

As a result one obtains for  $eB \gg \sigma$  (restoring  $e = |e_q| \equiv e_q$ )

$$f_{\pi^0}^2(e_q B) \cong N_c M^2(0) \frac{3^{3/4}}{2} \left(\frac{1}{2\pi}\right)^{3/2} \frac{e_q B}{\sigma} \cong f_{\pi^0}^2(0) \frac{3^{3/4}}{2} \cdot 3^{3/2} \frac{e_q B}{\sigma} \cong \frac{5.9e_q B}{\sigma} f_{\pi^0}^2(0), \quad (19)$$

and finally, since  $\pi^0 = \frac{1}{\sqrt{2}}(|\bar{u}u\rangle + |d\bar{d}\rangle)$ , one obtains

$$\overline{f_{\pi^0}^2(eB)} = \frac{1}{2} \left( f_{\pi^0}^2 \left( \frac{2}{3} eB \right) + f_{\pi^0}^2 \left( \frac{eB}{3} \right) \right). \quad (20)$$

The behavior of  $f_{\pi^0}(eB)$  according to (20) is shown in Fig.1 in comparison to the ChPT result of [6, 7, 8].

## 4 The case of charged pions

In this case one can use the so-called factorization approach, valid at large  $eB \gg \sigma$ , as shown in [5]. It is clear, that the states  $\rho^+(S_z = 0)$  and  $\pi^+$  are mixed by the hyperfine interaction and tend to their asymptotic states  $(+-)$  and  $(-+)$  of  $(u\bar{d})$  contribution. Considering  $u$  and  $\bar{d}$  states as independent, one has as in [5]

$$M_{+-}(B) = \left( \sqrt{m_u^2 + p_z^2} + \sqrt{m_d^2 + p_z^2 + 2|e_d|B} \right)_{P_z=0} \approx \sqrt{\frac{2}{3}} eB \quad (21)$$

$$M_{-+}(B) = \left( \sqrt{m_u^2 + p_z^2 + 2e_u B} + \sqrt{m_d^2 + p_z^2} \right)_{P_z=0} \approx \sqrt{\frac{4}{3}} eB. \quad (22)$$

Taking into account the hyperfine interaction, [5], one obtains the asymptotic behavior of  $\pi^+$  energy in m.f.

$$m_{as}(\pi^+) = M_{+-}(B) \approx \sqrt{\frac{2}{3}} eB. \quad (23)$$

One can also deduce from Eq. (29) of [5], that for the charges  $e_u = \frac{2}{3}e$  and  $e_{\bar{d}} = \frac{e}{3}$ , the resulting term in the hamiltonian const  $B^2\eta_{\perp}^2$  is equal to  $\frac{13e^2B^2}{18 \cdot 32\omega}\eta_{\perp}^2$ , which implies that  $e_q^2$  for  $\pi^+$  in (17), (18) should be replaced by approximately  $\frac{13}{36}e^2$ ,  $e_q = \sqrt{\frac{13}{36}}e$ .

Correspondingly, the  $\pi^+$  decay constant is given by Eq. (8), where masses can be taken from (23) and  $|\psi(0)|^2$  from (17), (18). Results will be discussed in the next section.

## 5 Discussion and results

We start with the case of  $\pi^0$  meson, where  $f_{\pi^0}^2$  is given in (8), (21),  $n = n_{\perp}, n_3$  and  $i = u, d$ , and  $M_{n,i}^{(-+)}$  are nonchiral pion masses, given by the eigenvalues of the Hamiltonian in Eqs. (9), (10), (11). As is was found above (see also [5]) the mass  $M_{oi}^{(+-)}$  changes from approximately  $\sqrt{3\sigma} - \Delta M_{coul} - \Delta M_{SE} - \Delta M_{ss} \cong \sqrt{\sigma}$  at  $B = 0$  to  $3^{-3/4}\sqrt{\sigma}$  at  $eB \gg \sigma$ , which can be approximated by the relation

$$M_{o,i}^{(+-)} = \sqrt{\sigma} \left( \frac{1 + \left(\frac{e_i B}{\sigma}\right)^2}{\left(\frac{e_i B}{\sigma}\right)^2 3^{3/2} + 1} \right)^{1/2} \equiv (M_{o,i}^{(+-)}(0))\mu^{(+-)}\left(\frac{e_i B}{\sigma}\right), \quad i = u, d, \quad (24)$$

with  $\mu^{(+-)}(x) = 1$  at  $x = 0$  and  $3^{-3/4}$  at  $x = \infty$ .

For the  $|\psi(0)|^2$  one has from (15)

$$|\psi_{o,i}^{(+-)}(0)|^2 = \left(\frac{\sigma}{2\pi}\right)^{3/2} \sqrt{c_{+-} + \left(\frac{e_i B}{\sigma}\right)^2}, \quad i = u, d, \quad (25)$$

and  $c_{+-}(eB)$  changes from 1 for  $eB = 0$  to  $3^{-1/12} \cong 0.91$  at  $eB \gg \sigma$ , and we take  $c_{+-} = 1$  with this accuracy for all  $eB$ .

For the  $(-+)$  states one has

$$M_{o,i}^{(-+)} = M_{o,i}^{(-+)}(0)\mu^{(-+)}\left(\frac{e_i B}{\sigma}\right), \quad \mu^{(-+)}(x) \cong (2^3 x + 1)^{1/2}. \quad (26)$$

Finally, for  $\psi^{(-+)}(0)$  one obtains from (15) (see also [5]. Eq.(71))

$$|\psi_{o,i}^{(-+)}(0)|^2 = (c_{-+})^{1/4} \left(\frac{\sigma}{2\pi}\right)^{3/2} \sqrt{c_{-+} + \left(\frac{e_i B}{\sigma}\right)^2}, \quad c_{-+} = \left(1 + \frac{8e_i B}{\sigma}\right)^{2/3}. \quad (27)$$

As a result one obtains for  $f_{\pi^0}$  (approximating the sum in (8) by the leading first term)

$$\frac{f_{\pi^0}^2(eB)}{f_{\pi^0}^2(0)} = \frac{1}{4} \left( \sum_{i=u,d} \frac{\sqrt{1+x_i^2}}{(\mu^{+-}(x_i))^3} + \sum_{i=u,d} \frac{c_{-+}^{1/4}(x_i)\sqrt{c_{-+}(x_i)+x_i^2}}{(\mu^{-+}(x_i))^3} \right) \quad (28)$$

where  $x_i = \frac{|e_i|B}{\sigma}$ . One can see, that at large  $eB \gg \sigma$  the behavior is

$$\frac{f_{\pi^0}^2(eB)}{f_{\pi^0}^2(0)} \cong 2.96 \frac{eB}{\sigma}, \quad e = e_u + |e_d|, \quad (29)$$

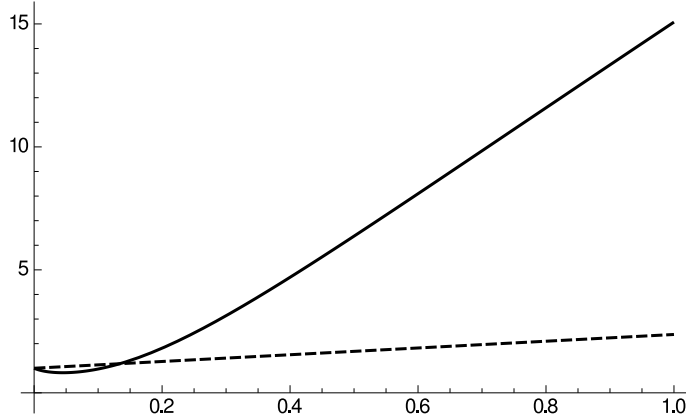


Figure 1: The ratio  $\frac{f_{\pi^0}^2(eB)}{f_{\pi^0}^2(0)}$  as a function of  $x = \frac{eB}{1 \text{ GeV}^2}$  in the ChPT (the lower curve (dashed)) and according to Eq.(28) (the upper curve)

The behavior of the ratio (28) is depicted in Fig. 1 together with the prediction of the ChPT [6, 7]. One can see, that the ratio (28) grows much

faster and exceeds the ChPT prediction more than 7 times at  $eB = 1 \text{ GeV}^2$ . However at small  $eB$ ,  $eB \ll \sigma$  the chiral ratio (1) grows  $1 + d_{ch} \left(\frac{eB}{\sigma}\right)$ , while the  $q\bar{q}$  answer (29) is  $1 + d_{q\bar{q}} \left(\frac{eB}{\sigma}\right)^2$ , and  $d_{ch} = \frac{2ln2\sigma^2}{(4\pi f_\pi(0))^2} \approx 0.04$ ,  $d_{q\bar{q}} \approx O(1)$ .

Hence there appears a region  $f_\pi^2(0) \sim eB \ll \sigma$ , where the result of ChPT is dominant, and the  $q\bar{q}$  structure of  $\pi^0$  is not yet displayed.

It is also interesting to follow the fate of the GOR relations. Writing it in the form, averaged over flavors

$$m_{\pi^0}^2 f_{\pi^0}^2 = \frac{m_n + m_d}{2} |\langle \bar{q}q \rangle|, \quad |\langle \bar{q}q \rangle| = \frac{1}{2} \sum_{i=u,d} (\bar{q}q_i) \quad (30)$$

where  $\langle \bar{q}q \rangle$  is [4, 5]

$$|\langle \bar{q}q \rangle_i(B)| = |\langle \bar{q}q \rangle_i(0)| \frac{1}{2} \left\{ \sqrt{1 + \left(\frac{e_q B}{\sigma}\right)^2} + \sqrt{1 + \left(\frac{e_q B}{\sigma}\right)^2 \frac{1}{c_{-+}}} \right\} \quad (31)$$

Comparing (30) and (8), one can see, that in(30) both l.h.s. and r.h.s. have the same form of the numerator,  $|\psi(0)|^2$ , and differ only in the power of  $M_{n,i}^{(+)}$  in the denominator, and since  $M_{\pi^0}(eB)$  is proportional to  $M_{o,i}^{(+)}$ , the GOR relation for  $\pi^0$  holds also large  $eB$ , in agreement with conclusions of [4, 5].

We now turn to the case of the charged pion. In this case at large  $eB$  one can use factorization technic [5], assuming both  $u$  and  $\bar{d}$  quark independent of each other, since at  $eB \gg \sigma$  the main interaction term  $\langle \sigma |\mathbf{r}_1 - \mathbf{r}_2| \rangle$  is subleading as compared to the m.f. contribution  $\sqrt{|e_i B|}$ .

As it was found in [5], the asymptotic form of the energy

$$M_{\pi^+}(eB) \approx M^{(+)}(B) \approx \sqrt{\frac{2}{3}} eB. \quad (32)$$

At this point it is interesting to compare this result with an exact solution, which obtains in the case  $e_u = e_{\bar{d}} = \frac{\epsilon}{2}$ , see [17] for details. In this case the mass can be written as (the  $(\frac{\epsilon}{2}, \frac{\epsilon}{2})$  approximation)

$$\frac{M_{\frac{\epsilon}{2}, \frac{\epsilon}{2}}(eB)}{M_{\frac{\epsilon}{2}, \frac{\epsilon}{2}}(0)} = \sqrt{1 + \chi(eB) \frac{eB}{\sigma}}, \quad (33)$$



where  $\chi(eB) \approx 1.22 \left( \frac{1+eB/\sigma}{1+2eB/\sigma} \right)$ . One can see, that at large  $\frac{eB}{\sigma}$  the ratio of (33) and (32) is 1.04, so that one can use the  $\left( \frac{e}{2}, \frac{e}{2} \right)$  approximation to calculate  $|\psi(0)|^2$ , which otherwise is difficult to do in the factorization scheme. In this case, using Hamiltonian (72) of [17], one obtains approximately

$$\frac{|\psi_{\frac{e}{2}, \frac{e}{2}}(0)|_{eB}^2}{|\psi_{\frac{e}{2}, \frac{e}{2}}(0)|_0^2} \cong \left( 1 + \frac{1}{16} \chi^2(eB) \frac{eB}{\sigma} \right)^{1/2}. \quad (34)$$

As a result the behavior of the  $f_{\pi^+}$  can be written as

$$\frac{f_{\pi^+}^2(eB)}{f_{\pi^+}^2(0)} = \frac{\left( 1 + \frac{1}{16} \chi^2(eB) \left( \frac{eB}{\sigma} \right)^2 \right)^{1/2}}{\left( 1 + \chi(eB) \frac{eB}{\sigma} \right)^{3/2}}. \quad (35)$$

From (35) one can deduce, that for  $eB \lesssim 1 \text{ GeV}^2$  the ratio (35) behaves as  $\frac{1}{\left( 3.4 \frac{eB}{1 \text{ GeV}^2} \right)^{3/2}}$ , which transforms into a slower decrease,  $\frac{1}{\sqrt{eB}}$  at large  $eB$ .

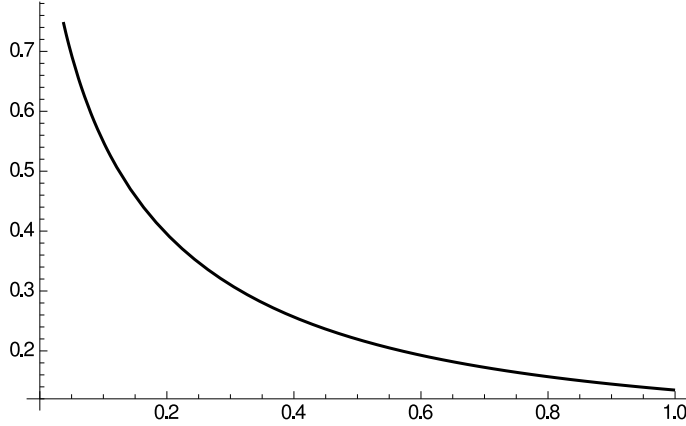


Figure 2: The ratio  $\frac{f_{\pi^+}^2(eB)}{f_{\pi^+}^2(0)}$  as a function of  $x = \frac{eB}{1 \text{ GeV}^2}$

In Fig. 2 we show <sup>1</sup> the ratio (35) as a function of  $eB$  up to  $eB = 1 \text{ GeV}^2$ . One can see a strong decrease of  $f_{\pi^+}^2(eB)$  with growing  $eB$ . At the same time the  $\pi^+$  mass is growing as  $\sqrt{\tilde{e}B}$ ,  $\tilde{e} = \frac{2}{3}e$  [5] (a similar behavior is found on the lattice [10] with  $\frac{2}{3}e \leq \tilde{e} \leq e$ , so that the l.h.s. of the GOR relation is kept constant, while in the r.h.s. the quark condensate  $\langle \bar{q}q \rangle$  is growing as  $eB$

<sup>1</sup>We have excluded the region  $eB \ll \sigma$ , where our result (35) is less accurate.

[4]. This is a clear manifestation of the fact, that GOR relations are violated for charged pions in m.f. – a conclusion, which was made before in ChPT [6, 7, 8].

Summarizing, we have found the m.f. dependence of the decay constants for the neutral and charged pions. We have compared the neutral pion constant behavior with that obtained in the ChPT, and found strong disagreement at large m.f.  $eB \gg \sigma$ , while for small m.f.,  $eB \lesssim f_{\pi^0}^2$ , the ChPT prevails. A moderate increase of  $\frac{f_{\pi^0}^2(eB)}{f_{\pi^0}^2(0)}$ , similar to the ChPT prediction, was also found in the NJL model in [18].

We have also found the decreasing behavior of  $f_{\pi^+} \sim (eB)^{-1/2}$  for large  $eB$  and confirmed the violation of GOR relations for charged pions in m.f.

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