# Equivalence of Wilson Actions

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## Abstract

We introduce the concept of equivalence among Wilson actions. Applying the concept to a real scalar theory on a euclidean space, we derive the exact renormalization group transformation of K. G. Wilson, and give a simple proof of universality of the critical exponents at any fixed point of the exact renormalization group transformation. We also show how to reduce the original formalism of Wilson to the simplified formalism by J. Polchinski.

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#### I. INTRODUCTION

The purpose of this paper is to introduce the concept of equivalence among Wilson actions. We consider a generic real scalar theory in D-dimensional euclidean space, and denote the Fourier component of the scalar field with momentum p by  $\phi(p)$ .

A Wilson action  $S[\phi]$  is a real functional of  $\phi(p)$ . A momentum cutoff is incorporated so that the exponentiated action  $e^{S[\phi]}$  can be integrated with no ultraviolet divergences. [\[1\]](#page-19-0) An example is given by

$$
S[\phi] = -\int_p \frac{p^2}{K\left(\frac{p}{\Lambda}\right)} \frac{1}{2} \phi(p)\phi(-p) + S_I[\phi] \tag{1}
$$

where the cutoff function  $K(\bar{p})$  is a positive function of  $\bar{p}^2$  that is 1 at  $\bar{p}^2 = 0$ , and decreases toward 0 rapidly for  $\bar{p}^2 \gg 1$ . The first term of the action can suppress the modes with momenta higher than  $\Lambda$  sufficiently that  $e^{S[\phi]}$  can be integrated over  $\phi(p)$  of all momenta. The second term consists of local interaction terms. In the continuum approach adopted here, correlation functions are defined for  $\phi(p)$  of all momenta, even those above  $\Lambda$ .

A Wilson action is meant to describe low momentum (energy) physics accurately, but not physics at or above the cutoff scale. If two Wilson actions describe the same low energy physics, we regard them as equivalent. It is the purpose of this paper to provide a concrete definition of equivalence using the continuum approach.

The paper is organized as follows. In sect. [II](#page-2-0) we introduce modified correlation functions, and then define equivalence of Wilson actions as the equality of the modified correlation functions. Basically, two Wilson actions are equivalent if their differences can be removed if we give them a massage at their respective cutoff scales. We derive two versions of explicit formulas that relate two equivalent Wilson actions. The concept of equivalence is applied in the rest of the paper.

In sect. [III](#page-5-0) we derive the exact renormalization group (ERG) transformation of Wilson [\[1](#page-19-0)] by considering a particular type of equivalence. We derive the ERG differential equation from our equivalence, which amounts to an integral solution to the differential equation. In sect. [IV](#page-6-0) we discuss the relation between the original formulation of ERG transformation by Wilson and the formulation by J. Polchinski [\[2\]](#page-19-1) which is more convenient for perturbation theory. In the previous literature only passing remarks have been given on this relation.[\[3](#page-19-2)][\[4\]](#page-19-3) Our short discussion of their relation is complete and hopefully illuminating. Sect. [V](#page-7-0) prepares us for the discussion of universality in sect. [VI.](#page-9-0) We generalize the definition of equivalence so that the exact renormalization group transformation can have fixed points. In sect. [VI,](#page-9-0) we assume a fixed point of the ERG transformation, and show that the critical exponents defined at the fixed point are independent of the choice of cutoff functions. This is what we mean by universality. We conclude the paper in sect. [VII.](#page-13-0)

Throughout the paper we work in D-dimensional euclidean momentum space. We use the following abbreviated notation

$$
\int_{p} \equiv \int \frac{d^{D}p}{(2\pi)^{D}}, \quad \delta(p) \equiv (2\pi)^{D} \delta^{(D)}(p) \tag{2}
$$

### <span id="page-2-0"></span>II. EQUIVALENCE

Given a Wilson action  $S[\phi]$ , we denote the correlation functions by

<span id="page-2-2"></span>
$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_S \equiv \int [d\phi] \, \phi(p_1) \cdots \phi(p_n) \, e^{S[\phi]} \tag{3}
$$

We consider modifying the correlation functions for high momenta without touching them for small momenta. We define modified correlation functions by

<span id="page-2-1"></span>
$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_S^{K,k} \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \exp\left(-\int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \tag{4}
$$

where  $K(p)$  and  $k(p)$  are non-negative functions of  $p^2$ . (We will call them cutoff functions.) As  $p^2 \to 0$ , we must find

$$
K(p) \longrightarrow 1, \quad k(p) \longrightarrow 0 \tag{5}
$$

so that the correlation functions are not modified at small momenta. In addition we constrain  $K(p)$  by

$$
K(p) \stackrel{p^2 \to \infty}{\longrightarrow} 0 \tag{6}
$$

In other words  $K(p)$  is small for  $p^2$  larger than the squared cutoff momentum of the Wilson action. The fluctuations of  $\phi(p)$  with p larger than the cutoff are suppressed, and we enhance their correlations by the large factor  $1/K(p)$  in [\(4\)](#page-2-1). The exponential on the right-hand side of [\(4\)](#page-2-1) amounts to mixing a free scalar with the propagator  $-k(p)/p^2$  to the original scalar field  $\phi$ . Since  $k(0) = 0$ , the free scalar has no dynamics of its own.

For example, we may take

$$
K(p) = e^{-\frac{p^2}{\Lambda^2}}\tag{7}
$$

and

$$
k(p) = \frac{p^2}{\Lambda^2}
$$
 or  $e^{-\frac{p^2}{\Lambda^2}} \left(1 - e^{-\frac{p^2}{\Lambda^2}}\right)$  (8)

where  $\Lambda$  is the momentum cutoff  $\Lambda$  of the Wilson action  $S$ .

In particular, for  $n = 2$  and  $n = 4$ , [\(4\)](#page-2-1) gives

$$
\langle \phi(p_1)\phi(p_2)\rangle_{S}^{K,k} = \frac{1}{K(p_1)^2} \left( \langle \phi(p_1)\phi(p_2)\rangle_{S} - \frac{k(p_1)}{p_1^2} \delta(p_1 + p_2) \right)
$$
(9)  

$$
\langle \phi(p_1)\cdots\phi(p_4)\rangle_{S}^{K,k} = \prod_{i=1}^{4} \frac{1}{K(p_i)} \left[ \langle \phi(p_1)\cdots\phi(p_4)\rangle_{S} - \langle \phi(p_1)\phi(p_2)\rangle_{S} \frac{k(p_3)}{p_3^2} \delta(p_3 + p_4) - \langle \phi(p_3)\phi(p_4)\rangle_{S} \frac{k(p_1)}{p_1^2} \delta(p_1 + p_2) + \frac{k(p_1)}{p_1^2} \delta(p_1 + p_2) \frac{k(p_3)}{p_3^2} \delta(p_3 + p_4) + \text{(t-, u-channels)} \right)
$$
(10)

For small momenta, the modified correlation functions [\(4\)](#page-2-1) reduce to the ordinary correlation functions [\(3\)](#page-2-2).

Now, we would like to introduce the concept of equivalence among Wilson actions. Let us regard two Wilson actions  $S_1, S_2$  as equivalent if, with an appropriate choice of  $K_{1,2}$  and  $k_{1,2}$ , their modified correlation functions become identical for any n and momenta:

<span id="page-3-0"></span>
$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_1}^{K_1, k_1} = \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_2}^{K_2, k_2}
$$
 (11)

Since the functions  $K_{1,2}, k_{1,2}$  keep the low energy physics intact,  $S_1$  and  $S_2$  describe the same low energy physics. In the following we solve [\(11\)](#page-3-0) to obtain an explicit relation between the two actions.

We first rewrite [\(11\)](#page-3-0) as

$$
\left\langle \exp\left(-\int_{p} \frac{k_{2}(p)}{p^{2}} \frac{1}{2} \frac{\delta^{2}}{\delta\phi(p)\delta\phi(-p)}\right) \phi(p_{1}) \cdots \phi(p_{n}) \right\rangle_{S_{2}}
$$
\n
$$
= \prod_{i=1}^{n} \frac{K_{2}(p_{i})}{K_{1}(p_{i})} \cdot \left\langle \exp\left(-\int_{p} \frac{k_{1}(p)}{p^{2}} \frac{1}{2} \frac{\delta^{2}}{\delta\phi(p)\delta\phi(-p)}\right) \phi(p_{1}) \cdots \phi(p_{n}) \right\rangle_{S_{1}} \tag{12}
$$

Since functional integration by parts gives

$$
\left\langle \exp\left(-\int_{p} \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S
$$
  
=  $\int [d\phi] e^{S[\phi]} \exp\left(-\int_{p} \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n)$   
=  $\int [d\phi] \phi(p_1) \cdots \phi(p_n) \exp\left(-\int_{p} \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) e^{S[\phi]}$  (13)

we obtain

$$
\int [d\phi] \left(\prod_{i=1}^{n} \phi(p_i)\right) \cdot \exp\left(-\int_p \frac{k_2(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right) e^{S_2[\phi]}
$$

$$
= \int [d\phi] \left(\prod_{i=1}^{n} \frac{K_2(p_i)}{K_1(p_i)} \phi(p_i)\right) \cdot \exp\left(-\int_p \frac{k_1(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right) e^{S_1[\phi]}
$$
(14)

This implies that

$$
\exp\left(-\int_{p} \frac{k_2(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) e^{S_2[\phi]} = \left[\exp\left(-\int_{p} \frac{k_1(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) e^{S_1[\phi]}\right]_{\text{subset}} (15)
$$

where the suffix "subst" denotes the substitution of

<span id="page-4-0"></span>
$$
\frac{K_1(p)}{K_2(p)}\phi(p) \tag{16}
$$

for  $\phi(p)$  on the right-hand side. Hence, we obtain an intermediate result

<span id="page-4-1"></span>
$$
e^{S_2[\phi]} = \exp\left(\int_p \frac{k_2(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) \left[\exp\left(-\int_p \frac{k_1(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) e^{S_1[\phi]}\right]_{\text{subst}} \quad (17)
$$

We can rewrite this in two ways. First, noting that under the substitution [\(16\)](#page-4-0), we must substitute

$$
\frac{K_2(p)}{K_1(p)} \frac{\delta}{\delta \phi(p)}\tag{18}
$$

for  $\frac{\delta}{\delta \phi(p)}$ , we obtain the first relation

<span id="page-4-2"></span>
$$
e^{S_2[\phi]} = \exp\left[\int_p \frac{1}{p^2} \left\{ k_2(p) - k_1(p) \left( \frac{K_2(p)}{K_1(p)} \right)^2 \right\} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] \exp\left(S_1\left[\frac{K_1}{K_2} \phi\right]\right) \tag{19}
$$

Note that the two actions are the same for  $\phi(p)$  with small p, since the function of  $p^2$  in the curly bracket above is negligible for small  $p^2$ . In this sense the two actions differ only by local terms.

Alternatively, we rewrite [\(17\)](#page-4-1) as

$$
e^{S_2[\phi]} = \left[ exp\left[\int_p \frac{1}{p^2} \left\{ k_2(p) \left( \frac{K_1(p)}{K_2(p)} \right)^2 - k_1(p) \right\} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_1[\phi]} \right]_{\text{subset}} \tag{20}
$$

Using the gaussian formula

$$
\exp\left(\int_p A(p) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right) \exp\left(S[\phi]\right)
$$
  
= 
$$
\int [d\phi'] \exp\left(-\int_p \frac{1}{2A(p)} \left(\phi'(-p) - \phi(-p)\right) \left(\phi'(p) - \phi(p)\right) + S[\phi']\right)
$$
(21)

(proven in Appendix [A\)](#page-13-1), we obtain the second relation

<span id="page-5-1"></span>
$$
e^{S_2[\phi]} = \int [d\phi'] \exp\left[ - \int_p \frac{p^2}{2\left(k_2(p)\left(\frac{K_1(p)}{K_2(p)}\right)^2 - k_1(p)\right)} \times \left(\phi'(p) - \frac{K_1(p)}{K_2(p)}\phi(p)\right) \left(\phi'(-p) - \frac{K_1(p)}{K_2(p)}\phi(-p)\right) + S_1[\phi']\right]
$$
(22)

We have thus obtained two explicit formulas [\(19,](#page-4-2) [22\)](#page-5-1) relating two equivalent Wilson actions. The remaining sections give applications of these formulas. In Appendix [B](#page-15-0) we give corresponding results for a Dirac fermion field.

## <span id="page-5-0"></span>III. EXACT RENORMALIZATION GROUP

Let us apply the results of the previous section to derive the exact renormalization group transformation of K. G. Wilson. (sect. 11 of [\[1\]](#page-19-0)) We choose

$$
\begin{cases}\nK_1(p) = K\left(\frac{p}{\Lambda}\right) , & k_1(p) = k\left(\frac{p}{\Lambda}\right) \\
K_2(p) = K\left(\frac{p}{\Lambda e^{-t}}\right), & k_2(p) = k\left(\frac{p}{\Lambda e^{-t}}\right)\n\end{cases}
$$
\n(23)

so that the two sets of cutoff functions differ only by the choice of a momentum cutoff. We demand that the modified correlation functions [\(4\)](#page-2-1) be independent of the momentum cutoff:

<span id="page-5-3"></span>
$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_2}^{K_2,k_2} = \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_1}^{K_1,k_1}
$$
 (24)

Now, using the first formula [\(19\)](#page-4-2), we obtain

$$
e^{S_2[\phi]} = \exp\left[\int_p \frac{1}{p^2} \left\{ k \left(\frac{p}{\Lambda} e^t\right) - k \left(\frac{p}{\Lambda}\right) \left(\frac{K\left(\frac{p}{\Lambda} e^t\right)}{K\left(\frac{p}{\Lambda}\right)}\right)^2 \right\} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right] \times \exp\left(S_1\left[\frac{K\left(\frac{p}{\Lambda}\right)}{K\left(\frac{p}{\Lambda} e^t\right)} \phi(p)\right]\right) \tag{25}
$$

By denoting  $S_1$  as  $S_\Lambda$  and  $S_2$  as  $S_{\Lambda e^{-t}}$ , and taking t infinitesimal, we obtain the exact renormalization group (ERG) differential equation

<span id="page-5-2"></span>
$$
-\Lambda \frac{\partial}{\partial \Lambda} e^{S_{\Lambda}[\phi]} = \int_{p} \left[ \frac{\Delta \left( \frac{p}{\Lambda} \right)}{K \left( \frac{p}{\Lambda} \right)} \phi(p) \frac{\delta}{\delta \phi(p)} \right] + \frac{1}{p^{2}} \left( 2 \frac{\Delta \left( \frac{p}{\Lambda} \right)}{K \left( \frac{p}{\Lambda} \right)} k \left( \frac{p}{\Lambda} \right) - \Lambda \frac{\partial}{\partial \Lambda} k \left( \frac{p}{\Lambda} \right) \right) \frac{1}{2} \frac{\delta^{2}}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_{\Lambda}[\phi]} \tag{26}
$$

where we define

$$
\Delta\left(\frac{p}{\Lambda}\right) \equiv \Lambda \frac{\partial}{\partial \Lambda} K\left(\frac{p}{\Lambda}\right) \tag{27}
$$

This amounts to (11.8) of ref. [\[1](#page-19-0)]. For the particular choice

<span id="page-6-2"></span>
$$
k(p) = K(p) (1 - K(p))
$$
\n(28)

[\(26\)](#page-5-2) gets simplified to

<span id="page-6-3"></span>
$$
-\Lambda \frac{\partial}{\partial \Lambda} e^{S_{\Lambda}[\phi]} = \int_{p} \left[ \frac{\Delta \left( \frac{p}{\Lambda} \right)}{K \left( \frac{p}{\Lambda} \right)} \phi(p) \frac{\delta}{\delta \phi(p)} + \frac{\Delta \left( \frac{p}{\Lambda} \right)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_{\Lambda}[\phi]}
$$
(29)

This was introduced first by J. Polchinski.[\[2](#page-19-1)]

Alternatively, we can use the second formula [\(22\)](#page-5-1) which gives

<span id="page-6-1"></span>
$$
e^{S_{\Lambda e^{-t}}[\phi]} = \int [d\phi'] \exp\left[ -\int_p \frac{p^2}{k\left(\frac{p}{\Lambda}e^t\right)\frac{K\left(\frac{p}{\Lambda}\right)^2}{K\left(\frac{p}{\Lambda}e^t\right)^2} - k\left(\frac{p}{\Lambda}\right)} \times \frac{1}{2} \left(\phi'(p) - \frac{K\left(\frac{p}{\Lambda}\right)}{K\left(\frac{p}{\Lambda}e^t\right)}\phi(p)\right) \left(\phi'(-p) - \frac{K\left(\frac{p}{\Lambda}\right)}{K\left(\frac{p}{\Lambda}e^t\right)}\phi(-p)\right) + S_{\Lambda}[\phi']\right] \tag{30}
$$

This is a well known integral solution of the ERG differential equation [\(26\)](#page-5-2). (This is discussed in details, for example, in [\[5\]](#page-19-4).) Though mathematically equivalent, our starting point [\(24\)](#page-5-3) of this section is easier to understand than the differential equation [\(26\)](#page-5-2) or its integral solution [\(30\)](#page-6-1).

Earlier in ref. [\[6\]](#page-20-0) it was observed that renormalized correlation functions of QED can be constructed out of correlation functions of its Wilson action: the modified correlation functions [\(4\)](#page-2-1) coincide with renormalized correlation functions. Cutoff independent correlation functions have been also discussed by O. J. Rosten.[\[7\]](#page-20-1)

## <span id="page-6-0"></span>IV. POLCHINSKI VS. WILSON

As a second application, we consider

$$
\begin{cases}\nK_1(p) = K\left(\frac{p}{\Lambda}\right), & k_1(p) = k\left(\frac{p}{\Lambda}\right) \\
K_2(p) = K'\left(\frac{p}{\Lambda}\right), & k_2(p) = k'\left(\frac{p}{\Lambda}\right) \equiv K'\left(\frac{p}{\Lambda}\right)\left(1 - K'\left(\frac{p}{\Lambda}\right)\right)\n\end{cases} (31)
$$

Note that  $k_2$  follows Polchinski's convention [\(28\)](#page-6-2), which is convenient for perturbative applications. Given a solution  $S_1$  of the ERG differential equation [\(26\)](#page-5-2) with  $K_1, k_1$ , we wish to construct an equivalent  $S_2$  that solves [\(29\)](#page-6-3) with  $K_2, k_2$ .

As has been shown in the previous section, the modified correlation functions are independent of  $\Lambda$ , if the Wilson action satisfies [\(26\)](#page-5-2). Hence, if  $S_1$  and  $S_2$  are equivalent at a particular  $\Lambda$ , they give the same modified correlation functions at any  $\Lambda$ . In the following let us choose  $\Lambda = 1$ , and demand  $S_1$  and  $S_2$  give the same modified correlation functions. Using [\(19\)](#page-4-2) and denoting  $S_1$  as S and  $S_2$  as S', we obtain

$$
e^{S'[\phi]} = \exp\left[\int_p \frac{1}{p^2} \left( k'(p) - k(p) \left( \frac{K'(p)}{K(p)} \right)^2 \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] \exp\left[S \left[ \frac{K}{K'} \phi \right] \right] \tag{32}
$$

A particularly simple result follows if we choose  $K'$  satisfying

$$
k'(p) = k(p) \left(\frac{K'(p)}{K(p)}\right)^2
$$
\n(33)

This gives

$$
K'(p) = \frac{K(p)}{K(p)^2 + k(p)} \cdot K(p)
$$
\n(34)

With this choice, we obtain

$$
S'[\phi] = S\left[\frac{K^2 + k}{K}\phi\right] \tag{35}
$$

so that the two equivalent actions are simply related by a linear change of field variables.

For the particular choice of made in sec. 11 of [\[1](#page-19-0)]

$$
K(p) = e^{-p^2}, \quad k(p) = p^2
$$
\n(36)

we obtain

$$
K'(p) = \frac{1}{1 + p^2 e^{2p^2}}\tag{37}
$$

### <span id="page-7-0"></span>V. ERG FOR FIXED POINTS

We now apply the results of sect. [II](#page-2-0) to show the universality of critical exponents at a fixed point of the ERG transformation, by which we mean the independence of critical exponents on the choice of cutoff functions  $K, k$ . For the ERG transformation to have a fixed point, we must change the transformation given in sect. [III](#page-5-0) in two ways:[\[1](#page-19-0)] first by adopting a dimensionless notation, and second by introducing an anomalous dimension to the scalar field. (We elaborate more on these points in Appendix [C.](#page-16-0)) After these changes, the Wilson action  $S_t$  depends on t such that

<span id="page-7-1"></span>
$$
\left\langle \phi(p_1 e^{\Delta t}) \cdots \phi(p_n e^{\Delta t}) \right\rangle_{S_{t+\Delta t}}^{K,k} = e^{\Delta t \cdot n \left(-\frac{D+2}{2} + \gamma\right)} \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_t}^{K,k}
$$
(38)

for the same cutoff functions  $K(p)$ ,  $k(p)$  independent of t. This is the new form of equivalence between  $S_t$  and  $S_{t+\Delta t}$ : their modified correlation functions are the same up to a scale transformation. On the right-hand side,  $-\frac{D+2}{2}$  $\frac{+2}{2}$  gives the canonical mass dimension of the field  $\phi(p)$  (since this is the Fourier transform, we obtain  $\frac{D-2}{2} - D = -\frac{D+2}{2}$  $\frac{+2}{2}$ , and  $\gamma$  is the anomalous dimension, taken for simplicity as a t-independent constant.

Let us solve [\(38\)](#page-7-1) to obtain  $S_{t+\Delta t}$  in terms of  $S_t$ . Following the same line of arguments given in sect. [II,](#page-2-0) we obtain

$$
e^{S_{t+\Delta t}[\phi]} = \exp\left(\int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right) \left[\exp\left(-\int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right) e^{S_t[\phi]}\right]_{\text{subset}} \quad (39)
$$

where "subst" stands for the substitution of

$$
e^{\Delta t \left(\frac{D+2}{2} - \gamma\right)} \frac{K(p)}{K(p e^{\Delta t})} \phi(p e^{\Delta t})\tag{40}
$$

for  $\phi(p)$ . Since this substitution implies the substitution of

$$
e^{\Delta t \left(D - \frac{D+2}{2} + \gamma\right)} \frac{K(p e^{\Delta t})}{K(p)} \frac{\delta}{\delta \phi(p e^{\Delta t})}
$$
(41)

for  $\frac{\delta}{\delta \phi(p)}$ , we obtain

$$
e^{S_{t+\Delta t}[\phi]} = \exp\left[\int_p \frac{1}{p^2} \left(k(p) - k(p e^{-\Delta t}) \frac{K(p)^2}{K(p e^{-\Delta t})^2} e^{\Delta t \cdot 2\gamma} \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right] \left[e^{S_t[\phi]}\right]_{\text{subset}} \quad (42)
$$

Taking  $\Delta t$  infinitesimal, we obtain the ERG differential equation

<span id="page-8-0"></span>
$$
\partial_t e^{S_t[\phi]} = \int_p \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma \right) \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} + \int_p \frac{1}{p^2} \left( 2 \frac{\Delta(p)}{K(p)} k(p) + 2p^2 \frac{dk(p)}{dp^2} - 2\gamma k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p)\delta \phi(-p)} e^{S_t[\phi]}
$$
(43)

For Polchinski's choice

$$
k(p) = K(p) (1 - K(p))
$$
\n(44)

this gets simplified to

$$
\partial_t e^{S_t[\phi]} = \int_p \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma \right) \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]}
$$

$$
+ \int_p \frac{1}{p^2} \left\{ \Delta(p) - 2\gamma K(p) \left( 1 - K(p) \right) \right\} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{S_t[\phi]}
$$
(45)

which is given in [\[5\]](#page-19-4). For Wilson's choice

$$
K(p) = e^{-p^2}, \quad k(p) = p^2
$$
\n(46)

[\(43\)](#page-8-0) gives

$$
\partial_t e^{S_t[\phi]} = \int_p \left[ \frac{D}{2} \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} + \int_p \left( 1 - \gamma + 2p^2 \right) \left( \phi(p) \frac{\delta}{\delta \phi(p)} + \frac{\delta^2}{\delta \phi(p)\delta \phi(-p)} \right) e^{S_t[\phi]}
$$
(47)

which reproduces (11.17) of [\[1\]](#page-19-0) under the identification

$$
\frac{d\rho(t)}{dt} = 1 - \gamma \tag{48}
$$

Now, the anomalous dimension  $\gamma$  is chosen for the existence of a fixed point action  $S^*$ that satisfies

<span id="page-9-1"></span>
$$
\int_{p} \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma \right) \phi(p) + p_{\mu} \frac{\partial \phi(p)}{\partial p_{\mu}} \right] \frac{\delta}{\delta \phi(p)} e^{S^*[\phi]} \n+ \int_{p} \frac{1}{p^2} \left( 2 \frac{\Delta(p)}{K(p)} k(p) + 2p^2 \frac{dk(p)}{dp^2} - 2\gamma k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p)\delta \phi(-p)} e^{S^*[\phi]} = 0
$$
\n(49)

At the fixed point, the correlation functions obey the scaling law:

$$
\left\langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \right\rangle_{S^*}^{K,k} = e^{tn\left(-\frac{D+2}{2} + \gamma\right)} \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S^*}^{K,k}
$$
\n
$$
(50)
$$

Only for specific choices of  $\gamma$ , [\(49\)](#page-9-1) has an acceptable solution. For example, if we assume  $S^*$  to be quadratic in  $\phi$ , the solution becomes non-local unless  $\gamma = 0, -1, -2, \cdots$ . We then obtain

$$
S^* = -\frac{1}{2} \int_p \frac{p^{2(1-\gamma)}}{ZK(p)^2 + k(p)p^{2(-\gamma)}} \phi(p)\phi(-p) \tag{51}
$$

which gives

$$
\langle \phi(p)\phi(q)\rangle_{S^*}^{K,k} = \frac{Z}{p^{2(1-\gamma)}}\delta(p+q)
$$
\n(52)

where  $Z$  is an arbitrary positive constant. (This is discussed in Appendix of [\[1](#page-19-0)].)

## <span id="page-9-0"></span>VI. UNIVERSALITY OF CRITICAL EXPONENTS

We now discuss universality of critical exponents at an arbitrary fixed point  $S^*$  of the ERG transformation, reviewed in the previous section. Universality within the ERG formalism has been shown in ref. [\[8](#page-20-2)]; our discussion below has the merit of conciseness. (In Appendix [D](#page-18-0) we derive those results of [\[8\]](#page-20-2) relevant to the present paper.)

 $S^*$  depends on  $K, k$ , but we know from sect. [II](#page-2-0) that for any choice of  $K, k$  there is an equivalent action that gives the same modified correlation functions. [\(19\)](#page-4-2) gives the equivalent action  $S^{\prime\ast}$  for  $K^{\prime}, k^{\prime}$  as

$$
e^{S'^{*}[\phi]} = \exp\left[\int_p \frac{1}{p^2} \left( k'(p) \frac{K(p)^2}{K'(p)^2} - k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] \exp\left(S^* \left[\frac{K}{K'}\phi\right]\right) \tag{53}
$$

Since the integrand of the exponent vanishes at  $p^2 = 0$ ,  $S^*$  and  $S'^*$  differ by local terms. Since

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S'^*}^{K',k'} = \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S^*}^{K,k}
$$
\n(54)

the anomalous dimension  $\gamma$  is independent of the choice of K, k.

Now, the anomalous dimension  $\gamma$  is not the only critical exponent defined at the fixed point  $S^*$ . The other exponents appear as scale dimensions of local composite operators. [\[1\]](#page-19-0) A composite operator  $\mathcal{O}_y(p)$  with momentum p is a functional of  $\phi$  satisfying

$$
\left\langle \!\! \left\langle \mathcal{O}_y(p\mathrm{e}^t)\phi(p_1\mathrm{e}^t)\cdots\phi(p_n\mathrm{e}^t) \right\rangle \!\! \right\rangle_{S^*}^{K,k} = \mathrm{e}^{t\left\{ -y+n\left( -\frac{D+2}{2}+\gamma \right) \right\}} \left\langle \!\! \left\langle \mathcal{O}_y(p)\phi(p_1)\cdots\phi(p_n) \right\rangle \!\! \right\rangle_{S^*}^{K,k} \tag{55}
$$

where the modified correlation functions are defined by

$$
\langle \mathcal{O}_y(p)\phi(p_1)\cdots\phi(p_n)\rangle_{S^*}^{K,k}
$$
  
\n
$$
\equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \langle \mathcal{O}_y(p) \exp\left(-\int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) \phi(p_1)\cdots\phi(p_n)\rangle_{S^*}
$$
(56)

and  $-y$  is the scale dimension of  $\mathcal{O}_y$ . (The scale dimension of  $\mathcal{O}_y(x) = \int_p e^{ipx} \mathcal{O}_y(p)$  in coordinate space is  $D-y$ .) For the equivalent fixed point action  $S'^*$  with  $K', k'$ , the corresponding composite operator has the same modified correlation functions:

$$
\langle \! \langle \mathcal{O}_y(p)\phi(p_1)\cdots\phi(p_n) \rangle \! \rangle_{S^*}^{K,k} = \langle \! \langle \mathcal{O}'_y(p)\phi(p_1)\cdots\phi(p_n) \rangle \! \rangle_{S'^*}^{K',k'} \tag{57}
$$

This gives  $\mathcal{O}'_y(p)$  as

$$
\mathcal{O}'_y(p) e^{S'^*[\phi]} = \exp\left(\int_p \frac{1}{p^2} \left( k'(p) \frac{K(p)^2}{K'(p)^2} - k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \left[ \mathcal{O}_y(p) e^{S^*[\phi]} \right]_{\text{subst}} \tag{58}
$$

where "subst" implies substitution of

$$
\frac{K(p)}{K'(p)}\phi(p) \tag{59}
$$

into  $\phi(p)$ . The scale dimension y is thus independent of the choice of K, k. We conclude that all the critical exponents are independent of  $K, k$ .

Before closing this section, we would like to discuss two issues related to the fixed point action  $S^*$ .

#### <span id="page-11-0"></span>A. Ambiguity of the fixed point action

Given K, k, and an appropriate choice of  $\gamma$ , the fixed point solution  $S^*$  of the ERG differential equation is still not unique. This is because normalization of the scalar field can be arbitrary.

Given  $S^*$ , we can construct  $S^*_{Z}$  $Z$  satisfying

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_Z^*}^{K,k} = Z^{\frac{n}{2}} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S^*}^{K,k}
$$
 (60)

To obtain  $S_Z^*$  $Z_Z^*$ , we set  $K_2 = \sqrt{Z}K_1$  and  $k_2 = k_1$  in [\(19\)](#page-4-2). We then get

$$
e^{S_Z^*[\phi]} = \exp\left(-\left(Z - 1\right) \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right) e^{S^* \left[\frac{\phi}{\sqrt{Z}}\right]}
$$
(61)

For example, the Z-dependence of the gaussian fixed point  $(\gamma = 0)$  is given by

$$
S_{G,Z}[\phi] = -\frac{1}{2} \int_p \frac{p^2}{ZK(p)^2 + k(p)} \phi(p)\phi(-p) \tag{62}
$$

Taking  $Z = 1 + 2\epsilon$ , where  $\epsilon$  is infinitesimal, we obtain

$$
S_{1+2\epsilon}^*[\phi] - S^*[\phi] = \epsilon \mathcal{N}^*[\phi]
$$
\n(63)

where

$$
\mathcal{N}^*[\phi] \equiv -\int_p \left\{ \phi(p) \frac{\delta S^*}{\delta \phi(p)} + \frac{k(p)}{p^2} \left( \frac{\delta S^*}{\delta \phi(p)} \frac{\delta S^*}{\delta \phi(-p)} + \frac{\delta^2 S^*}{\delta \phi(p) \delta \phi(-p)} \right) \right\} \tag{64}
$$

is a local composite operator satisfying

$$
\langle \mathcal{N}^*[\phi] \phi(p_1) \cdots \phi(p_n) \rangle_{S^*}^{K,k} = n \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S^*}^{K,k}
$$
(65)

Obviously,  $\mathcal{N}^*[\phi]$ , called an equation-of-motion operator in [\[5](#page-19-4)], has scale dimension 0.

#### B. Universal fixed point action?

We have shown that the modified correlation functions are universal up to normalization of the scalar field. We now ask if there is a universal Wilson action  $S_{\text{univ}}^*$  that gives the universal modified correlation functions as its unmodified correlation functions:

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S^*_{\text{univ}}} = \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S^*}^{K,k}
$$
\n(66)

This implies

<span id="page-12-0"></span>
$$
e^{S_{\text{univ}}^*[\phi]} = \left[ \exp\left(-\frac{1}{2} \int_p \frac{k(p)}{p^2} \frac{\delta^2}{\delta \phi(p)\delta \phi(-p)}\right) e^{S^*[\phi]} \right]_{\text{subst}} \tag{67}
$$

where  $S^*$  is the fixed point action for  $K, k$ , and "subst" denotes substitution of

$$
K(p)\phi(p) \tag{68}
$$

for  $\phi(p)$ . The above result is obtained from [\(19\)](#page-4-2) by setting

<span id="page-12-1"></span>
$$
K_2 = 1, \quad k_2 = 0 \tag{69}
$$

We expect that the right-hand side is independent of  $K, k$ , i.e.,  $S_{\text{univ}}^*$  has no cutoff. But we know that the use of a cutoff is essential for Wilson actions, and there must be something wrong with  $S_{\text{univ}}^*$ .

Let us first consider the example of the gaussian fixed point given by

$$
S_G[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K(p)^2 + k(p)} \phi(p)\phi(-p) \tag{70}
$$

This gives the modified two-point function

$$
\langle \phi(p)\phi(q)\rangle_{S_G}^{K,k} = \frac{1}{p^2}\delta(p+q)
$$
\n(71)

[\(67\)](#page-12-0) indeed gives an action free from a cutoff:

$$
S_{G,\text{univ}}[\phi] = -\frac{1}{2} \int_p p^2 \phi(p)\phi(-p) \tag{72}
$$

For interacting theories, though, we expect [\(67\)](#page-12-0) makes no sense. Let us look at this a little more closely. As  $K_2, k_2$ , we choose

$$
\begin{cases}\nK_2(p) = K_{-t}(p) \equiv K(pe^{-t}) \\
k_2(p) = k_{-t}(p) \equiv k(pe^{-t})\n\end{cases}
$$
\n(73)

In the limit  $t \to +\infty$ , we obtain [\(69\)](#page-12-1):

$$
\lim_{t \to +\infty} K_{-t}(p) = 1, \quad \lim_{t \to +\infty} k_{-t}(p) = 0
$$
\n(74)

We then define  $S_{-t}^*$  so that

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_{-t}^*}^{K_{-t}, k_{-t}} = \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S^*}^{K, k}
$$
\n(75)

 $S_{-t}^*$  is related to  $S^*$  by the ERG transformation of sect. [III.](#page-5-0) Since the momentum cutoff of  $S^*$  is of order 1 (we are using the dimensionless convention), that of  $S_{-t}^*$  is of order  $e^t$ . Hence,  $S_{\text{univ}}^*$  has the infinite momentum cutoff. We then expect that the terms of  $S_{\text{univ}}^*$  to have divergent coefficients.

Thus, there is no fixed point action  $S_{\text{univ}}^*$  that gives the correlation functions without modification.

### <span id="page-13-0"></span>VII. CONCLUDING REMARKS

We have introduced the concept of equivalence among Wilson actions. Our equivalence is physically more transparent than the other formulations of the exact renormalization group via differential equations or integral formulas. In particular we have applied our equivalence to obtain a simple proof of universality of critical exponents within the ERG formalism.

#### <span id="page-13-1"></span>Appendix A: Gaussian Formula

In this appendix we prove the formula

<span id="page-13-2"></span>
$$
\exp\left[\int_p A(p)\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right] \exp\left[S[\phi]\right]
$$
  
= 
$$
\int [d\phi'] \exp\left[-\int_p \frac{1}{2A(p)}\phi'(-p)\phi'(p) + S[\phi + \phi']\right]
$$
(A1)

Though the following proof requires the positivity of  $A(p)$ , we expect the formula to remain valid as long as both hand sides make sense. The left-hand side makes sense for any  $A(p)$ , and the right-hand side makes sense even if  $A(p) < 0$  for some p as long as convergence of functional integration is provided by the Wilson action S.

It is easy to understand this formula in terms of Feynman graphs. The right-hand side implies the coupling of a scalar field  $\phi'$  whose propagator is  $A(p)$ . Contracting the pairs of  $\phi'$ , we obtain the left-hand side. More formally, we can prove the equality by comparing the generating functionals of both hand sides for arbitrary source  $J(p)$ . Let us first compute the generating functional of the left-hand side:

$$
e^{W_L[J]} \equiv \int [d\phi] \exp\left[\int_p J(-p)\phi(p)\right] \exp\left[\int_p A(p)\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right] \exp\left[S[\phi]\right] \tag{A2}
$$

Integrating this by parts, we obtain

$$
e^{W_L[J]} = \int [d\phi] \exp\left[S[\phi]\right] \exp\left[\int_p A(p) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right] \exp\left[\int_p J(-p) \phi(p)\right]
$$
  
= 
$$
\int [d\phi] \exp\left[S[\phi] + \int_p \left(J(-p) \phi(p) + \frac{1}{2} J(-p) A(p) J(p)\right)\right]
$$
(A3)

We next compute the generating functional of the right-hand side:

$$
e^{W_R[J]} = \int [d\phi] \exp\left[\int_p J(-p)\phi(p)\right]
$$
  
 
$$
\times \int [d\phi'] \exp\left[-\frac{1}{2} \int_p \frac{1}{A(p)} \phi'(p)\phi'(-p) + S[\phi + \phi']\right]
$$
 (A4)

We first shift  $\phi'$  by  $-\phi$ , and then shift  $\phi$  by  $+\phi'$  to obtain

$$
e^{W_R[J]} = \int [d\phi][d\phi'] \exp\left[\int_p J(-p) (\phi(p) + \phi'(p))\right]
$$
  
\n
$$
\times \exp\left[-\frac{1}{2} \int_p \frac{1}{A(p)} \phi(p)\phi(-p) + S[\phi']\right]
$$
  
\n
$$
= \int [d\phi] \exp\left[\int_p \left(-\frac{1}{2A(p)} \phi(p)\phi(-p) + J(-p)\phi(p)\right)\right]
$$
  
\n
$$
\times \int [d\phi'] \exp\left[\int_p J(-p)\phi'(p) + S[\phi']\right]
$$
(A5)

If  $A(p)$  is positive, we can perform the gaussian integral over  $\phi$  to obtain

$$
e^{W_R[J]} = \int [d\phi'] \exp\left[\frac{1}{2} \int_p J(-p)A(p)J(p) + \int_p J(-p)\phi'(p) + S[\phi']\right]
$$
(A6)

We thus obtain

$$
W_L[J] = W_R[J] \tag{A7}
$$

for arbitrary J. This proves the gaussian formula [\(A1\)](#page-13-2).

Finally, shifting  $\phi'$  by  $-\phi$ , we rewrite [\(A1\)](#page-13-2) as

$$
\exp\left[\int_p A(p)\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right] \exp\left[S[\phi]\right]
$$
  
= 
$$
\int [d\phi'] \exp\left[-\int_p \frac{1}{2A(p)} \left(\phi'(-p) - \phi(-p)\right) \left(\phi'(p) - \phi(p)\right) + S\left[\phi'\right]\right]
$$
 (A8)

This is the form used in section [II.](#page-2-0)

#### <span id="page-15-0"></span>Appendix B: Equivalence of Fermionic Wilson Actions

For a Dirac spinor field  $\psi$  and its complex conjugate  $\bar{\psi}$ , we define modified correlation functions by

$$
\langle \langle \psi(p_1) \cdots \psi(p_n) \bar{\psi}(q_n) \cdots \bar{\psi}(q_1) \rangle \rangle_S^{K,k} \equiv \prod_{i=1}^n \frac{1}{K(p_i)K(q_i)} \times \langle \psi(p_1) \cdots \psi(p_n) \exp \left( - \int_p \frac{\overleftarrow{\delta}}{\delta \psi(p)} \frac{k(p)}{p} \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} \right) \bar{\psi}(q_n) \cdots \bar{\psi}(q_1) \rangle_S
$$
(B1)

so that

$$
\langle \! \langle \psi(p)\bar{\psi}(q) \rangle \! \rangle_{S}^{K,k} = \frac{1}{K(p)^2} \left[ \langle \psi(p)\bar{\psi}(q) \rangle_{S} - \frac{k(p)}{\not p} \delta(p+q) \right]
$$
(B2)

Two Wilson actions  $S_{1,2}$  are equivalent if  $K_{1,2}$  and  $k_{1,2}$  exist so that

$$
\left\langle \psi(p_1)\cdots\psi(p_n)\bar{\psi}(q_n)\cdots\bar{\psi}(q_1)\right\rangle_{S_1}^{K_1,k_1} = \left\langle \psi(p_1)\cdots\psi(p_n)\bar{\psi}(q_n)\cdots\bar{\psi}(q_1)\right\rangle_{S_2}^{K_2,k_2}
$$
(B3)

The formula analogous to [\(22\)](#page-5-1) is given by

$$
e^{S_2[\psi,\bar{\psi}]} = \int [d\psi'd\bar{\psi}'] \exp \left[ - \int_p \left( \bar{\psi}'(-p) - \frac{K_1(p)}{K_2(p)} \bar{\psi}(-p) \right) \frac{\not p}{k_2(p) \frac{K_1(p)^2}{K_2(p)^2} - k_1(p)} \times \left( \psi'(p) - \frac{K_1(p)}{K_2(p)} \psi(p) \right) + S[\psi',\bar{\psi}'] \right]
$$
(B4)

The formula analogous to [\(19\)](#page-4-2) is somewhat more complicated to write down. Denoting

$$
A_{ab}(p) \equiv \left(\frac{1}{p}\right)_{ab} \left(k_2(p) - k_1(p)\frac{K_2(p)^2}{K_1(p)^2}\right)
$$
 (B5)

we obtain

$$
e^{S_2[\psi,\bar{\psi}]} = \text{Tr}\left[\exp\left(\int_p \frac{\overleftarrow{\delta}}{\delta\psi(p)} A(p) \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}\right) \exp\left(S_1\left[\frac{K_1}{K_2}\psi, \frac{K_1}{K_2}\bar{\psi}\right]\right)\right]
$$
\n
$$
\equiv \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_{p_1,\dots,p_n} \prod_{i=1}^n A_{a_i b_i}(p_i)
$$
\n
$$
\times \frac{\overrightarrow{\delta}}{\delta\bar{\psi}_{b_n}(-p_n)} \cdots \frac{\overrightarrow{\delta}}{\delta\bar{\psi}_{b_1}(-p_1)} \exp\left(S_1\left[\frac{K_1}{K_2}\psi, \frac{K_1}{K_2}\bar{\psi}\right]\right) \frac{\overleftarrow{\delta}}{\delta\psi_{a_1}(p_1)} \cdots \frac{\overleftarrow{\delta}}{\delta\psi_{a_n}(p_n)}\tag{B7}
$$

 $K_{2}$ 

 $K_{2}$ 

 $\delta \psi_{a_n}(p_n)$ 

where the spinor indices are summed over. The exponential implies contraction of  $\psi(p)\bar{\psi}(q)$ by  $A(p)\delta(p+q)$ .

#### <span id="page-16-0"></span>Appendix C: Derivation of [\(38\)](#page-7-1)

In this appendix we provide more details behind the new form of equivalence [\(38\)](#page-7-1). Starting from the original equivalence [\(24\)](#page-5-3), we obtain [\(38\)](#page-7-1) in two steps: first by rescaling dimensionful quantities, and second by introducing an anomalous dimension of the scalar field.

#### 1. Rescaling

We first rewrite the equivalence  $(24)$  by rescaling dimensionful quantities such as momenta and field variables. Note that  $K_1$  and  $K_2$  differ only by a rescaling of momentum

<span id="page-16-3"></span>
$$
K_2(p e^{-t}) = K_1(p).
$$
 (C1)

Likewise, we have

<span id="page-16-1"></span>
$$
k_2(pe^{-t}) = k_1(p).
$$
 (C2)

We wish to rewrite  $S_2$  in such a way that its cutoff functions become  $K_1, k_1$ .

For this purpose, we introduce a rescaled field variable

$$
\bar{\phi}(p) \equiv e^{-t\frac{D+2}{2}}\phi(p e^{-t})
$$
\n(C3)

so that

<span id="page-16-2"></span>
$$
\frac{\delta}{\delta \bar{\phi}(p)} = e^{-t\frac{D-2}{2}} \frac{\delta}{\delta \phi(p e^{-t})}.
$$
\n(C4)

We then define

<span id="page-16-4"></span>
$$
\bar{S}_2[\bar{\phi}] \equiv S_2[\phi]. \tag{C5}
$$

In other words  $\bar{S}_2[\phi]$  is obtained from  $S_2[\phi]$  by substituting  $e^{t\frac{D+2}{2}}\phi(pe^t)$  for  $\phi(p)$ . For example, given

$$
S_2[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K_2(p)} \phi(p)\phi(-p) , \qquad (C6)
$$

we obtain

$$
\bar{S}_2[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K_2(p)} e^{t(D+2)} \phi(pe^t) \phi(-pe^t)
$$
  
= 
$$
-\frac{1}{2} \int_p \frac{p^2}{K_2(pe^{-t})} \phi(p)\phi(-p) = -\frac{1}{2} \int_p \frac{p^2}{K_1(p)} \phi(p)\phi(-p).
$$
 (C7)

We rewrite the left-hand side of [\(24\)](#page-5-3) as

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_2}^{K_2, k_2} = e^{nt \frac{D+2}{2}} \langle \langle \bar{\phi}(p_1 e^t) \cdots \bar{\phi}(p_n e^t) \rangle_{S_2}^{K_2, k_2}
$$
  
\n
$$
= e^{nt \frac{D+2}{2}} \prod_{i=1}^n \frac{1}{K_2(p_i)} \cdot \langle \exp\left(-\int_p \frac{k_2(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}\right)
$$
  
\n
$$
\times \bar{\phi}(p_1 e^t) \cdots \bar{\phi}(p_n e^t) \rangle_{S_2[\phi]}.
$$
 (C8)

Using  $(C2)$  and  $(C4)$ , we obtain

$$
\int_{p} \frac{k_2(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} = e^{-t(D-2)} \int_{p} \frac{k_2(pe^{-t})}{p^2} \frac{\delta^2}{\delta \phi(pe^{-t}) \delta \phi(-pe^{-t})}
$$
\n
$$
= \int_{p} \frac{k_1(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)}.
$$
\n(C9)

Hence, using [\(C1\)](#page-16-3), we obtain

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_2}^{K_2, k_2} = e^{nt \frac{D+2}{2}} \prod_{i=1}^n \frac{1}{K_1(p_i e^t)} \cdot \left\langle \exp\left(-\int_p \frac{k_1(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)}\right) \right\rangle \times \bar{\phi}(p_1 e^t) \cdots \bar{\phi}(p_n e^t) \right\rangle_{S_2[\phi]} \tag{C10}
$$

Using [\(C5\)](#page-16-4) and rewriting integration variables  $\bar{\phi}$  as  $\phi$ , we obtain

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_2}^{K_2, k_2} = e^{nt \frac{D+2}{2}} \langle \langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle \rangle_{\bar{S}_2}^{K_1, k_1}
$$
 (C11)

Thus, by rescaling,  $S_2$  has been converted to  $\bar{S}_2$  with the cutoff functions  $K_1, k_1$ .

We can now write [\(24\)](#page-5-3) as

$$
e^{nt\frac{D+2}{2}} \left\langle \left\langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \right\rangle \right\rangle_{\bar{S}_2}^{K_1,k_1} = \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_1}^{K_1,k_1}
$$
 (C12)

Replacing t by  $\Delta t$ , we obtain

<span id="page-17-0"></span>
$$
\left\langle \phi(p_1 e^{\Delta t}) \cdots \phi(p_n e^{\Delta t}) \right\rangle_{\bar{S}_2}^{K_1, k_1} = e^{-\Delta t \cdot n \frac{D+2}{2}} \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_1}^{K_1, k_1}
$$
\n(C13)

By writing  $S_1$  as  $S_t$  and  $\bar{S}_2$  as  $S_{t+\Delta t}$ , we obtain [\(38\)](#page-7-1) for  $\gamma = 0$ .

#### 2. Anomalous dimension

Given a Wilson action  $S[\phi]$ , we can construct an action  $S_Z[\phi]$  whose modified correlation functions differ only by normalization of the field:

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_Z}^{K,k} = Z^{\frac{n}{2}} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S}^{K,k}
$$
 (C14)

To obtain  $S_Z$ , we set  $K_2 = \sqrt{Z} K_1$  and  $k_2 = k_1$  in [\(19\)](#page-4-2). We then get

$$
\exp\left(S_Z[\phi]\right) = \exp\left(-(Z-1)\int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) \exp\left(S\left[\frac{\phi}{\sqrt{Z}}\right]\right) \tag{C15}
$$

(We have used the same transformation for the fixed point action in sect. [VI A.](#page-11-0))

Given  $\bar{S}_2$ , we construct  $S'_2$  $y_2$  such that

$$
\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S'_2}^{K_1, k_1} = e^{n\gamma \Delta t} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_2}^{K_1, k_1}
$$
 (C16)

where  $\gamma$  is an arbitrary constant. Then, [\(C13\)](#page-17-0) becomes

$$
\left\langle \phi(p_1 e^{\Delta t}) \cdots \phi(p_n e^{\Delta t}) \right\rangle_{S'_2}^{K_1, k_1} = e^{\Delta t \cdot n \left(-\frac{D+2}{2} + \gamma\right)} \left\langle \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_1}^{K_1, k_1} \tag{C17}
$$

This gives [\(38\)](#page-7-1), which defines the renormalization group transformation with an anomalous dimension.

Note that we have introduced an anomalous dimension  $\gamma$  by hand. A particular  $\gamma$  must be chosen for the new renormalization group transformation to have a fixed point.

### <span id="page-18-0"></span>Appendix D: Relation to the results of Latorre and Morris

In [\[8](#page-20-2)] Latorre and Morris have shown that the change of a cutoff function can be compensated by a change of field variables. We would like to explain briefly how their result can be reproduced from the results of the present paper.

The relation between two equivalent actions  $S_1$  (with  $K_1, k_1$ ) and  $S_2$  (with  $K_2, k_2$ ) has been given by [\(19\)](#page-4-2). Choosing

$$
\begin{cases}\nK_1 = K, & k_1 = k, \\
K_2 = K + \delta K, & k_2 = k + \delta k,\n\end{cases}
$$
\n(D1)

where  $\delta K$  and  $\delta k$  are infinitesimal, we obtain from [\(19\)](#page-4-2)

<span id="page-18-1"></span>
$$
(S_2[\phi] - S_1[\phi]) e^{S_1[\phi]} \simeq \int_p \frac{\delta}{\delta \phi(p)} \left[ \theta(p) e^{S_1[\phi]} \right], \qquad (D2)
$$

where

<span id="page-18-2"></span>
$$
\theta(p) \equiv -\frac{\delta K(p)}{K(p)} \phi(p) + \frac{1}{p^2} \left( \frac{1}{2} \delta k(p) - k(p) \frac{\delta K(p)}{K(p)} \right) \frac{\delta S_1}{\delta \phi(-p)}.
$$
 (D3)

In deriving [\(D2\)](#page-18-1), we have taken only the terms first order in  $\delta K$  or  $\delta k$ , and we have ignored a field independent constant. [\(D2\)](#page-18-1) gives the change of the action under an infinitesimal change of  $\phi(p)$  by  $\theta(p)$ . Upon the choice of the Polchinski convention  $k = K(1 - K)$ , [\(D3\)](#page-18-2) reduces to

$$
\theta(p) = -\frac{\delta K(p)}{2p^2} \left( \frac{\delta S_1}{\delta \phi(-p)} + \frac{2p^2}{K(p)} \phi(p) \right) \tag{D4}
$$

which reproduces (3.15) of [\[8\]](#page-20-2).

In addition Latorre and Morris have shown that the ERG transformation is also a change of variables. Our ERG differential equation [\(43\)](#page-8-0) can be rewritten as

$$
\partial_t e^{S_t[\phi]} = \int_p \frac{\delta}{\delta \phi(p)} \left[ \Psi_t(p) e^{S_t[\phi]} \right], \qquad (D5)
$$

where

$$
\Psi_t(p) \equiv \left(\frac{D+2}{2} - \gamma + \frac{\Delta(p)}{K(p)}\right)\phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} + \frac{1}{p^2} \left(\frac{\Delta(p)}{K(p)} k(p) + p^2 \frac{dk(p)}{dp^2} - \gamma k(p)\right) \frac{\delta S_t}{\delta \phi(-p)}.
$$
\n(D6)

Thus,  $\partial_t S_t$  is the change of the action by an infinitesimal change of  $\phi(p)$  by  $\Psi_t(p)$ . Upon the choice  $k = K(1 - K)$ , the above reduces to

$$
\Psi_t(p) = \left(\frac{D+2}{2} - \gamma + \frac{\Delta(p)}{K(p)}\right)\phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} + \frac{1}{p^2} \left(\frac{1}{2}\Delta(p) - \gamma K(p)(1 - K(p))\right) \frac{\delta S_t}{\delta \phi(-p)}
$$
(D7)

which reproduces (2.3) of [\[8\]](#page-20-2) if  $\gamma = 0$ .

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