## Krylov-Veretennikov Formula for Functionals from the Stopped Wiener Process

G. V. Riabov

Institute of Mathematics, NAS of Ukraine

#### Abstract

We consider a class of measures absolutely continuous with respect to the distribution of the stopped Wiener process  $w(\cdot \wedge \tau)$ . Multiple stochastic integrals, that lead to the analogue of the Itô-Wiener expansions for such measures, are described. An analogue of the Krylov-Veretennikov formula for functionals  $f = \varphi(w(\tau))$  is obtained.

Keywords and phrases. Wiener process, stochastic integral, Itô-Wiener expansion.

2010 Mathematics Subject Classification. Primary 60J60; Secondary 60J50, 60H30.

### 1 Introduction

Let  $\{w(t)\}_{t\geq 0}$  be a standard Wiener process in  $\mathbb{R}^d$ , starting from the point  $u \in \mathbb{R}^d$ . Consider an open connected set  $G \ni u$ , the exit time

$$
\tau = \inf\{t > 0 : w(t) \notin G\},\
$$

and a Borel function  $\rho : \mathbb{R}^d \to (0,1)$ .

The main object of the investigation in the present paper is the orthogonal structure of the space  $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$ , where the measure Q is given by the density

$$
\frac{dQ}{d\mathbb{P}} = \frac{1_{\tau < \infty} \rho(w(\tau))}{\mathbb{E}1_{\tau < \infty} \rho(w(\tau))}.
$$

In [\[11,](#page-8-0) L. 2.4] it was proved that the space  $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$  possesses an orthogonal structure similar to the Itô-Wiener decomposition in the Gaussian case  $[1, 5, 8]$  $[1, 5, 8]$  $[1, 5, 8]$ . Namely, consider functions

$$
\beta(v) = \mathbb{E}1_{\tau(w-u+v)<\infty}\rho(w(\tau(w-u+v)-u+v)), \ v \in G,
$$
  

$$
\alpha(s,v) = \beta^{-1}(v)\mathbb{E}1_{s<\tau(w-u+v)<\infty}\rho(w(\tau(w-u+v)-u+v)), \ s > 0, v \in G,
$$

and processes

$$
\tilde{w}(s) = w(s \wedge \tau) - \int_0^{s \wedge \tau} \nabla \log \beta(w(r)) dr, \ s \ge 0;
$$
  

$$
\hat{w}_t(s) = \tilde{w}(s) - \int_0^{s \wedge \tau} \nabla \log \alpha(t - r, w(r)) dr, \ 0 \le s \le t.
$$

<span id="page-1-1"></span>**Theorem 1.1.** [\[11,](#page-8-0) L. 2.4] Each random variable  $f \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$  can be uniquely represented as a series of pairwise orthogonal stochastic integrals

<span id="page-1-0"></span>
$$
f = \sum_{n=0}^{\infty} \int \ldots \int a_n(t_1,\ldots,t_n) d\hat{\tilde{w}}_{t_n}(t_1)\ldots d\hat{\tilde{w}}_{t_n}(t_{n-1}) d\tilde{w}(t_n).
$$
 (1.1)

Conversely, given a sequence of Borel functions  $a_n : (0, \infty)^n \to \mathbb{R}^{d^n}$ ,  $n \geq 0$ , such that

$$
\sum_{n=0}^{\infty}\int_{0
$$

the series in the right-hand side of [\(1.1\)](#page-1-0) converges in  $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$ , and its sum f satisfies

$$
\mathbb{E}f^2 = \sum_{n=0}^{\infty} \int \ldots \int \limits_{0 < t_1 < \ldots < t_n} \alpha(t_n, u) |a_n(t_1, \ldots, t_n)|^2 dt_1 \ldots dt_n.
$$

In this paper we derive the explicit form of the expansion [\(1.1\)](#page-1-0) for random variables of the kind  $f = \varphi(w(\tau))$ . The resulting formula is similar to the well-known Krylov-Veretennikov formula [\[6\]](#page-7-3). It is written in terms of the transition semigroup  $\{T_t^k\}_{t\geq0}$  of a certain diffusion, killed at the boundary of G. Indeed, the process  $\tilde{w}$  is a stopped Wiener process relatively to the measure  $Q$  [\[11,](#page-8-0) L. 2.4]. Respectively, the initial Wiener process  $w$ is a diffusion process relatively to the measure  $Q$ . Then  $\{T_t^k\}_{t\geq0}$  is the transition semigroup of the process  $w$  killed at the boundary of  $G$ . Let  $T$  denote the integration with respect to the exit distribution of w from  $G$  (precise expressions for these operators are given in the section 2). The main result of the present paper is the following formula, proved in the theorem [2.1:](#page-4-0)

for every random variable  $\varphi(w(\tau)) \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$  the expansion [\(1.1\)](#page-1-0) has the form

$$
\varphi(w(\tau)) = \sum_{n=0}^{\infty} \int \ldots \int \alpha(t_n, u)^{-1} \bigg( T_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla \big( \alpha(t_2 - t_1, \cdot)^{-1} T_{t_2 - t_1}^k \big) \ldots
$$
  
\n
$$
\alpha(t_n - t_{n-1}, \cdot) \nabla \big( \alpha(t_n - t_{n-1}, \cdot)^{-1} T_{t_n - t_{n-1}}^k \big) \nabla T \varphi \bigg)(u) d\hat{\tilde{w}}_{t_n}(t_1) \ldots d\hat{\tilde{w}}_{t_n}(t_{n-1}) d\tilde{w}(t_n).
$$
\n(1.2)

Expansions of the kind  $(1.1)$  appeared in  $[4]$  in connection with the problem of studying the behaviour of Gaussian measures under nonlinear transformations. Such expansions have two main features:

- 1. the summands in [\(1.1\)](#page-1-0) are pairwise orthogonal;
- 2. the summands in [\(1.1\)](#page-1-0) are  $\sigma(w(\cdot \wedge \tau))$ –measurable.

Of course, there are other possibilities to organize series expansions for random variables from  $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$ . For simplicity, consider the case  $Q = \mathbb{P}$ . The most straightforward approach comes from the obvious inclusion  $\sigma(w(\cdot \wedge \tau)) \subset \sigma(w)$ . It means that each

random variable  $f \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), \mathbb{P})$  possesses an Itô-Wiener expansion with respect to the Wiener process  $w$ :

<span id="page-2-0"></span>
$$
f = \sum_{n=0}^{\infty} \int \ldots \int b_n(t_1, \ldots, t_n) dw(t_1) \ldots dw(t_n).
$$
 (1.3)

The summands in the expansion are not  $\sigma(w(\cdot \wedge \tau))$  –measurable. While the left-hand side of [\(1.3\)](#page-2-0) is  $\sigma(w(\cdot \wedge \tau))$ –measurable, one can condition (1.3) with respect to  $w(\cdot \wedge \tau)$ and get another expansion

<span id="page-2-1"></span>
$$
f = \sum_{n=0}^{\infty} \int \ldots \int_{0 < t_1 < \ldots < t_n < \tau} b_n(t_1, \ldots, t_n) dw(t_1) \ldots dw(t_n). \tag{1.4}
$$

Now the stochastic integrals of different degree are not orthogonal. This causes known inconveniences: the expansion  $(1.4)$  is not unique (an example is given in [\[4\]](#page-7-4)); the conditions for the expression in the right-hand side of [\(1.4\)](#page-2-1) to converge are complicated. An application of the Gram-Shmidt orthogonalization procedure to expansions [\(1.4\)](#page-2-1) was considered in [\[2\]](#page-7-5). However, in our framework it seems to be too complicated either to obtain the orthogonalized form of [\(1.4\)](#page-2-1), or to find the orthogonalized expansion [\(1.4\)](#page-2-1) for a concrete random variable f. The expansion [\(1.1\)](#page-1-0) overcomes all these problems.

Motivation for the  $\sigma(w(\cdot \wedge \tau))$ −measurability of the summands in [\(1.1\)](#page-1-0) comes from B. S. Tsirelson's theory of black noise. It is well-known that Brownian coalescing flows produce filtrations with trivial Gaussian parts [\[12,](#page-8-1) [7\]](#page-7-6). So, to get a unified description of functionals measurable with respect to such flows, it is reasonable to use the noise generated by the flow itself. The results from [\[4,](#page-7-4) [11\]](#page-8-0) show that this idea works: in [\[4\]](#page-7-4) an orthogonal expansion of the kind [\(1.1\)](#page-1-0) was obtained for the stopped Brownian motion; in [\[11\]](#page-8-0) the same was done for the n−point motions of the Arratia flow. We refer to [\[4,](#page-7-4) [11\]](#page-8-0) for the detailed discussion of this and related questions.

Generalization of the Krylov-Veretennikov formula to the wide class of dynamical systems driven by the additive Gaussian noise was obtained in  $[3]$ . Our formula  $(2.7)$  is similar to the one obtained in [\[3\]](#page-7-7) despite the additional multipliers  $\alpha$ . They occure to normalize operators  $T_t^k$ , as  $T_t^k 1 = \alpha(t, \cdot)$ .

The article is organized in the following way. In the section 2 we introduce all the needed notions and constructions. Also, it contains the reduction of the main theorem [2.1](#page-4-0) to lemmata [2.1](#page-5-0) and [2.2.](#page-5-1) Sections 3 and 4 are devoted to the proof of these auxiliary results.

### 2 Notations and Main Results

To formulate our results, we will use the following notations.

 $\{w(t)\}_{t\geq0}$  is the Wiener process in  $\mathbb{R}^d$ . Without loss of generality, we will assume that  $w$  is constructed in a canonical way:

 $\Omega = C([0,\infty), \mathbb{R}^d)$  is a space of continuous functions equipped with a metric of uniform convergence on compacts;

 $\mathcal F$  is the Borel  $\sigma$ −field on  $\Omega$ ;

 $w(t, \omega) = \omega(t)$  is the canonical process on  $(\Omega, \mathcal{F})$ ,  $\mathcal{F}_t = \sigma(w(s) : 0 \le s \le t)$  is the natural filtration of w;

 $(\mathbb{P}_{v})_{v\in\mathbb{R}^{d}}$  is a family of probability measures on  $(\Omega,\mathcal{F})$ , such that relatively to  $\mathbb{P}_{v}$ , w is a d−dimensional Wiener process starting from v. The expectation with respect to certain probability measure Q on  $(\Omega, \mathcal{F})$  will be denoted by  $\mathbb{E}_{Q}$ .  $\mathbb{E}_{\mathbb{P}_{v}}$  will be abbreviated to  $\mathbb{E}_{v}$ .

Let  $G \subset \mathbb{R}^d$  be an open connected set,  $\tau$  be the exit time of w from the set  $G$ :

$$
\tau = \inf\{t > 0 : w(t) \notin G\}.
$$

We will assume that for all  $v \in G$ ,  $\mathbb{P}_v(\tau < \infty) > 0$ . Fix a Borel function  $\rho : \mathbb{R}^d \to (0,1)$ and consider the function

$$
\beta(v) = \mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)), \ v \in G.
$$

It is a harmonic function in  $G$  [\[9,](#page-7-8) Ch. 4, Prop. 2.1]:

$$
\Delta_v \beta(v) = 0, \ v \in G.
$$

Denote  $Q_u$  the probability measure on  $(\Omega, \mathcal{F})$ , defined via the density

$$
\frac{dQ_u}{d\mathbb{P}_u} = \beta(u)^{-1} 1_{\tau < \infty} \rho(w(\tau)).
$$

We will need another probability measure corresponding to the process  $w$  killed at the moment  $\tau$ . Consider the function

$$
\alpha(s,v) = Q_v(\tau > s), \ s > 0, v \in G.
$$

In the section 1 following processes were introduced.

<span id="page-3-0"></span>
$$
\tilde{w}(s) = w(s \wedge \tau) - \int_0^{s \wedge \tau} \nabla_v \log \beta(w(\tau)) d\tau, \ s \ge 0; \tag{2.5}
$$

<span id="page-3-1"></span>
$$
\hat{\tilde{w}}_t(s) = \tilde{w}(s) - \int_0^{s \wedge \tau} \nabla_v \log \alpha(t - r, w(r)) dr, \ 0 \le s \le t.
$$
\n(2.6)

Throughout the paper derivatives will be taken in  $v \in G$ , so we will omit the index v in the derivatives' notation.

Consider a probability measure  $Q_{t,u}$  on  $(\Omega, \mathcal{F}_t)$ , defined via the density

$$
\frac{dQ_{t,u}}{dQ_u} = \alpha(t,u)^{-1}1_{\tau>t}.
$$

The key observation leading to the theorem [1.1](#page-1-1) is that on the probability space  $(\Omega, \mathcal{F}_t, Q_{t,u})$ the process  $\hat{\tilde{w}}_t$  is a Wiener process [\[10,](#page-8-2) Ch. VIII, Th. (1.4)].

Introduce following operators:

1.  $T\psi(v) = \mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)) \psi(w(\tau)), v \in G.$ 

Denote  $\mu_v$  the distribution of  $w(\tau)$  relatively to the measure  $1_{\tau<\infty}d\mathbb{P}_v$ . Then the action of the operator T reduces to the integration with respect to  $\mu_v$ :

$$
T\psi(v) = \int \psi(x)\mu_v(dx).
$$

2.  $\widetilde{T}\psi(v) = \beta(v)^{-1}T\psi(v), v \in G.$ 

The operator  $\widetilde{T}$  is the expectation relatively to the probability measure  $Q_v$ :

$$
\widetilde{T}\psi(v) = \mathbb{E}_{Q_v} \rho(w(\tau)).
$$

3.  $T_s^k \psi(v) = \mathbb{E}_{Q_v} 1_{\tau > s} \psi(w(s)), \ s > 0, v \in G.$ From equations  $(2.5)$ ,  $(2.6)$  it follows that

$$
dw(s) = (\nabla \log \alpha(t - s, w(s)) + \nabla \log \beta(w(s)))ds + d\hat{\tilde{w}}(s),
$$

where  $\hat{\hat{w}}$  is a Wiener process on  $(\Omega, \mathcal{F}_t, Q_{t,u})$ . So, relatively to the measure  $Q_{t,u}$ the process w satisfies (degenerate) SDE. Respectively,  $\{T_t^k\}_{t\geq 0}$  is the transition semigroup of a killed diffusion process w. Denote  $\mu_{s,v}$  the distribution of  $w(s)$ relatively to the measure  $1_{\tau > s} dQ_v$ . Then the action of the operator  $T_s^k$  reduces to the integration with respect to  $\mu_{s,v}$ :

$$
T_s^k \psi(v) = \int \psi(x) \mu_{s,v}(dx).
$$

4.  $T_s^k \psi(v) = \alpha(s, v)^{-1} T_s^k \psi(v), \ s > 0, v \in G.$ 

The operator  $T_s^k$  is the expectation relatively to the probability measure  $Q_{s,v}$ :

$$
\widetilde{T_s^k}\psi(v) = \mathbb{E}_{Q_{s,v}}\psi(w(s)).
$$

The following theorem is the main result of the paper.

<span id="page-4-0"></span>**Theorem 2.1.** For every  $\varphi \in L^2(\rho d\mu_u)$  the expansion [\(1.1\)](#page-1-0) has the form

<span id="page-4-1"></span>
$$
\varphi(w(\tau)) = \sum_{n=0}^{\infty} \int \ldots \int \alpha(t_n, u)^{-1} \left( \alpha(t_1, \cdot) \widetilde{T}_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla \widetilde{T}_{t_2 - t_1}^k \ldots \right. \\
\left. \alpha(t_n - t_{n-1}, \cdot) \nabla \widetilde{T}_{t_n - t_{n-1}}^k \nabla \widetilde{T} \varphi \right) (u) d\hat{\tilde{w}}_{t_n}(t_1) \ldots d\hat{\tilde{w}}_{t_n}(t_{n-1}) d\tilde{w}(t_n).
$$
\n(2.7)

The proof is divided into two lemmas, which are proved in the next sections. At first we derive the Clark representation for  $\varphi(w(\tau))$  with respect to the stopped Wiener process  $\tilde{w}$  [\[10,](#page-8-2) Ch. V, Th.  $(3.5)$ ]

<span id="page-5-0"></span>**Lemma 2.1.** For every  $\varphi \in L^2(\rho d\mu_u)$ , one has the representation

<span id="page-5-3"></span>
$$
\varphi(w(\tau)) = \widetilde{T}\varphi(u) + \int_0^{\tau} \nabla \widetilde{T}\varphi(w(t))d\widetilde{w}(t), \ Q_u - a.s. \tag{2.8}
$$

Subsequently, we find the Itô-Wiener expansion for the random variable  $\psi(w(t))$  with respect to the Wiener process  $\tilde{w}$ .

<span id="page-5-1"></span>**Lemma 2.2.** For every  $\psi \in L^2(\mu_{t,u})$  the Itô-Wiener expansion of  $\psi(w(t))$  has the form

<span id="page-5-2"></span>
$$
\psi(w(t)) = \sum_{n=0}^{\infty} \int \ldots \int \alpha(t, u)^{-1} \left( \alpha(t_1, \cdot) \widetilde{T}_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla \widetilde{T}_{t_2 - t_1}^k \ldots \right. \\
\left. \alpha(t - t_n, \cdot) \nabla \widetilde{T}_{t - t_n}^k \varphi \right) (u) d\hat{\tilde{w}}_t(t_1) \ldots d\hat{\tilde{w}}_t(t_n).
$$
\n(2.9)

The theorem [2.1](#page-4-0) follows by substituting  $\psi = \nabla \tilde{T} \varphi$  in [\(2.9\)](#page-5-2) and inserting the right-hand side of [\(2.9\)](#page-5-2) into [\(2.8\)](#page-5-3).

# 3 Clark Representation Formula with respect to the Measure  $Q_u$ . Proof of the Lemma [2.1](#page-5-0)

*Proof.* 1) At first we will prove that the function  $\widetilde{T}\varphi$  is smooth and satisfies the equation

<span id="page-5-4"></span>
$$
(\nabla \widetilde{T}\varphi, \nabla \log \beta) + \frac{1}{2}\Delta \widetilde{T}\varphi = 0
$$
\n(3.10)

in G. Indeed,

<span id="page-5-5"></span>
$$
\widetilde{T}\varphi(v) = \frac{\mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)) \varphi(w(\tau))}{\beta(v)}\tag{3.11}
$$

is the ratio of two harmonic functions [\[9,](#page-7-8) Ch. 4, Th. 3.7] (for the numerator the condition  $\varphi \in L^2(\rho d\mu_u)$  is used). The equation [\(3.10\)](#page-5-4) is checked by straightforward calculation.

2) We will prove the relation [\(2.8\)](#page-5-3) for bounded and continuous functions  $\varphi$  and  $\rho$ , the other cases being covered by the usual limiting procedure. Let  ${G_n}_{n>1}$  be a sequence of open relatively compact sets, such that  $\overline{G_n} \subset G$  and  $G = \bigcup_{n=1}^{\infty} G_n$ . Denote  $\tau_n$  be the exit time from  $G_n$ :

$$
\tau_n = \inf\{t \ge 0 : w(t) \notin G_n\}.
$$

The convergence  $\tau_n \to \tau$ ,  $n \to \infty$ , holds.

From the relation [\(2.5\)](#page-3-0) it follows that the stopped process  $w(\cdot \wedge \tau_n)$  satisfies the SDE

$$
dw(s) = \nabla \log \beta(w(s))ds + d\tilde{w}(s), 0 \le s \le \tau_n.
$$

Applying the Itô formula to the function  $\widetilde{T}\varphi$  and the process  $w(\cdot \wedge \tau_n)$ , and using [\(3.10\)](#page-5-4), one gets the representation

$$
\widetilde{T}\varphi(w(\tau_n)) = \widetilde{T}\varphi(u) + \int_0^{\tau_n} \nabla \widetilde{T}\varphi(w(s))d\widetilde{w}(s), Q_u - \text{a.s.}
$$

It remains to check that  $\widetilde{T}\varphi(w(\tau_n)) \to \widetilde{T}\varphi(w(\tau_n))$ . As the function  $\varphi$  is bounded, one has

$$
\sup_{n\geq 1}\int_0^\infty \mathbb{E}_u 1_{\tau_n>s}(\nabla \widetilde{T}\varphi(w(s)))^2ds < \infty.
$$

Now, the convergence  $\tau_n \to \tau$ ,  $n \to \infty$ , implies the convergence

$$
\int_0^{\tau_n} \nabla \widetilde{T} \varphi(w(s)) d\tilde{w}(s) \xrightarrow{L^2(Q_u)} \int_0^{\tau} \nabla \widetilde{T} \varphi(w(s)) d\tilde{w}(s), \ n \to \infty.
$$

It remains to check that  $\widetilde{T}\varphi(w(\tau_n)) \to \varphi(w(\tau))$ ,  $n \to \infty$ . By [\[9,](#page-7-8) Ch. 4, Th. 2.3] the point  $w(\tau)$  is the regular point for the Dirichlet problem on G. The needed convergence follows from the representation [\(3.11\)](#page-5-5).  $\Box$ 

# 4 The Krylov-Veretennikov Formula. Proof of the Lemma [2.2](#page-5-1)

*Proof.* The kernels  $a_n$  in the expansion

$$
\psi(w(t)) = \sum_{n=0}^{\infty} \int \ldots \int a_n(t_1,\ldots,t_n) d\hat{\tilde{w}}_t(t_1)\ldots d\hat{\tilde{w}}_t(t_n)
$$

will be recovered from the expression

$$
\mathbb{E}_{Q_{t,u}}\psi(w(t))\int\limits_{0
$$
\left(\alpha(t_1,\cdot)\widetilde{T}_{t_1}^k\alpha(t_2-t_1,\cdot)\nabla\widetilde{T}_{t_2-t_1}^k...\alpha(t-t_n,\cdot)\nabla\widetilde{T}_{t-t_n}^k\varphi\right)(u)b_n(t_1,...,t_n)dt_1...dt_n,
$$
$$

in which  $b_n$  is a deterministic square integrable function. By induction, it is enough to check that for any square integrable  $\hat{\tilde{w}}$ –adapted process  ${g(s)}_{0 \le s \le t}$ , one has

<span id="page-6-0"></span>
$$
\mathbb{E}_{Q_{t,u}}\psi(w(t))\int_0^t g(s)d\hat{\tilde{w}}_t(s) = \int_0^t \frac{\alpha(s,u)}{\alpha(t,u)}\mathbb{E}_{Q_{s,u}}\alpha(t-s,w(s))\nabla \widetilde{T}_{t-s}^k\psi(w(s))g(s)ds. \tag{4.12}
$$

To do it note the equalities, which follow from [\(2.6\)](#page-3-1) and lemma [2.1](#page-5-0)

$$
\int_0^t g(s)d\hat{\tilde{w}}_t(s) = \int_0^t g(s)d\tilde{w}(s) - \int_0^t g(s)\nabla \log \alpha(t-s, w(s))ds, Q_{t,u} - \text{a.s.},
$$
  

$$
1_{\tau>t}\psi(w(t)) = T_t^k \psi(u) + \int_0^{t \wedge \tau} \nabla T_{t-s}^k \psi(w(s))d\tilde{w}(s), Q_u - \text{a.s.}
$$

Consequently,

$$
\mathbb{E}_{Q_{t,u}}\psi(w(t))\int_0^t g(s)d\hat{\tilde{w}}_t(s)=
$$

$$
= \mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_0^t g(s) d\tilde{w}_t(s) - \mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) ds =
$$
  
\n
$$
= \alpha(t, u)^{-1} \Big( \mathbb{E}_{Q_u} 1_{\tau > t} \psi(w(t)) \int_0^t g(s) d\tilde{w}_t(s) -
$$
  
\n
$$
- \mathbb{E}_{Q_u} 1_{\tau > t} \psi(w(t)) \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) ds \Big) =
$$
  
\n
$$
= \alpha(t, u)^{-1} \Big( \int_0^t \mathbb{E}_{Q_u} 1_{\tau > s} \nabla T_{t-s}^k \psi(w(s)) g(s) ds -
$$
  
\n
$$
- \int_0^t \mathbb{E}_{Q_u} 1_{\tau > s} T_{t-s}^k \psi(w(s)) g(s) \nabla \log \alpha(t-s, w(s)) ds \Big) =
$$
  
\n
$$
= \alpha(t, u)^{-1} \int_0^t \mathbb{E}_{Q_u} 1_{\tau > s} \Big( \nabla T_{t-s}^k \psi(w(s)) - T_{t-s}^k \psi(w(s)) \nabla \log \alpha(t-s, w(s)) \Big) g(s) ds =
$$
  
\n
$$
= \int_0^t \frac{\alpha(s, u)}{\alpha(t, u)} \mathbb{E}_{Q_{s,u}} \alpha(t-s, w(s)) \nabla \widetilde{T}_{t-s}^k \psi(w(s)) g(s) ds.
$$

The equality [\(4.12\)](#page-6-0) is proved.

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 $\Box$ 

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