Partial regularity of viscosity solutions for a class of Kolmogorov equations arising from mathematical finance

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Abstract

We study value functions which are viscosity solutions of certain Kolmogorov equations. Using PDE techniques we prove that they are $C^{1+\alpha}$ regular on special finite dimensional subspaces. The problem has origins in hedging derivatives of risky assets in mathematical finance.

Keywords: viscosity solution, Kolmogorov equation, stochastic differential equation, delay problem, hedging problem.

AMS 2010 subject classification: 35R15, 49L25, 60H15, 91G80.

1 Introduction

In this paper we study partial regularity of viscosity solutions for a class of Kolmogorov equations. Our motivation comes from mathematical finance, more precisely from hedging a derivative of a risky asset whose volatility as well as the claim may depend on the past history of the asset. Our Kolmogorov equations are thus associated to stochastic delay problems. They are linear second order partial differential equations in an infinite dimensional Hilbert space with a drift term which contains an unbounded operator and a second order term which only depends on a finite dimensional component of the Hilbert space. Such equations are typically investigated using the notion of the so-called B-continuous viscosity solutions (see [11, 14, 23]). We impose conditions under which our Kolmogorov equations have unique B-continuous viscosity solutions. However general Hamilton-Jacobi-Bellman equations associated to stochastic delay optimal control problems which are rewritten as optimal control problems for stochastic differential equations (SDEs) in an infinite dimensional Hilbert space are difficult, not well studied yet, and few results are available in the literature.

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We work directly with the value function here since its partial regularity is of interest in the hedging problem and it is well known that under our assumptions the value function is the unique B-continuous viscosity solution of the Kolmogorov equation (see e.g. [11, 14]). We thus never use the theory of B-continuous viscosity solutions. Instead our strategy for proving partial regularity of the value function is the following. We consider SDEs with smoothed out coefficients and the unbounded operator replaced by its Yosida approximations and study the corresponding value functions with smoothed out payoff function. The new value functions are Gâteaux differentiable and converge on compact sets to the original value functions. They also satisfy their associated Kolmogorov equations. We then prove that their finite dimensional sections are viscosity solutions of certain linear finite dimensional parabolic equations for which we establish $C^{1,\alpha}$ estimates. Passing to the limit with the approximations, these estimates are preserved giving $C^{1,\alpha}$ partial regularity for finite dimensional sections of the original value function.

Partial regularity results for first order unbounded HJB equations in Hilbert spaces associated to certain deterministic optimal control problems with delays have been obtained in [12]. The technique of [12] relied on arguments using concavity of the data and strict convexity of the Hamiltonian and provided C^1 regularity on one-dimensional sections corresponding to the so-called "present" variable. Here the equations are of second order, we rely on approximations and parabolic regularity estimates, and we obtain regularity on m-dimensional sections. The reader can also consult [18] for various global and partial regularity results for bounded HJB equations in Hilbert spaces (see also [22]).

We refer the reader to [11, 18, 19] for the theory of viscosity solutions for bounded second order HJB equations in Hilbert spaces and to [11, 14, 23] for the theory of the so-called B-continuous viscosity solutions for unbounded second order HJB equations in Hilbert spaces. A fully nonlinear equation with a similar separated structure to our Kolmogorov equation (3.14) but with a nonlinear unbounded operator A was studied in [15]. For classical results about Kolmogorov equation in Hilbert spaces we refer the reader to [8].

The plan of the paper is the following. In the rest of the Introduction we explain the financial motivation of our problem. Section 2 contains notation and various results about mild solutions of the SDE, their extensions to a bigger space with a weaker topology related to the original unbounded operator A, and various approximation results. In Section 3 we study viscosity solutions of the approximating equations, investigate finite dimensional sections of viscosity solutions, and prove their regularity.

1.1 Motivation from finance

One motivation for the present study comes from the classical problem in financial mathematics of hedging a derivative of some risky assets.

Let us consider a financial market composed of two assets: a risk free asset P (a bond price), and a risky asset R (a stock price). We assume that P follows the deterministic dynamics $dP_s = rP_s ds$, where r is the (constant) spot interest reate, and that R follows the dynamics

$$\begin{cases} dR_s = rR_s ds + \nu(s,R_s)dW_s & s \in (t,T], \\ R_t = x, \end{cases} \tag{1.1}$$

where $(\Omega, \mathscr{F}, \mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space, $T > 0$ is the maturity date, $x \in \mathbb{R}$, $t \in [0,T)$, and v satisfies the usual Lipschitz assumptions. Denote by $R^{t,x}$ the unique strong solution of SDE (1.1).

Given a function $\varphi\colon \mathbb{R} \to \mathbb{R}$, the problem of hedging the derivative $\varphi(R_T^{0,x})$ consists in finding a self-financing portfolio strategy replicating $\varphi(R_T^{0,x})$, i.e. a couple of real-valued processes $\{(h_s^P, h_s^R)\}_{s\in[0,T]}$ such that the portfolio $V_s := h_s^P P_s + h_s^R R_s^{0,x}$, composed of h_s^P shares of P and h_s^R shares of $R^{0,x}$, satisfies

$$\begin{cases} dV_s = h_s^P dP_s + h_s^R dR_s^{0,x} & s \in [0,T) \\ V_T = \varphi(R_T^{0,x}). \end{cases} \tag{1.2}$$

The hedging problem can be solved as follows (see e.g. [1, Ch. 8] for the financial argument and [8, Ch. 7] for the mathematical details). We begin by introducing the function

$$u(t,x) := e^{-r(T-t)}\mathbb{E}\left[\varphi(R_T^{t,x})\right] \qquad \forall (t,x) \in [0,T] \times \mathbb{R}. \tag{1.3}$$

Notice that, by Markov property of R, we have

$$u(t,x) = e^{-rh}\mathbb{E}\left[u(t+h, R_{t+h}^{t,x})\right] \qquad 0 \le t, h, t+h \le T. \tag{1.4}$$

If $u(t,x)$ is Fréchet differentiable up to order 2 with respect to x, with derivatives which are bounded and continuous jointly in (t,x), then Itô's formula and (1.4) permit to show that u is actually $C^{1,2}$ and solves to the following Kolmogorov-type partial differential equation

$$\begin{cases} u_t + rxD_x u + \frac{1}{2}v^2(t,x)D_x^2 u - ru = 0 & (t,x) \in (0,T) \times \mathbb{R}, \\ u(T,x) = \varphi(x) & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

By using (1.5) and applying Itô's formula to $u(s, X_s^{0,x})$, we find the following representation formula

$$u(s, R_s^{0,x}) = u(0,x) + \int_0^s ru(w, R_w^{0,x})dw + \int_0^s D_x u(w, R_w^{0,x})v(w, R_w^{0,x})dW_w. \tag{1.6}$$

Finally, by recalling the definition of u and considering formula (1.6), we can see that the portfolio strategy

$$h_s^P = \frac{u(s, R_s^{0,x}) - D_x u(s, R_s^{0,x})R_s^{0,x}}{P_s} \qquad \text{and} \qquad h_s^R = D_x u(s, R_s^{0,x}) \qquad \forall s \in [0,T), \tag{1.7}$$

solves the hedging problem. Indeed, we have

$$V_s := h_s^P P_s + h_s^R R_s^{0,x} = u(s, X_s^{0,x}) \qquad \forall s \in [0,T],$$

hence in particular $V_T = u(T, X_T^{0,x}) = \varphi(R_T^{0,x})$. Moreover, by (1.6), we have the self-financing condition

$$dV_s = h_s^P dP_s + h_s^R dR_s^{0,x} \qquad \forall s \in [0,T).$$

There are three essential features of the model that allow to implement the program above:

(1) The Markov property of R, which makes (1.4) possible.

(2) The existence of $D_x u$, which lets the portfolio strategy be defined by (1.7)

(3) The availability of Itô's formula and the fact that u solves to (1.5), in order to derive (1.6), hence to see that (1.7) is the hedging strategy.

Let us now consider a slightly more general risky asset R, in which the volatility depends not only on the value R_s of R at time s, but also on the entire past values of R. That is, the dynamics of R has the following form

$$\begin{cases} dR_s = rR_s ds + \nu(s, R_s, \{R_{s+s'}\}_{s'\in(-\infty,0)})dW_s & s \in (t,T] \\ R_t = x_0 \\ R_{t'} = x_1(t') & t' \in (-\infty, t), \end{cases} \tag{1.8}$$

where $x_0 \in \mathbb{R}$ and $x_1 : (-\infty, 0) \to \mathbb{R}$ is a given deterministic funtion belonging to $L^2(\mathbb{R}^-, \mathbb{R})$, expressing the past history of the stock price R up to time t. We also would like to face the case in which the European claim depends itself on the history of R, i.e. it has the form $\varphi(R_T, \{R_t\}_{t\in(-\infty,T)})$.

We point out that model (1.8) can also include the case in which the path-dependency is only relative to a finite past window $[-d, 0]$, i.e. ν is defined as a function of the past history of R only from the past date $t-d$ up to the present t. To fit this case into (1.8), it is sufficient to replace the coefficient ν in (1.8) by a ν' defined by

$$\nu'(s, R_s, \{R_{s+s'}\}_{s'\in(-\infty,0)}) := \nu(s, R_s, \mathbf{1}_{[-d,0)}(\cdot)\{R_{s+s'}\}_{s'\in(-\infty,0)}).$$

In such a case, it is easily seen that R does not depend on the tail $\mathbf{1}_{(-\infty,-d)}(\cdot)x_1$ of the initial datum. Hence a delay model with a finite delay window can be rewritten in the form (1.8).

A natural question is if we can solve the hedging problem for the delay case by implementing the standard arguments outlined above for the case in which R is given by (1.1). We now see that this can be done, if we take into account the three features mentioned above which make the machinery work.

If $R^{t,(x_0,x_1)}$ solves (1.8), then in general it is not Markovian. Moreover, since both the claim φ and the function u, now defined by

$$u(t, x_0, x_1) := e^{-r(T-t)}\mathbb{E}\left[\varphi\left(R_T^{t,x_0,x_1}, \{R_{t'}^{t,x_0,x_1}\}_{t'\in(-\infty,T)}\right)\right] \quad \forall (t, x_0, x_1) \in [0,T] \times \mathbb{R} \times L^2(\mathbb{R}^-, \mathbb{R}),$$

are path-dependent, the analogous PDE (1.5) would now be path-dependent, and it would be necessary to employ a stochastic calculus for path-dependent functionals of Itô processes in order to relate u with the PDE, as done for the non-path-dependent case.

A classical workaround tool to regain Markovianity and avoid the complications of a path-dependent stochastic calculus consists in rephrasing the model in a functional space setting. What we lose by doing so is that the dynamics will evolve in an infinite dimensional space. We briefly recall how the rephrasing works. We refer the reader to [2] for the case with finite delay. The argument extends without difficulty to the case with infinite delay.

We first introduce the Hilbert space $H := \mathbb{R} \times L^2(\mathbb{R}^-, \mathbb{R})$, the functions

$$\begin{aligned} F &: [0,T] \times H \to H,\ (x_0, x_1) \mapsto (r x_0, 0) \\ \Sigma &: [0,T] \times H \to H,\ (x_0, x_1) \mapsto (\nu(t, x_0, x_1), 0), \end{aligned} \tag{1.9}$$

and the strongly continuous semigroup of translations on H, i.e. the family $\hat{S} := \{\hat{S}_t\}_{t\in\mathbb{R}^+}$ of linear continuous operators defined by

$$\hat{S}_t : H \to H,\ (x_0, x_1) \mapsto (x_0, x_1(t+\cdot)\mathbf{1}_{(-\infty,-t)}(\cdot) + x_0 \mathbf{1}_{[-t,0]}(\cdot)).$$

The infinitesimal generator $\hat{A}$ of $\hat{S}$ is given by

$$\hat{A} : D(\hat{A}) \to H,\ (x_0, x_1) \mapsto (0, x_1'),$$

where

$$D(\hat{A}) = \left\{(x_0, x_1) \in H : x_1 \in W^{1,2}(\mathbb{R}^-) \text{ and } x_0 = x_1(0)\right\}.$$

Then we consider the H-valued dynamics

$$\begin{cases} d\hat{X}_s = \left(\hat{A}\hat{X}_s + F\left(s, \hat{X}_s\right)\right) ds + \Sigma\left(s, \hat{X}_s\right) dW_s & s \in (t,T], \\ \hat{X}_t = (x_0, x_1), \end{cases} \tag{1.10}$$

where $(x_0, x_1) \in H$, $t \in [0,T)$. Under usual Lipschitz assumptions on ν, it can be shown that (1.10) has a unique mild solution $\hat{X}^{t,(x_0,x_1)}$ (we refer to [7] for stochastic differential equations in Hilbert spaces). The link between (1.8) and (1.10) is given by the following equation:

$$\text{for all } s \in [t,T],\ \hat{X}_s^{t,(x_0,x_1)} = \left(R_s^{t,(x_0,x_1)}, \{R_{s'+s}^{t,(x_0,x_1)}\}_{s'\in(-\infty,0)}\right) \ \mathbb{P}\text{-a.s.}, \tag{1.11}$$

where $R^{t,(x_0,x_1)}$ denotes the unique strong solution of (1.8). Observe that $\hat{X}$ is Markovian and no path-dependency appears in the coefficients F, Σ. This is the natural rephrasing of the dynamics of R to get a Markovian setting for which the basic tools of stochastic calculus in Hilbert spaces (such as Itô's formula) are available.

We need an additional step to let the model studied in the paper apply to the financial problem we are considering. We rephrase (1.10) as an SDE in the same Hilbert space H, but with a maximal dissipative unbounded operator. To this goal, we observe that $A := \hat{A} - \frac{1}{2}$ is a maximal dissipative operator generating the semigroup of contractions $S := \{S_t := e^{-t/2}\hat{S}_t\}_{t\in\mathbb{R}^+}$. Let us define $G(t,x) := F(t,x) + \frac{x}{2}$, $(t,x) \in [0,T] \times H$. Denote by $X^{t,(x_0,x_1)}$ the unique mild solution of the SDE

$$\begin{cases} dX_s = (AX_s + G(s, X_s))ds + \Sigma(s, X_s) dW_s & s \in (t,T], \\ X_t = (x_0, x_1). \end{cases} \tag{1.12}$$

It is not difficult to see that $\hat{X}^{t,(x_0,x_1)} = X^{t,(x_0,x_1)}$. Indeed, if $\{\hat{A}_\lambda\}_{\lambda>1/2}$ denote the Yosida approximations of $\hat{A}$, then the strong solution of

$$\begin{cases} dX_{\lambda,s} = \left(\hat{A}_\lambda X_{\lambda,s} + F\left(s, X_{\lambda,s}\right)\right)ds + \Sigma\left(s, X_{\lambda,s}\right)dW_s & s \in (t,T], \\ X_{\lambda,t} = (x_0, x_1), \end{cases} \tag{1.13}$$

coincides with the strong solution $X_\lambda^{t,(x_0,x_1)}$ of

$$\begin{cases} X_{\lambda,s} = \left(\left(\hat{A}_\lambda - \frac{1}{2}\right) X_{\lambda,s} + G\left(s, X_{\lambda,s}\right)\right) ds + \Sigma\left(s, X_{\lambda,s}\right) dW_s \quad s \in (t,T], \\ X_{\lambda,t} = (x_0, x_1), \end{cases} \tag{1.14}$$

by the very definition and by uniqueness of strong solutions. Recalling that strong and mild solutions coincide when the linear operator appearing in the drift is bounded[1], $X_\lambda^{t,(x_0,x_1)}$ solves (1.14) in the mild sense. Now observe that $\hat{A}_\lambda - \frac{1}{2}$ generates the semigroup $\hat{S}_\lambda := \{\hat{S}_{\lambda,t} := e^{-t/2} e^{\hat{A}_\lambda t}\}_{t \in \mathbb{R}^+}$. Since $e^{\hat{A}_\lambda t} \to \hat{S}_t$ strongly as $\lambda \to +\infty$, we have also $\hat{S}_{\lambda,t} \to S_t$ strongly. Then the mild solution $X_\lambda^{t,(x_0,x_1)}$ converges to the mild solution $X^{t,(x_0,x_1)}$ as $\lambda \to +\infty$ (see e.g. the argument used to show Proposition 2.10-*(ii)*). Similarly, $X_\lambda^{t,(x_0,x_1)}$ solves (1.13) in the mild sense and then $X_\lambda^{t,(x_0,x_1)} \to \hat{X}^{t,(x_0,x_1)}$ as $\lambda \to +\infty$. We thus conclude that $\hat{X}^{t,(x_0,x_1)} = X^{t,(x_0,x_1)}$ in a suitable space of processes where the well-posedness of the SDEs and the convergences above are considered.

It follows that equation (1.11) can be rewritten as:

$$\text{for all } s \in [t,T], \ X_s^{t,(x_0,x_1)} = \left(R_s^{t,(x_0,x_1)}, \{R_{s'+s}^{t,(x_0,x_1)}\}_{s' \in (-\infty,0)}\right) \ \mathbb{P}\text{-a.s..} \tag{1.15}$$

Having (1.15), the function u can be written as

$$u(t,x_0,x_1) = e^{-r(T-t)} \mathbb{E}\left[\varphi\left(X_T^{t,(x_0,x_1)}\right)\right] \qquad \forall (t,(x_0,x_1)) \in [0,T] \times H. \tag{1.16}$$

Thanks to the special structure of Σ in SDE (1.12), if u has enough regularity to perform the computations, it turns out that, for $s \in [0,T]$,

$$\begin{aligned} u\left(s, X_s^{0,(x_0,x_1)}\right) = u\left(0,(x_0,x_1)\right) + \int_0^s ru\left(w, X_w^{0,(x_0,x_1)}\right) dw \\ + \int_0^s D_{x_0} u\left(w, X_w^{0,(x_0,x_1)}\right) \nu\left(w, X_w^{0,(x_0,x_1)}\right) dW_w, \end{aligned} \tag{1.17}$$

and the only derivative of u appearing in the above formula is the directional derivative $D_{x_0} u$ with respect to the variable x_0, representing the "present", according to the rephrasing $R \rightsquigarrow X$. Once (1.17) is available, one can verify, as it is done for the case without delay, that

$$h_s^P = \frac{u\left(s, X_s^{0,(x_0,x_1)}\right) - D_{x_0} u\left(s, X_s^{0,(x_0,x_1)}\right) X_{0,s}^{0,(x_0,x_1)}}{P_s} \quad \text{and} \quad h_s^R = D_{x_0} u\left(s, X_s^{0,(x_0,x_1)}\right) \ \forall s \in [0,T)$$

solve the hedging problem in the delay case.

The goal of this paper is to show the regularity of the function u, defined by (1.16), with respect to the component x_0, when all the data are assumed to be Lipschitz with respect to a particular norm associated to the operator A.

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[1] This can be seen by an easy application of Ito's formula, together with uniqueness of mild solutions.

2 Preliminaries

2.1 Notation

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, let $T > 0$, and let $\mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,T]}$ be a filtration on $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying the usual conditions. Define $\Omega_T = \Omega \times [0,T]$. Denote by $\mathscr{P}$ the σ-algebra in Ω_T generated by the sets $A_s \times (s,t]$, where $A_s \in \mathscr{F}_s$, $0 \le s < t \le T$, and $A_0 \times \{0\}$, where $A_0 \in \mathscr{F}_0$. An element of $\mathscr{P}$ is called a predictable set. We denote $\mathbb{R}^- = (-\infty, 0]$, $\mathbb{R}^+ = [0, +\infty)$.

Let $(F, |\cdot|_F)$[2] be a real separable Banach space. We define the following spaces:

(i) For $p \ge 1$, $L^p_{\mathscr{P}}(F) := L^p_{\mathscr{P}}(\Omega_T, F)$ is the Banach space of F-valued predictable processes X such that
$$|X|_{L^p_{\mathscr{P}}(F)} := \left(\mathbb{E}\left[\int_0^T |X_t|_F^p dt\right]\right)^{1/p} < +\infty.$$

(ii) $\mathscr{H}^p_{\mathscr{P}}(F)$ is the subspace of elements X of $L^p_{\mathscr{P}}(F)$ such that
$$|X|_{\mathscr{H}^p_{\mathscr{P}}(F)} := \sup_{t\in[0,T]} \left(\mathbb{E}\left[|X_t|_F^p\right]\right)^{1/p} < +\infty,$$
and, for all $t' \in [0,T]$,
$$\lim_{t\to t'} \mathbb{E}\left[|X_t - X_{t'}|_F^p\right] = 0.$$
$\mathscr{H}^p_{\mathscr{P}}(F)$, when endowed with the norm $|\cdot|_{\mathscr{H}^p_{\mathscr{P}}(F)}$, is a Banach space.

We will consider $\mathscr{H}^p_{\mathscr{P}}(F)$ also with other norms. For $\gamma > 0$, define
$$|X|_{\mathscr{H}^p_{\mathscr{P}}(F),\gamma} := \sup_{t\in[0,T]} \left(e^{-\gamma t}\left(\mathbb{E}\left[|X_t|_F^p\right]\right)^{1/p}\right).$$

The norms $|\cdot|_{\mathscr{H}^p_{\mathscr{P}}(F)}$ and $|\cdot|_{\mathscr{H}^p_{\mathscr{P}}(F),\gamma}$ are equivalent.

Let $n \ge 0$, $k \ge 0$, $T > 0$, and let E, F be real separable Banach spaces.

(iii) $\mathscr{G}^1_s(E,F)$ denotes the space of continuous functions $f: E \to F$ such that the Gâteaux derivative $\nabla f(x)$ exists for every $x \in E$, the function
$$\nabla f : E \to L(E,F)$$
is strongly continuous and
$$\sup_{x\in E} |\nabla f(x)|_{L(E,F)} < +\infty.$$
When E is a Hilbert space and $F = \mathbb{R}$, we will identify ∇f with an element of E through the Riesz representation $E^* = E$.

[2] We use the same symbol $|\cdot|$ to denote the norm of a normed space when the space is clear from the context. If not, we will clarify the space of reference with a subscript.

(iv) $\mathscr{G}_s^{0,1}([0,T]\times E,F)$ denotes the space of continuous functions $f\colon[0,T]\times E\to F$, such that the Gâteaux derivative in the x variable $\nabla_x f(t,x)$ exists for every $x\in E$, the function

$$\nabla_x f\colon[0,T]\times E\to L(E,F)$$

is strongly continuous and

$$\sup_{(t,x)\in[0,T]\times E}|\nabla_x f(t,x)|_{L(E,F)}<+\infty.$$

(v) $C_b^1(E,F)$ denotes the space of continuous functions $f\colon E\to F$, continuously Fréchet differentiable, and such that

$$\sup_{x\in E}|Df(x)|_{L(E,F)}<+\infty,$$

where Df denotes the Fréchet derivative of f.

(vi) $C^{0,1}([0,T]\times E,F)$ denotes the space of continuous functions $f\colon[0,T]\times E\to F$, continuously Fréchet differentiable with respect to the second variable.

(vii) $C_b^{0,1}([0,T]\times E,F)$ denotes the space of functions $f\in C^{0,1}([0,T]\times E,F)$ such that

$$\sup_{(t,x)\in[0,T]\times E}|D_x f(t,x)|_{L(E,F)}<+\infty,$$

where $D_x f$ denotes the Fréchet derivative of f with respect to x.

When $F=\mathbb{R}$, we drop $\mathbb{R}$ and simply write $L^p_{\mathscr{P}}$, $\mathscr{H}^p_{\mathscr{P}}$, $\mathscr{G}_s^1(E)$, $\mathscr{G}_s^{0,1}(E)$, $C_b^{0,1}([0,T]\times E)$, and $C_b^{0,1}([0,T]\times E)$.

Though the notation could appear to be misleading, observe that if $f\in C_b^{0,1}([0,T]\times E,F)$ or $f\in C_b^1(E,F)$, then f is not supposed to be bounded.

Let $m>0$ be a positive integer, and let U be an open subset of $\mathbb{R}^m$. Let a,b be real numbers such that $a<b$. Define $Q:=[a,b)\times U$ and $\partial_P Q:=[a,b]\times\partial U\cup\{b\}\times U$.

(viii) For $\alpha\in(0,1)$, $C^{1+\alpha}(Q)$ denotes the space of continuous functions $f\colon Q\to\mathbb{R}$ such that $D_x f(t,x)$ exists classically for every $(t,x)\in Q$, and such that

$$|f|_{C^{1+\alpha}(Q)}:=|f|_\infty+|D_x f|_\infty+\sup_{\substack{(t,x),(s,y)\in Q\\(t,x)\neq(s,y)}}\frac{|u(s,y)-u(t,x)-\langle D_x f(t,x),y-x\rangle_m|}{\left(|t-s|+|x-y|_m^2\right)^{(1+\alpha)/2}}<+\infty,$$

where $|\cdot|_\infty$ is the supremum norm, and $|\cdot|_m$ and $\langle\cdot,\cdot\rangle_m$ are the Euclidean norm and scalar product in $\mathbb{R}^m$ respectively.

(ix) For $\alpha\in(0,1)$, $C^{1+\alpha}_{\mathrm{loc}}((0,T)\times\mathbb{R}^m)$ denotes the space of continuous functions $f\colon(0,T)\times\mathbb{R}^m\to\mathbb{R}$ such that, for every point $(t,x)\in(0,T)\times\mathbb{R}^m$, there exists $\varepsilon>0$ and $a,b\in(0,T)$, with $a<b$, such that $f\in C^{1+\alpha}([a,b)\times B(x,\varepsilon))$[3].

(x) For $p\geq 1$, $W^{1,2,p}(Q)$ denotes the usual Sobolev space of functions $f\in L^p(Q)$, whose weak partial derivatives u_t, f_{x_i} and $f_{x_ix_j}$ belong to $L^p(Q)$. $W^{1,2,p}(Q)$ is equipped with the norm

$$|f|_{W^{1,2,p}(Q)}:=\left(|f|^p_{L^p(Q)}+|f_t|^p_{L^p(Q)}+|D_x f|^p_{L^p(Q)}+|D_x^2 f|^p_{L^p(Q)}\right)^{1/p}.$$

[3] $B(x,\varepsilon)$ denotes the open ball centered at x of radius ε.

2.2 H_B-extensions of mild solutions of SDEs

Let $m \geq 1$, and let H_1 be a real separable Hilbert space with scalar product $\langle \cdot,\cdot \rangle_{H_1}$. Define $H := \mathbb{R}^m \times H_1$. Whenever x is a point of H, we will denote by x_0 the component of x in $\mathbb{R}^m$ and by x_1 the component of x in H_1. We endow H with the natural scalar product

$$\langle\langle (x_0,x_1),(y_0,y_1)\rangle\rangle := \langle x_0,y_0\rangle_m + \langle x_1,y_1\rangle_{H_1} \qquad \forall (x_0,x_1),(y_0,y_1) \in H.$$

Let $G\colon [0,T]\times H \to H$ and $\sigma\colon [0,T]\times H \to L(\mathbb{R}^m)$. We will consider the following assumptions on them.

Assumption 2.1. *The functions G and σ are continuous, and there exists $M>0$ such that*

$$|G(t,x)-G(t,y)|_H + |\sigma(t,x)-\sigma(t,y)|_{L(\mathbb{R}^m)} \leq M|x-y|_H \qquad \forall (t,x),(t,y) \in [0,T]\times H.$$

We associate to σ the following function:

$$\Sigma\colon [0,T]\times H \to L(\mathbb{R}^m, H),$$

defined by

$$\Sigma(t,x)y = (\sigma(t,x)y, 0_1) \tag{2.1}$$

for $(t,x) \in [0,T]\times H$, $y \in \mathbb{R}^m$, and where 0_1 denotes the origin in H_1.

The following assumption will be standing for the remaining part of the work.

Assumption 2.2. *S is a strongly continuous semigroup of contractions, with A as its infinitesimal generator.*

We remark that Assumption 2.2 implies that A is a linear densely defined maximal dissipative operator on H. In the rest of the paper A is an abstract operator which may be different from the operator A introduced in Section 1.1.

Let W be a standard m-dimensional Brownian motion with respect to the filtration $\mathbb{F}$. For $t \in [0,T)$ and $x \in H$, consider the SDE

$$\begin{cases} dX_s = (AX_s + G(s,X_s))ds + \Sigma(s,X_s)dW_s & s \in (t,T] \\ X_t = x. \end{cases} \tag{2.2}$$

It is well known (see [7, Ch. 7]) that, under Assumption 2.1, for $p \geq 2$, there exists a unique mild solution in $\mathcal{H}^p_{\mathscr{P}}(H)$ to (2.2), i.e. a unique process $X^{t,x} \in \mathcal{H}^p_{\mathscr{P}}(H)$ such that

$$X^{t,x}_s = \begin{cases} x & s \in [0,t] \\ S_{s-t}x + \int_t^s S_{s-w}G(w,X^{t,x}_w)dw + \int_t^s S_{s-w}\Sigma(w,X^{t,x}_w)dW_w & s \in (t,T]. \end{cases}$$

Moreover, for every $t \in [0,T]$, the map

$$H \to \mathcal{H}^p_{\mathscr{P}}(H), \quad x \mapsto X^{t,x} \tag{2.3}$$

is continuous and Lipschitz.

For future reference, we state existence and uniqueness of mild solution in the following proposition, where we also show continuity in t, and we introduce tools useful for later proofs.

Proposition 2.3. *For any $p \geq 2$, under Assumption 2.1, there exists a unique mild solution $X^{t,x} \in \mathcal{H}^p_{\mathcal{P}}(H)$ to SDE (2.2), and the map*

$$[0,T] \times (H, |\cdot|) \to \mathcal{H}^p_{\mathcal{P}}(H), (t,x) \mapsto X^{t,x} \tag{2.4}$$

is continuous in (t,x), and Lipschitz in x, uniformly in t.

Proof. Since the arguments are standard, we just give a sketch of proof. Let $t \in [0,T]$. Define the map

$$\Phi(t;\cdot,\cdot) \colon H \times \mathcal{H}^p_{\mathcal{P}}(H) \to \mathcal{H}^p_{\mathcal{P}}(H)$$

by

$$\Phi(t;x,Z)_s := \begin{cases} x & s \in [0,t) \\ S_{s-t}x + \int_t^s S_{s-w}G(w,Z_w)dw + \int_t^s S_{s-w}\Sigma(w,Z_w)dW_w & s \in [t,T]. \end{cases}$$

Let $\gamma > 0$. By Assumption 2.1, we have

$$\sup_{t\in[0,T]} e^{-\gamma p t}\mathbb{E}\left[|G(t,Z_t) - G(t,Z'_t)|^p\right] \leq M^p |Z - Z'|^p_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \quad \forall Z, Z' \in \mathcal{H}^p_{\mathcal{P}}(H) \tag{2.5}$$

$$\sup_{t\in[0,T]} e^{-\gamma p t}\mathbb{E}\left[|\sigma(t,Z_t) - \sigma(t,Z'_t)|^p_{L(\mathbb{R}^m)}\right] \leq M^p |Z - Z'|^p_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \quad \forall Z, Z' \in \mathcal{H}^p_{\mathcal{P}}(H). \tag{2.6}$$

By (2.5), (2.6), the linearity of $\Phi(t;x,Z)$ in x, and [8, Ch. 7, Proposition 7.3.1], there exists $\gamma > 0$, depending only on p, T, M, such that

$$\sup_{(t,x)\in[0,T]\times H} |\Phi(t;x,Z) - \Phi(t;x,Z')|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \leq \frac{1}{2}|Z - Z'|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \quad \forall Z, Z' \in \mathcal{H}^p_{\mathcal{P}}(H). \tag{2.7}$$

This shows that, for every $(t,x) \in [0,T]\times H$, there exists a unique fixed point $X^{t,x} \in \mathcal{H}^p_{\mathcal{P}}(H)$ of $\Phi(t;x,\cdot)$. Such a fixed point is the mild solution of (2.2).

The continuity of (2.4) is also standard. We sketch a slightly different argument. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence converging to t in $[0,T]$. By standard estimates on the integrals defining Φ (for the stochastic integral using Burkholder-Davis-Gundy's inequality), by sublinear growth of $G(t,x)$ and $\sigma(t,x)$ in x uniformly in t, and by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to+\infty} \Phi(t_n;x,Z) = \Phi(t;x,Z) \text{ in } \mathcal{H}^p_{\mathcal{P}}(H), \ \forall (x,Z) \in H \times \mathcal{H}^p_{\mathcal{P}}(H). \tag{2.8}$$

Then, by (2.8), (2.7), and [8, Theorem 7.1.5], we have

$$\lim_{n\to+\infty} X^{t_n,x} = X^{t,x} \quad \text{in } \mathcal{H}^p_{\mathcal{P}}(H), \ \forall x \in H. \tag{2.9}$$

This shows the continuity in t of $X^{t,x}$. We notice that

$$\sup_{(t,Z)\in[0,T]\times\mathcal{H}^p_{\mathcal{P}}(H)} |\Phi(t;x,Z) - \Phi(t;x',Z)|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \leq |x - x'|_H \quad \forall x, x' \in H. \tag{2.10}$$

By applying [13, inequality $(***)$ on p. 13], we obtain

$$\sup_{t\in[0,T]} |X^{t,x} - X^{t,x'}|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \leq 2|x - x'|_H \quad \forall x, x' \in H. \tag{2.11}$$

By (2.9) and by (2.11) we conclude that the map

$$[0,T]\times H \to \mathcal{H}^p_{\mathcal{P}}(H),\ (t,x)\mapsto X^{t,x}$$

is continuous and Lipschitz continuous in x uniformly in t. ■

We are going to endow H with a weaker norm, and give conditions such that the above continuity in (t,x) of $X^{t,x}$ extends to the new norm. We will also make assumptions which will guarantee the Gâteaux differentiability of the mild solution with respect to the initial datum x in the space with the weaker norm and the strong continuity of the the Gâteaux derivative.

Let $R\colon D(R)\to H$ be a densely defined linear operator such that $R\colon D(R)\to H$ has inverse $R^{-1}\in L(H)$. Then $B=(R^*)^{-1}R^{-1}\in L(H)$ is selfadjoint and positive. For $x\in H$, define

$$|x|_B^2 = \langle Bx,x\rangle = \left|R^{-1}x\right|_H^2 \tag{2.12}$$

Such norms have been introduced in the context of the so-called B-continuous viscosity solutions of HJB equations in [5, 6] and used in many later works on HJB equations in infinite dimensional spaces (see [11, Ch. 3] for more on this). The space H endowed with the norm $|\cdot|_B$ is pre-Hilbert, since $|\cdot|_B$ is inherited by the scalar product $\langle x,y\rangle_B=\langle B^{1/2}x,B^{1/2}y\rangle$, where $B^{1/2}$ is the unique positive self-adjoint continuous linear operator such that $B=B^{1/2}B^{1/2}$. Denote by H_B the completion of the pre-Hilbert space $(H,|\cdot|_B)$. With some abuse of notation, we also denote by $|\cdot|_B$ the extension of $|\cdot|_B$ to H_B.

By definition of $|\cdot|_B$, $R\colon (D(R),|\cdot|_H)\to(H,|\cdot|_B)$ is a full-range isometry. This implies the following facts:

(1) there exists a unique extension $\widetilde{R}\colon H\to H_B$;

(2) $\widetilde{R}$ and $\widetilde{R}^{-1}$ are isometries;

(3) $\widetilde{R}^{-1}=\widetilde{R^{-1}}$, where $\widetilde{R^{-1}}\colon H_B\to H$ is the unique continuous extension of R^{-1}.

Denote by $\overline{R}$ the operator $\widetilde{R}$ considered as an operator $H_B\supset H=D(\overline{R})\to H_B$. The above facts imply that $\overline{R}$ is a densely-defined full-range closed linear operator in H_B, and that $D(R)$ is a core for $\overline{R}$.

We will need the following proposition.

Proposition 2.4. *Let $R\colon D(R)\subset H\to H$ be a densely defined linear operator such that $R^{-1}\in L(H)$. Let H_B be the Hilbert space defined above as the completion of H with respect to the norm $|\cdot|_B$ given by (2.12).*

(i) Suppose that

$$S_tR\subset RS_t \qquad \forall t\in\mathbb{R}^+. \tag{2.13}$$

Then, for every $t\in\mathbb{R}^+$, there exists a unique continuous extension $\overline{S}_t$ of S_t to H_B, the family $\overline{S}:=\{\overline{S}_t\}_{t\in\mathbb{R}^+}$ is a strongly continuous semigroup of contractions on H_B, and

$$\overline{S}_t\overline{R}\subset\overline{R}\overline{S}_t \qquad \forall t\in\mathbb{R}^+, \tag{2.14}$$

$$\overline{A}=\overline{R}A\overline{R}^{-1}, \tag{2.15}$$

where $\overline{A}$ is the infinitesimal generator of $\overline{S}$.

(ii) Suppose that

$$AR = RA, \tag{2.16}$$
$$D(A) \subset D(R). \tag{2.17}$$

Then (2.13) *is satisfied and the Yosida approximations* $\{\overline{A}_n\}_{n\geq 1}$ *of the infinitesimal generator* $\overline{A}$ *of* $\overline{S}$ *are given by the unique continuous extensions to* H_B *of the Yosida approximations* $\{A_n\}_{n\geq 1}$ *of* A*, i.e.*

$$\overline{A}_n = \overline{A_n} \qquad \forall n \geq 1.$$

Proof. *(i)* Suppose that (2.13) holds true. Observe that (2.13) implies

$$AR \subset RA. \tag{2.18}$$

Since $R^{-1}S_t = S_t R^{-1}$, we have

$$|S_t x|_B = |R^{-1}S_t x|_H = |S_t R^{-1}x|_H \leq |R^{-1}x|_H = |x|_B \qquad \forall t \in \mathbb{R}^+, x \in H.$$

We can then extend each S_t to an operator $\overline{S}_t \in L(H_B)$ with the operator norm less than or equal to 1. By density of H in H_B, it is clear that the family $\{\overline{S}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of contractions. Moreover, for $x \in H$,

$$\lim_{t\to 0^+} |\overline{S}_t x - x|_B = \lim_{t\to 0^+} |R^{-1}(S_t x - x)|_H = \lim_{t\to 0^+} |S_t R^{-1}x - R^{-1}x|_H = 0.$$

The above observations imply that the family $\{\overline{S}_t\}_{t\in\mathbb{R}^+}$ is uniformly bounded and strongly continuous on a dense subspace of H_B. Thus, by [10, Proposition 5.3], $\overline{S}$ is a strongly continuous semigroup on H_B.

We now prove (2.14). Let $(x, \overline{R}x) \in \Gamma(\overline{R})$, where $\Gamma(\overline{R})$ is the graph of $\overline{R}$. We noticed that $D(R)$ is a core for $\overline{R}$. Then we can choose a sequence $\{(x_n, Rx_n)\}_{n\in\mathbb{N}} \in \Gamma(R)$ such that $(x_n, Rx_n) \to (x, \overline{R}x)$ in $H_B \times H_B$. Hence, using (2.13), we can write

$$\overline{S}_t \overline{R}x = \lim_{n\to+\infty} \overline{S}_t R x_n = \lim_{n\to+\infty} S_t R x_n = \lim_{n\to+\infty} R S_t x_n = \lim_{n\to+\infty} \overline{R}\,\overline{S}_t x_n,$$

where all the limits are considered in H_B. This means that $\{\overline{R}\,\overline{S}_t x_n\}_{n\in\mathbb{N}}$ is convergent in H_B. We recall that $\overline{R}$ is closed in H_B and we observe that $\overline{S}_t x_n \to \overline{S}_t x$ in H_B by continuity. Thus we conclude that $\overline{R}\,\overline{S}_t x_n \to \overline{R}\,\overline{S}_t x$ in H_B. This proves (2.14).

Now let $\overline{A}$ be the generator of the semigroup $\{\overline{S}_t : H_B \to H_B\}_{t\in\mathbb{R}^+}$. Obviously $\overline{A}$ is an extension of A, i.e. $\overline{A}x = Ax$ for $x \in D(A)$. We will show that $D(\overline{A}) = \overline{R}(D(A))$. Using (2.14) we have for $x \in H_B$,

$$\lim_{t\to 0^+} \frac{\overline{S}_t - I}{t} x = \lim_{t\to 0^+} \frac{\overline{S}_t - I}{t} \overline{R}\,\overline{R}^{-1} x = \text{(by (2.14))} = \lim_{t\to 0^+} \overline{R} \frac{\overline{S}_t - I}{t} \overline{R}^{-1} x = \lim_{t\to 0^+} \overline{R} \frac{S_t - I}{t} \overline{R}^{-1} x.$$

The last limit exists in H_B if and only if the limit

$$\lim_{t\to 0^+} \frac{S_t - I}{t} \overline{R}^{-1} x$$

exists in H. Therefore we conclude that

$$D(\overline{A}) = \overline{R}(D(A)) \qquad \text{and} \qquad \overline{A}x = \overline{R}A\overline{R}^{-1}x \quad \forall x \in D(\overline{A}), \tag{2.19}$$

which can be written as (2.15).

(ii) Let $\{A_n\}_{n\geq 1}$ be the Yosida approximations of A. We begin by showing that

$$(n-A)^{-1}R \subset R(n-A)^{-1} \quad \forall n \geq 1. \tag{2.20}$$

By (2.17), it follows that

$$D((n-A)^{-1}R) = D(R) \subset H = D(R(n-A)^{-1}).$$

By (2.17), we have, for $x \in D(R)$,

$$A(n-A)^{-1}x = n(n-A)^{-1}x - x \subset D(A) + D(R) \subset D(R), \tag{2.21}$$

hence $(n-A)^{-1}x \in D(RA)$. Then, by using (2.16), we can write, for $x \in D(R)$,

$$(n-A)^{-1}Rx = (n-A)^{-1}R(n-A)(n-A)^{-1}x = (n-A)^{-1}(n-A)R(n-A)^{-1}x = R(n-A)^{-1}x.$$

This shows (2.20).

We now claim that

$$e^{tA_n}R \subset Re^{tA_n}, \tag{2.22}$$

where e^{tA_n} is the semigroup generated by A_n. By (2.20), we have

$$A_nRx = n^2(n-A)^{-1}Rx - nRx = n^2R(n-A)^{-1}x - nRx = RA_nx \qquad \forall x \in D(R),$$

that is

$$A_nR \subset RA_n. \tag{2.23}$$

Let $x \in D(R)$. By (2.21) and (2.23),

$$A_n^kRx = RA_n^kx \qquad \forall k \in \mathbb{N}. \tag{2.24}$$

For $t \in \mathbb{R}^+$, define

$$y_m := \sum_{k=0}^{m} \frac{t^k}{k!}A_n^kx.$$

By (2.21), $y_m \in D(R)$. Moreover, $\lim_{m\to+\infty} y_m = e^{tA_n}x$ and, by (2.24),

$$\lim_{m\to+\infty} Ry_m = \lim_{m\to+\infty} \sum_{k=0}^{m} \frac{t^k}{k!}A_n^kRx = e^{tA_n}Rx.$$

Since R is closed, it follows that $e^{tA_n}x \in D(R)$, and $Re^{tA_n}x = e^{tA_n}Rx$. Since this holds for every $x \in D(R)$, we conclude $e^{tA_n}R \subset Re^{tA_n}$.

We can now prove that (2.13) is satisfied. Let $x \in D(R)$. By (2.22),

$$\lim_{n\to\infty} Re^{tA_n}x = \lim_{n\to\infty} e^{tA_n}Rx = S_tRx.$$

Since R is closed, we have $\lim_{n\to\infty} e^{tA_n}x = S_t x \in D(R)$ and $RS_t x = S_t Rx$. Then (2.13) is verified.

We can now conclude the proof. By (2.22), arguing as it was done for S, we obtain that every S_n can be uniquely extended to the semigroup $e^{t\overline{A_n}}$ on H_B generated by $\overline{A_n}$. Similarly to (2.19), we have

$$D(\overline{A_n}) = \overline{R}(D(A_n)) \qquad \text{and} \qquad \overline{A_n}x = \overline{R}A_n\overline{R}^{-1}x \quad \forall x \in D(\overline{A_n}). \tag{2.25}$$

We observe that $\overline{R}(D(A_n)) = \overline{R}(H) = H_B$. If $x \in H$, by (2.17), (2.19), (2.20), and (2.25), we have

$$\begin{aligned}\overline{A_n}x = \overline{R}A_n\overline{R}^{-1}x = \overline{R}nA(n-A)^{-1}\overline{R}^{-1}x = RnA(n-A)^{-1}R^{-1}x \\ = n(RAR^{-1})(R(n-A)^{-1}R^{-1})x = n(RAR^{-1})(n-A)^{-1}x = nA(n-A)^{-1}x,\end{aligned}$$

which can be written as

$$\overline{A_n}x = n\overline{A}(n-\overline{A})^{-1}x = \overline{A}_n x \qquad \forall x \in H,$$

where $\overline{A}_n$ is the Yosida approximation of $\overline{A}$. Finally, since both $\overline{A_n}$ and $\overline{A}_n$ are continuous on H_B, and since H is dense in H_B, we obtain

$$\overline{A_n} = \overline{A}_n,$$

and then $e^{t\overline{A}_n} = e^{t\overline{A_n}}$, where $e^{t\overline{A}_n}$ is the semigroup generated by $\overline{A}_n$. ■

In the remaining of this section we will assume that (2.16) and (2.17) hold true.

Assumption 2.5. *The functions G and Σ are Lipschitz with respect to the norm $|\cdot|_B$, with respect to the second variable and uniformly in the first one, that is there exists $M > 0$ such that*

$$|G(t,x) - G(t,y)|_B + |\Sigma(t,x) - \Sigma(t,y)|_{L(\mathbb{R}^m, H_B)} \le M|x-y|_B \tag{2.26}$$

for all $t \in [0,T]$, $x,y \in H$. Denote by $\overline{G}$ (resp. $\overline{\Sigma}$) the unique extension of G (resp. Σ) to a function from $[0,T]\times H_B$ into H_B (resp. from $[0,T]\times H_B$ into $L(\mathbb{R}^m, H_B)$).

Remark 2.6. *It is obvious that Assumptions 2.1 and 2.5 are satisfied if*

$$|G(t,x) - G(t,y)|_H + |\sigma(t,x) - \sigma(t,y)|_{L(\mathbb{R}^m)} \le M|x-y|_B. \tag{2.27}$$

It is then easy to see that the functions $G_0(t,x) = G(t,Rx)$ and $\sigma_0(t,x) = \sigma(t,Rx)$ defined on $[0,T]\times D(R)$ satisfy

$$|G_0(t,x) - G_0(t,y)|_H + |\sigma_0(t,x) - \sigma_0(t,y)|_{L(\mathbb{R}^m)} \le M|x-y|_H \tag{2.28}$$

for $t \in [0,T]$ and $x,y \in D(R)$, and hence they uniquely extend to functions defined on $[0,T]\times H$ satisfying (2.28) for all $t \in [0,T]$ and $x,y \in H$. The converse is also true, i.e. (2.28) implies (2.27). Thus (2.27) is satisfied if and only if $G(t,x) = G_0(t,R^{-1}x)$, $\sigma(t,x) = \sigma_0(t,R^{-1}x)$, for $(t,x) \in [0,T]\times H$, for some G_0, σ_0 which satisfy (2.28) for all $t \in [0,T]$ and $x,y \in H$. We notice that for σ, (2.27) is also necessary for Assumptions 2.1 and 2.5.

For instance, focusing on σ (which corresponds to ν in the financial problem considered in Section 1.1), this condition is easily seen to be satisfied if

$$\sigma(t,x) = f(t, \langle x, \bar{y}^1\rangle, \dots, \langle x, \bar{y}^n\rangle)$$

for some $f: [0,T]\times\mathbb{R}^n \to L(\mathbb{R}^m)$ Lipschitz continuous in the last n variables (uniformly for $t\in[0,T]$) and $\bar{y}^1,\dots,\bar{y}^n \in D(R^*)$. Indeed, in such a case we can write

$$\sigma(t,x) = f(t, \langle x, \bar{y}^1\rangle, \dots, \langle x, \bar{y}^n\rangle) = f(t, \langle R^{-1}x, R^*\bar{y}^1\rangle, \dots, \langle R^{-1}x, R^*\bar{y}^n\rangle) = \sigma_0(t, R^{-1}x), \quad (2.29)$$

where $\sigma_0(t,x) = f(t, \langle x, R^*\bar{y}^1\rangle, \dots, \langle x, R^*\bar{y}^n\rangle)$. Since later in (2.59) we take $R = A - I$, in applications to our financial problem (Section 1.1) this would mean that

$$\bar{y}^i = (\bar{y}^i_0, \bar{y}^i_1) \in \mathbb{R}\times W^{1,2}(\mathbb{R}^-) \qquad i = 1,\dots,n.$$

Thus a function of the form

$$\sigma(t,x) = f\left(t, x^1_0\bar{y}^1_0, \int_{-\infty}^0 x^1_1(s)\bar{y}^1_1(s)ds, \dots, x^n_0\bar{y}^n_0, \int_{-\infty}^0 x^n_1(s)\bar{y}^n_1(s)ds\right), \quad (2.30)$$

where $f: [0,T]\times\mathbb{R}^{2n} \to L(\mathbb{R}^m)$ is continuous in the $2n+1$ variables and Lipschitz continuous in the last $2n$ variables, uniformly for $t\in[0,T]$, satisfies Assumptions 2.1 and 2.5.

One can also give an equivalent condition which may be easier to check. We can only require that $G(t,x) = G_0(t,Kx)$, $\sigma(t,x) = \sigma_0(t,Kx)$, for some G_0, σ_0 satisfying (2.28) for all $t\in[0,T]$ and $x,y\in H$, and a bounded operator K on H such that $|Kx|_H \le C|R^{-1}x|_H$ for all $x\in H$. The last requirement (see e.g. [7, p. 429, Proposition B.1]) is equivalent to $K^*(H)\subset(R^{-1})^*(H) = D(R^*)$. In particular, if K is the orthogonal projection onto a finite dimensional subspace H_0 of H, then we need $H_0\subset D(R^*)$. By assuming without loss of generality that $\bar{y}^1,\dots,\bar{y}^n$ in (2.29) are orthonormal, then the previous example is readily reduced to the present if K is the orthogonal projection onto $\mathrm{span}\{\bar{y}^1,\dots,\bar{y}^n\}$.

Though functions like σ in (2.30) are of a very special form (they are cylindrical as functions of $x^1_1,\dots,x^n_1$), it should be noticed that they are in general not smooth, since $f(t,\cdot)$ is only assumed to be Lipschitz continuous.

Under Assumption 2.5 we can consider the following SDE on H_B

$$\begin{cases} d\overline{X}_s = \left(\overline{AX}_s + \overline{G}(s,\overline{X}_s)\right)ds + \overline{\Sigma}(s,\overline{X}_s)dW_s, & s\in(t,T], \\ \overline{X}_t = x, \end{cases} \quad (2.31)$$

where $x\in H_B$. By changing the reference Hilbert space from H to H_B, we can apply Proposition 2.3 and say that SDE (2.31) has a unique mild solution $\overline{X}^{t,x}$ in $\mathcal{H}^p_{\mathcal{P}}(H_B)$, and $[0,T]\times H_B \to \mathcal{H}^p_{\mathcal{P}}(H_B)$, $(t,x)\mapsto\overline{X}^{t,x}$, is continuous and $|\cdot|_B$-Lipschitz with respect to x, uniformly in t.

Proposition 2.7. *For any $p\ge 2$, under Assumptions 2.1 and 2.5, there exists a unique mild solution $\overline{X}^{t,x}\in\mathcal{H}^p_{\mathcal{P}}(H_B)$ of SDE (2.31), and the map*

$$[0,T]\times H_B \to \mathcal{H}^p_{\mathcal{P}}(H_B), (t,x)\mapsto\overline{X}^{t,x} \quad (2.32)$$

is continuous in (t,x), and Lipschitz in x, uniformly in t. If $x\in H$, $\overline{X}^{t,x}\in\mathcal{H}^p_{\mathcal{P}}(H)$ and $\overline{X}^{t,x} = X^{t,x}$, where $X^{t,x}\in\mathcal{H}^p_{\mathcal{P}}(H)$ is the unique mild solution of SDE (2.2).

Proof. The first part follows from Proposition 2.3. It remains to comment on the fact that $X^{t,x} = \overline{X}^{t,x}$ if $x \in H$. The space $\mathcal{H}^p_{\mathcal{P}}(H)$ is continuously embedded in $\mathcal{H}^p_{\mathcal{P}}(H_B)$. Thus, if G and Σ satisfy Assumptions 2.1 and 2.5, and if the initial value x belongs to H, the mild solution $X^{t,x}$ of (2.2) is also a mild solution of (2.31), and then, by uniqueness of mild solutions, $X^{t,\overline{x}} = \overline{X}^{t,x}$ in $\mathcal{H}^p_{\mathcal{P}}(H_B)$. ■

In order to obtain an a-priori estimate giving the regularity in which we are interested, we will need to approximate mild solutions with other mild solutions of SDEs with smoother coefficients.

Proposition 2.8. *Let G and σ satisfy Assumptions 2.1 and 2.5. There exist sequences $\{G_n\}_{n\in\mathbb{N}} \subset C^{0,1}_b([0,T]\times H, H)$, $\{\Sigma_n\}_{n\in\mathbb{N}} \subset C^{0,1}_b([0,T]\times H, L(\mathbb{R}^m, H))$, with $\Sigma_n(t,x)y = (\sigma_n(t,x)y, 0_1)$ for some $\sigma_n \in C^{0,1}_b([0,T]\times H, L(\mathbb{R}^m))$, satisfying:*

(i) For every $n \in \mathbb{N}$, G_n and Σ_n have extensions $\overline{G}_n \in C^{0,1}_b([0,T]\times H_B, H_B)$ and $\overline{\Sigma}_n \in C^{0,1}_b([0,T]\times H_B, L(\mathbb{R}^m, H_B))$.

(ii) For all $(t,x), (t,y) \in [0,T]\times H_B$,

$$\sup_{n\in\mathbb{N}} |\overline{G}_n(t,x) - \overline{G}_n(t,y)|_B \leq M|x-y|_B \tag{2.33}$$

$$\sup_{n\in\mathbb{N}} |\overline{\Sigma}_n(t,x) - \overline{\Sigma}_n(t,y)|_{L(\mathbb{R}^m,H_B)} \leq M|x-y|_B. \tag{2.34}$$

(iii) For every compact set $K \subset H_B$,

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times K} |\overline{G}(t,x) - \overline{G}_n(t,x)|_B = 0 \tag{2.35}$$

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times K} |\overline{\Sigma}(t,x) - \overline{\Sigma}_n(t,x)|_{L(\mathbb{R}^m,H_B)} = 0. \tag{2.36}$$

Remark 2.9. *We remark that, due to the fact that the range of Σ is finite-dimensional (see (2.1)), once the above continuity/differentiability/approximation conditions for $\overline{\Sigma}_n$ are satisfied with respect to H_B, they automatically hold for Σ_n with respect to H.*

Proof of Proposition 2.8. The proof uses approximations similar to those in [20]. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of H_B contained in H. For $n \in \mathbb{N}$, let us define the functions

$$I_n \colon \mathbb{R}^n \to H_B,\ y \mapsto \sum_{k=1}^n y_k e_k$$

and

$$P_n \colon H_B \to \mathbb{R}^n,\ x \mapsto (\langle x, e_1\rangle_B, \dots, \langle x, e_n\rangle_B).$$

It is clear that $|I_n|_{L(\mathbb{R}^n,H_B)} = 1$ and $|I_nP_n|_{L(H_B)} = 1$. We observe also that, for every $n \in \mathbb{N}$, the linear operator

$$H_B \to H,\ x \mapsto I_nP_nx = \sum_{k=1}^n \langle x, e_k\rangle_B e_k$$

is well defined and continuous. Denote $c_n := |I_nP_n|_{L(H_B,H)}$.

Let

$$\varphi(r) := \begin{cases} e^{-\frac{1}{1-r^2}} & \text{if } r \in (-1,1) \\ 0 & \text{otherwise,} \end{cases}$$

and, for every $n \in \mathbb{N}$,

$$C_n := \left(\int_{\mathbb{R}^n} \varphi(n|y|_n) dy \right)^{-1},$$

where $|\cdot|_n$ denotes the Euclidean norm in $\mathbb{R}^n$. Define

$$g_n : [0,T] \times \mathbb{R}^n \to H$$

by standard mollification

$$g_n(t,y) := C_n \left(G(t, I_n \cdot) * \varphi(n|\cdot|_n) \right)(y) = C_n \int_{\mathbb{R}^n} G\left(t, \sum_{k=0}^{n-1} z_k e_k \right) \varphi(n|y-z|_n) dz,$$

for all $(t,y) \in [0,T] \times \mathbb{R}^n$. We observe that g_n is well-defined, because G is H-valued and continuous, and φ has compact support. By Lebesgue's dominated convergence theorem, g_n is continuous.

Since the map $\mathbb{R}^n \to \mathbb{R}$, $z \mapsto \varphi(n|z|)$, is continuously differentiable and has compact support and since G is continuous, by a standard argument we can differentiate under the integral sign to obtain g_n is differentiable with respect to y and

$$D_y g_n(t,y)v = nC_n \int_{\mathbb{R}^n} G(t, I_n z) \varphi'(n|y-z|_n) \frac{\langle y-z, v \rangle_n}{|y-z|_n} dz.$$

By Lebesgue's dominated convergence theorem, the map

$$[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to H, \ (t,y,v) \mapsto D_y g_n(t,y)v$$

is continuous. Thus $g_n \in C^{0,1}([0,T] \times \mathbb{R}^n, H)$. Define

$$\overline{G}_n : [0,T] \times H_B \to H_B$$

by

$$\overline{G}_n(t,x) := g_n(t, P_n x) = C_n \int_{\mathbb{R}^n} G(t, I_n P_n x - I_n z) \varphi(n|z|_n) dz \qquad \forall (t,x) \in [0,T] \times H_B.$$

Since $\overline{G}_n([0,T] \times H_B) \subset H$, we can also define $G_n : [0,T] \times H \to H$ by $G_n(t,x) := \overline{G}_n(t,x)$ for every $(t,x) \in [0,T] \times H$. Then $G_n \in C^{0,1}([0,T] \times H, H)$ and $\overline{G}_n \in C^{0,1}([0,T] \times H_B, H_B)$. Moreover, by Assumption 2.1,

$$\begin{aligned} |G_n(t,x) - G_n(t,x')|_H &= |g_n(t, P_n x) - g_n(t, P_n x')|_H \\ &\le C_n \int_{\mathbb{R}^n} \left| G(t, I_n P_n x - I_n z) - G\left(t, I_n P_n x' - I_n z\right) \right|_H \varphi(n|z|_n) dz \\ &\le M \left| I_n P_n x - I_n P_n x' \right|_H \le M c_n |x - x'|_B \le M c_n |R^{-1}|_{L(H)} |x - x'|_H, \end{aligned} \tag{2.37}$$

for every $t \in [0,T]$ and $x, x' \in H$. Similarly, by Assumption 2.5,

$$\begin{aligned} |\overline{G}_n(t,x) - \overline{G}_n(t,x')|_B &= |g_n(t, P_n x) - g_n(t, P_n x')|_B \\ &\le M \left| I_n P_n x - I_n P_n x' \right|_B \le M \left| x - x' \right|_B, \end{aligned} \tag{2.38}$$

for every $t \in [0,T]$ and $x,x' \in H_B$. Thus $G_n \in C_b^{0,1}([0,T]\times H,H)$ and $\overline{G}_n \in C_b^{0,1}([0,T]\times H_B,H_B)$.

To prove (2.35) for every compact $K \subset H_B$, we first notice that

$$\sup_{x\in K}|I_nP_nx - x|_B = \varepsilon_n \to 0 \quad \text{as } n\to+\infty.$$

Thus by (2.26),

$$\lim_{n\to+\infty}\sup_{(t,x)\in[0,T]\times K}|\overline{G}(t,I_nP_nx)-\overline{G}(t,x)|_B \le \lim_{n\to+\infty} M\varepsilon_n = 0. \tag{2.39}$$

Moreover, for $(t,x)\in[0,T]\times H_B$,

$$\begin{aligned}|\overline{G}(t,I_nP_nx)-\overline{G}_n(t,x)|_B &\le C_n\int_{\mathbb{R}^n}|G(t,I_nP_nx-I_nz)-G(t,I_nP_nx)|_B\,\varphi(n|z|_n)dz\\ &\le MC_n\int_{\mathbb{R}^n}|I_nz|_B\varphi(n|z|_n)dz \le MC_n\int_{\mathbb{R}^n}|z|_n\varphi(n|z|_n)dz\le\frac{M}{n}.\end{aligned}$$

This, together with (2.39), gives (2.35).

We have thus proved that $\{G_n\}_{n\in\mathbb{N}}\subset C_b^{0,1}([0,T]\times H,H)$, that $\{\overline{G}_n\}_{n\in\mathbb{N}}\subset C_b^{0,1}([0,T]\times H_B,H_B)$, and that (2.33) and (2.35) hold true.

The other half of the proof, regarding Σ, is similar. We only make a few comments. For $n\in\mathbb{N}$, define

$$\zeta_n\colon \mathbb{R}^n \to L(\mathbb{R}^m)$$

by

$$\zeta_n(t,y) := C_n\big(\sigma(t,I_n\cdot)*\varphi(n|\cdot|_n)\big)(y) = C_n\int_{\mathbb{R}^n}\sigma\left(t,\sum_{k=1}^n z_ke_k\right)\varphi(n|y-z|_n)dz,$$

for all $(t,y)\in[0,T]\times\mathbb{R}^n$, and $\overline{\sigma}_n\colon[0,T]\times H_B\to L(\mathbb{R}^m)$ by $\overline{\sigma}_n(t,x) := \zeta_n(t,I_nP_nx)$ for all $(t,y)\in[0,T]\times\mathbb{R}^n$, and $n\in\mathbb{N}$. Arguing as it was done done for g_n, we have that $\zeta_n\in C^{0,1}([0,T]\times\mathbb{R}^n,L(\mathbb{R}^m))$, and then $\overline{\sigma}_n\in C^{0,1}([0,T]\times H_B,L(\mathbb{R}^m))$. Moreover,

$$|\overline{\sigma}_n(t,x)-\overline{\sigma}_n(t,x')|_B \le M\left|x-x'\right|_B, \tag{2.40}$$

and hence $\overline{\sigma}_n\in C_b^{0,1}([0,T]\times H_B,L(\mathbb{R}^m))$. The proof of (2.36) is done in the same way as for $\overline{G}_n$. Finally we define

$$\overline{\Sigma}_n(t,x)y := (\overline{\sigma}_n(t,x)y,0_1)\quad \forall (t,x)\in[0,T]\times H_B,\ \forall y\in\mathbb{R}^n,\ \forall n\in\mathbb{N}$$

and

$$\Sigma_n(t,x)y := \overline{\Sigma}_n(t,x)y\quad \forall(t,x)\in[0,T]\times H,\ \forall y\in\mathbb{R}^n,\ \forall n\in\mathbb{N}.$$

This concludes the proof. ■

Unless specified otherwise, Assumptions 2.1 and 2.5 will be standing for the remaining part of the manuscript, and $\{G_n\}_{n\in\mathbb{N}}$, $\{\Sigma_n\}_{n\in\mathbb{N}}$, $\{\overline{G}_n\}_{n\in\mathbb{N}}$, $\{\overline{\Sigma}_n\}_{n\mathbb{N}}$ will denote the sequences introduced in Proposition 2.8.

Let $\{A_n\}_{n\ge1}$ be the Yosida approximation of A. We recall that for every $n\ge1$, by Proposition 2.4, A_n has a unique continuous extension $\overline{A_n}$ to H_B, and $\overline{A_n} = \overline{A}_n$, where

$\{\overline{A}_n\}_{n\geq 1}$ is the Yosida approximation of the infinitesimal generator $\overline{A}$ of $\overline{S}$. We remind that we denote by $e^{t\overline{A}_n}$ the semigroup generated by $\overline{A}_n$. For $t\in[0,T)$ and $n\geq 1$, we denote by $X_n^{t,x}$, $\overline{X}_n^{t,x}$ respectively the unique mild solutions to

$$\begin{cases} dX_{n,s} = \left(A_n X_{n,s} + G_n\left(s, X_{n,s}\right)\right) ds + \Sigma_n\left(s, X_{n,s}\right) dW_s & s\in(t,T] \\ X_{n,t} = x \in H, \end{cases} \tag{2.41}$$

$$\begin{cases} d\overline{X}_{n,s} = (\overline{A}_n \overline{X}_{n,s} + \overline{G}_n(s, \overline{X}_{n,s})) ds + \overline{\Sigma}_n(s, \overline{X}_{n,s}) dW_s & s\in(t,T] \\ \overline{X}_{n,t} = x \in H_B. \end{cases} \tag{2.42}$$

For any $p\geq 2$, existence and uniqueness of mild solution are provided by Propositions 2.3 and 2.7, together with the continuity of the maps

$$[0,T]\times H \to \mathcal{H}^p_{\mathcal{P}}(H), (t,x)\mapsto X_n^{t,x} \qquad [0,T]\times H_B \to \mathcal{H}^p_{\mathcal{P}}(H_B), (t,x)\mapsto \overline{X}_n^{t,x}. \tag{2.43}$$

Proposition 2.10. *Let Assumptions 2.1 and 2.5 hold and let $p\geq 2$. Then:*

(i) For every $n\in\mathbb{N}$ and $x\in H$, $X_n^{t,x} = \overline{X}_n^{t,x}$ (in $\mathcal{H}^p_{\mathcal{P}}(H_B)$).

(ii) $\lim_{n\to+\infty} \overline{X}_n^{t,x} = \overline{X}^{t,x}$ in $\mathcal{H}^p_{\mathcal{P}}(H_B)$ uniformly for (t,x) on compact sets of $[0,T]\times H_B$.

(iii) For every $n\in\mathbb{N}$ the map

$$[0,T]\times H_B \to \mathcal{H}^p_{\mathcal{P}}(H_B), (t,x)\mapsto \overline{X}_n^{t,x} \tag{2.44}$$

belongs to $\mathcal{G}^{0,1}_s([0,T]\times H_B, \mathcal{H}^p_{\mathcal{P}}(H_B))$.

(iv) The set $\{\nabla_x \overline{X}_n^{t,x}\}_{n\in\mathbb{N}}$ is bounded in $L(H_B, \mathcal{H}^p_{\mathcal{P}}(H_B))$, uniformly for $(t,x)\in[0,T]\times H_B$.

Proof. *(i)* Let $(t,x)\in[0,T]\times H$. Since $A_n = \overline{A}_n$ on H, we have $e^{sA_n} = e^{s\overline{A}_n}$ on H for all $s\in\mathbb{R}^+$. Recalling that $\mathcal{H}^p_{\mathcal{P}}(H)$ is continuously embedded in $\mathcal{H}^p_{\mathcal{P}}(H_B)$, we then have that the mild solution $X_n^{t,x}$ is also a mild solution of (2.42) in $\mathcal{H}^p_{\mathcal{P}}(H_B)$. By uniqueness we conclude that $X_n^{t,x} = \overline{X}_n^{t,x}$ in $\mathcal{H}^p_{\mathcal{P}}(H_B)$.

(ii) For $t\in[0,T]$, $x\in H_B$, $n\geq 1$, similarly to what was done in the proof of Proposition 2.3, we define the maps

$$\overline{\Phi}(t;\cdot,\cdot)\colon H_B\times \mathcal{H}^p_{\mathcal{P}}(H_B)\to \mathcal{H}^p_{\mathcal{P}}(H_B)$$

by

$$\overline{\Phi}(t;x,Z)_s := \begin{cases} x & s\in[0,t) \\ \overline{S}_{s-t}x + \int_t^s \overline{S}_{s-w}\overline{G}(w,Z_w)dw + \int_t^s \overline{S}_{s-w}\overline{\Sigma}(w,Z_w)dW_w & s\in[t,T] \end{cases}$$

and

$$\overline{\Phi}_n(t;\cdot,\cdot)\colon H_B\times \mathcal{H}^p_{\mathcal{P}}(H_B)\to \mathcal{H}^p_{\mathcal{P}}(H_B)$$

by

$$\overline{\Phi}_n(t;x,Z)_s := \begin{cases} x & s\in[0,t) \\ e^{(s-t)\overline{A}_n}x + \int_t^s e^{(s-w)\overline{A}_n}\overline{G}_n(w,Z_w)dw + \int_t^s e^{(s-w)\overline{A}_n}\overline{\Sigma}_n(w,Z_w)dW_w & s\in[t,T]. \end{cases}$$

The mild solutions $\overline{X}^{t,x}$ and $\overline{X}_n^{t,x}$ are the fixed points of $\overline{\Phi}(t;x,\cdot)$ and $\overline{\Phi}_n(t;x,\cdot)$ respectively. Since the operators $\overline{A}_n$, $n\geq 1$, are the Yosida approximations of $\overline{A}$, they generate semigroups of contractions on H_B. Recalling (2.33) and (2.34), and arguing for $\overline{\Phi}$ and $\overline{\Phi}_n$ as in the proof of Proposition 2.3 for Φ, we find $\gamma>0$, depending only on p, T, M, such that

$$\sup_{(t,x)\in[0,T]\times H} |\overline{\Phi}(t;x,Z)-\overline{\Phi}(t;x,Z')|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq \frac{1}{2}|Z-Z'|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \quad \forall Z,Z'\in\mathcal{H}^p_{\mathcal{P}}(H_B) \tag{2.45}$$

and

$$\sup_{\substack{(t,x)\in[0,T]\times H\\ n\in\mathbb{N}}} |\overline{\Phi}_n(t;x,Z)-\overline{\Phi}_n(t;x,Z')|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq \frac{1}{2}|Z-Z'|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \quad \forall Z,Z'\in\mathcal{H}^p_{\mathcal{P}}(H_B). \tag{2.46}$$

Let $\{t_n\}_{n\geq 1}$ be a sequence converging to t in $[0,T]$. We claim that

$$\lim_{n\to+\infty} \overline{\Phi}_n(t_n;x,Z) = \overline{\Phi}(t;x,Z) \text{ in } \mathcal{H}^p_{\mathcal{P}}(H_B),\ \forall (x,Z)\in H_B\times\mathcal{H}^p_{\mathcal{P}}(H_B). \tag{2.47}$$

Once (2.47) is proved, we can conclude, again invoking [8, Theorem 7.1.5], that

$$\lim_{n\to+\infty} \overline{X}_n^{t_n,x} = \overline{X}^{t,x} \text{ in } \mathcal{H}^p_{\mathcal{P}}(H_B),\ \forall x\in H_B. \tag{2.48}$$

But (2.47) is easily obtained by combining strong convergence of $e^{t\overline{A}_n}$ to $\overline{S}_t$ uniformly for $t\in[0,T]$, sublinear growth of $\overline{G}_n(t,x)$ and $\overline{\Sigma}_n(t,x)$ in x uniformly on $t\in[0,T]$ and $n\geq 1$ (obtained by (2.33), (2.34), (2.35), (2.36), and by continuity of $\overline{G}(\cdot,0)$ and of $\overline{\Sigma}(\cdot,0)$), Burkholder-Davis-Gundy's inequality, Lebesgue's dominated convergence theorem, and pointwise convergence of $\{\overline{G}_n\}_{n\in\mathbb{N}}$ to $\overline{G}$ and of $\{\overline{\Sigma}_n\}_{n\in\mathbb{N}}$ to $\overline{\Sigma}$.

By linearity of $\overline{\Phi}(t;x,Z)$ and $\overline{\Phi}_n(t;x,Z)$ in x, we have

$$\sup_{(t,Z)\in[0,T]\times\mathcal{H}^p_{\mathcal{P}}(H_B)} |\overline{\Phi}(t;x,Z)-\overline{\Phi}(t;x',Z)|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq |x-x'|_{H_B} \quad \forall x,x'\in H_B. \tag{2.49}$$

and

$$\sup_{\substack{(t,Z)\in[0,T]\times\mathcal{H}^p_{\mathcal{P}}(H_B)\\ n\in\mathbb{N}}} |\overline{\Phi}_n(t;x,Z)-\overline{\Phi}_n(t;x',Z)|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq |x-x'|_{H_B} \quad \forall x,x'\in H_B. \tag{2.50}$$

Thus, using (2.45), (2.46), and [13, inequality $(***)$ on p. 13], we have

$$\sup_{t\in[0,T]} |\overline{X}^{t,x}-\overline{X}^{t,x'}|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq 2|x-x'|_{H_B} \quad \forall x,x'\in H_B. \tag{2.51}$$

and

$$\sup_{\substack{t\in[0,T]\\ n\in\mathbb{N}}} |\overline{X}_n^{t,x}-\overline{X}_n^{t,x'}|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq 2|x-x'|_{H_B} \quad \forall x,x'\in H_B. \tag{2.52}$$

Now (2.48), (2.51), (2.52), and the continuity of $\overline{X}^{t,x}$ in t (Proposition 2.7), yield the convergence of $\overline{X}_n^{t,x}$ to $\overline{X}^{t,x}$ in $\mathcal{H}^p_{\mathcal{P}}(H_B)$ as $n \to +\infty$, uniformly for (t,x) on compact subsets of $[0,T] \times H_B$.

(iii) Let $n \geq 1$. By [7, p. 243, Th. 9.8], for every $t \in [0,T]$, the map (2.44) is Gâteaux differentiable and, for every $x, y \in H_B$, the directional derivate $\nabla_x \overline{X}_n^{t,x} y$ is the unique fixed point of $\Psi_n(t,x;y,\cdot)$, where $\Psi_n(t,\cdot;\cdot)$ is defined by

$$\Psi_n(t,x;\cdot,\cdot)\colon H_B \times \mathcal{H}^p_{\mathcal{P}}(H_B) \to \mathcal{H}^p_{\mathcal{P}}(H_B),$$

$$\Psi_n(t,x;y,Z) := \begin{cases} y & s \in [0,t) \\ \overline{S}_{s-t} y + \int_t^s \overline{S}_{s-w} \nabla_x \overline{G}_n(w, \overline{X}_{n,w}^{t,x}) Z_w dw + \int_t^s \overline{S}_{s-w} \nabla_x \overline{\Sigma}_n(w, \overline{X}_{n,w}^{t,x}) Z_w dW_w & s \in [t,T]. \end{cases}$$

To show strong continuity and uniform boundedness of $\nabla_x \overline{X}_n$, we argue similarly as in the proof of Proposition 2.3. By (2.33) and (2.34),

$$|\nabla_x \overline{G}_n(t,x) y|_B \leq M |y|_B \tag{2.53}$$

$$|\nabla_x \overline{\Sigma}_n(t,x) y|_{L(\mathbb{R}^m, H_B)} \leq M |y|_B, \tag{2.54}$$

for all $(t,x,y) \in [0,T] \times H_B \times H_B$. Then, by linearity of $\Psi_n(t,x;\cdot,\cdot)$ and by [8, Ch. 7, Proposition 7.3.1], there exists $\gamma > 0$, depending only on p, T, M, b, such that

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times H} |\Psi_n(t,x;y,Z) - \Psi_n(t,x;y,Z')|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} &= \sup_{(t,x)\in[0,T]\times H} |\Psi_n(t,x;0,Z-Z')|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \\ &\leq \frac{1}{2} |Z - Z'|_{\mathcal{H}^p_{\mathcal{P}}(H),\gamma} \quad \forall Z, Z' \in \mathcal{H}^p_{\mathcal{P}}(H). \end{aligned} \tag{2.55}$$

We also have

$$\sup_{(t,x,Z)\in[0,T]\times H_B \times \mathcal{H}^p_{\mathcal{P}}(H_B)} |\Psi_n(t,x;y,Z) - \Psi_n(t,x;y',Z)|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq |y - y'|_{H_B} \tag{2.56}$$

for all $y, y' \in H_B$. By (2.55), (2.56), and [13, inequality $(***)$ on p. 13], we thus obtain

$$\sup_{(t,x)\in[0,T]\times H_B} |\nabla_x \overline{X}_n^{t,x} y|_{\mathcal{H}^p_{\mathcal{P}}(H_B),\gamma} \leq 2 |y|_{H_B} \quad \forall y \in H_B. \tag{2.57}$$

Hence $\nabla_x \overline{X}_n^{t,x}$ is bounded in $L\left(H_B, \mathcal{H}^p_{\mathcal{P}}(H_B)\right)$, uniformly for $(t,x) \in [0,T] \times H_B$ and $n \geq 1$.

Let now $\{(t_k, x_k)\}_{k\in\mathbb{N}} \subset [0,T] \times H_B$ be a sequence converging to $(t,x) \in [0,T] \times H_B$. We claim that

$$\lim_{k\to+\infty} \Psi_n(t_k, x_k; y, Z) = \Psi_n(t,x;y,Z) \quad \forall (y,Z) \in H_B \times \mathcal{H}^p_{\mathcal{P}}(H_B). \tag{2.58}$$

Once (2.58) is proved, using (2.55) and applying [8, Theorem 7.1.5], we obtain

$$\lim_{k\to+\infty} \nabla_x \overline{X}_n^{t_k,x_k} y = \nabla_x \overline{X}_n^{t,x} y \quad \forall y \in H_B,$$

which provides the strong continuity of $\nabla_x \overline{X}_n^{t,x}$. Recalling that $\lim_{k\to+\infty} \overline{X}_n^{t_k,x_k} = \overline{X}_n^{t,x}$ in $\mathcal{H}^p_{\mathcal{P}}(H_B)$, we can consider a subsequence, again denoted by $\{(t_k,x_k)\}_{k\in\mathbb{N}}$, such that $\lim_{n\to+\infty} \overline{X}_n^{t_k,x_k} = \overline{X}_n^{t,x}$ $\mathbb{P} \otimes dt$-a.e. on Ω_T. Then (2.58) is obtained by applying Lebesgue's dominated convergence theorem, together with Burkholder-Davis-Gundy's inequality, for the stochastic integral.

(iv) This follows immediately from (2.57). ■

We will make a particular choice of R and thus B. Recall that $(0,+\infty)$ is contained in the resolvent set of A (and hence of A^*). For $\lambda > 0$, let $A_\lambda := A - \lambda$, $A^*_\lambda := A^* - \lambda = (A-\lambda)^*$. If $R = A_\lambda$, then (2.13), (2.16), and (2.17), are satisfied. We can then apply all of the above arguments with

$$B = B_{A,\lambda} := (A^*_\lambda)^{-1} A_\lambda^{-1}.$$

Notice that

$$|x|_{B_{A,\lambda}} \le \left(1 + |\lambda - \lambda'| |A_\lambda^{-1}|_{L(H)}\right) |x|_{B_{A,\lambda'}} \qquad \forall \lambda, \lambda' \in (0,+\infty), x \in H,$$

hence the norms $|\cdot|_{B_{A,\lambda}}$ and $|\cdot|_{B_{A,\lambda'}}$ are equivalent. We will thus pick $\lambda = 1$ and from now on we set

$$B := B_{A,1} = (A^*_1)^{-1} A_1^{-1}. \tag{2.59}$$

We observe that with this choice of B we have

$$|x|_B = |(\overline{A} - I)^{-1} x|_H \quad \text{for all } x \in H_B,$$

and

$$\langle x, y \rangle_B = \langle (\overline{A} - I)^{-1} x, (\overline{A} - I)^{-1} y \rangle \quad \text{for all } x, y \in H_B.$$

In particular

$$\langle x, y \rangle_B = \langle (A^* - I)^{-1} (\overline{A} - I)^{-1} x, y \rangle \quad \text{if } x \in H_B, y \in H.$$

3 Viscosity solutions of Kolmogorov PDEs in Hilbert spaces with finite-dimensional second-order term

We remind that throughout the rest of the paper B is defined by (2.59). For this B, Assumptions 2.1 and 2.5 will be standing for the remaining part of the manuscript, $\{G_n\}_{n\in\mathbb{N}}$, $\{\Sigma_n\}_{n\in\mathbb{N}}$, $\{\overline{G}_n\}_{n\in\mathbb{N}}$, $\{\overline{\Sigma}_n\}_{n\mathbb{N}}$ denote the sequences introduced in Proposition 2.8, the operators $A_n, n \ge 1$ are the Yosida approximations of A, and $X_n^{t,x}$, $\overline{X}_n^{t,x}$ are respectively the mild solutions of (2.41), (2.42), with $B = B_{A,1}$, $n \ge 1$. We recall that, by Proposition 2.10, $X^{t,x} = \overline{X}^{t,x}$ and $X_n^{t,x} = \overline{X}_n^{t,x}$ for every $(t,x) \in [0,T] \times H$, $n \ge 1$.

3.1 Existence and uniqueness of solution

The following assumption will be standing for the remaining part of the work.

Assumption 3.1. *The function $h\colon H_B \to \mathbb{R}$ is such that there is a constant $M \ge 0$ such that*

$$|h(x) - h(y)| \le M|x - y|_B \quad \forall x, y \in H. \tag{3.1}$$

The function h extends uniquely to $\overline{h} : H_B \to \mathbb{R}$ which also satisfies (3.1). Taking the inf-sup convolutions of $\overline{h}$ in H_B (see [11, 17]) we can obtain a sequence of functions $\{\overline{h}_n\}_{n\in\mathbb{N}} \subset C_b^1(H_B)$ (and even more regular) such that

$$\sup_{\substack{n\in\mathbb{N}\\ x\in H_B}} |D\overline{h}_n(x)|_B < +\infty \qquad \text{and} \qquad \lim_{n\to+\infty} \sup_{x\in H_B} \left|\overline{h}(x) - \overline{h}_n(x)\right| = 0. \tag{3.2}$$

The restriction of $\overline{h}_n$ to H will be denoted by h_n.

We define the functions

$$u\colon [0,T]\times H \to \mathbb{R},\ (t,x)\mapsto \mathbb{E}\left[h(X_T^{t,x})\right], \tag{3.3}$$

$$u_n\colon [0,T]\times H \to \mathbb{R},\ (t,x)\mapsto \mathbb{E}\left[h_n(X_{n,T}^{t,x})\right], \quad n\geq 1. \tag{3.4}$$

By sublinear growth of h and h_n, u and u_n are well defined. Each of the above functions has an associated Kolmogorov equation in $(0,T]\times H$. However we will only need to consider the equation satisfied by u_n. We also define

$$\overline{u}_n\colon [0,T]\times H_B \to \mathbb{R},\ (t,x)\mapsto \mathbb{E}\left[\overline{h}_n(\overline{X}_{n,T}^{t,x})\right], \quad n\geq 1.$$

We observe that $u_n=\overline{u}_{n|[0,T]\times H}$.

Proposition 3.2. *Let $p\geq 2$. Then:*

(i) u_n is uniformly continuous on bounded sets of $[0,T]\times(H,|\cdot|_B)$ and, for every $t\in[0,T]$, $u_n(t,\cdot)$ is $|\cdot|_B$-Lipschitz continuous, with a Lipschitz constant uniform in $t\in[0,T]$ and $n\geq 1$.

(ii) The sequence $\{u_n\}_{n\geq 1}$ converges to u uniformly on compact sets of $[0,T]\times H$.

(iii) For every $n\geq 1$, $u_n\in\mathscr{G}_s^{0,1}([0,T]\times H)$, and

$$\sup_{\substack{(t,x)\in[0,T]\times H\\ n\geq 1}} |\nabla_x u_n(t,x)|_H<+\infty, \tag{3.5}$$

$$\sup_{\substack{(t,x)\in[0,T]\times H\\ n\geq 1}} \left|A_n^*\nabla_x u_n(t,x)\right|_H<+\infty. \tag{3.6}$$

Proof. *(i)* From (3.2) and Proposition 2.10-*(i)*,*(iii)*,*(iv)*, it follows that u_n is continuous and $|\cdot|_B$-Lipschitz continuous in x with a Lipschitz constant uniform in $t\in[0,T]$ and $n\geq 1$.

The uniform continuity of u_n on bounded sets is standard since we are dealing with bounded evolution and can be deduced from a more general result, see e.g. [7, Theorem 9.1], however we present a short argument. We first notice that it follows from Proposition 2.10-*(iii)*,*(iv)* that, for any $r>0$ and $n\geq 1$, there exists $K>0$ such that

$$|\overline{X}_n^{t,x}|_{\mathscr{H}^2_{\mathscr{P}}(H_B)}\leq K \qquad \forall t\in[0,T],\ \forall x\in H_B,\ |x|_B\leq r.$$

Secondly, we recall that, for $t\in[0,T]$ and $x\in H_B$, $\overline{X}_n^{t,x}$ is a strong solution to (2.42), because $\overline{A}_n$ is bounded (see footnote 1 on p. 6). Then if $0\leq t\leq t'\leq T$ and $x\in H_B$, $|x|_B\leq r$, for some constants C_1, C_2 depending only on T, K, $|\overline{A}_n|_{L(H_B)}$, and on the Lipschitz and the linear-growth constants of $\overline{G}_n$ and $\overline{\Sigma}_n$, by standard estimates we have

$$\mathbb{E}\left[\left|\overline{X}_{n,s}^{t,x}-\overline{X}_{n,s}^{t',x}\right|_B^2\right]\leq C_1(t'-t)+C_2\int_{t'}^s \mathbb{E}\left[\left|\overline{X}_{n,w}^{t,x}-\overline{X}_{n,w}^{t',x}\right|_B^2\right]dw \qquad \forall s\in[t',T].$$

By Gronwall's lemma, the inequality above provides

$$\mathbb{E}\left[\left|\overline{X}_{n,T}^{t,x}-\overline{X}_{n,T}^{t',x}\right|_B^2\right]\leq C_1e^{C_2T}(t'-t). \tag{3.7}$$

The uniform continuity of u_n on $[0,T]\times\{x\in H : |x|_B\le r\}$ is then obtained by (3.2), (3.7), and by the $|\cdot|_B$-Lipschitz continuity of $\overline{X}_n^{t,x}$ in x with a Lipschitz constant uniform in $t\in[0,T]$.

(ii) Part *(ii)* is a consequence of Proposition 2.10-*(i)*,*(ii)* and (3.2).

(iii) Let $n\ge 1$. By [8, Ch. 7, Proposition 7.3.3], the map

$$\Xi_n : \mathcal{H}^p_{\mathcal{P}}(H_B)\to\mathcal{H}^p_{\mathcal{P}}(H_B),\ Z\mapsto\overline{h}_n(Z)$$

belongs to $\mathcal{G}^1_s(\mathcal{H}^p_{\mathcal{P}}(H_B),\mathcal{H}^p_{\mathcal{P}}(H_B))$, and

$$(\nabla_Z\Xi_n(Z)Y)_t = D\overline{h}_n(Z_t)Y_t \quad \forall t\in[0,T],\ \forall Z,Y\in\mathcal{H}^p_{\mathcal{P}}(H_B). \tag{3.8}$$

By Proposition 2.10-*(iii)*, linearity and continuity of the expected valued $\mathbb{E}$ on $L^p_{\mathcal{P}}(H_B)$, linearity and continuity of the T-evaluation map $\mathcal{H}^p_{\mathcal{P}}(H_B)\to L^p(H_B)$, $Z\mapsto Z_T$, formula (3.8), composition of strongly continuously Gâteaux differentiable functions, we obtain $\overline{u}_n\in\mathcal{G}^{0,1}_s([0,T]\times H_B)$ and

$$\langle\nabla_x\overline{u}_n(t,x),y\rangle_B = \mathbb{E}\left[D\overline{h}_n(\overline{X}^{t,x}_{n,T})\left(\nabla_x\overline{X}^{t,x}_n y\right)_T\right] \qquad \forall(t,x,y)\in[0,T]\times H_B\times H_B. \tag{3.9}$$

By Proposition 2.10-*(iv)*, (3.2), (3.9),

$$\sup_{\substack{(t,x)\in[0,T]\times H_B\\ n\ge1}} |\nabla_x\overline{u}_n(t,x)|_B < +\infty. \tag{3.10}$$

By continuous embedding $H\to H_B$ and by (3.10) we have also

$$u_n\in\mathcal{G}^{0,1}_s([0,T]\times H), \qquad \sup_{\substack{(t,x)\in[0,T]\times H\\ n\ge1}} |\nabla_x u_n(t,x)|_H < +\infty, \tag{3.11}$$

which shows (3.5). Moreover, since

$$\nabla_x u_n(t,x) = (A^*-1)^{-1}(\overline{A}-1)^{-1}\nabla_x\overline{u}_n(t,x),$$

we obtain from (3.10) that

$$\sup_{\substack{n\ge1\\ (t,x)\in[0,T]\times H}} |A^*\nabla_x u_n(t,x)|_H < +\infty. \tag{3.12}$$

Therefore, recalling that S is a semigroup of contractions, we have

$$|A_n^*\nabla_x u_n(t,x)|_H \le |n(n-A)^{-1}|_{L(H)}|A^*\nabla_x u_n(t,x)|_H \le |A^*\nabla_x u_n(t,x)|_H$$

for all $(t,x)\in[0,T]\times H$ which, together with (3.12), shows (3.6). ■

We now define for $n\ge1$

$$L_n\colon [0,T]\times H\times H\times\mathbf{S}_m\to\mathbb{R},\ (t,x,p,P)\mapsto\langle p,G_n(t,x)\rangle+\frac{1}{2}\mathrm{Tr}(\sigma_n(t,x)\sigma_n^*(t,x)P)$$

where $\mathbf{S}_m$ is the set of $m\times m$ symmetric matrices.

We consider the following terminal value problems

$$\begin{cases} -v_t-\langle A_n x,D_x v\rangle - L_n(t,x,D_x v,D^2_{x_0x_0}v)=0 & (t,x)\in(0,T)\times H\\ v(T,x)=h_n(x) & x\in H.\end{cases} \tag{3.13}$$

Since the operator A_n is bounded we will use the definition of viscosity solution from [19].

Definition 3.3. *A locally bounded*[4] *upper semi-continuous function* v *on* $(0,T]\times H$ *is a viscosity subsolution of* (3.13) *if* $v(T,x)\le h_n(x)$ *for all* $x\in H$*, and whenever* $v-\varphi$ *has a local maximum at a point* $(\hat t,\hat x)\in(0,T)\times H$*, for some* $\varphi\in C^{1,2}((0,T)\times H)$*, then*

$$-\varphi_t(\hat t,\hat x)-\langle A_n\hat x, D_x\varphi(\hat t,\hat x)\rangle - L_n(\hat t,\hat x, D_x\varphi(\hat t,\hat x), D^2_{x_0}\varphi(\hat t,\hat x))\le 0.$$

A locally bounded lower semi-continuous function v *on* $(0,T]\times H$ *is a viscosity supersolution of* (3.13) *if* $v(T,x)\ge h_n(x)$ *for all* $x\in H$*, and whenever* $v-\varphi$ *has a local minimum at a point* $(\hat t,\hat x)\in(0,T)\times H$*, for some* $\varphi\in C^{1,2}((0,T)\times H)$*, then*

$$-\varphi_t(\hat t,\hat x)-\langle A_n\hat x, D_x\varphi(\hat t,\hat x)\rangle - L_n(\hat t,\hat x, D_x\varphi(\hat t,\hat x), D^2_{x_0}\varphi(\hat t,\hat x))\ge 0.$$

A viscosity solution of (3.13) *is a function which is both a viscosity subsolution and a viscosity supersolution of* (3.13).

Theorem 3.4. *For* $n\ge 1$*, the function* u_n *is the unique (within the class of, say locally uniformly continuous functions with at most polynomial growth) viscosity solution of* (3.13).

Proof. Since A_n is a bounded operator this is a standard result, see e.g. [11, 14, 19]. Notice that Proposition 3.2-*(i)* guarantees that the function u_n is locally uniformly continuous on $[0,T]\times H$ and is Lipschitz continuous in x. ∎

Remark 3.5. *This is not needed here however it is worth noticing that the function* u *is the unique so called* $B_{A,1}$*-continuous viscosity solution (unique within the class of, say* $B_{A,1}$*-continuous functions with at most polynomial growth which attain the terminal condition locally uniformly), of the equation*

$$\begin{cases} -u_t-\langle Ax, D_xu\rangle - L(t,x,D_xu,D^2_{x_0x_0}u)=0 & (t,x)\in(0,T)\times H\\ u(T,x)=h(x) & x\in H,\end{cases}\tag{3.14}$$

where

$$L\colon [0,T]\times H\times H\times \mathbf{S}_m\to\mathbb{R},\ (t,x,p,P)\mapsto \langle p, G(t,x)\rangle+\frac12\operatorname{Tr}(\sigma(t,x)\sigma^*(t,x)P).$$

For the proof of this we refer the reader to [11, Theorem 3.64].

3.2 Space sections of viscosity solutions

We skip the proof of the following basic lemma (for a very similar version, see [3, Proposition 3.7].

Lemma 3.6. *Let* D *be a set, and* $f,g\colon D\to\mathbb{R}$ *be functions, with* $g\ge 0$*. Let*

$$Z=\{y\in D\colon g(y)=0\}$$

be the set of zeros of g*. Suppose that* $Z\neq\emptyset$*. Let* $\{h_i\colon D\to\mathbb{R}\}_{i\in\mathbb{N}}$ *be a sequence of functions converging uniformly to* 0 *in* D *as* $i\to+\infty$*. Let* $\{\varepsilon_i\}_{i\in\mathbb{N}}$ *be a sequence of positive numbers decreasing to* 0*. Define*

$$\psi_i(y):=f(y)-\frac{g(y)}{\varepsilon_i}+h_i(y)\qquad \forall i\in\mathbb{N},\ \forall y\in D.$$

[4] By "locally bounded" we mean "bounded on bounded subsets of the domain", and by "locally uniformly continuous" we mean "uniformly continuous on bounded subsets of the domain".

Suppose that $\{y_i\}_{i\in\mathbb{N}} \subset D$ *is a sequence such that*

$$\lim_{i\to\infty}\left[\sup_{y\in D}\psi_i(y)-\psi_i(y_i)\right]=0.$$

Then $\displaystyle\lim_{i\to\infty}\frac{g(y_i)}{\varepsilon_i}=0.$

Fix $\overline{x}_1 \in H_1$. Let $\varphi \in C^{1,2}((0,T)\times\mathbb{R}^m)$ and let $(\hat{t},\hat{x}_0)\in(0,T)\times\mathbb{R}^m$ be a maximum point of $u_n(\cdot,(\cdot,\overline{x}_1))-\varphi(\cdot,\cdot)$ over $[0,T]\times\mathbb{R}^m$. Without loss of generality we can assume that $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^m)$ and that the maximum is strict and global.

For $\varepsilon>0$, define the function

$$\Phi_\varepsilon(t,x_0,x_1)=\varphi(t,x_0)+\frac{1}{\varepsilon}|(0,x_1-\overline{x}_1)|_H^2, \tag{3.15}$$

where $t\in(0,T)$, $(x_0,x_1)\in H$. Observe that $\Phi_\varepsilon\in C^{1,2}([0,T]\times H)$, and

$$\begin{aligned} D_t\Phi_\varepsilon(t,x)&=\varphi_t(t,x_0)\\ D_x\Phi_\varepsilon(t,x)&=\big(D_{x_0}\varphi(t,x_0),0\big)+\frac{2}{\varepsilon}(0,x_1-\overline{x}_1)\\ D^2_{x_0}\Phi_\varepsilon(t,x)&=D^2_{x_0}\varphi(t,x_0). \end{aligned} \tag{3.16}$$

Lemma 3.7. *For each* $n\geq 1$*, there exist real sequences* $\{a_i\}_{i\in\mathbb{N}}$*,* $\{\varepsilon_i\}_{i\in\mathbb{N}}$ *converging to* 0*, and a sequence* $\{p_i\}_{i\in\mathbb{N}}$ *converging to the origin in* H*, such that the function*

$$(0,T)\times H\to\mathbb{R},\ (t,x)\mapsto u_n(t,x)-\Phi_{\varepsilon_i}(t,x)-\langle p_i,x\rangle-a_it \tag{3.17}$$

has a strict global maximum at (t_i,x_i) *and the sequence* $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ *converges to* $(\hat{t},(\hat{x}_0,\overline{x}_1))$*.*

Proof. Let $R>|(\hat{x}_0,\overline{x}_1)|_H$ and $\mathbf{B}_R:=\{x\in H\colon |x|_H\leq R\}$. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence converging to 0. Applying the classical result of Ekeland and Lebourg [9, 21], there exist sequences $\{a_i\}_{i\in\mathbb{N}}\subset\mathbb{R}$ and $\{p_i\}_{i\in\mathbb{N}}\subset H$ such that $|a_i|\leq 1/i$, $|p_i|_H\leq 1/i$, and such that the function

$$[0,T]\times\mathbf{B}_R\to\mathbb{R},\ u_n(t,x)-\Phi_{\varepsilon_i}(t,x)-\langle p_i,x\rangle-a_it$$

has a strict global maximum at some point $(t_i,x_i)\in[0,T]\times\mathbf{B}_R$. By applying Lemma 3.6 with $D=[0,T]\times\mathbf{B}_R$, $f(t,x)=u_n(t,x)-\varphi(t,x_0)$, $g(t,x)=|(0,x_1-\overline{x}_1)|_H^2$, $h_i(t,x)=-\langle p_i,x\rangle-a_it$, $y_i=(t_i,x_i)$, we obtain

$$\lim_{i\to\infty}|(0,x_{i,1}-\overline{x}_1)|_H=0. \tag{3.18}$$

To conclude the proof it is then sufficient to show that $(t_i,x_{i,0})\to(\hat{t},\hat{x}_0)$. Indeed, suppose that this does not hold. Up to a subsequence, we can suppose that $(t_i,x_{i,0})\to(\widetilde{t},\widetilde{x}_0)\neq(\hat{t},\hat{x}_0)$. Since, by assumption, $(\hat{t},\hat{x}_0)$ is a strict global maximum point of $u_n(\cdot,(\cdot,\overline{x}_1))-\varphi(\cdot,\cdot)$, there exists $\eta>0$ such that, for i sufficiently large, we have

$$\begin{aligned} u_n(\hat{t},(\hat{x}_0,\overline{x}_1))-\varphi(\hat{t},\hat{x}_0)\geq&\,\eta+u_n(t_i,(x_{i,0},\overline{x}_1))-\varphi(t_i,x_{i,0})\\ \geq&\,\eta+u_n(t_i,(x_{i,0},\overline{x}_1))-\Phi_{\varepsilon_i}(t_i,x_i)\\ =&\,\eta+\big(u_n(t_i,(x_{i,0},\overline{x}_1))-u_n(t_i,x_i)\big)+u_n(t_i,x_i)-\Phi_{\varepsilon_i}(t_i,x_i)\\ \geq&\,\eta+\big(u_n(t_i,(x_{i,0},\overline{x}_1))-u_n(t_i,x_i)\big)+u_n(\hat{t},(\hat{x}_0,\overline{x}_1))-\varphi(\hat{t},\hat{x}_0)\\ &+\langle p_i,x_i-(\hat{x}_0,\overline{x}_1)\rangle+a_i(t_i-\hat{t}). \end{aligned} \tag{3.19}$$

By (3.18), $\lim_{i\to\infty}(x_{i,0},x_{i,1}) = (\tilde{x}_0,\overline{x}_1)$. Thus by continuity of u_n, for i sufficiently large, we have

$$|u_n(t_i,(x_{i,0},\overline{x}_1)) - u_n(t_i,x_i)| \le \frac{\eta}{2}$$

and then if follows from 3.19 that

$$u_n(\hat{t},(\hat{x}_0,\overline{x}_1)) - \varphi(\hat{t},\hat{x}_0) \ge \frac{\eta}{2} + u_n(\hat{t},(\hat{x}_0,\overline{x}_1)) - \varphi(\hat{t},\hat{x}_0) + \langle p_i, x_i - (\hat{x}_0,\overline{x}_1)\rangle + a_i(t_i - \hat{t}).$$

This produces a contradiction by letting $i \to +\infty$, recalling that $p_i \to 0$ and $a_i \to 0$. Thus we must have $\lim_{i\to\infty}(t_i,x_{i,0}) = (\hat{t},\hat{x}_0)$. ■

For any $\overline{x}_1 \in H_1$ and $n \in \mathbb{N}$, we define the following functions

$$v_{n,\overline{x}_1} : [0,T]\times\mathbb{R}^m \to \mathbb{R}, (t,x_0) \mapsto v_{n,\overline{x}_1}(t,x_0) := u_n(t,(x_0,\overline{x}_1)), \tag{3.20}$$

$$a_{n,\overline{x}_1} : [0,T]\times\mathbb{R}^m \to \mathbf{S}_m, (t,x_0) \mapsto \sigma_n(t,(x_0,\overline{x}_1))\sigma_n^*(t,(x_0,\overline{x}_1)) \tag{3.21}$$

and

$$\beta_{n,\overline{x}_1} : [0,T]\times\mathbb{R}^m \to \mathbb{R}, (t,x_0) \mapsto \langle A_n(x_0,\overline{x}_1) + G_n(t,(x_0,\overline{x}_1)), \nabla_x u_n(t,(x_0,\overline{x}_1))\rangle. \tag{3.22}$$

We associate to (3.13) the following terminal value problem

$$\begin{cases} -v_t(t,x_0) - \dfrac{1}{2}\mathrm{Tr}(a_{n,\overline{x}_1}(t,x_0)D^2_{x_0}v(t,x_0)) - \beta_{n,\overline{x}_1}(t,x_0) = 0 & (t,x_0)\in(0,T)\times\mathbb{R}^m \\ v(T,x_0) = h_n(x_0,\overline{x}_1) & x_0 \in \mathbb{R}^m. \end{cases} \tag{3.23}$$

We recall that it follows from Proposition 3.2-*(iii)* that for every $\overline{x}_1 \in H_1$ the function $\beta_{n,\overline{x}_1}$ is continuous and for every compact set $K \subset \mathbb{R}^m$,

$$\sup_{\substack{n\ge1 \\ (t,x_0)\in[0,T]\times K}} |\beta_{n,\overline{x}_1}(t,x_0)| < +\infty. \tag{3.24}$$

In the following proposition we show that the section functions $v_{n,\overline{x}_1}$ are the viscosity solutions of (3.23). For the definition of viscosity solution in finite dimensions, we refer to [3].

Proposition 3.8. *For every $\overline{x}_1 \in H_1$ and $n \ge 1$, $v_{n,\overline{x}_1}$ is a viscosity solution of* (3.23).

Proof. We prove that $v_{n,\overline{x}_1}$ is a subsolution. The supersolution case is similar. The continuity of u_n implies the continuity of $v_{n,\overline{x}_1}$. Let $\varphi \in C^{1,2}((0,T)\times\mathbb{R}^m)$ be such that $v_{n,\overline{x}_1} - \varphi$ has a local maximum at $(\hat{t},\hat{x}_0) \in (0,T)\times\mathbb{R}^m$. Without loss of generality, we can assume that the maximum is strict and global and that $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^m)$. By Lemma 3.7, there exist real sequences $\{\varepsilon_i\}_{i\in\mathbb{N}}, \{a_i\}_{i\in\mathbb{N}}$ converging to 0, and a sequence $\{p_i\}_{i\in\mathbb{N}}$ in H converging to 0, such that the functions

$$[0,T]\times\mathbb{R}^m \to \mathbb{R}, (t,x) \mapsto u_n(t,x) - \Phi_{\varepsilon_i}(t,x) - \langle p_i, x\rangle - a_i t$$

have local maxima at (t_i,x_i) and the sequence $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ converges to $(\hat{t},(\hat{x}_0,\overline{x}_1))$. Since u_n is a viscosity solution of (3.13), we have

$$\begin{aligned} -D_t\Phi_{\varepsilon_i}(t_i,x_i)-a_i-\langle A_n x_i, D_x\Phi_{\varepsilon_i}(t_i,x_i)+p_i\rangle \\ -L_n\left(t_i,x_i,D_x\Phi_{\varepsilon_i}(t_i,x_i)+p,D^2_{x_0}\Phi_{\varepsilon_i}(t_i,x_i)\right)\le 0. \end{aligned} \tag{3.25}$$

Since $u_n\in\mathscr{G}^{0,1}_s([0,T]\times H,\mathbb{R})$, we must have

$$\nabla_x u_n(t_i,x_i)=D_x\Phi_{\varepsilon_i}(t_i,x_i)+p_i. \tag{3.26}$$

Thus, by recalling (3.16), we have

$$-D_t\Phi_{\varepsilon_i}(t_i,x_i)-a_i-\langle A_n x_i,\nabla_x u_n(t_i,x_i)\rangle-L_n\left(t_i,x_i,\nabla_x u_n(t_i,x_i),D^2_{x_0}\varphi(t_i,x_{i,0})\right)\le 0. \tag{3.27}$$

We now pass to the limit $i\to+\infty$ and, by (3.16) and the strong continuity of $\nabla_x u_n$, we obtain

$$-\varphi_t\left(\hat{t},\hat{x}_0\right)-\langle A_n(\hat{x}_0,\overline{x}_1),\nabla_x u_n(\hat{t},(\hat{x}_0,\overline{x}_1))\rangle-L_n\left(\hat{t},(\hat{x}_0,\overline{x}_1),\nabla_x u_n(\hat{t},(\hat{x}_0,\overline{x}_1)),D^2_{x_0}\varphi(\hat{t},\hat{x}_0)\right)\le 0,$$

which can be written, by using the definition of $\beta_{n,\overline{x}_1}$,

$$-\varphi_t(\hat{t},\hat{x}_0)-\frac{1}{2}\mathrm{Tr}\left(\left(D^2_{x_0}\varphi(\hat{t},\hat{x}_0)\right)\sigma_n(\hat{t},(\hat{x}_0,\overline{x}_1))\sigma_n^*(\hat{t},(\hat{x}_0,\overline{x}_1))\right)-\beta_{n,\overline{x}_1}(\hat{t},\hat{x}_0)\le 0.$$

Thus $v_{n,\overline{x}_1}$ is a viscosity subsolution of (3.23). ∎

3.3 Regularity with respect to the finite dimensional component

In this section we show that, if σ is non-degenerate, then the function u defined by (3.3) is differentiable with respect to x_0 and $D_{x_0}u$ enjoys some Hölder continuity.

Theorem 3.9. *Suppose that, for every $(t,x)\in[0,T]\times H$ and $y\in\mathbb{R}^m$, $\sigma(t,x)y\neq 0$. Then, for every $\overline{x}_1\in H_1$, the function $v_{\overline{x}_1}$ defined by $v_{\overline{x}_1}(t,x_0):=u(t,(x_0,\overline{x}_1))$ belongs to $C^{\alpha+1}_{loc}((0,T)\times\mathbb{R}^m)$, for every $\alpha\in(0,1)$.*

Proof. Let $(t,x_0)\in(0,T)\times\mathbb{R}^m$. Let $Q:=[c,d]\times B(x_0,\varepsilon)$ be a neighborhood of (t,x_0) in $(0,T)\times\mathbb{R}^m$ such that, for some $M>0$ and $\delta>0$, $\delta<a_{\overline{x}_1}(s,y):=\sigma(s,(y,\overline{x}_1))\sigma^*(s,(y,\overline{x}_1))<M$ for all $(s,y)\in Q$. Since $\Sigma_n(s,(y,\overline{x}_1))z=(\sigma_n(s,(y,\overline{x}_1))z,0_1)$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to σ uniformly on compact sets (Remark 2.9), we can suppose that $\delta<a_{n,\overline{x}_1}(s,y)<M$ for all $n\in\mathbb{N}$ and $(s,y)\in Q$ and that the family $\{a_{n,\overline{x}_1}\}_{n\in\mathbb{N}}$ is equi-uniformly continuous.

By Proposition 3.8, for $n\ge 1$, $v_{n,\overline{x}_1}$ is a viscosity solution of (3.23), in particular it is a viscosity solution of the terminal boundary value problem

$$\begin{cases} -v_t(s,y)-\dfrac{1}{2}\mathrm{Tr}(a_{n,\overline{x}_1}(s,y)D^2_y v(s,y))-\beta_{n,\overline{x}_1}(s,y)=0 & (s,y)\in Q \\ v(s,y)=u_n(s,(y,\overline{x}_1)) & (s,y)\in\partial_P Q \end{cases} \tag{3.28}$$

Thus, for instance by [4, Lemma 2.9, Proposition 2.10, and Theorem 9.1], $v_{n,\overline{x}_1}$ is the unique viscosity solution (in particular also a unique L^p-viscosity solution[5]) of (3.28), and

$$|v_{n,\overline{x}_1}|_{W^{1,2,p}(Q')}\le C\left(\sup_{(s,y)\in Q}|u_n(s,(y,\overline{x}_1))|+\sup_{(s,y)\in Q}|\beta_n(s,(y,\overline{x}_1))|\right) \tag{3.29}$$

[5] See [4] for the definition of L^p-viscosity solution.

for all $m+1 \le p < +\infty$ and for all $Q' = [c', d') \times B(x, \varepsilon')$, with $c < c' < d' < d$ and $0 < \varepsilon' < \varepsilon$, and where C depends only on m, p, δ, M, Q, Q', and the uniform modulus of continuity of the functions $a_{n,\overline{x}_1}$. Thus, by Proposition 3.2 and (3.24), the set $\{v_{n,\overline{x}_1}\}_{n\ge 1}$ is uniformly bounded in $W^{1,2,p}(Q')$. Therefore applying an embedding theorem, see e.g. [16, Lemma 3.3, p. 80], we obtain that for every $\alpha \in (0,1)$

$$|v_{n,\overline{x}_1}|_{C^{1+\alpha}(Q')} \le C_\alpha$$

for some constant C_α independent of n. Since the sequence $\{v_{n,\overline{x}_1}\}_{n\ge 1}$ converges uniformly on compact sets to the function $v_{\overline{x}_1}$ as $n \to +\infty$, it follows that the function $v_{\overline{x}_1}$ satisfies the above estimate too. This completes the proof. ■

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