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Efficient formulas for efficiency correction of cumulants

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We derive formulas which connect cumulants of particle numbers observed with efficiency losses with the original ones based on the binomial model. These formulas can describe the case with multiple efficiencies in a compact form. Compared with the presently suggested ones based on factorial moments, these formulas would drastically reduce the numerical cost for efficiency corrections when the order of the cumulant and the number of different efficiencies are large. The efficiency correction with realistic p_T-dependent efficiency would be carried out with the aid of these formulas.

I. INTRODUCTION

Fluctuations, especially those of conserved charges, are promising observables for the analysis of primordial thermodynamics in relativistic heavy-ion collisions [1–3]. In particular, non-Gaussian fluctuations characterized by higher order cumulants acquire much attention recently [4–6]. Active studies have been carried out experimentally [7–12] and by lattice QCD numerical simulations [13, 14]. See for reviews [15, 16] and latest progress [17, 18].

Experimentally, the fluctuations are measured by the event-by-event analysis. In this analysis, the number of particles arriving at some range of detectors are counted in each event. The event-by-event distribution characterized by the histogram of the particle number is called the event-by-event fluctuations. The cumulants of the particle number are constructed from the histogram.

In real experiments, however, particle numbers in each event cannot be measured accurately, because the detectors can measure particles with some probability called efficiency which is less than unity [19]. Due to the imperfect measurement, the event-by-event histogram and accordingly the cumulants constructed from the histogram are modified in a nontrivial way.

The effect of the efficiency on cumulants can be understood if it is assumed that the efficiency for individual particles are uncorrelated, i.e. the probability to observe different particles in an event is not correlated with one another. In this case, the probability distribution of experimentally-observed particle numbers can be related to the original one without efficiency loss using binomial distribution function [20]. In this paper, we call this relation the binomial model. The binomial model enables us to relate the cumulants of the original and observed particle number distributions. It has been recognized that the imperfect efficiency can significantly modify the values of the cumulants especially for higher order ones [20–22]. The efficiency correction in experimental analyses thus is an important procedure to obtain the relevant values of the cumulants. The binomial model is then extended to the case with multiple values of local efficiencies for different particle species and phase spaces [22, 23].

In Refs. [22, 23], the formulas for the efficiency correction with multiple efficiencies are obtained using factorial moments. These formulas consist of M^m factorial moments for m-th order cumulants with M different efficiencies. In practical analyses for efficiency correction, therefore, the numerical cost for the efficiency correction increases rapidly as M becomes larger. The number of efficiencies M thus is limited to small values in the present experimental analyses [24].

In the present study, we derive a set of formulas which relate the cumulants of original and observed distribution functions with multiple efficiencies. They are derived by a straightforward extension of the method used in Ref. [20]. In these formulas, the original cumulants before the efficiency loss are represented by the mixed cumulants of observed particles. The number of the cumulants in these relations does not depend on M. They thus would enable one to carry out the efficiency correction more effectively for large M and higher order cumulants. Even the realistic p_T-dependent efficiency [19] would be treated with these formulas.

This paper is organized as follows. In Sec. II, we first present our main results, i.e. the formulas to relate the original and observed cumulants. Their derivation are then discussed in later sections. Section III is devoted to reviews on the definition and properties of cumulants. We then present the derivation for single-variable distribution functions in Sec. IV, as a simple illustration of the full derivation addressed in Sec. V. Section VI is devoted to discussions and a short summary.

II. MAIN RESULT

In this section, we clarify the problem considered in this paper and show the answer, which is the main result of this paper.

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A. Problem

We consider a probability distribution function

$$P(N_1, N_2, \cdots, N_M) = P(\vec{N}), \tag{1}$$

for M integer stochastic variables, $N_1,\ N_2, \cdots, N_M$, with $\vec{N} = (N_1, N_2, \cdots, N_M)$ being the vectorical representation of the stochastic variables. In the following we call N_i the number of i-th particle in an event. The purpose of the present study is to obtain the cumulants of a linear combination of N_i,

$$Q = \sum_{i=1}^{M} a_i N_i, \tag{2}$$

with a_i being numerical numbers.

If $P(\vec{N})$ is obtained directly in an experiment as an event-by-event histogram, the cumulants of Q can, of course, be constructed straightforwardly from the histogram. In real experiments, however, the detectors miss the measurement of particles with some efficiency. The number of the i-th particles observed by the detector, n_i, is thus different from and smaller than N_i. The probability distribution function of observed particle numbers obtained by the imperfect experiment

$$\tilde{P}(n_1, n_2, \cdots, n_M) = \tilde{P}(\vec{n}), \tag{3}$$

differs from Eq. (1).

If the efficiencies for the observation of individual particles are assumed to be independent with one another, one can relate $\tilde{P}(\vec{n})$ and $P(\vec{N})$ in a simple form. In this case, the probability distribution of n_i for a fixed N_i obeys the binomial distribution function

$$B_{p,N}(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^n, \tag{4}$$

with $N = N_i$ and $p = \epsilon_i$ being the efficiency of the i-th particle. Therefore, $P(\vec{N})$ and $\tilde{P}(\vec{n})$ are related with each other as [20, 22, 23]

$$\tilde{P}(\vec{n}) = \sum_{N_1, \cdots, N_M} P(\vec{N}) \Big(\prod_i B_{\epsilon_i, N_i}(n_i) \Big). \tag{5}$$

In this paper we refer to Eq. (5) as the binomial model. The purpose of the present study is to represent the cumulants of Q using those of $\tilde{P}(\vec{n})$ in the binomial model in a compact form.

B. Answer

The cumulants of Q up to fourth order are given by

$$\langle Q \rangle_{\rm c} = \langle\!\langle q_{(1)} \rangle\!\rangle_{\rm c}, \tag{6}$$

$$\langle Q^2 \rangle_{\rm c} = \langle\!\langle q_{(1)}^2 \rangle\!\rangle_{\rm c} - \langle\!\langle q_{(2)} \rangle\!\rangle_{\rm c}, \tag{7}$$

$$\langle Q^3 \rangle_{\rm c} = \langle\!\langle q_{(1)}^3 \rangle\!\rangle_{\rm c} - 3\langle\!\langle q_{(2)} q_{(1)} \rangle\!\rangle_{\rm c} + \langle\!\langle 3q_{(2,1|2)} - q_{(3)} \rangle\!\rangle_{\rm c}, \tag{8}$$

$$\begin{aligned} \langle Q^4 \rangle_{\rm c} =& \langle\!\langle q_{(1)}^4 \rangle\!\rangle_{\rm c} - 6\langle\!\langle q_{(2)} q_{(1)}^2 \rangle\!\rangle_{\rm c} + 12\langle\!\langle q_{(2,1|2)} q_{(1)} \rangle\!\rangle_{\rm c} \\ &+ 6\langle\!\langle q_{(1,1|2)} q_{(2)} \rangle\!\rangle_{\rm c} - 4\langle\!\langle q_{(3)} q_{(1)} \rangle\!\rangle_{\rm c} - 3\langle\!\langle q_{(2)}^2 \rangle\!\rangle_{\rm c} \\ &+ \langle\!\langle -18q_{(2,1,1|2,2)} + 6q_{(2,1,1|3)} + 4q_{(3,1|2)} \\ &+ 3q_{(2,2|2)} - q_{(4)} \rangle\!\rangle_{\rm c}, \end{aligned} \tag{9}$$

where the cumulants $\langle \cdot \rangle_{\rm c}$ and $\langle\!\langle \cdot \rangle\!\rangle_{\rm c}$ are taken for $P(\vec{N})$ and $\tilde{P}(\vec{n})$, respectively. $q_{(\cdots)}$ are linear combinations of n_i defined by

$$q_{(s)} = \sum_{i=1}^{M} c_{(s)}^{(i)} n_i, \tag{10}$$

$$q_{(s_1,\cdots,s_j|t_1,\cdots,t_k)} = \sum_{i=1}^{M} c_{(s_1,\cdots,s_j|t_1,\cdots,t_k)}^{(i)} n_i. \tag{11}$$

The coefficients $c_{(\cdots)}^{(i)}$ are numerical numbers which depend on a_i and ϵ_i defined by

$$c_{(s)}^{(i)} = \tilde{a}_i^s \tilde{\xi}_s^{(i)}, \tag{12}$$

$$c_{(s_1,\cdots,s_j|t_1,\cdots,t_k)}^{(i)} = \tilde{a}_i^{s_1+\cdots+s_j} \tilde{\xi}_{s_1}^{(i)} \cdots \tilde{\xi}_{s_j}^{(i)} \tilde{\xi}_{t_1}^{(i)} \cdots \tilde{\xi}_{t_k}^{(i)}, \tag{13}$$

with $\tilde{a}_i = a_i/\epsilon_i$, $\tilde{\xi}_m^{(i)} = \xi_m^{(i)}/\epsilon_i$, and $\xi_m^{(i)} = \xi_m(\epsilon_i)$ being coefficients of the binomial cumulants with probability ϵ_i; explicit forms of ξ_m up to sixth order are given in Eqs. (25) – (30).

When one applies Eqs. (6) – (9) to the efficiency correction in event-by-event analyses, the required procedure is as follows:

1. Calculate $q_{(\cdots)}$ in each event from n_i observed experimentally in the event.
2. Using the event-by-event distribution of $q_{(\cdots)}$, obtain the (mixed) cumulants of $q_{(\cdots)}$ which appear in Eqs. (6) – (9).

Remark that the numerical cost for the latter procedure does not depend on M.

In Ref. [20], the relation of the cumulants for the "net-particle number" with common efficiencies for particles and anti-particles, respectively, are derived. This result is reproduced by substituting $M = 2$, $a_1 = 1$ and $a_2 = -1$ in Eqs. (6) – (9).

In the rest of this paper, we derive Eqs. (6) – (9).

III. CUMULANTS

In this section we first summarize properties of cumulants required for the derivation of Eqs. (6) – (9).

A. Definition

The cumulants are defined from the cumulant generating function. For the probability distribution function Eq. (1), the generating function is defined by

$$K(\theta_1,\cdots,\theta_M)=\ln\left[\sum_{N_1,\cdots,N_M}P(\vec{N})\exp(\sum_{i=1}^{M}N_i\theta_i)\right]. \tag{14}$$

The cumulants are then defined by the derivatives of Eq. (14) as

$$\langle N_m^s\rangle_{\rm c}=\frac{\partial^s}{\partial\theta_m^s}K(\vec{\theta})|_{\vec{\theta}=0}\equiv\partial_m^sK(\vec{0}), \tag{15}$$

and

$$\langle N_{m_1}^{s_1}\cdots N_{m_j}^{s_j}\rangle_{\rm c}=\partial_{m_1}^{s_1}\cdots\partial_{m_j}^{s_j}K(\vec{0}). \tag{16}$$

From these definitions, one immediately obtains the "distributive property" of the cumulants such as

$$\begin{aligned}&\langle(a_1N_1+a_2N_2)^2\rangle_{\rm c}\\&=a_1^2\langle N_1^2\rangle_{\rm c}+2a_1a_2\langle N_1N_2\rangle_{\rm c}+a_2^2\langle N_2^2\rangle_{\rm c}.\end{aligned} \tag{17}$$

Other properties of cumulants, such as their relation with moments, are found in Ref. [16].

B. Cumulant expansion

Consider a (non-homogeneous) linear function of $\vec{N}$,

$$L(\vec{N})=d_0+\sum_i d_{i=1}^{M}N_i, \tag{18}$$

with numerical numbers d_i. The cumulants of $L(\vec{N})$ are given by

$$\langle(L(\vec{N}))^m\rangle_{\rm c}=\frac{\partial^m}{\partial\bar{\theta}^m}K_L(\bar{\theta})|_{\bar{\theta}=0}, \tag{19}$$

with

$$K_L(\bar{\theta})=\ln\sum_{N_1,\cdots,N_M}P(\vec{N})e^{\bar{\theta}L(\vec{N})}=\ln\langle e^{\bar{\theta}L(\vec{N})}\rangle. \tag{20}$$

From Eq. (19), the generating function $K_L(\bar{\theta})$ is expanded as

$$K_L(\bar{\theta})=\sum_{m=1}^{\infty}\frac{\bar{\theta}^m}{m!}\langle(L(\vec{N}))^m\rangle_{\rm c}. \tag{21}$$

Note that the sum starts from $m=1$ because $K_L(0)=0$ which is ensured from the fundamental property of probability $\sum_{\vec{N}}P(\vec{N})=1$.

By substituting $\bar{\theta}=1$ in Eqs. (20) and (21) we obtain,

$$\ln\langle e^{L(\vec{N})}\rangle=\sum_{m=1}^{\infty}\frac{1}{m!}\langle(L(\vec{N}))^m\rangle_{\rm c}. \tag{22}$$

Equation (22) is referred to as the cumulant expansion [16], and plays a crucial role in the following derivations.

C. Binomial distribution function

The cumulant generating function of the binomial distribution function $B_{p,N}(n)$ is given by

$$k_N(\theta)=\ln\sum_n e^{\theta n}B_{p,N}(n). \tag{23}$$

By taking derivatives, the m-th order cumulant is given by $\langle n^m\rangle_{\rm c,binomial}=\xi_m(p)N$ with

$$\xi_m(p)=\frac{1}{N}\frac{\partial^m}{\partial\theta^m}k_N(0). \tag{24}$$

Explicit forms of $\xi_m(p)$ up to sixth order are given by

$$\xi_1(p)=p, \tag{25}$$
$$\xi_2(p)=p(1-p), \tag{26}$$
$$\xi_3(p)=p(1-p)(1-2p), \tag{27}$$
$$\xi_4(p)=p(1-p)(1-6p+6p^2), \tag{28}$$
$$\xi_5(p)=p(1-p)(1-2p)(1-12p+12p^2), \tag{29}$$
$$\xi_6(p)=p(1-p)(1-30p+150p^2-240p^3+120p^4). \tag{30}$$

Using $\xi_m(p)$, Eq. (23) is written as

$$k_N(\theta)=\sum_m\frac{\theta^m}{m!}\xi_m(p)N, \tag{31}$$

which shows that $k_N(\theta)$ is proportional to N. In this study, we fully make use of this property of $k_N(\theta)$.

IV. SINGLE VARIABLE CASE

Before the full derivation of Eqs. (6) – (9), in this section we first deal with a simplified problem with $M=1$, as this analysis would become a good exercise for the full derivation addressed in the next section.

In this section, we consider probability distribution functions $P(N)$ and $\tilde{P}(n)$ for single stochastic variables N and n, respectively, which are related with each other as

$$\tilde{P}(n)=\sum_N P(N)B_{p,N}(n). \tag{32}$$

The cumulant generating function of $\tilde{P}(n)$ is calculated to be

$$\begin{aligned}\tilde{K}(\theta)&=\ln\sum_n e^{\theta n}\tilde{P}(n)\\&=\ln\sum_n e^{\theta n}\sum_N P(N)B_{p,N}(n)\\&=\ln\sum_N P(N)\sum_n e^{\theta n}B_{p,N}(n)\\&=\ln\sum_N P(N)e^{k_N(\theta)}\\&=\ln\langle e^{k_N(\theta)}\rangle,\end{aligned} \tag{33}$$

where in the fourth equality we used Eq. (23). The expectation value in the last line is taken for $P(N)$.

Using the fact that $k_N(\theta)$ is linear in N, Eq. (33) is expressed by the cumulant expansion Eq. (22) as

$$\tilde{K}(\theta) = \sum_m \frac{1}{m!} \langle (k_N(\theta))^m \rangle_c, \tag{34}$$

while $\tilde{K}(\theta)$ also defines the cumulants of $\tilde{P}(n)$ by their derivatives with $\theta = 0$,

$$\langle\!\langle n^m \rangle\!\rangle_c = \partial^n \tilde{K}, \tag{35}$$

where $\partial = \partial/\partial\theta$ and we introduced the following notations:

1. The cumulants of single and double brackets are taken for $P(\vec{N})$ and $\tilde{P}(\vec{n})$, respectively.

2. When the argument of generating functions $\tilde{K}(\theta)$ and $k_N(\theta)$ is suppressed, it is understood that $\theta = 0$ is substituted.

These notations are used throughout this paper.

From Eqs. (34) and (35), the j-th order cumulant $\langle\!\langle n^j \rangle\!\rangle_c$ is represented by the cumulant expansion

$$\langle\!\langle n^j \rangle\!\rangle_c = \partial^j \sum_m \frac{1}{m!} \langle k_N^m \rangle_c. \tag{36}$$

We now represent the right-hand side of Eq. (36) by the cumulants of N. To proceed this analysis, there are two convenient rules.

Rule 1: θ derivatives on $\langle k_N^m \rangle_c$ act on k_N's as if $\langle \cdot \rangle_c$ were a standard bracket; for example,

$$\partial \langle k^m \rangle_c = m \langle k^{m-1} (\partial k) \rangle_c. \tag{37}$$

$$\partial^2 \langle k^m \rangle_c = m(m-1) \langle k^{m-2} (\partial k)^2 \rangle_c + m \langle k^{m-1} (\partial^2 k) \rangle_c. \tag{38}$$

This is because $k_N(\theta)$ is proportional to N and only the coefficient in front of N depends on θ.

Rule 2: By substituting $\theta = 0$, all $k_N(\theta)$'s which do not receive θ derivative vanish. Therefore, all $k(\theta)$ must receive at least one differentiation so that the term gives nonzero contribution to $\langle n^j \rangle_c$ in Eq. (36). This immediately means that the m-th order term in Eq. (36) can affect $\langle n^j \rangle_c$ only if $m \le j$.

To see the manipulation for the right-hand side of Eq. (36) with these rules, let us consider the $j = 3$ case. In this case, the terms with $m = 1,\ 2$, and 3 have nonvanishing contribution because of Rule 2. These terms are calculated to be

$$\partial^3 \langle k_N \rangle_c = \langle \partial^3 k_N \rangle_c = \xi_3 \langle N \rangle_c, \tag{39}$$

$$\frac{1}{2} \partial^3 \langle k_N^2 \rangle_c = 3 \langle (\partial^2 k_N)(\partial k_N) \rangle_c = 3\xi_2 \xi_1 \langle N^2 \rangle_c, \tag{40}$$

$$\frac{1}{3!} \langle \partial^3 k_N^3 \rangle_c = \langle (\partial k_N)^3 \rangle_c = \xi_1^3 \langle N^3 \rangle_c, \tag{41}$$

where all terms with k_N without derivatives are neglected from Rule 2. Substituting them in Eq. (36) we obtain

$$\begin{aligned} \langle\!\langle n^3 \rangle\!\rangle_c =& \partial^3 \tilde{K} \\ =& \partial^3 \langle k_N \rangle_c + \frac{1}{2} \partial^3 \langle k_N^2 \rangle_c + \frac{1}{3!} \partial^3 \langle k_N^3 \rangle_c \\ =& \xi_3 \langle N \rangle_c + 3\xi_2\xi_1 \langle N^2 \rangle_c + \xi_1^3 \langle N^3 \rangle_c. \end{aligned} \tag{42}$$

Similar manipulations up to 6-th order lead to

$$\langle\!\langle n \rangle\!\rangle_c = \xi_1 \langle N \rangle_c, \tag{43}$$

$$\langle\!\langle n^2 \rangle\!\rangle_c = \xi_2 \langle N \rangle_c + \xi_1^2 \langle N^2 \rangle_c, \tag{44}$$

$$\langle\!\langle n^3 \rangle\!\rangle_c = \xi_3 \langle N \rangle_c + 3\xi_2\xi_1 \langle N^2 \rangle_c + \xi_1^3 \langle N^3 \rangle_c, \tag{45}$$

$$\langle\!\langle n^4 \rangle\!\rangle_c = \xi_4 \langle N \rangle_c + (4\xi_3\xi_1 + 3\xi_2^2) \langle N^2 \rangle_c + 6\xi_2\xi_1^2 \langle N^3 \rangle_c + \xi_1^4 \langle N^4 \rangle_c, \tag{46}$$

$$\langle\!\langle n^5 \rangle\!\rangle_c = \xi_5 \langle N \rangle_c + (5\xi_4\xi_1 + 10\xi_3\xi_2) \langle N^2 \rangle_c + (10\xi_3\xi_1^2 + 15\xi_2^2\xi_1) \langle N^3 \rangle_c + 10\xi_2\xi_1^3 \langle N^4 \rangle_c + \xi_1^5 \langle N^5 \rangle_c, \tag{47}$$

$$\begin{aligned} \langle\!\langle n^6 \rangle\!\rangle_c =& \xi_6 \langle N \rangle_c + (6\xi_5\xi_1 + 15\xi_4\xi_2 + 10\xi_3^2) \langle N^2 \rangle_c + (15\xi_4\xi_1^2 + 60\xi_3\xi_2\xi_1 + 15\xi_2^3) \langle N^3 \rangle_c, \\ &+ (20\xi_3\xi_1^3 + 45\xi_2^2\xi_1^2) \langle N^4 \rangle_c + 15\xi_2\xi_1^4 \langle N^5 \rangle_c + \xi_1^6 \langle N^6 \rangle_c. \end{aligned} \tag{48}$$

These are the formulas which represent the cumulants of observed particles numbers, $\langle\!\langle n^m \rangle\!\rangle_c$, using the original ones $\langle N^m \rangle_c$.

For the efficiency correction, we have to represent $\langle N^m \rangle_c$ using $\langle\!\langle n^m \rangle\!\rangle_c$. These relations are most straightforwardly obtained by representing Eqs. (43) – (48) in a matrix form,

$$\vec{V}_n = \mathbb{M} \vec{V}_N, \tag{49}$$

with

$$\vec{V}_n = (\langle\!\langle n \rangle\!\rangle_c, \cdots, \langle\!\langle n^m \rangle\!\rangle_c)^T, \quad \vec{V}_N = (\langle N \rangle_c, \cdots, \langle N^m \rangle_c)^T,$$

and taking the inverse. We note that the matrix $\mathbb{M}$ in Eq. (49) is lower triangular. Accordingly, the inverse of $\mathbb{M}$ is also lower triangular. The m-th order cumulant $\langle N^m\rangle_c$ thus is represented by $\langle\!\langle n^l\rangle\!\rangle_c$ with $l \leq m$. The results correspond to a special case of Eqs. (6) – (9) with $M = 1$ and $a_1 = 1$. The explicit forms up to fourth order are found in Ref. [16].

V. MULTI-VARIABLE CASE

Next, we derive Eqs. (6) – (9). We start from the cumulant generating function of $\tilde{P}(\vec{n})$ in Eq. (3),

$$\begin{aligned}\tilde{K}(\vec{\theta}) =& \sum_{n_1,\cdots,n_M} \tilde{P}(\vec{n}) \exp(\sum_i \theta_i n_i)\\ =&\ln \sum_{N_1,\cdots,N_M} P(\vec{N}) \sum_{n_1,\cdots,n_M} \prod_i \big(e^{\theta_i n_i} B_{\epsilon_i,N_i}(n_i)\big)\\ =&\ln \sum_{N_1,\cdots,N_M} P(\vec{N}) \prod_i \big(\sum_{n_i} e^{\theta_i n_i} B_{\epsilon_i,N_i}(n_i)\big)\\ =&\ln \sum_{N_1,\cdots,N_M} P(\vec{N}) e^{\kappa(\vec{\theta})}\\ =&\ln\langle e^{\kappa(\vec{\theta})}\rangle_c,\end{aligned} \tag{50}$$

where

$$\kappa(\vec{\theta}) = \sum_{i=1}^{M} k_{N_i}(\theta_i), \tag{51}$$

with $k_{N_i}(\theta_i)$ being the cumulant generating function of $B_{\epsilon_i,N_i}(n_i)$ defined in Eq. (23). In the following, we use the notations 1 and 2 introduced in the previous section.

From the definition, it is clear that κ is a linear function of N_i. Derivatives of $\kappa(\vec{\theta})$ for $\theta_i = 0$ are given by

$$\partial_i^m \kappa = \xi_m^{(i)} N_i, \tag{52}$$

with $\xi_m^{(i)} = \xi_m(\epsilon_i)$, while derivatives of $\kappa(\vec{\theta})$ with different θ_i's vanish, i.e.

$$\partial_j^m \partial_k^l \kappa = 0, \tag{53}$$

for all $j \neq k$ and nonzero m and l, and so forth, which is trivial from Eq. (51). We also note that $\kappa(\vec{0}) = 0$.

Because of the linearity of $\kappa(\vec{\theta})$ on $\vec{N}$, $\tilde{K}(\vec{\theta})$ can be written in the cumulant expansion as

$$\tilde{K}(\vec{\theta}) = \sum_m \frac{1}{m!}\langle(\kappa(\vec{\theta}))^m\rangle_c. \tag{54}$$

Equations (6) – (9) are obtained by taking appropriate derivatives of Eq. (54) similarly to the previous section. In this manipulation, one can again apply the Rules 1 and 2 introduced in the previous section because of the linearity of $\kappa(\vec{\theta})$.

In order to obtain the cumulants of Q defined in Eq. (2), we apply a differential operator,

$$D = \sum_i \tilde{a}_i \partial_i, \tag{55}$$

to both sides in Eq. (54), with $\tilde{a}_i = a_i/\epsilon_i$. The left-hand side is then given by

$$D^m \tilde{K} = \langle\!\langle(\sum_i \tilde{a}_i n_i)^m\rangle\!\rangle_c = \langle\!\langle q_{(1)}^m\rangle\!\rangle_c, \tag{56}$$

where $q_{(1)}$ is defined in Eq. (10).

Next, we see derivatives of the right-hand side order by order. For the first derivative, only the first term in Eq. (54) has nonzero contribution and we obtain

$$D\tilde{K} = \langle D\kappa(\vec{\theta})\rangle_c = \langle Q\rangle_c. \tag{57}$$

From Eq. (56) with $m = 1$ and Eq. (57), we obtain Eq. (6).

The second derivative is calculated to be

$$D^2\tilde{K} = D^2\big(\langle\kappa\rangle_c + \frac{1}{2}\langle\kappa^2\rangle_c\big) = \langle D^2\kappa\rangle_c + \langle(D\kappa)^2\rangle_c, \tag{58}$$

where we used Rules 1 and 2. The second term in the far right-hand side is $\langle Q^2\rangle_c$ as a special case of the relation

$$\langle(D\kappa)^m\rangle_c = \langle Q^m\rangle_c. \tag{59}$$

The first term, on the other hand, needs a further manipulation. For this calculation, we note the relation

$$\begin{aligned}D^m\kappa =& \big(\sum_i \tilde{a}_i\partial_i\big)^m\kappa = \sum_i \tilde{a}_i^m \partial_i^m\kappa = \sum_i \tilde{a}_i^m \xi_m^{(i)} N_i\\ =& \sum_i \tilde{a}_i^m \tilde{\xi}_m^{(i)} \partial_i\kappa = D_{(m)}\kappa,\end{aligned} \tag{60}$$

where we have defined a differential operator

$$D_{(s)} = \sum_i \tilde{a}^s \tilde{\xi}_s^{(i)} \partial_i, \tag{61}$$

with $\tilde{\xi}_s^{(i)} = \xi_s^{(i)}/\xi_1^{(i)}$. Note that $D = D_{(1)}$. In the second equality in Eq. (60) we have used Eq. (53). We then take $D_{(2)}$ derivative of $\tilde{K}$ as

$$D_{(2)}\tilde{K} = \langle D_{(2)}\kappa\rangle_c. \tag{62}$$

Substituting Eqs. (60) and (62) in Eq. (58) and

$$D_{(m)}\tilde{K} = \langle\!\langle q_{(m)}\rangle\!\rangle_c, \tag{63}$$

we obtain Eq. (7).

To extend the calculation to third and higher orders, we need another differential operator

$$D_{(s_1,\cdots,s_j|t_1,\cdots,t_k)} = \sum_i c^{(i)}_{(s_1,\cdots,s_j|t_1,\cdots,t_k)} \partial_i, \tag{64}$$

which appears in differentiations

$$D_{(s_1)}D_{(s_2)}\kappa = D_{(s_1,s_2|2)}\kappa, \tag{65}$$

$$D_{(s_1)}D_{(s_2)}D_{(s_3)}\kappa = D_{(s_1,s_2,s_3|3)}\kappa, \tag{66}$$

and

$$D_{(\vec{s}_1|\vec{t}_1)}D_{(\vec{s}_2|\vec{t}_2)}\kappa = D_{(\vec{s}_1,\vec{s}_2|\vec{t}_1,\vec{t}_2,2)}\kappa, \tag{67}$$

$$D_{(\vec{s}_1|\vec{t}_1)}D_{(\vec{s}_2|\vec{t}_2)}D_{(\vec{s}_3|\vec{t}_3)}\kappa = D_{(\vec{s}_1,\vec{s}_2,\vec{s}_3|\vec{t}_1,\vec{t}_2,\vec{t}_3,3)}\kappa, \tag{68}$$

and etc., where the vectorical representations for subscripts are understood. On the other hand, the operations of these operators on $\tilde{K}$ give

$$D_{(\vec{s}|\vec{t})}\tilde{K} = \langle\!\langle q_{(\vec{s}|\vec{t})}\rangle\!\rangle_{\rm c}, \tag{69}$$

$$D_{(\vec{s}_1|\vec{t}_1)}D_{(\vec{s}_2|\vec{t}_2)}\tilde{K} = \langle\!\langle q_{(\vec{s}_1|\vec{t}_1)}q_{(\vec{s}_2|\vec{t}_2)}\rangle\!\rangle_{\rm c}, \tag{70}$$

and so forth.

For third order, all equations required to obtain $\langle Q^3\rangle_{\rm c}$ are

$$\begin{aligned} D^3\tilde{K} =& \langle\!\langle q_{(1)}^3\rangle\!\rangle_{\rm c} \\ =& \langle D_{(3)}\kappa\rangle_{\rm c} + 3\langle (D_{(2)}\kappa)(D_{(1)}\kappa)\rangle_{\rm c} \\ &+ \langle (D_{(1)}\kappa)^3\rangle_{\rm c}, \end{aligned} \tag{71}$$

$$\begin{aligned} D_{(2)}D_{(1)}\tilde{K} =& \langle\!\langle q_{(2)}q_{(1)}\rangle\!\rangle_{\rm c} \\ =& \langle D_{(2,1|2)}\kappa\rangle_{\rm c} + \langle (D_{(2)}\kappa)(D_{(1)}\kappa)\rangle_{\rm c}, \end{aligned} \tag{72}$$

$$D_{(2,1|2)}\tilde{K} = \langle\!\langle q_{(2,1|2)}\rangle\!\rangle_{\rm c} = \langle D_{(2,1|2)}\kappa\rangle_{\rm c} \tag{73}$$

$$D_{(3)}\tilde{K} = \langle\!\langle q_{(3)}\rangle\!\rangle_{\rm c} = \langle D_{(3)}\kappa\rangle_{\rm c}. \tag{74}$$

Using these results with Eq. (59), we obtain Eq. (8). Note that the calculation to obtain Eq. (8) from Eqs. (71) – (74) is straightforwardly carried out by representing these equations in a matrix form,

$$\langle\!\langle V_q\rangle\!\rangle_{\rm c} = \mathbb{M}_3\langle V_\kappa\rangle_{\rm c}, \tag{75}$$

with

$$V_q = (q_{(1)}^3, q_{(2)}q_{(1)}, q_{(2,1|2)}, q_{(3)})^T, \tag{76}$$

$$V_\kappa = ((D_{(1)}\kappa)^3, (D_{(2)}\kappa)(D_{(1)}\kappa), D_{(2,1|2)}\kappa, D_{(3)}\kappa)^T, \tag{77}$$

and taking the inverse of $\mathbb{M}_3$.

Similarly, the m-th order relation for $m \geq 4$ is obtained by the following procedures:

1. Calculate $D^m\tilde{K}$. The result contains $\langle (D_{(1)}\kappa)^m\rangle_{\rm c} = \langle Q^m\rangle_{\rm c}$.
2. The remaining terms in the above result consists of derivatives of κ. Calculate the derivatives of $\tilde{K}$ with the same differential operator. For example, when one obtains $\langle (D_{(2)}\kappa)(D_{(1)}\kappa)^2\rangle_{\rm c}$ in the first process, calculate $D_{(2)}D_{(1)}^2\tilde{K}$. The result contains the original term (in the example, $\langle (D_{(2)}\kappa)(D_{(1)}\kappa)^2\rangle_{\rm c}$).
3. Repeat this procedure until all derivatives of κ are represented by derivatives of $\tilde{K}$.
4. Unify them in a matrix form like Eq. (75), and take the inverse.

For fourth order, Eq. (9) is obtained by calculating the following 11 differentiations of $\tilde{K}$:

$$\begin{aligned} &D^4,\ D_{(3)}D_{(1)},\ D_{(2)}^2,\ D_{(3,1|2)},\ D_{(2,2|2)} \\ &D_{(2)}D_{(1)}^2,\ D_{(2,1|2)}D_{(1)},\ D_{(1,1|2)}D_{(2)}, \\ &D_{(2,1,1|2,2)},\ D_{(2,1,1|3)},\ D_{(4)}. \end{aligned} \tag{78}$$

VI. DISCUSSIONS

In this paper we have presented the formulas representing the cumulants of Q up to fourth order. These results can straightforwardly be extended to much higher orders and to mixed cumulants, though the calculation becomes more lengthy as the order becomes higher.

In this paper, we considered the binomial model Eq. (4). Only a property of the binomial distribution function $B_{p,N}(n)$ used in the derivations of Eqs. (6) – (9) is the fact that the cumulant generating function $k_N(\theta)$ is proportional to N. Therefore, $B_{p,N}(n)$ can be replaced by other distribution functions satisfying this condition by replacing the values of ξ_i.

In this paper we discussed the efficiency correction assuming the binomial model Eq. (4). As discussed already, this model can be justified when the efficiencies for the individual particles can be regarded independent. In real detectors, however, efficiencies of individual particles can be correlated. The effect of such correlations are discussed recently [25], and it is suggested that a small correlation can give rise to large discrepancy of the reconstructed values especially for higher orders. The efficiency corrected cumulants based on the binomial model have to be interpreted with this caveat.

In typical detectors in heavy ion collisions, neutrons cannot be observed. Because of this problem, the proton number cumulants are experimentally analyzed and compared with the theoretical studies on the baryon number cumulants. As suggested in Refs. [20, 26], the reconstruction of the baryon number cumulants is possible in the binomial model, since the measurement of protons among nucleons can be regarded as the 50% efficiency loss. In this case, the assumption of the independence of efficiencies is well justified for high-energy collisions because of the isospin randomization [20, 26]. It is an important subject to analyze the baryon number cumulants experimentally in this method, and compare their values directly with theoretical studies.

In real experiments, particle misidentifications and secondary particles also affect the event-by-event analysis [27]. These effects are another issue which has to be taken care of besides the problem of the efficiency correction.

In this paper, we derived the relations Eqs. (6) – (9) which relate the "original" and "observed" cumulants of particle numbers in event-by-event analyses. In these formula, the cumulants of original particle numbers are represented by the mixed cumulants of observed particles. The number of cumulants does not depend on the number of different efficiencies, M. These formulas thus would effectively be applied to practical analyses of efficiency correction of the cumulants with large M.

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