The Faddeev Model and Scaling in Quantum Chromodynamics

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The Faddeev two body bound state model is discussed as an example of a QCD inspired model thought by some to exhibit dimensional transmutation. This simple model is solved exactly and the growth of a specified dimensional energy scale is shown to be an illusion.

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I. INTRODUCTION

Consider a two spatial dimensional non-relativistic attractive potential scattering model[1, 2] described by the Faddeev Hamiltonian

$$\mathcal{H} = -\left(\frac{\hbar^2}{2\mu}\right)\left[\Delta + \epsilon\delta(\mathbf{r})\right] \tag{1}$$

wherein μ is the reduced mass of the two particles, Δ is the two dimensional Laplacean, ϵ is a dimensionless coupling strength and $\delta(\mathbf{r})$ is a pure s-wave short ranged potential. In principle, one solves the Schrödinger equation for scattering states

$$\mathcal{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}) = \left(\frac{\hbar^2 k^2}{2\mu}\right)\psi(\mathbf{r}) \tag{2}$$

which behaves as

$$\psi(\mathbf{r}) \to e^{ikx} + \left(e^{ikr}\sqrt{\frac{i}{r}}\right) f(E) + \cdots \quad \text{as} \quad r\to\infty. \tag{3}$$

The differential s-wave target length is thereby

$$\left(\frac{dL}{d\theta}\right) = |f(E)|^2, \tag{4}$$

with a total target length given by the two dimenisonal optical theorem

$$L = \sqrt{\frac{8\pi}{k}}\,\Im m f(E). \tag{5}$$

Introducing a complex energy in the upper half plane $\Im m\ z > 0$, the dimensionless analytic scattering amplitude $\tau(z)$ determines $f(E)$ via

$$f(E) = -\sqrt{\frac{1}{8\pi k}}\,\tau(E+i0^+),$$
$$L = -\left(\frac{1}{k}\right)\Im m\tau(E+i0^+). \tag{6}$$

The solution to the Faddeev model is thought to be described in terms of a bound state two particle energy of $-E_B$ by the so-called[3, 9] renormalized expression

$$\tau(z) = \left[\frac{4\pi}{\ln(-E_B/z)}\right] \text{(dimensional transmutation).} \tag{7}$$

What is remarkable about Eq.(7), is that the energy scale needed to make the logarithm argument dimensionless cannot be uniquely determined by the Hamiltonian in Eq.(1) since there is no combination of $\hbar$ and μ that has the physical dimensions of an energy. Dimensional transmutation is thought by some[5] to exist in a pure Yang-Mills quantum field theory of glue. But it is difficult to construct theories that are reasonable that do not obey the usual Abelian multiplicative group involved in changing physical units. To conclude that Eq.(7) has some validity one must introduce in some form a length scale which smears out the $\delta(\mathbf{r})$ in Eq.(1). Without such smearing, the rigorous solution to the Faddeev model is $\tau(z) = 0$, i.e. *the two particles neither bind nor scatter*.

In Sec.II, the general theory of scattering amplitudes are reviewed with an eye toward describing how such amplitudes vary with energy. The particular case of a separable interaction is discussed in Sec.III. For the Faddeev model, it is shown that there is no binding and no scattering in Sec.IV. The idea of renormalizing the model to get a bound state energy is also discussed. The actual value of the *dimensionally transmuted energy* is not very well defined., i.e. the renormalized solution is not rigorously valid. This point has also been discussed in [6] who suggest that any apparent anomalous breaking of scale invariance in the problem is in fact an explicit symmetry breaking due to the introduction of a regulator which breaks it. Our exact solution of the problem confirms this. In the concluding Sec.V the notion of dimensional transmutation is further discussed.

II. GENERAL SCATTERING THEORY

With z as the complex energy in the upper half complex plane $\Im m\ z > 0$, the scattering matrix for a Hamiltonian decomposition

$$\mathcal{H} = H + V \tag{8}$$

is given by

$$G(z) = \left[\frac{1}{z-H}\right],$$
$$T(z) = V + VG(z)T(z). \tag{9}$$

If we fix the scattering amplitude at a reference energy z_0, then the scattering potential

$$V = \left[\frac{1}{1+T(z_0)G(z_0)}\right] T(z_0) \tag{10}$$

can be eliminated from the scattering Eq.(9) so that

$$T(z) = T(z_0) + T(z_0)\left\{G(z) - G(z_0)\right\} T(z). \tag{11}$$

Eq.(11) shows in a general manner how it is possible to slide the scattering amplitude from one energy scale z_0 to another energy scale z.

III. A SEPARABLE INTERACTION

A separable two body interaction model producing a possible bound state may be taken to be

$$V = \mathcal{V}\left|v\right\rangle\left\langle v\right|. \tag{12}$$

The scattering amplitude thereby has the form

$$T(z) = \mathcal{T}(z)\left|v\right\rangle\left\langle v\right|. \tag{13}$$

For the separable interaction, Eq.(9) reads

$$g(z) = \left\langle v\right| G(z)\left|v\right\rangle,$$
$$\mathcal{T}(z) = \left[\frac{\mathcal{V}}{1-\mathcal{V}g(z)}\right]. \tag{14}$$

Employing

$$\mathcal{G}(z,z_0) = \left\langle v\right| G(z) - G(z_0)\left|v\right\rangle = g(z) - g(z_0), \tag{15}$$

the energy scaling of the scattering amplitude is given by Eq.(11) in the separable interaction form

$$\mathcal{T}(z) = \left[\frac{1}{1-\mathcal{T}(z_0)\mathcal{G}(z,z_0)}\right]\mathcal{T}(z_0). \tag{16}$$

Finally we may define the spectral weight of the separable potential via

$$Q(E) = \left\langle v\right|\delta(E-H)\left|v\right\rangle,$$
$$q(t) = \int_0^\infty Q(E)e^{-iEt/\hbar}dE = \left\langle v\right| e^{-iHt/\hbar}\left|v\right\rangle, \tag{17}$$

so that

$$g(z) = -\left(\frac{i}{\hbar}\right)\int_0^\infty e^{izt/\hbar}q(t)dt,$$
$$g(z) = \int_0^\infty \left[\frac{Q(E)dE}{z-E}\right], \tag{18}$$

and

$$\mathcal{G}(z,z_0) = \int_0^\infty Q(E)\left[\left(\frac{1}{z-E}\right) - \left(\frac{1}{z_0-E}\right)\right]dE,$$
$$\mathcal{G}(z,z_0) = \left(\frac{i}{\hbar}\right)\int_0^\infty \left[e^{iz_0t/\hbar} - e^{izt/\hbar}\right]q(t)dt, \tag{19}$$

determines the sliding energy scale.

IV. THE FADDEEV MODEL

In two spatial dimensions, the Faddeev model is described as the potential model

$$\mathcal{H} = -\left(\frac{\hbar^2}{2\mu}\right)\left[\Delta + \epsilon\delta(\mathbf{r})\right] = H + \mathcal{V}\delta(\mathbf{r}), \tag{20}$$

wherein ϵ is a dimensionless coupling strength. The short ranged potential is thereby

$$V(\mathbf{r}) = \mathcal{V}\delta(\mathbf{r}) = -\left(\frac{\hbar^2}{2\mu}\right)\epsilon\delta(\mathbf{r}),$$
$$V(\mathbf{r})\psi(\mathbf{r}) = \mathcal{V}\delta(\mathbf{r})\psi(\mathbf{0}) = \int \left\langle\mathbf{r}\right| V\left|\mathbf{r}'\right\rangle\psi(\mathbf{r}')d^2\mathbf{r}',$$
$$\left\langle\mathbf{r}\right| V\left|\mathbf{r}'\right\rangle = \mathcal{V}\delta(\mathbf{r})\delta(\mathbf{r}') = \mathcal{V}\left\langle\mathbf{r}|v\right\rangle\left\langle v|\mathbf{r}'\right\rangle,$$
$$\left\langle\mathbf{r}|v\right\rangle = \delta(\mathbf{r}) = \int \left\langle\mathbf{r}|\mathbf{k}\right\rangle\left\langle\mathbf{k}|v\right\rangle\left[\frac{d^2\mathbf{k}}{(2\pi)^2}\right],$$
$$\left\langle\mathbf{r}|\mathbf{k}\right\rangle = e^{i\mathbf{k}\cdot\mathbf{r}} \quad\Rightarrow\quad \left\langle\mathbf{k}|v\right\rangle = 1. \tag{21}$$

One may now compute the rigorously exact solution for this Faddeev problem.

A. Sliding Energy Scale

Employing Eqs.(17) and (20) yields the decay amplitude

$$q(t) = \int \exp\left(-\frac{i\hbar k^2 t}{2\mu}\right)\left|\left\langle\mathbf{k}|v\right\rangle\right|^2\left[\frac{d^2\mathbf{k}}{(2\pi)^2}\right]. \tag{22}$$

From Eqs.(21) and (22)

$$q(t) = \left(\frac{\mu}{2\pi i t\hbar}\right). \tag{23}$$

In virtue of Eqs.(19) and (23) one computes

$$\mathcal{G}(z,z_0) = \left(\frac{\mu}{2\pi\hbar^2}\right)\int_0^\infty \left[e^{iz_0t/\hbar} - e^{izt/\hbar}\right]\frac{dt}{t},$$
$$\mathcal{G}(z,z_0) = \left(\frac{\mu}{2\pi\hbar^2}\right)\ln\left[\frac{z}{z_0}\right]. \tag{24}$$

Let

$$\tau(z) = \left(\frac{2\mu}{\hbar^2}\right)\mathcal{T}(z). \tag{25}$$

Eqs.(16), (24) and (25) now read

$$\tau(z) = \left[\frac{\tau(z_0)}{1-\left[\tau(z_0)/4\pi\right]\ln(z/z_0)}\right]. \tag{26}$$

Eq.(26) is central to the solution of this Faddeev model.

$$\frac{1}{\tau(z)} = \frac{1}{\tau(z_0)} - \left[\frac{1}{4\pi}\right]\ln\left(\frac{z}{z_0}\right). \tag{27}$$

Having chosen the energy scale z_0, the scattering amplitude at any energy z may be found from its value at that chosen scale. There is no guarantee, however, that that value, at z_0 is nonzero. The characteristic logarithm appears in the denominator of Eq.(26).

B. Fixed Energy

To evaluate the scattering amplitude at an initial complex energy z_0, one must evaluate Eqs.(14), (21), (22) and (25) to arrive at

$$\tau(z_0) = -\left[\frac{\epsilon}{1+(\hbar^2/2\mu)\epsilon g(z_0)}\right] = -\left[\frac{\epsilon}{1+\epsilon I(z_0)}\right],$$
$$I(z_0) = -i\left(\frac{\hbar^2}{2\mu}\right)\int_0^\infty q(t)e^{iz_0 t/\hbar}dt. \quad (28)$$

From Eqs.(23) and (28), one has for the Faddeev model

$$I(z_0) = -\left[\frac{1}{4\pi}\right]\int_0^\infty e^{iz_0 t/\hbar}\left(\frac{dt}{t}\right)$$
$$|I(z_0)| = \infty \quad \text{for} \quad \Im m\ z_0 > 0 \quad (29)$$

due to the logarithmic divergence in the time integral as $t \to 0$. Thus, we have $\tau(z_0) = 0$ and then in virtue of Eq.(26) the following[7]:

Theorem: *The Faddeev singular potential does not scatter nor does it bind the particles, i.e.*

$$\tau(z) = 0 \quad \text{for} \quad \Im m\ z > 0. \quad (30)$$

Since the divergence in $I(z_0)$ is merely logarithmic, one might seek to *evade this rigorous theorem* by means of an intuitive *renormalization viewpoint.*

C. Theorem Evasion

In order to evade a rigorous theorem, it is required to relax standards of mathematics. A formal derivative of the $I(z_0)$ function in Eq.(29) yields

$$I'(z_0) = -\left[\frac{i}{4\pi\hbar}\right]\int_0^\infty e^{iz_0 t/\hbar}dt = \left[\frac{1}{4\pi z_0}\right]. \quad (31)$$

The general solution of Eq.(31) can be expressed in terms of a *large energy cut-off* Λ as

$$I(z_0,\Lambda) = \left(\frac{1}{4\pi}\right)\ln\left[\frac{-z_0}{\Lambda}\right]. \quad (32)$$

Note that $|I(z_0,\Lambda)| \to \infty$ as $\Lambda \to \infty$ in agreement with Eq.(29). Now the scattering amplitude with a high energy cut-off follows from Eqs,(27) and (28). It is

$$\frac{1}{\tau(z)} = -\left(\frac{1}{\epsilon}\right) - \left(\frac{1}{4\pi}\right)\ln\left[\frac{-z_0}{\Lambda}\right] - \left(\frac{1}{4\pi}\right)\ln\left[\frac{z}{z_0}\right]. \quad (33)$$

Now we may consider the so-called renormalization prescription:

(i) Choose a bound state energy $z_0 = -E_B$ so that the sum of the first two terms on the right hand side of Rq.(33) vanish; i.e.

$$-z_0 = \Lambda e^{-4\pi/\epsilon} \equiv E_B. \quad (34)$$

(ii) The scattering amplitude has a singularity at the bound state energy

$$\tau(z) = \left[\frac{4\pi}{\ln(-E_B/z)}\right]. \quad (35)$$

(iii) The bound state energy appearing in the denominator logarithm of Eq.(35) has appeared from nowhere since there is nothing with dimensions of energy in the Faddeev model. This paradox is known as *dimensional transmutation.* However the bound state energy can be whatever one wishes it to be. The renormalization limiting process,

$$E_B = \left[\lim_{\Lambda\to\infty \text{ and } \epsilon\to 0^+}\right]\left[\Lambda\exp\left(-\frac{4\pi}{\epsilon}\right)\right], \quad (36)$$

gives one no idea of the actual value of the bound state energy E_B.

The paradox is resolved when it is realized that for the rigorous solution of the Faddeev model in Sec.IV B *there is no bound state energy and there is no scattering.* A mathematical theorem is very difficult to evade.

Even in very recent literature this seems to have remained a point of confusion. Padmanabhan in his otherwise delightful book[8] of 2015, writing in his discussion of this problem in Chapter 10, describes the problem as "ill-defined", but then "taking a clue from what is done in quantum field theory" goes on to try to find the scattering cross section for an attractive delta function potential in 2 dimensions anyway. This is eventually expressed formally in terms of the "bound state energy". The determination of this putative bound state energy requires an additional process outside the mathematics of the problem where "one performs an experiment to measure some observable quantity (like the binding energy) of the system as well as some of the parameters describing the system (like the coupling constant)". Of course this project is not going to work if there is no bound state at all, so his formal solution is in fact unphysical, in accord with his initial intuition. The theory is as incapable of delivering a cross section as it is of delivering a bound state energy – and for the same reason: any quantity with dimensions of length or energy must be found by extending the model to some larger theory which can provide quantities with the needed dimensions. The one given simply does not suffice. Similar arguments are given throughout the literature[9–12], though rarely with Padmanabhan's clarity in showing how an unphysical result can arise.

V. CONCLUSION

The Faddeev two body bound state model has been discussed as an example of a QCD inspired model thought by some to exhibit *dimensional transmutation.* This simple model was solved exactly and the growth of a specified dimensional energy scale is shown to be something of an illusion.

The argument concerning quantities with physical dimensions again occurs in the pure Yang-Mills quantum field theory of pure glue. It does not seem possible to generate physical quantities with physical dimensions if such dimensions do not occur in the microscopic Hamiltonian. The Faddeev model is a particularly simple model that can be employed to discuss the general problem.

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[1] F.D. Faddeev, *Bull. Braz. Math. Soc.* **33**, 201 (2002).
[2] R. Jackiw, "*What Good are Quantum Field Theory Infinities*", Mathematical Physics 2000, ed. A. Fokas et al., Imperial College Press, London (2000).
[3] F.A. Berezin and L.D. Faddeev, *Soviet Math. Dokl.* **2** 372 (1961).
[4] L.R. Mead and J. Godines *Am. J. of Phys.* **59**, 935 (1991).
[5] S. Coleman, "*Secret Symmetries: An Introduction to Spontaneous Symmetry Breakdown and Gauge Fields*", Lecture given at 1973 Intern. Summer School in *Phys. Ettore Majorana*, Erice Sicily,(1973).
[6] A. Cabo, J. L. Lucio, and H. Mercado, *Am. J. Phys.* **66** (1998) 240.
[7] C.N. Friedman, *Journ. Funct. Anal.* **10**, 346 (1972).
[8] T. Padmanabhan, "*Sleeping Beauties in Theoretical Physics*", Springer Lectures Notes in Physics 895, Springer-Verlag, Heidelberg(2015).
[9] L. R. Mead and J. Godines, *Am. J. Phys.* **59**, 935 (1991).
[10] P. Gosdzinsky and R. Tarrach, *Am. J. Phys.* **59**, 70 (1991).
[11] B. R. Holstein, *Am. J. Phys.* **61**, 142 (1993).
[12] M. Hans, *Am. J. Phys.* **51**, 694 (1983).