Collinearity constraints for on-shell massless particle three-point functions, and implications for allowed-forbidden $n+1$-point functions

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A simple collinearity argument implies that the massless particle three-point function of helicities h_1, h_2, h_3 with corresponding real-valued four-momenta k_1, k_2, k_3 taken as all incoming or all outgoing (i.e., $k_1 + k_2 + k_3 = 0$), vanishes by helicity conservation unless $h_1 + h_2 + h_3 = 0$. When any one particle with four-momentum k is off mass shell, this constraint no longer applies; a forbidden amplitude with $h_1 + h_2 + h_3 \neq 0$ on-shell can be nonzero off-shell, but vanishes proportionally to k^2 as k approaches mass shell. When an on-shell forbidden amplitude is coupled to an allowed n-point amplitude to form an $n+1$ point function, this k^2 factor in the forbidden amplitude cancels the k^2 in the propagator, leading to a $n+1$-point function that has no pole at $k^2 = 0$. We relate our results for real-valued four-momenta to the corresponding selection rules that have been derived in the on-shell literature for complexified four-momenta.

A number of recent papers [1], [2], [3], [4] have studied the properties of on-mass-shell three- and four-point functions for massless particles by employing factorization and pole counting constraints on the four-point S-matrix, together with complex continuation of four-momenta. Our purpose in this note is to show that strong constraints governing on-mass-shell three-point functions for massless particles with real-valued four-momenta can be obtained by using a collinearity argument that appeared in the context of high energy neutrino reactions [5] and photon splitting in a constant external magnetic field [6].

Consider the amplitude $\mathcal{A}(k_1 h_1 k_2 h_2 | k_3 h_3)$ for a particle of four-momentum k_1 and helicity h_1 combining with a particle of four-momentum k_2 and helicity h_2 to give an outgoing particle of four-momentum k_3 and helicity h_3. We take all four-momenta to be real-valued rather than complexified as in [1], [2], [3], [4]. Since all particles are propagating forward in time, their energies are non-negative,

$$k_1^0 = |\vec{k}_1| \ , \quad k_2^0 = |\vec{k}_2| \ , \quad k_3^0 = |\vec{k}_3| \quad , \tag{1}$$

and the on-mass-shell condition states that

$$k_1^2 = k_2^2 = k_3^2 = 0 \quad . \tag{2}$$

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We assume that the amplitude $\mathcal{A}$ depends on no variables other than the ones explicitly shown. Squaring the four-momentum conservation condition $k_3 = k_1 + k_2$ and using Eqs. (1) and (2) gives

$$0 = k_3^2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 = 2k_1 \cdot k_2 = 2|\vec{k}_1||\vec{k}_2|\big(1 - \cos\theta\big) \quad , \tag{3}$$

with θ the angle between the three-vector momenta $\vec{k}_1$ and $\vec{k}_2$. Equation (3) implies that $\cos\theta = 1$, that is, the three-vectors $\vec{k}_1$ and $\vec{k}_2$, and hence also $\vec{k}_3$, are collinear. Rotational invariance around the common direction of propagation then implies the helicity conservation condition

$$h_1 + h_2 = h_3 \quad . \tag{4}$$

When the four-momentum k_3 is taken as incoming, so that $k_1 + k_2 + k_3 = 0$, its energy is non-positive and its helicity is reversed in sign, in agreement with the standard convention [7] used in the spinor helicity formalism. Equation (4) then becomes

$$h_1 + h_2 + h_3 = 0 \quad ; \tag{5}$$

clearly the same constraint also holds if all three particles are taken as outgoing. Thus amplitudes $\mathcal{A}(k_1h_1k_2h_2k_3h_3) \equiv \mathcal{A}(k_1h_1k_2h_2| - k_3 - h_3)$ with $h_1 + h_2 + h_3 \neq 0$ must vanish. This implies, for example, that the amplitude for a massless spin $\frac{3}{2}$ particle to absorb a real photon must vanish, since $\pm\frac{3}{2} \pm 1$ can never give $\pm\frac{3}{2}$.

The constraint that we have derived no longer holds when any one of the three particles is off-shell. If the off-shell four-momentum is denoted by k, an amplitude that is forbidden on-shell by the helicity constraint is no longer forbidden, but develops a kinematic zero proportional to k^2. Consider now an amplitude in which an allowed n-point function $B(..., kh)$, which can involve massless or massive particles with momenta and helicities denoted by ..., is linked by exchange of a massless particle with four-momentum k and helicity h to a helicity-forbidden three-point function $A(khk_3h_3k_4h_4)$ with $h + h_3 + h_4 \neq 0$. The corresponding $n+1$-point amplitude is proportional to

$$\sum_h B(..., kh)\frac{1}{k^2}A(khk_3h_3k_4h_4) \quad . \tag{6}$$

But since the helicity-forbidden amplitude on the right has a kinematic zero,

$$A(khk_3h_3k_4h_4) = k^2 R(khk_3h_3k_4h_4) \tag{7}$$

with R regular as k approaches mass shell, Eq. (6) reduces to the form

$$\sum_h B(..., kh)R(khk_3h_3k_4h_4) \quad . \tag{8}$$

This has no pole at $k^2 = 0$ in the variable k^2, and so is part of the background analytic in k^2 that is not determined by polology arguments. This scenario applies to the scattering of a charged spin $\frac{3}{2}$ particle in a non-constant electric or magnetic field, and so the amplitude [8] for this scattering need not vanish kinematically.

We conclude by comparing our results with the on-shell rules for massless particle three-point functions obtained by McGady and Rodina [1]. They give a range of cases for which three-point functions with $h_1+h_2+h_3 \neq 0$ are non-vanishing for complexified four-momenta. Our results show that in the limit of real four-momenta, all of these amplitudes must vanish. Conversely, McGady and Rodina show that all $h_1 + h_2 + h_3 = 0$ amplitudes with complexified momenta vanish except for the case $0+0+0 = 0$ corresponding to the three scalar meson coupling of ϕ^3 theory. Taking the real four-momentum limit of their result, this shows that all helicity-allowed amplitudes other than the ϕ^3 vertex vanish for on-shell massless particles. For the cases of a massless spin-1/2 particle scattering off a massless spin 0 or spin 1 particle to another massless spin-1/2 particle, this can be verified directly from the Feynman rules for the vertex. For a massless Dirac particle, the Lorentz spinor $u(k, h)$ is given by

$$u(k,h) = N(k) \begin{pmatrix} 1_2 \\ \vec{\sigma} \cdot \hat{k} \end{pmatrix} \chi(h) \tag{9}$$

with $N(k)$ a normalization constant, $\hat{k} = \vec{k}/|\vec{k}|$ a unit vector, $\vec{\sigma}$ the Pauli matrices, 1_2 a 2×2 unit matrix, and $\chi(h)$ a 2 component spinor carrying the helicity information. From this formula, together with the Dirac gamma matrix $\gamma^0 = \mathrm{diag}(1, -1)$, one immediately sees that $\overline{u}(k_1, h_1)u(k_2, h_2) = 0$ in the collinear case $\hat{k}_1 = \hat{k}_2$, which shows vanishing of the vertex in which a massless spin-1/2 particle absorbs a massless spin-0 particle. Similarly, one sees that $\overline{u}(k_1, h_1)\,\vec{\gamma} \cdot \vec{e}\,u(k_2, h_2) = 0$ when $\vec{k}_1 = \vec{k}_2$ and $\vec{k}_1 \cdot \vec{e} = 0$, which shows vanishing of the vertex in which a massless spin-1/2 particle absorbs a massless spin-1 particle with transverse polarization vector $\vec{e}$. In both cases, the contribution from the "small" components of the Dirac spinor exactly cancels the contribution from the "large" components of the Dirac spinor, something that does not happen when the spinor describes a massive spin-1/2 particle.

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