

Weak and strong coupling limits of the Boltzmann equation in the relaxation-time approximation

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We consider a momentum dependent relaxation time for the Boltzmann equation in the relaxation time approximation. We employ a power law parametrization for the momentum dependence of the relaxation time, and calculate the shear and bulk viscosity, as well as, the charge and heat conductivity. We show, that for the two popular parametrizations, referred to as the linear and quadratic ansatz, one can obtain transport coefficients which corresponds to the weak and strong coupling regimes, respectively. We also show that, for a system of massless particles with vanishing chemical potential, the off-equilibrium corrections to the phase-space distribution function calculated with the quadratic ansatz are identical with those of the Grad's 14-moment method.

I. INTRODUCTION

In ultra-relativistic heavy-ion collisions, the energy density in the initial state can exceed the critical value, predicted by Lattice QCD, for the existence of the hadronic matter [1]. At such conditions, quarks and gluons are deconfined and form a new state of matter called “quark-gluon plasma” (QGP). It is now well established, that the QGP is indeed formed in nucleus-nucleus collisions, already at energies accessible at the BNL Relativistic Heavy Ion Collider (RHIC) [2, 3] and the CERN Large Hadron Collider (LHC) [4–6]. It is also confirmed experimentally, that the QGP behaves as a nearly perfect fluid with a very small shear viscosity-to-entropy density ratio, η/s [7–13]. Consequently, relativistic dissipative hydrodynamics has been quite successful in describing the space-time evolution of the QGP and its transport properties [14].

The Boltzmann equation has been used to derive the dissipative hydrodynamic equations [15–41]. It is a transport equation which governs the space-time evolution of the single particle phase-space distribution function, and is capable to accurately describe the microscopic dynamics of a system in the dilute limit. Moreover, in the limit of small mean free path, the Boltzmann equation starts to describe hydrodynamics. Therefore, derivation of the equations of dissipative hydrodynamics and its associated transport coefficients from the Boltzmann equation, is of importance, to characterize the non-equilibrium dynamics of a system.

Despite its advantages, the Boltzmann equation is difficult to solve directly because its collision integral depends on the product of the distribution functions. Simpler approximations for the collision term have been proposed, of which the relaxation-time approximation by Anderson and Witting, is the most commonly used model [18]. The relaxation-time approximation for the collision term assumes, that the collisions between particles tend to restore the distribution function to its local equilibrium

value, exponentially. This is an excellent approximation when the system is close to local thermodynamic equilibrium.

In the Anderson-Witting model, the Boltzmann relaxation time is assumed to be independent of the particle momenta. However, in general, the relaxation time can be momentum dependent and might show different functional dependence for different theories [21]. In this paper, we consider a power law parametrization for the momentum dependence of the relaxation time of the Boltzmann equation. We derive expressions for transport coefficients, such as shear and bulk viscosity, as well as, charge and heat conductivity. We show, that for two popular parametrizations, referred to as the linear and quadratic ansatz, the first viscous correction to the distribution function leads to identical expressions as that obtained using the Chapman-Enskog, and Grad's 14-moment method, respectively. We also demonstrate that the ratios of transport coefficients in these two cases corresponds to the weak and strong coupling regimes.

II. RELATIVISTIC HYDRODYNAMICS

The conserved energy-momentum tensor and particle four-current can be expressed in terms of the single particle phase-space distribution function $f(x, p)$ [42], as

$$T^{\mu\nu} = \int dp p^\mu p^\nu (f + \bar{f}) = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (1)$$

$$N^\mu = \int dp p^\mu (f - \bar{f}) = nu^\mu + n^\mu. \quad (2)$$

Here $dp = g d\mathbf{p}/[(2\pi)^3 \sqrt{\mathbf{p}^2 + m^2}]$, g and m are the degeneracy factor and particle rest mass, p^μ is the particle four-momentum, and f and \bar{f} are the phase-space distribution functions for particles and anti-particles, respectively. Here we consider a system consisting of a single species of particles. In the tensor decompositions, ϵ , P

and n are the energy density, pressure and net number density, respectively, and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projection operator orthogonal to the hydrodynamic four-velocity u^μ defined in the Landau frame: $T^{\mu\nu} u_\nu = \epsilon u^\mu$. The bulk viscous pressure Π , the shear stress tensor $\pi^{\mu\nu}$, and the charge diffusion current n^μ , are dissipative quantities. We work with the Minkowskian metric tensor $g^{\mu\nu} \equiv \text{diag}(+, -, -, -)$.

The fundamental conservation equations of energy-momentum $\partial_\mu T^{\mu\nu} = 0$, and particle current $\partial_\mu N^\mu = 0$, yields the evolution equations for ϵ , u^μ and n , as

$$\dot{\epsilon} + (\epsilon + P + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \quad (3)$$

$$(\epsilon + P + \Pi)\dot{u}^\alpha - \nabla^\alpha(P + \Pi) + \Delta_\nu^\alpha \partial_\mu \pi^{\mu\nu} = 0, \quad (4)$$

$$\dot{n} + n\theta + \partial_\mu n^\mu = 0. \quad (5)$$

Here we use the standard notation: $\dot{A} \equiv u^\mu \partial_\mu A$ for the co-moving derivative, $\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$ for the space-like derivative, $\theta \equiv \partial_\mu u^\mu$ for the expansion scalar, and $\sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\theta \Delta^{\mu\nu}$ for the velocity stress tensor.

The equilibrium quantities such as the energy density, the thermodynamic pressure and the net number density, can be defined in terms of the equilibrium distribution function, as

$$\epsilon_0 \equiv u_\mu u_\nu T_0^{\mu\nu} = u_\mu u_\nu \int dp p^\mu p^\nu (f_0 + \bar{f}_0), \quad (6)$$

$$P_0 \equiv -\frac{1}{3}\Delta_{\mu\nu} T_0^{\mu\nu} = -\frac{1}{3}\Delta_{\mu\nu} \int dp p^\mu p^\nu (f_0 + \bar{f}_0), \quad (7)$$

$$n_0 \equiv u_\mu N_0^\mu = u_\mu \int dp p^\mu (f_0 - \bar{f}_0), \quad (8)$$

where the suffix "0" denotes the corresponding values in equilibrium.

In this work we consider a system of Boltzmann gas for which the equilibrium distribution function is given by

$$f_0 = \exp(-\beta u \cdot p + \alpha), \quad (9)$$

$$\bar{f}_0 = \exp(-\beta u \cdot p - \alpha). \quad (10)$$

Here $\beta \equiv 1/T$ is the inverse temperature, $\alpha \equiv \mu/T$ is the ratio of chemical potential to temperature, and $u \cdot p \equiv u_\mu p^\mu$. For such a system, the integrals in Eqs. (6)-(8) can be solved analytically, to obtain

$$\epsilon_0 = g \frac{T^4 z^2}{\pi^2} [3K_2(z) + zK_1(z)] \cosh(\alpha), \quad (11)$$

$$P_0 = g \frac{T^4 z^2}{\pi^2} K_2(z) \cosh(\alpha), \quad (12)$$

$$n_0 = g \frac{T^3 z^2}{\pi^2} K_2(z) \sinh(\alpha), \quad (13)$$

where $z \equiv m/T$ is the ratio of the particle mass to temperature and K_n are the modified Bessel functions of the second kind.

For a dissipative system, the thermodynamic temperature and the chemical potential is defined by the matching condition $\epsilon = \epsilon_0$ and $n = n_0$. The Navier-Stokes expressions for the dissipative quantities can be written in terms of the first-order gradients, as

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \quad (14)$$

$$\Pi = -\zeta\theta, \quad (15)$$

$$n^\mu = \kappa_n \nabla^\mu \alpha. \quad (16)$$

Here the transport coefficients η , ζ and κ_n , denote the shear and bulk viscosity, and the charge conductivity, respectively.

It is well known, that the first-order relativistic Navier-Stokes theory suffers from acasality and instabilities. These issues are solved by considering second-order corrections to the dissipative equations. On the other hand, the form of the first-order transport coefficients are sensitive to the nature of the microscopic interactions, and can be used to distinguish between a weakly and strongly coupled field theory.

For a system close to local thermodynamic equilibrium, the phase-space distribution function can be decomposed into equilibrium and non-equilibrium parts, $f = f_0 + \delta f$, where $|\delta f|/f_0 \ll 1$. Therefore, from Eqs. (1) and (2), the shear stress tensor $\pi^{\mu\nu}$, the bulk viscous pressure Π , and the particle diffusion current n^μ , can be expressed in terms of δf , as

$$\pi^{\mu\nu} = \Delta_{\alpha\beta}^{\mu\nu} \int dp p^\alpha p^\beta (\delta f + \delta \bar{f}), \quad (17)$$

$$\Pi = -\frac{\Delta_{\alpha\beta}}{3} \int dp p^\alpha p^\beta (\delta f + \delta \bar{f}), \quad (18)$$

$$n^\mu = \Delta_\alpha^\mu \int dp p^\alpha (\delta f - \delta \bar{f}), \quad (19)$$

where $\Delta_{\alpha\beta}^{\mu\nu} \equiv \frac{1}{2}(\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu) - \frac{1}{3}\Delta^{\mu\nu} \Delta_{\alpha\beta}$ is a traceless symmetric projection operator which is orthogonal to u_μ and $\Delta_{\mu\nu}$. In the following, we derive the Navier-Stokes expressions for the dissipative quantities, by iteratively solving the Boltzmann equation in the relaxation-time approximation to obtain δf , up to first order in gradients.

III. RELAXATION-TIME APPROXIMATION

Within kinetic theory, the evolution of the phase-space distribution function is governed by the Boltzmann equation. In the dilute limit, the Boltzmann equation provides a complete description of the microscopic dynamics of a system. In the present work, we consider a simplified version of the Boltzmann equation, where the collision term is written in the relaxation-time approximation [18],

$$p^\mu \partial_\mu f = -\frac{u \cdot p}{\tau_R} (f - f_0). \quad (20)$$

Here, τ_R is the relaxation time for the Boltzmann equation which, in general, can be a function of space-time, as well as, the particle momenta.

For different microscopic theories, τ_R , can exhibit a distinct functional dependence on particle momenta [21]. Therefore, to obtain the correct functional dependence, one should in general, consider the details of the microscopic dynamics. In the present work, however, we parametrize the momentum dependence of the relaxation time, with the following power law,

$$\tau_R(x, p) = \tau_0(x) \left(\frac{u \cdot p}{T} \right)^a, \quad (21)$$

and consider two limiting cases:

1. $a = 0$ (linear ansatz).
2. $a = 1$ (quadratic ansatz).

Most microscopic theories lie between these two extreme limits [21].

In the following, we demonstrate, that the transport coefficients obtained by using the linear ansatz, corresponds to weakly coupled microscopic theories, whereas those obtained with the quadratic ansatz, corresponds to strongly coupled theories.

In order to obtain the transport coefficients from Eqs. (17)-(19), one needs to calculate δf . To that end, we solve Eq. (20) iteratively, by employing a Chapman-Enskog like expansion [24, 25]. The first-order solution is obtained as

$$\delta f_1 = -\frac{\tau_R}{u \cdot p} p^\mu \partial_\mu f_0, \quad (22)$$

which translates to

$$\delta f_L = -\frac{\tau_0}{u \cdot p} p^\mu \partial_\mu f_0, \quad (23)$$

$$\delta f_Q = -\frac{\tau_0}{T} p^\mu \partial_\mu f_0, \quad (24)$$

for the linear and quadratic ansatz, respectively. In the next section, we employ the above results for the linear and quadratic ansatzes for δf , to obtain expressions for the relativistic Navier-Stokes equations by evaluating the integrals in Eqs. (17)-(19).

IV. DISSIPATIVE EQUATIONS

The first theoretical formulations of relativistic dissipative hydrodynamics were proposed by Eckart [43] and Landau-Lifshitz [44]. These formulations were the relativistic analogues of the Navier-Stokes theory and involved first-order gradients. In the following, we derive relativistic Navier-Stokes equations for the dissipative quantities. We consider three different scenarios:

1. Both bulk viscous pressure and dissipative charge current vanishes but shear stress tensor remains non-zero. Mathematically this amounts to setting $m = 0$ and $\mu = 0$.

2. The dissipative charge current vanishes but shear stress tensor and bulk viscous pressure remains non-zero. This is equivalent to $m \neq 0$ and $\mu = 0$.
3. The bulk viscous pressure vanishes but shear stress tensor and dissipative charge current are non-vanishing. This translates to $m = 0$ and $\mu \neq 0$.

In each of the above three cases, we obtain the relativistic Navier-Stokes equations by using both linear and quadratic ansatz for δf and compare the results.

A. Case 1: $m = 0, \mu = 0$

Since the bulk viscous pressure is proportional to m^2 , and current conservation equation vanishes for $\mu = 0$, thus the only non-vanishing dissipative quantity in this case is the shear stress tensor. In order to derive first-order expression for $\pi^{\mu\nu}$, we need to obtain the derivatives of α and β . Considering the $z \rightarrow 0$ and $\alpha \rightarrow 0$ limits of Eqs. (11)-(13) and substituting in Eqs. (3)-(5), one gets

$$\dot{\beta} = \frac{\beta}{3}\theta - \frac{\beta}{3(\epsilon + P)}\pi^{\mu\nu}\sigma_{\mu\nu}, \quad (25)$$

$$\nabla^\alpha \beta = -\beta \dot{u}^\alpha - \frac{\beta}{\epsilon + P} \Delta_\mu^\alpha \partial_\nu \pi^{\mu\nu}. \quad (26)$$

Using the above relations, Eqs. (23) and (24) can be written as

$$\delta f_L = \frac{\tau_0 \beta}{u \cdot p} f_0 p^\mu p^\nu \sigma_{\mu\nu}, \quad (27)$$

$$\delta f_Q = \frac{\tau_0 \beta}{T} f_0 p^\mu p^\nu \sigma_{\mu\nu}. \quad (28)$$

It is now apparent, that while the coefficient of f_0 in Eq. (27) is linear in momenta, in Eq. (28) it is quadratic; hence the nomenclature.

The first-order expression for $\pi^{\mu\nu}$, in the case of linear and quadratic ansatzes, can be obtained by substituting Eqs. (27) and (28) into Eq. (17). We get the relativistic Navier-Stokes equation,

$$\pi^{\mu\nu} = 2\tau_0 \beta_\pi \sigma^{\mu\nu}, \quad (29)$$

where

$$\beta_\pi = \begin{cases} \frac{1}{5}(\epsilon + P) & \text{(linear ansatz),} \\ \epsilon + P & \text{(quadratic ansatz).} \end{cases} \quad (30)$$

Comparing Eqs. (29) and Eq. (14), one gets, $\eta = \tau_0 \beta_\pi$.

The difference in the expressions of β_π in Eq. (30), obtained for the two ansatzes on the momentum dependent relaxation time, has some interesting consequences. Indeed, using Eqs. (29) and (30), one can rewrite Eqs. (27) and (28), as

$$\delta f_L = \frac{5f_0}{2(\epsilon + P)(u \cdot p)T} p^\mu p^\nu \pi_{\mu\nu}, \quad (31)$$

$$\delta f_Q = \frac{f_0}{2(\epsilon + P)T^2} p^\mu p^\nu \pi_{\mu\nu}. \quad (32)$$

Consequently, as could be expected, the δf in Eq. (31) is the same as that obtained using the iterative Chapman-Enskog method [32, 33]. On the other hand, the δf in Eq. (32) is identical to that of the Grad's 14-moment method¹. This is indeed a very interesting and rather unexpected result, indicating that with a suitable choice of the momentum dependence of the relaxation time, the iterative Chapman-Enskog method can reproduce the δf obtained using the moment method. A detailed analysis of this finding is left for a future work.

From the phenomenological perspective, it is interesting to note, that the experimental results for the Hanbury-Brown-Twiss (HBT) radii favor the linear, rather than quadratic, momentum dependence of the viscous correction to the distribution function [32]. Moreover, the transport results for the anisotropic flow and transverse momentum spectra also show agreement with the linear ansatz [45].

B. Case 2: $m \neq 0, \mu = 0$

In this case, the non-vanishing dissipative quantities are the shear stress tensor and the bulk viscous pressure. Considering the $\alpha \rightarrow 0$ limits of Eqs. (11)-(13), and substituting them into Eqs. (3)-(5), we get

$$\dot{\beta} = \frac{\beta(\epsilon + P)}{3\epsilon + (3 + z^2)P}\theta + \frac{\beta(\Pi\theta - \pi^{\mu\nu}\sigma_{\mu\nu})}{3\epsilon + (3 + z^2)P}, \quad (33)$$

$$\nabla^\alpha \beta = -\beta \dot{u}^\alpha - \frac{\beta}{\epsilon + P} (\Pi \dot{u}^\alpha - \nabla^\alpha \Pi + \Delta_\nu^\alpha \partial_\mu \pi^{\mu\nu}). \quad (34)$$

Using the above relations, Eqs. (23) and (24) become

$$\delta f_L = \frac{\beta\tau_0}{u \cdot p} f_0 \left[\frac{1}{3} \{p^2 - (1 - 3c_s^2)(u \cdot p)^2\} \theta + p^\mu p^\nu \sigma_{\mu\nu} \right], \quad (35)$$

$$\delta f_Q = \frac{\beta\tau_0}{T} f_0 \left[\frac{1}{3} \{p^2 - (1 - 3c_s^2)(u \cdot p)^2\} \theta + p^\mu p^\nu \sigma_{\mu\nu} \right], \quad (36)$$

where $c_s^2 \equiv dP/d\epsilon$ is the velocity of sound squared.

Substituting Eqs. (35) and (36) in Eqs. (17) and (18), one gets the following Navier-Stokes equations for the shear stress tensor and the bulk viscous pressure,

$$\pi^{\mu\nu} = 2\tau_0\beta_\pi \sigma^{\mu\nu}, \quad (37)$$

$$\Pi = -\tau_0\beta_\Pi \theta. \quad (38)$$

The first-order transport coefficient of the shear stress tensor for these two ansatzes, is then obtained in the following form

$$\beta_\pi = \begin{cases} \beta I_{42}^{(1)} & \text{(linear ansatz),} \\ \epsilon + P & \text{(quadratic ansatz),} \end{cases} \quad (39)$$

where

$$I_{42}^{(1)} = \frac{gT^5 z^5}{30\pi^2} \left[\frac{1}{16}(K_5 - 7K_3 + 22K_1) - K_{i,1} \right]. \quad (40)$$

Here the z -dependence of K_n is implicitly understood, and the function $K_{i,1}$ is defined by the integral

$$K_{i,1}(z) = \int_0^\infty \frac{d\theta}{\cosh \theta} \exp(-z \cosh \theta), \quad (41)$$

which can be evaluated as a Taylor series expansion up to any given order in z .

The first-order transport coefficient of the bulk viscous pressure, for the two ansatzes, is obtained as

$$\beta_\Pi = \begin{cases} \frac{5}{3}\beta_\pi - (\epsilon + P)c_s^2 & \text{(linear ansatz),} \\ 2(\epsilon + P) \left(\frac{1}{3} - c_s^2 \right) & \text{(quadratic ansatz).} \end{cases} \quad (42)$$

Note, that a comparison of Eqs. (37) and (38) with Eqs. (14) and (15) gives $\eta = \tau_0\beta_\pi$ and $\zeta = \tau_0\beta_\Pi$. Therefore, the ratio of the coefficient of the bulk viscosity to that of shear viscosity, $\zeta/\eta = \beta_\Pi/\beta_\pi$, is independent of τ_0 , and can be written, as

$$\frac{\zeta}{\eta} = \begin{cases} 75 \left(\frac{1}{3} - c_s^2 \right)^2 & \text{(linear ansatz),} \\ 2 \left(\frac{1}{3} - c_s^2 \right) & \text{(quadratic ansatz).} \end{cases} \quad (43)$$

The expression for the linear ansatz is obtained by considering a small- z expansion up to $\mathcal{O}(z^4)$. On the other hand, the result for the quadratic ansatz is exact.

The difference in the functional dependence of ζ/η on the sound velocity, obtained for the linear and quadratic ansatzes in Eq. (43), have an interesting interpretation [46, 47]. Indeed, the change of ζ/η with c_s^2 , for the linear ansatz, is the same as that found in a weakly coupled theory [48, 49]. On the other hand, the quadratic ansatz leads to a qualitative behaviour similar to that of the strongly coupled theories [50]. Moreover, for quadratic ansatz, the result in Eq. (43) is exactly the same as the lower bound for ζ/η found in Ref. [50]. This is a very interesting and quite intriguing result.

C. Case 3: $m = 0, \mu \neq 0$

In this case, the non-vanishing dissipative quantities are the shear stress tensor and the dissipative charge current. Considering the $z \rightarrow 0$ limits of Eqs. (11)-(13) and substituting in Eqs. (3)-(5), we get

$$\dot{\beta} = \frac{\beta}{3}\theta + \mathcal{O}(\delta^2), \quad \dot{\alpha} = \mathcal{O}(\delta^2), \quad (44)$$

$$\nabla^\mu \beta = -\beta \dot{u}^\mu + \frac{n}{\epsilon + P} \nabla^\mu \alpha - \frac{\beta}{\epsilon + P} \Delta_\mu^\nu \partial_\nu \pi^{\mu\nu}. \quad (45)$$

¹ For a detailed comparison see Ref. [32]

Using the above relations, Eqs. (23) and (24) becomes

$$\delta f_L = \frac{\beta\tau_0}{u \cdot p} f_0 \left[\left\{ \frac{n(u \cdot p)}{\beta(\epsilon + P)} - \frac{1}{\beta} \right\} p^\mu \nabla_\mu \alpha + p^\mu p^\nu \sigma_{\mu\nu} \right], \quad (46)$$

$$\delta f_Q = \frac{\beta\tau_0}{T} f_0 \left[\left\{ \frac{n(u \cdot p)}{\beta(\epsilon + P)} - \frac{1}{\beta} \right\} p^\mu \nabla_\mu \alpha + p^\mu p^\nu \sigma_{\mu\nu} \right], \quad (47)$$

up to first order in gradients.

Substituting Eqs. (46) and (47) in Eqs. (17) and (19), we obtain the Navier-Stokes equations for the shear stress tensor and dissipative charge current, as

$$\pi^{\mu\nu} = 2\tau_0\beta_\pi \sigma^{\mu\nu}, \quad (48)$$

$$n^\mu = \tau_0\beta_n \nabla^\mu \alpha. \quad (49)$$

The first-order transport coefficient of the shear stress tensor for the two ansatzes, is obtained as

$$\beta_\pi = \begin{cases} \frac{1}{5}(\epsilon + P) & \text{(linear ansatz),} \\ \epsilon + P & \text{(quadratic ansatz),} \end{cases} \quad (50)$$

This is the same result as obtained in the Case 1 where $m = \mu = 0$.

The first-order transport coefficient for the dissipative charge current is obtained as

$$\beta_n = \begin{cases} \frac{\beta P}{3} - \frac{n^2 T}{\epsilon + P} & \text{(linear ansatz),} \\ \beta P - \frac{4n^2 T}{\epsilon + P} & \text{(quadratic ansatz).} \end{cases} \quad (51)$$

In this case, a comparison of Eqs. (48) and (49) with Eqs. (14) and (16) gives, $\eta = \tau_0\beta_\pi$ and $\kappa_n = \tau_0\beta_n$. Therefore, the ratio of the charge conductivity and the shear viscosity coefficients, $\kappa_n/\eta = \beta_n/\beta_\pi$, is independent of τ_0 . Moreover, the ratio of the heat conductivity and the shear viscosity, $\kappa_q/\eta = (\beta_n/\beta_\pi)[(\epsilon + P)/nT]^2$. In the small α limit, this ratio is calculated as

$$\frac{\kappa_q}{\eta} = \begin{cases} \frac{20}{3} \frac{T}{\mu^2} & \text{(linear ansatz),} \\ 4 \frac{T}{\mu^2} & \text{(quadratic ansatz).} \end{cases} \quad (52)$$

The above equations are similar to the Wiedemann-Franz law [51, 52]. We also note that the ratio of the heat conductivity and shear viscosity exhibit an identical qualitative behaviour for the linear and quadratic ansatz. This property is in accordance with the previous results obtained for the strongly [52] and weakly coupled [53] theories in the limit of small chemical potential.

V. CONCLUSIONS AND OUTLOOK

We have considered the momentum dependent relaxation time $\tau_R(x, p)$ of the Boltzmann equation in

the relaxation time approximation. The power law parametrization, $\tau_R(x, p) \sim p^a$, have been applied with the linear ($a = 0$) and quadratic ($a = 1$) ansatz. The main focus was to calculate the influence of the momentum dependent τ_R on the properties of the transport coefficients. We have employed the iterative Chapman-Enskog method to obtain the first-order solution of the Boltzmann equation with the momentum dependent relaxation time. We then derived expressions for transport coefficients such as the shear and bulk viscosity as well as the charge and heat conductivity.

We have shown that the first viscous correction to the distribution function derived using the linear and quadratic ansatz leads to identical expressions as those obtained from the Chapman-Enskog and Grad's 14-moment method, respectively. We also demonstrated, that the ratios of transport coefficients in these two cases corresponds to the weak and strong coupling regimes. In particular, in the case of quadratic ansatz, we found that the ratio of the bulk and shear viscosity is exactly the same as the lower limit obtained for a system of a strongly coupled gauge theory plasma [50]. We also found, that in the limit of small chemical potential, the ratio of the heat conductivity to shear viscosity, for linear and quadratic ansatz, has a similar qualitative behavior, which is in agreement with the previous results obtained in the weak and strong coupling regimes.

Although we have considered a system of Boltzmann gas of single species to reduce cumbersome calculations, the present treatment can be rather easily generalized to a more complex system. We are also considering the extension of our results to the second-order dissipative hydrodynamic equations with the quadratic ansatz on the momentum dependent relaxation time to compare the transport coefficients with those obtained by using the moment method.

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