

Evaluation of the CHY Gauge Amplitude

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Abstract

The Cachazo-He-Yuan (CHY) formula for n -gluon scattering is known to give the same amplitude as the one obtained from Feynman diagrams, though the former contains neither vertices nor propagators explicitly. The equivalence was shown by indirect means, not by a direct evaluation of the $(n-3)$ -dimensional integral in the CHY formula. The purpose of this paper is to discuss how such a direct evaluation can be carried out. There are two basic difficulties in the calculation: how to handle the large number of terms in the reduced Pfaffian, and how to carry out the integrations in the presence of a σ -dependence much more complicated than the Parke-Taylor form found in a CHY double-color scalar amplitude. We have solved both of these problems, and have formulated a method that can be applied to any n . Many examples are provided to illustrate these calculations.

I. INTRODUCTION

Cachazo, He, and Yuan (CHY) proposed a formula for the n -gluon scattering amplitude in the tree approximation [1–4]. Unlike the textbook amplitude which is made up of a sum of Feynman diagrams, each built from a collection of vertices and propagators, the CHY amplitude is a single global formula given by an $(n-3)$ -dimensional complex integral. See equation (1) in Sec. II for the exact expression. It is not immediately obvious how to break it down into a sum of Feynman amplitudes, much less to obtain the local structure of vertices and propagators within each Feynman diagram. To simplify description, ‘Feynman amplitudes’ will be used throughout this paper to stand for amplitudes obtained in the text-book manner by summing Feynman diagrams with Feynman rules. Yet, it is known since the beginning [1–4] that it factorizes correctly in the collinear and the soft-gluon limits, just like Feynman tree amplitudes do, and its equivalence to the Feynman amplitude was proven [5] by showing that it satisfies the Britto-Cachazo-Feng-Witten (BCFW) on-shell factorization formula [6]. Like the Feynman scattering amplitude, it is gauge invariant in the sense that the same result is obtained by choosing any set of gauge-equivalent external polarization vectors. But unlike the Feynman amplitude, there is no need to fix a gauge for the propagators, simply because there are no propagators in the CHY formalism. Instead, a new kind of ‘gauge’ emerges from the Möbius invariance of the CHY amplitude. For concrete computations, it is necessary to choose three Möbius constants $\sigma_r, \sigma_s, \sigma_t$, as well as the Pfaffian lines λ and ν in the CHY formula to be discussed in the next section, in spite of the fact that the final scattering amplitude is independent of these choices. In this respect it is similar to the necessity of choosing a gauge for the propagators in evaluating a Feynman amplitude.

Although we know that the CHY amplitude must be the same as the Feynman amplitude, it is not easy to calculate the amplitude directly from the CHY formula. $(n-3)$ complex integrations must be performed, from which the momentum poles given by the Feynman propagators must emerge. It must also give the Feynman-amplitude numerators without the benefit of triple-gluon and four-gluon vertices which are absent in the CHY formalism. The purpose of this paper is to discuss how the calculation can be carried out for any n . It turns out that the CHY amplitude arranges its terms quite differently than the Feynman amplitude. Whereas the latter is grouped according to Feynman diagrams with a fixed set

of propagators in the denominators, the former is grouped according to a fixed pattern of its numerators. The numerators of a Feynman amplitude, assembled from the vertices, have a complicated form after a straight forward expansion. It depends on the topology of the Feynman diagram and seems to have no discernible pattern between diagrams. Its expansion into a function of dot products of external momenta and polarization vectors can only be obtained through tedious algebra. The denominators of a CHY amplitude corresponding to a fixed pattern of the numerator has to be calculated by integration from its specific σ -dependence, but a set of rules can be developed to determine the associated propagators.

Before going into details of how computations are to be carried out, it might be useful to have a rough idea of the role played by each part of the CHY integrand. Other than the integration measure and a normalization factor, the integrand consists of two main parts. The first contains the product of scattering functions f_i , present universally in all the CHY amplitudes. It is the source of all propagators in the amplitude. Every propagator of every Feynman diagram originates here, so it is like a stem cell before differentiation. The differentiation or selection control comes from the σ -dependent factors in the rest of the integrand. It picks out one or several dominant regions of integration, from which one or several Feynman diagrams emerge. The second part consisting of the reduced Pfaffian $\text{Pf}'(\Psi)$ and a Parke-Taylor factor is where these controlling σ factors lie. The reduced Pfaffian consists of many terms having different polarizations and momenta, each with a different σ dependence, and it is possible to group together terms in the reduced Pfaffian with the same σ dependence. As a result, the CHY amplitudes can be arranged according to numerator factors coming from terms of the reduced Pfaffians with the same σ dependence, rather than identical denominator factors like in a Feynman amplitude. Integration over its σ dependence then produces products of propagators corresponding to one or several Feynman diagrams.

The groupings of the reduced Pfaffian with the same σ factors will be discussed in Sec. III. There is a very useful ‘shift invariance’ for the reduced Pfaffian which can be used to check calculations. That will be discussed in Sec. IV.

As in the Feynman amplitude evaluation, simplification can be obtained by choosing a suitable representation for the transverse polarization vectors of the gluons. Polarization vectors used in the spinor-helicity technique offers great simplifications, resulting for example in the celebrated Parke-Taylor formula [7] for n -gluon amplitudes when all but two legs carry

the same helicity. The same choice also simplifies CHY amplitude calculations. For easy reference, we will refer to this choice of polarization vectors as the ‘helicity gauge’. Its general properties are reviewed and how it is useful in calculating the CHY amplitude is explained in Sec. V.

Sec. VI discusses how the complex integrations in (1) can be carried out using a technique developed in a previous paper [8] for the CHY double-color scalar amplitude. Dominant regions in the $(n-3)$ -dimensional complex space are picked out to present a pole in each of the $(n-3)$ successive integrations, to enable residue calculus to be used for evaluating the integral. This technique is applied to the evaluation of the $n = 3$, $n = 4$, and $n = 5$ amplitudes respectively in Secs. VII, VIII, and IX, and to the evaluation of amplitudes of a general n in Sec. X. Double poles are discussed in Sec. XII.

The expansion of the reduced Pfaffian depends on the choice of λ and ν , and the selection of the dominant integration regions depends on r, s, t , though not on the values $\sigma_r, \sigma_s, \sigma_t$. This freedom is illustrated in the explicit examples in Secs. VII and VIII. Calculation details differ from one choice to another, so with experience one can exploit this freedom to make a choice best suited for the problem. This independence can also be used to check the result of the calculations, as is done in Secs. VII and VIII.

In the CHY approach, the triple-gluon vertex of the Feynman amplitude can be reproduced in a $n = 3$ calculation, as is done in Sec. VII. To get the four-gluon vertex, one must carry out an $n = 4$ calculation, without choosing the helicity gauge or any other gauge, then subtract out the contributions coming from the triple-gluon vertices. This is carried out in Sec. XI. The calculation is somewhat lengthy, but it does explicitly demonstrate the equivalence between the $n = 4$ CHY amplitude and the Feynman amplitude in any gauge.

Finally a summary is provided in Sec. XIII.

II. CHY AMPLITUDE

A color-stripped n -gluon scattering amplitude is given by the CHY formula [2] to be

$$M^\alpha = \left(-\frac{1}{2\pi i}\right)^{n-3} \oint_\Gamma \sigma_{(rst)}^2 \left(\prod_{i=1, i \neq r, s, t}^n \frac{d\sigma_i}{f_i}\right) \frac{\text{Pf}'\Psi}{\sigma_{(\alpha)}}, \quad (1)$$

where $\alpha = (\alpha_1\alpha_2\cdots\alpha_n)$ describes the color, expressed as a permutation of S_n , in cycle notation. The scattering functions and the σ -factors are

$$\begin{aligned} f_i &= \sum_{j=1, j \neq i}^n \frac{2k_i \cdot k_j}{\sigma_{ij}}, \quad (1 \leq i \leq n) \\ \sigma_{ij} &= \sigma_i - \sigma_j, \quad \sigma_{(rst)} = \sigma_r \sigma_s \sigma_t, \\ \sigma_{(\alpha)} &= \sigma_{(\alpha_1\alpha_2\cdots\alpha_n)} = \prod_{i=1}^n \sigma_{\alpha_i\alpha_{i+1}} = \sigma_{[\alpha_1\alpha_2\cdots\alpha_n]} \sigma_{\alpha_n\alpha_1}, \quad n+1 \equiv 1. \end{aligned} \quad (2)$$

The three lines r, s, t for the Möbius constants $\sigma_r, \sigma_s, \sigma_t$ will be referred to as constant lines, the rest variable lines. The reduced Pfaffian $\text{Pf}'\Psi$ is invariant under any permutation of the external particles. It is related to the Pfaffian of a matrix $\Psi_{\lambda\nu}^{\lambda\nu}$ by

$$\text{Pf}'\Psi = \frac{(-1)^{\lambda+\nu+\frac{1}{2}n(n+1)} 2^{n-3}}{\sigma_{\nu\lambda}} \text{Pf}(\Psi_{\lambda\nu}^{\lambda\nu}), \quad (\lambda < \nu), \quad (3)$$

where $\Psi_{\lambda\nu}^{\lambda\nu}$ is obtained from the matrix Ψ with its λ th and ν th columns and rows removed. We need this normalization of $\text{Pf}'\Psi$, rather than the one used in Ref. [2], to reproduce the Parke-Taylor formula [7]. Different normalizations may be more convenient for other purposes. The antisymmetric matrix Ψ is made up of three $n \times n$ matrices A, B, C ,

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}. \quad (4)$$

The non-diagonal elements of these three sub-matrices are

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_{ij}}, \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}}, \quad C_{ij} = \frac{\epsilon_i \cdot k_j}{\sigma_{ij}}, \quad -C_{ij}^T = \frac{k_i \cdot \epsilon_j}{\sigma_{ij}}, \quad (1 \leq i \neq j \leq n), \quad (5)$$

where ϵ_i is the polarization of the i th gluon, satisfying $\epsilon_i \cdot k_i = 0$. The diagonal elements of A and B are zero, and that of C is defined by

$$C_{ii} = - \sum_{j=1}^n C_{ij}, \quad (6)$$

so that the column and row sums of C is zero. A similar property is true for A if the scattering equations $f_i = 0$ are obeyed. This is the case because the integration contour Γ encloses these zeros anticlockwise.

For massless particles satisfying momentum conservation, the amplitude M^α is independent on the choice of λ and ν . It is also gauge invariant, in the sense that when any ϵ_i is replaced by k_i , then the amplitude is zero.

If i is a number between 1 and $2n$, its *complement* i' will be defined to be $i - n$ if $i > n$, and $i + n$ if $i \leq n$. One of i, i' is between 1 and n , and that number will be denoted by \bar{i} . With this notation, the matrix elements of Ψ for $\bar{i} \neq \bar{j}$ can be summarized in one line as

$$\Psi_{ij} = \frac{v_i \cdot v_j}{\sigma_{ij}}, \quad v_{\bar{i}} = k_{\bar{i}}, \quad v_{\bar{i}+n} = \epsilon_{\bar{i}}, \quad \text{if } \bar{i} \neq \bar{j}. \quad (7)$$

For $\bar{i} = \bar{j}$, we have

$$\Psi_{\bar{i}\bar{i}} = \Psi_{\bar{i}+n, \bar{i}+n} = 0, \quad \Psi_{\bar{i}+n, \bar{i}} = C_{\bar{i}\bar{i}} = -\Psi_{\bar{i}, \bar{i}+n}. \quad (8)$$

The momenta k_λ and k_ν do not seem to appear in $\text{Pf}'\Psi$ because the rows and columns containing them are absent in $\Psi_{\lambda\nu}^{\lambda\nu}$. Yet, the numerator of a Feynman amplitude generally depends on all momenta, so how can these two momenta be absent? The answer is, they are there but hidden inside C_{ii} . Recall that $C_{ii} = -\sum_{j \neq i} \epsilon_i \cdot k_j / \sigma_{ij}$, and the sum includes $j = \lambda$ and ν . Whereas every other matrix element in Ψ consists of only one term, with simple and regular dependence on ϵ, k , and σ , the element C_{ii} is given by a sum so it has a complicated dependence on these variables. This fact complicate the evaluation of the amplitude as we shall see later.

III. PFAFFIAN DECOMPOSITION

Let Ψ be a $2n \times 2n$ antisymmetric matrix, and $p = [p_1 p_2 \cdots p_{2n}] \in S_{2n}$ be a permutation of the $2n$ numbers $[1 2 \cdots 2n]$, with signature $(-)^p$. Let

$$\Psi_p = (-)^p \prod_{\ell=1}^n \Psi_{p_{2\ell-1} p_{2\ell}}. \quad (9)$$

Two permutations which differ by interchanging their $(2\ell - 1)$ th and (2ℓ) th elements give the same Ψ_p , because Ψ is antisymmetric, and two permutations which permute the factors of (9) also yield the same Ψ_p . In what follows we would use \sum' to indicate the sum over those independent permutations in S_{2n} which give rise to distinct Ψ_p . Then a Pfaffian of Ψ is defined by

$$\text{Pf}(\Psi) = \sum'_p \Psi_p = \frac{1}{2^n n!} \sum_{p \in S_{2n}} \Psi_p. \quad (10)$$

The number of terms in $\text{Pf}(\Psi)$ is $(2n)!/2^n n! = (2n - 1)!!$.

The denominators of the non-diagonal matrix elements of A, B, C in (5) are the same. That property allows a simplification of $\text{Pf}(\Psi)$ in (4) from a sum over $p \in S_{2n}$ to a sum over $\bar{p} \in S_n$. To do so, take the term Ψ_p in (9) and arrange the order of its factors $\Psi_{p_{2\ell-1}p_{2\ell}}$ so that every Ψ_{pq} is followed by a $\Psi_{q'r}$, where q' is the complement of q defined at the end of the last section. That may require the use of antisymmetry to convert $\Psi_{rq'}$ to $-\Psi_{q'r}$, but even so, this arrangement cannot continue if q' has already been used up, because every number between 1 and $2n$ can occur only once. In that case start over again with any other number not already used. In that way, up to a sign, Ψ_p can finally be written in the form

$$\Psi_p = \pm (\Psi_{i'_1 i_2} \Psi_{i'_2 i_3} \cdots \Psi_{i'_x i_1}) (\Psi_{j'_1 j_2} \Psi_{j'_2 j_3} \cdots \Psi_{j'_y j_1}) \cdots (\Psi_{k'_1 k_2} \Psi_{k'_2 k_3} \cdots \Psi_{k'_z k_1}). \quad (11)$$

If we interchange i_a and i'_a and sum up the two terms, for $2 \leq a \leq x$, then the first factor in (11) can be written as

$$\begin{aligned} k_{\bar{i}_1} \cdot U_{\bar{i}_2} \cdots U_{\bar{i}_x} \cdot \epsilon_{\bar{i}_1} & , \quad \text{if } i'_1 = \bar{i}_1, \\ -\epsilon_{\bar{i}_1} \cdot U_{\bar{i}_2} \cdots U_{\bar{i}_x} \cdot k_{\bar{i}_1} & , \quad \text{if } i'_1 = (\bar{i}_1)', \end{aligned} \quad (12)$$

where U is a dyadic with respect to Lorentz indices, defined by

$$U_{\bar{i}} = k_{\bar{i}} \epsilon_{\bar{i}} - \epsilon_{\bar{i}} k_{\bar{i}}. \quad (13)$$

Summing also over $i'_1 = \bar{i}_1$ and \bar{i}'_1 , the first factor of (11) becomes $\text{Tr}(U_{\bar{i}_1} U_{\bar{i}_2} \cdots U_{\bar{i}_x})$. Doing the same thing for the other factors of (11) would result in an expression for $\text{Pf}(\Psi)$ in which the sum over $p \in S_{2n}$ is reduced to a sum over $\bar{p} \in S_n$:

$$\text{Pf}\Psi = (-1)^{\frac{1}{2}n(n+1)} \sum_{\bar{p} \in S_n} (-)^{\bar{p}} \Psi_{\bar{p}} = (-1)^{\frac{1}{2}n(n+1)} \sum_{\bar{p} \in S_n} (-)^{\bar{p}} \Psi_I \Psi_J \cdots \Psi_K, \quad (14)$$

where

$$I = (\bar{i}_1 \bar{i}_2 \bar{i}_3 \cdots \bar{i}_x), \quad J = (\bar{j}_1 \bar{j}_2 \bar{j}_3 \cdots \bar{j}_y), \quad \cdots \quad K = (\bar{k}_1 \bar{k}_2 \bar{k}_3 \cdots \bar{k}_z) \quad (15)$$

are the cycles of the permutation $\bar{p} \in S_n$. The cycle factors are given by

$$\Psi_I = \frac{U_I}{\sigma_I} = \frac{\frac{1}{2} \text{Tr}(U_{\bar{i}_1} U_{\bar{i}_2} \cdots U_{\bar{i}_x})}{\sigma_{\bar{i}_1 \bar{i}_2} \sigma_{\bar{i}_2 \bar{i}_3} \cdots \sigma_{\bar{i}_x \bar{i}_1}} \quad (16)$$

when the cycle length $x > 1$, and for a 1-cycle, the cycle factor is

$$\Psi_{(\bar{i})} = C_{\bar{i}} = - \sum_{\bar{j} \neq \bar{i}} \frac{C_{\bar{i}\bar{j}}}{\sigma_{\bar{i}\bar{j}}}. \quad (17)$$

Similar expressions apply to all the other cycles including J and K .

The factor $\frac{1}{2}$ in (16) comes about because of double counting. For $x > 1$, the cycles $(\bar{i}_1\bar{i}_2\cdots\bar{i}_x)$ and $(\bar{i}_x\cdots\bar{i}_2\bar{i}_1)$ have identical cycle factors, so if we sum up all permutations $\bar{p} \in S_n$ in (14), a factor $\frac{1}{2}$ is called for.

To illustrate the formulas and notations, suppose $n = 4$. Consider the permutation $p = [15247368]$ of $[12345678]$ whose signature is $(-)^p = +1$. Its matrix element is

$$\begin{aligned}\Psi_p &= \Psi_{15}\Psi_{24}\Psi_{73}\Psi_{68} = \Psi_{51}\Psi_{24}\Psi_{86}\Psi_{73} \\ &= C_{11} \frac{(k_2 \cdot k_4)(\epsilon_4 \cdot \epsilon_2)}{\sigma_{(24)}} C_{33}, \quad \sigma_{(24)} = \sigma_{24}\sigma_{42}.\end{aligned}\tag{18}$$

The associated $\bar{p} \in S_4$ is $\bar{p} = (1)(24)(3)$, with $I = (1)$, $J = (24)$, $K = (3)$ and signature $(-)^{\bar{p}} = -1$. The associated cycle factor

$$\begin{aligned}(-)^{\bar{p}}\Psi_{\bar{p}} &= -\Psi_I\Psi_J\Psi_K = -C_{11} \frac{\frac{1}{2}\text{Tr}[U_2U_4]}{\sigma_{(24)}} C_{33} = -C_{11} \frac{\frac{1}{2}\text{Tr}[(k_2\epsilon_2 - \epsilon_2k_2)(k_4\epsilon_4 - \epsilon_4k_4)]}{\sigma_{(24)}} C_{33} \\ &= -C_{11} \frac{[(\epsilon_2 \cdot k_4)(\epsilon_4 \cdot k_2) - (\epsilon_2 \cdot \epsilon_4)(k_2 \cdot k_4)]}{\sigma_{(24)}} C_{33}\end{aligned}\tag{19}$$

does include (18) as it should.

The Pfaffian $\text{Pf}(\Psi)$ for $n = 4$ has $7!! = 105$ terms, which can be grouped into a sum over the cycles of S_4 . These cycles are

$$\begin{aligned}\mathcal{C} = \{ & (1)(234), (1)(243), (2)(134), (2)(143), (3)(124), (3)(142), (4)(123), (4)(132), \\ & (1)(2)(34), (1)(3)(24), (1)(4)(23), (2)(3)(14), (2)(4)(13), (3)(4)(12), (1)(2)(3)(4), \\ & (12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432) \}.\end{aligned}\tag{20}$$

Back to the general discussion. What we really need for the scattering amplitude is not $\text{Pf}(\Psi)$, but $\text{Pf}(\Psi_{\lambda\nu}^{\lambda\nu})$, where $\Psi_{\lambda\nu}^{\lambda\nu}$ is Ψ with the λ th and ν th columns and rows removed, $1 \leq \lambda < \nu \leq n$. Note that although λ and ν are absent, the columns and rows λ', ν' are still present. In the first group of terms in (11), suppose we let $i'_1 = \lambda'$. Since λ, ν are not present in $\Psi_{\lambda\nu}^{\lambda\nu}$, the arrangement in this group could only end with a factor of the form $\Psi_{i_x\nu'}$. This is the only change for $\text{Pf}(\Psi_{\lambda\nu}^{\lambda\nu})$, the other groupings J, \dots, K remain the same as in $\text{Pf}(\Psi)$. Since an extra factor $1/\sigma_{\nu\lambda}$ is present in $\text{Pf}'\Psi = (-1)^{\lambda+\nu+\frac{1}{2}n(n+1)}2^{n-3}\text{Pf}(\Psi_{\lambda\nu}^{\lambda\nu})/\sigma_{\nu\lambda}$, as long as we stipulate that the cycle I must be of the form $(\lambda\bar{i}_2\cdots\bar{i}_{x-1}\nu)$, the σ factors in $\text{Pf}'\Psi$ are identical to those in $\text{Pf}\Psi$. The trace in U_I is replaced by the matrix element

$$W_I = \epsilon_\lambda \cdot (U_{i_2}U_{i_3}\cdots U_{i_{x-1}}) \cdot \epsilon_\nu,\tag{21}$$

and everything else remains essentially the same. In this way we get

$$\text{Pf}'\Psi = -2^{n-3} \sum'_{p \in S_n} (-)^p \frac{W_I U_J \cdots U_K}{\sigma_p}, \quad \sigma_p = \sigma_I \sigma_J \cdots \sigma_K, \quad (22)$$

where the prime on the summation sign indicates that the sum is taken over all $p \in S_n$ such that ν is changed into λ . There are $(n-1)!$ such permutations in S_n so the sum consists of $(n-1)!$ terms. Note that we have dropped the bar on top of p because from now on we always talk about cycles in S_n , never S_{2n} .

We shall refer to the cycle bounded at two ends by λ and ν as an *open cycle*, and the rest as *closed cycles*. Open cycles are denoted by a square bracket, and closed cycles are denoted by a round bracket. Thus $I = [\lambda i_2 i_3 \cdots i_{x-1} \nu]$, but $J = (j_1 j_2 \cdots j_y)$. Every matrix element in a closed cycle has the same number of ϵ and k , hence the same number of $\epsilon\epsilon$ and $k\cdot k$, whereas the matrix element in the open cycle has two more ϵ than k , hence there is one more $\epsilon\epsilon$ than $k\cdot k$. In particular, there must be at least one $\epsilon\epsilon$ in this matrix element. This property will be crucial in rendering many cycle factors zero in the ‘helicity gauge’.

There is another simple thing to note. Every number a between 1 and n must appear in every $p \in S_n$ once. If it resides in U_a , then both ϵ_a and k_a must appear. However, if $a = \lambda$ or ν , then only ϵ_a is present, not k_a .

Gauge invariance is the statement that $\text{Pf}'\Psi = 0$ if an ϵ_i is replaced by k_i . This is easy to see in the present formalism because such replacement renders $U_i = \epsilon_i k_i - k_i \epsilon_i$ zero.

The gauge amplitude (1) is invariant under a Möbius transformation, $\sigma_i \rightarrow (\alpha\sigma_i + \beta)/(\gamma\sigma_i + \delta)$, $\alpha\delta - \beta\gamma = 1$. This is so because under such a transformation,

$$\begin{aligned} \sigma_{ij} &\rightarrow \sigma_{ij}/(\gamma\sigma_i + \delta)(\gamma\sigma_j + \delta), \\ C_{ii} &\rightarrow C_{ii}(\gamma\sigma_i + \delta)^2, \\ \sigma_{(\alpha)} &\rightarrow L\sigma_{(\alpha)}, \\ \text{Pf}'\Psi &\rightarrow \text{Pf}'\Psi/L, \\ \left(\sigma_{(rst)}^2 \prod_{a=1, a \neq r, s, t}^n \frac{d\sigma_a}{f_a} \right) &\rightarrow L^2 \left(\sigma_{(rst)}^2 \prod_{a=1, a \neq r, s, t}^n \frac{d\sigma_a}{f_a} \right), \\ L &\equiv 1/\prod_{j=1}^n (\gamma\sigma_j + \delta)^2. \end{aligned} \quad (23)$$

It is this invariance of (1) that allows $\sigma_r, \sigma_s, \sigma_t$ to be chosen freely. Now it can be verified that every one of the $(n-1)!$ terms in (22) also transforms like $\text{Pf}'\Psi$, hence when (22) is

used to expand the amplitude, every term of the amplitude is also Möbius invariant. This allows the values of $\sigma_r, \sigma_s, \sigma_t$ to be chosen differently for different terms, a freedom which may be useful in simplifying calculation.

Let us illustrate the difference between the cycles in $\text{Pf}(\Psi)$ and $\text{Pf}'(\Psi)$ with the $n = 4$ example, assuming $\lambda = 1$ and $\nu = 2$. The allowed cycles for $\text{Pf}'\Psi$ are now reduced to

$$\{[142](3), [132](4), [12](3)(4), [12](34), [1342], [1432]\}, \quad (24)$$

a total of $3! = 6$ terms, far smaller than the number of terms of $\text{Pf}(\Psi)$ appearing in (20).

It is convenient to give the factors encountered in (22) names to make them simpler to write. Accordingly, the following abbreviations will be used throughout this article:

$$a_{ij} = a_{ji} = k_i \cdot k_j, \quad b_{ij} = b_{ji} = \epsilon_i \cdot \epsilon_j, \quad c_{ij} = \epsilon_i \cdot k_j. \quad (25)$$

We may also add a superscript to indicate the helicity of a polarization vector ϵ . For example, $b_{ij}^{+-} = \epsilon_i^+ \cdot \epsilon_j^-$.

IV. SHIFT INVARIANCE

Add a multiple of the $(i+n)$ th column of Ψ to the i th column, and the same multiple of the $(i+n)$ th row to the i th row. This action changes the matrix Ψ to a ‘shifted matrix’ $\hat{\Psi}$ by replacing k_i with the shifted momentum $\hat{k}_i = k_i + z_i \epsilon_i$, where z_i is an arbitrary complex constant. This action does not change the Pfaffian, $\text{Pf}\Psi = \text{Pf}\hat{\Psi}$, nor the reduced Pfaffians, $\text{Pf}\Psi_{\lambda\nu}^{\lambda\nu} = \text{Pf}\hat{\Psi}_{\lambda\nu}^{\lambda\nu}$, provided $i \neq \lambda, \nu$. The latter condition is necessary because k_λ, k_ν are not contained in $\text{Pf}'\Psi$ except through C_{aa} , so a shift in either of them is meaningless. This invariance, which is true in any gauge, will be referred to as the *shift invariance*.

Shift invariance is a useful consequence of the CHY theory because it can be used to check a calculation of $\text{Pf}'\Psi$. Examples of such checks will be given in the next few sections. The shift $k_i \rightarrow \hat{k}_i = k_i + \epsilon_i$ leads to the shifts

$$\begin{aligned} a_{ji} &\rightarrow a_{j\hat{i}} = a_{ji} + c_{ij}, & a_{ij} &\rightarrow a_{i\hat{j}} = a_{ij} + c_{ij}, \\ c_{ji} &\rightarrow c_{j\hat{i}} = c_{ji} + b_{ji}, & c_{ij} &\rightarrow c_{i\hat{j}} = c_{ij}, & (i \neq \lambda, \nu); \\ b_{ij} &\rightarrow b_{i\hat{j}} = b_{ij}, & b_{ji} &\rightarrow b_{j\hat{i}} = b_{ji}, \\ C_{ii} &\rightarrow C_{i\hat{i}} = C_{ii}, & (\text{all } i). \end{aligned} \quad (26)$$

Similarly, if we add the i th column to the $(i+n)$ th column, and the i th row to the $(i+n)$ th row, the reduced Pfaffian also remains invariant, provided $i \neq \lambda, \nu$. In terms of a, b, c , this shift invariance is

$$\begin{aligned}
c_{ji} &\rightarrow c_{j\hat{i}} = c_{ji}, & c_{ij} &\rightarrow c_{\hat{i}j} = c_{ij} + a_{ij}, \\
b_{ij} &\rightarrow b_{\hat{i}j} = b_{ij} + c_{ji}, & b_{ji} &\rightarrow b_{j\hat{i}} = b_{ji} + c_{ji}, & (i \neq \lambda, \nu); \\
a_{ji} &\rightarrow a_{j\hat{i}} = a_{ji}, & a_{ij} &\rightarrow a_{\hat{i}j} = a_{ij}, \\
C_{ii} &\rightarrow C_{\hat{i}\hat{i}} = C_{ii}, & & (\text{all } i).
\end{aligned} \tag{27}$$

We shall refer to (26) as shift invariance of the first kind, and (27) as shift invariance of the second kind. Shift invariance of the second kind is the same as Yang-Mills gauge invariance.

Let δ_i refer to the change when $k_i \rightarrow k_i + \epsilon_i$, and δ' refer to the change when $\epsilon_i \rightarrow \epsilon_i + k_i$, then

$$\begin{aligned}
\delta_i a_{ji} &= \delta_i a_{ij} = c_{ij}, & \delta_i c_{ji} &= b_{ji}, & \delta_i c_{ij} &= \delta_i b_{ij} = \delta_i b_{ji} = 0, \\
\delta'_i b_{ji} &= \delta'_i b_{ij} = c_{ji}, & \delta'_i c_{ij} &= a_{ij}, & \delta'_i c_{ji} &= \delta'_i a_{ij} = \delta'_i a_{ji} = 0, & (i \neq \lambda, \nu); \\
\delta_i C_{ii} &= \delta_i C_{jj} = \delta'_i C_{ii} = \delta'_i C_{jj} = 0, & & (\text{all } i).
\end{aligned} \tag{28}$$

Shift invariance can also be seen from the fact that $U_i = k_i \epsilon_i - \epsilon_i k_i$ in (13) is invariant under $k_i \rightarrow k_i + \epsilon_i$ and $\epsilon_i \rightarrow \epsilon_i + k_i$.

Terms of the reduced Pfaffian with different σ -dependence must be separately shift invariant.

If the Pfaffian is calculated in the helicity gauge where the constraints (31), (32), and (33) are used, then only those changes respecting these constraints are shift invariant. In other words, any shift that changes any of the b_{ij} or c_{ij} in (31), (32), and (33) should not be applied to $\text{Pf}'\Psi$.

V. HELICITY GAUGE

The number of terms in $\text{Pf}'\Psi$ for an n -particle amplitude is $(2n - 3)!!$, which is 15 for $n = 4$, and already 105 for $n = 5$. In an attempt to reduce the number of terms to make the calculation manageable, we adopt the *helicity gauge*, defined by having the polarization

vectors used in the spinor-helicity technique. Every particle i carrying a $+$ helicity is required to satisfy $\epsilon_i^+ \cdot k_+ = 0$, and every particle i carrying a $-$ helicity is required to satisfy $\epsilon_i^- \cdot k_- = 0$. The *reference momenta* k_\pm are the momentum of a line labelled \pm and carry helicity \mp . In spinor-helicity notation,

$$\epsilon_i^+ = \frac{|+\rangle[i]}{\langle+i\rangle}, \quad \epsilon_i^- = -\frac{|i\rangle[-]}{[i-]}, \quad k_\pm = |\pm\rangle[\pm|, \quad k_i = |i\rangle[i|. \quad (29)$$

They lead to the dot products

$$\begin{aligned} \epsilon_i^\pm \cdot \epsilon_j^\pm &= 0, \quad b_{ij}^{+-} := \epsilon_i^+ \cdot \epsilon_j^- = \frac{\langle j+ \rangle [i-]}{\langle i+ \rangle [j-]}, \quad a_{ij} := k_i \cdot k_j = \langle ij \rangle [ji], \\ c_{ij}^+ &:= \epsilon_i^+ \cdot k_j = -[ij] \frac{\langle j+ \rangle}{\langle i+ \rangle}, \quad c_{ij}^- := \epsilon_i^- \cdot k_j = -\langle ij \rangle \frac{[j-]}{[i-]}. \end{aligned} \quad (30)$$

For simplicity in writing, the superscripts of b and c are often omitted. From these relations, we see that many dot products are zero, which can be used to reduce the number of terms in $\text{Pf}'\Psi$. For instance,

$$0 = c_{i\pm}^\pm = \epsilon_i^\pm \cdot k_\pm, \quad (31)$$

$$0 = b_{+i}^- = \epsilon_+^- \cdot \epsilon_i^+, \quad (32)$$

$$0 = b_{i-}^+ = \epsilon_i^- \cdot \epsilon_-^+, \quad (33)$$

$$0 = b_{ij}^{+-} a_{ji} - c_{ij}^+ c_{ji}^- = (\epsilon_i^+ \cdot \epsilon_j^-)(k_j \cdot k_i) - (\epsilon_i^+ \cdot k_j)(\epsilon_j^- \cdot k_i) = -\frac{1}{2} \text{Tr}(U_i^+ U_j^-), \quad (34)$$

where U_a^\pm is (13) with ϵ_a replaced by ϵ_a^\pm , and the subscripts \pm stands for the reference lines $j = \pm$. We also have

$$\begin{aligned} U_+ U_- &= (\epsilon_+^- k_+ - k_+ \epsilon_+^-) \cdot (\epsilon_-^+ k_- - k_- \epsilon_-^+) = \frac{1}{\langle + - \rangle [- +]} (\ell k_+ - k_+ \ell) \cdot (\ell k_- - k_- \ell) \\ &= -\ell \ell = U_- U_+. \end{aligned} \quad (35)$$

In terms of the complex light-like momentum

$$\ell = |+\rangle[-] = \epsilon_+^- [-+] = \epsilon_-^+ \langle + - \rangle, \quad (36)$$

(29) also implies

$$\ell \cdot \ell = \epsilon_+^- \cdot \epsilon_-^+ = k_\pm \cdot \ell = \epsilon_a^\pm \cdot \ell = 0 \quad (\forall a). \quad (37)$$

Moreover, since $\epsilon_+ = \epsilon_- \frac{\langle + - \rangle}{[- +]} := \epsilon_- J$, we also have

$$c_{+j} = J c_{-j}. \quad (38)$$

Recall that the factors of $\text{Pf}'\Psi$ in the open cycle in (22) contains at least one $\epsilon \cdot \epsilon$. Thus

1. *the amplitude vanishes if every particle has the same helicity*, because then all $\epsilon_i \cdot \epsilon_j = 0$;
2. *the amplitude also vanishes if all but one particle have the same helicity*. If we take a to be the reference line for the majority helicity, then all $\epsilon_i \cdot \epsilon_j = 0$, hence the amplitude vanishes.

VI. EVALUATION OF THE AMPLITUDE

The gauge amplitude M^α in (1) differs from the double-color scalar amplitude $m(\alpha|\beta)$ in only one aspect: the dynamical factor $1/\sigma_{(\alpha)}\sigma_{(\beta)}$ in $m(\alpha|\beta)$ is replaced by the dynamical factor $\text{Pf}'\Psi/\sigma_{(\alpha)}$ in M^α . Nevertheless, this simple replacement makes the gauge amplitude much more difficult to evaluate. To start with, $\text{Pf}'\Psi$ contains $(2n-3)!!$ terms compared to the single term in $1/\sigma_{(\beta)}$. This can be greatly simplified in the helicity gauge, for example,

- i). (32) implies that the open cycle must carry both helicities because it must contain at least one non-zero $\epsilon \cdot \epsilon$ factor;
- ii). (33) implies that if λ or ν is $+$ or $-$, then the W -factor in the open cycle (see (21)) must contain at least one U_i ;
- iii). (34) implies that a closed 2-cycle must have the same helicity.

The following relation of momentum conservation is also very useful in calculations,

- iv).

$$\sum_{j=1}^n c_{ij}^\pm = \sum_{j \neq i} c_{ij}^\pm = 0, \quad (39)$$

especially when coupled with equation (31).

Besides the large number of terms in $\text{Pf}'\Psi$, the other problem is the complicated σ -dependence of $\text{Pf}'\Psi$. As shown in (16) and (22), it is given by many cycles of different lengths, unlike the double-color $\sigma_{(\beta)}$ which only consists of a single n -cycle for the permutation $\beta \in S_n$. Among these cycles, the 1-cycles are hard to handle because they consist of many terms with different σ 's, and the higher cycles may lead to double poles making the residue difficult to calculate. All of these make the integrations of M^α far more difficult to evaluate than the integrations in $m(\alpha|\beta)$.

One way to deal with the complicated cycles in $\text{Pf}'\Psi$ is to algebraically convert them to a combination of simple n -cycles [9, 10], so that the trivalent [3], polygon [11], or pairing [8] rules designed for double-color scalar amplitudes can be used to evaluate the gauge amplitude. We shall illustrate this method below for $n = 3, 4$, and 5. However, there is no guarantee that the procedure will succeed, at least not easily. In particular, the matrix element $C_{\bar{i}}$ for a 1-cycle (\bar{i}) has a complicated σ -dependence which makes it difficult to be so converted for $n > 5$, for reasons to be explained later.

Another way is to calculate (1) directly as it is, using the ‘multi-crystal’ method developed in [8] for the CHY double-color scalar amplitude. That method is simply a systematic way to locate regions in the $(n-3)$ -dimensional complex space that dominate the integral, then use residue calculus to evaluate the $(n-3)$ integrals one at a time. It will be adapted here to compute the CHY gauge amplitude. It works well if only simple poles are present, because residues are then easy to calculate. The technique discussed in this section and in Sec. X are designed with that in mind. Terms with double poles, if present, must be computed separately. Fortunately, by suitably choosing the gauge constants $r, s, t, \lambda, \nu, +, -$, double poles can often be avoided. This is for example the case for $n = 3, 4, 5$, which will be dealt with in Secs. VII, VIII, IX. Otherwise double poles can still be handled. Details can be found in Sec. XII.

To make writings simple but without scarifying generality, we shall assume for convenience $\alpha = (123 \cdots n)$ from now on. Different α 's can be obtained simply by substitution.

In the present context, a *crystal* is a set $S = \{i+1, i+2, \cdots, i+m, i+m+1\}$ of consecutive lines (line additions are understood to be mod n) that contains one and only one ‘defect’. A *defect* is a line r without the corresponding scattering function f_r present in the integrand of the remaining integrations. Originally, in (1), only the constant lines qualify as defects, but later on new defects will appear from old ‘triggers’.

To explain what a ‘trigger’ is, it is necessary to know how crystals are related to integrations. Each crystal gives rise to one integration, with the integration variable s defined by a scaling change $\sigma_{x,x+1} = s\sigma'_{x,x+1}$, ($i+1 \leq x \leq i+m$). To ensure the right number of new variables, a constraint has to be put on $\sigma'_{x,x+1}$, and we shall do so by picking a variable line p in S and demand $\sigma'_{pr} = 1$. This line p is the *trigger*, and the relation $\sigma'_{pr} = 1$ is referred to as the *triggering relation*. For a motivation of these names please see Ref. [8].

The integration in s consists of computing the residue at the $s = 0$ pole. To see when a pole appears, note that on the one hand, every $1/f_x$ for $x \in S$ scale like s , so $\prod_{x \in S, x \neq r} (d\sigma_x/f_x) \sim s^{2m-1}$, and on the other hand, $1/\sigma_\alpha$ scales like $1/s^m$, so a pole at $s = 0$ occurs only when $\text{Pf}'\Psi \sim 1/s^m$ or a higher power. The most efficient way to produce a high power of $1/s$ in $\text{Pf}'\Psi$ is to have all the lines in S appear adjacently in some cycle of $\text{Pf}'\Psi$, in which case $\text{Pf}'\Psi \sim 1/s^m$ if the cycle also contains some other lines, and $\text{Pf}'\Psi \sim 1/s^{m+1}$ if the cycle contains no other lines. In particular, a 1-cycle $(x) = C_{xx} \sim 1/s$ when $x \in S$. Putting these together, we see that the integrand of (1) produces a simple pole at $s = 0$ if the lines of the crystal all appear adjacently in a cycle which contains also other lines, and it produces a double pole if the cycle contains no other line.

Double poles will be discussed in Sec. XII. For the rest of this section we will assume the poles to be simple poles.

After the integration, f_p turns into the inverse propagator $s_S = s_{i+1, i+2, \dots, i+m+1}$ [8]. As a result, f_p is removed from the integrand so p becomes a defect in subsequent integrations.

Two crystals are said to be *compatible* if they do not intersect, or one is completely inside the other. Originally there are only three defects, coming from the three constant lines r, s, t in (1). Since each crystal is allowed only one defect, the external lines of the amplitude can be divided into three non-intersecting crystals. One of these three crystals may not contain any variable line in which case it can be ignored in all the integrations. Each integration brings along a new defect p which allows more ways to create smaller compatible crystals. After all the integrations, $(n-3)$ propagators appear which can be represented by a Feynman diagram. Variable lines are replaced by defects, so using the triggering relations, every σ_{ij} becomes either $\pm 1, \pm\sigma_{rs}, \pm\sigma_{st}$, or $\pm\sigma_{ts}$. The numerator of the gauge amplitude is then essentially $\text{Pf}'(\Psi)$ with these substitutions of σ_{ij} .

The collection of a set of compatible crystals is referred to as a *multi-crystal*. The CHY amplitude is given by a sum over all possible independent multi-crystal sets, or a *complete set of multi-crystals*.

Owing to Möbius symmetry, the amplitude (1) contains a large number of invariance, making it independent of the choice of the constant lines r, s, t , the Möbius constants $\sigma_r, \sigma_s, \sigma_t$, the Pfaffian lines λ and ν , and the reference lines $+, -$ in the helicity gauge. Different choices give rise to different surviving cycle structures in $\text{Pf}'\Psi$ and different crystals, making the calculations different, though at the end the result must be independent of

the choice. One might attempt to exploit this freedom to pick a choice most convenient for the calculation, but before having the confidence to do so and the knowledge to know what to pick, and what difference do they make, some explicit examples should be carried out. This is done in the next two sections for the $n = 3$ and $n = 4$ CHY amplitudes, to show how different choices cause their computations to differ, and how at the end they all agree with the known result given by the Parke-Taylor formula [7]. The amplitude for $n \geq 5$ is computed in Sec. X, whose outcome also agrees with the Parke-Taylor formula.

To simplify notations, we shall use a superscript to indicate the helicity of a particle i in an amplitude, and a subscript(s), if present, to indicate whether it is line $\lambda, \nu, +$, or $-$. As in Sec. III, a square bracket $[\dots]$ is used to indicate the open cycle, and a round bracket (\dots) is used to indicate closed cycles. We will also use the Mandelstam invariants defined by

$$s_{ij\dots k} = (k_i + k_j + \dots + k_k)^2 \quad (40)$$

in the calculations.

VII. $n = 3$

There is no integration in $n = 3$, so the three-point amplitude is simply $M = \sigma_{(123)}^2 \text{Pf}'(\Psi)/\sigma_{(123)} = \sigma_{(123)} \text{Pf}'\Psi$. This amplitude is supposed to be independent of the Möbius constants $\sigma_{1,2,3}$, nor the choice of the Pfaffian lines λ, ν in (22). In what follows, explicit calculations of $1_\lambda 2_\nu 3$, $1_\nu 23_\lambda$ and $12_\lambda 3_\nu$ are carried out to verify this independence.

A. Three-point vertex

Table 1 shows the three choices of λ and ν , the permutation $p \in S_3$ used in the calculation, and the resulting amplitude M . These three expressions are identical because $b_{ij} = b_{ji} = \epsilon_i \cdot \epsilon_j$, and because $c_{12} + c_{13} = c_{21} + c_{23} = c_{31} + c_{32} = 0$ on account of momentum conservation.

λ, ν	$p = \text{permutation}$	M
$1_\lambda 2_\nu 3$	$[132], [12](3)$	$b_{13}c_{23} - b_{32}c_{13} + b_{12}c_{31}$
$1_\lambda 23_\nu$	$[123], [13](2)$	$-b_{12}c_{32} + b_{23}c_{12} - b_{13}c_{21}$
$12_\lambda 3_\nu$	$[213], [23](1)$	$b_{21}c_{31} - b_{13}c_{21} + b_{23}c_{12}$

Table 1. Three-point vertex

Here is the calculation of the first row using (22):

$$\begin{aligned}
M &= -W_{[132]} \frac{\sigma_{(123)}}{\sigma_{(132)}} + W_{[12]} \frac{\sigma_{(123)}}{\sigma_{(12)}} C_{33}, \\
\frac{1}{\sigma_{(12)}} C_{33} &= \frac{1}{\sigma_{(12)}} \left(-\frac{c_{31}}{\sigma_{31}} - \frac{c_{32}}{\sigma_{32}} \right) = \frac{1}{\sigma_{12}\sigma_{21}} c_{31} \frac{\sigma_{21}}{\sigma_{32}\sigma_{31}} = -\frac{c_{31}}{\sigma_{(123)}}, \Rightarrow \\
M &= -W_{[132]} + c_{31} W_{[12]} = b_{13}c_{23} - c_{13}b_{32} + c_{31}b_{12}.
\end{aligned} \tag{41}$$

M in the second row can be obtained from M of the first row with $2 \leftrightarrow 3$, times -1 , and M of the third row can be obtained from that of the second row with $2 \leftrightarrow 1$, times -1 . The factor -1 comes from the ratio $\sigma_{(123)}/\sigma_{(132)}$ and $\sigma_{(123)}/\sigma_{(213)}$.

We can use (28) to check the shift invariance of the second column in Table 1:

$$\begin{aligned}
\delta_3[132] &= \delta_3(b_{13}c_{23} - b_{32}c_{13}) = b_{13}b_{23} - b_{32}b_{13} = 0, & \delta_3([12](3)) &= 0, \\
\delta_2[123] &= \delta_2(-b_{12}c_{32} + b_{23}c_{12}) = -b_{12}b_{32} + b_{23}b_{12} = 0, & \delta_2([13](2)) &= 0, \\
\delta_1[213] &= \delta_1(b_{21}c_{31} - b_{13}c_{21}) = b_{21}b_{31} - b_{13}b_{21} = 0, & \delta_1([23](1)) &= 0, \\
\delta'_3[132] &= \delta'_3(b_{13}c_{23} - b_{32}c_{13}) = c_{13}c_{23} - c_{23}c_{13} = 0, & \delta'_3([12](3)) &= 0, \\
\delta'_2[123] &= \delta'_2(-b_{12}c_{32} + b_{23}c_{12}) = -c_{12}c_{32} + c_{32}c_{12} = 0, & \delta'_2([13](2)) &= 0, \\
\delta'_1[213] &= \delta'_1(b_{21}c_{31} - b_{13}c_{21}) = c_{21}c_{31} - c_{31}c_{21} = 0, & \delta'_1([23](1)) &= 0,
\end{aligned} \tag{42}$$

The Feynman triple-gluon vertex with the color stripped off is

$$g [(k_1 - k_2)_\gamma g_{\alpha\beta} + (k_2 - k_3)_\alpha g_{\beta\gamma} + (k_3 - k_1)_\beta g_{\gamma\alpha}], \tag{43}$$

where α, β, γ are the Lorentz index for momentum vectors k_1, k_2, k_3 . If we contract this vertex with the polarization vectors $\epsilon_1, \epsilon_2, \epsilon_3$, and use momentum conservation, we get

$$2g [(\epsilon_3 \cdot k_1)(\epsilon_1 \cdot \epsilon_2) + (\epsilon_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3) + (\epsilon_2 \cdot k_3)(\epsilon_3 \cdot \epsilon_1)] = 2g [c_{31}b_{12} + c_{12}b_{23} + c_{23}b_{31}] = 2gM, \tag{44}$$

where M is given by the last column of Table 1. Hence the CHY three-point amplitude is the usual triple-gluon vertex with the coupling constant $g = \frac{1}{2}$.

B. Spinor-helicity expression

The three-point amplitude $M = b_{13}c_{23} - b_{32}c_{13} - b_{12}c_{32}$ will now be expressed in spinor-helicity language for the two different helicity configurations, $1^+2^+3^-$, and $1^-2^-3^+$. Note

that in this simple case there is no need to specify what the reference lines \pm are. They may very well be lines outside of 1,2,3.

1. $1^+2^+3^-$

In this case $b_{12} = 0$, hence

$$\begin{aligned} M &= b_{13}c_{23} - b_{23}c_{13} = \frac{\langle 3+ \rangle [1-] [32] \langle 3+ \rangle}{\langle 1+ \rangle [3-] \langle 2+ \rangle} - \frac{\langle 3+ \rangle [2-] [31] \langle 3+ \rangle}{\langle 2+ \rangle [3-] \langle 1+ \rangle} \\ &= \frac{\langle 3+ \rangle^2}{\langle 1+ \rangle \langle 2+ \rangle [3-]} ([1-][32] - [2-][31]) = -\frac{\langle 3+ \rangle^2 [21] [3-]}{\langle 1+ \rangle \langle 2+ \rangle [3-]}, \end{aligned} \quad (45)$$

where Schouten identity was used in the last step. Now using momentum conservation, $\langle 3+ \rangle [31] = -\langle 2+ \rangle [21]$ and $\langle 3+ \rangle [32] = -\langle 1+ \rangle [12]$, hence

$$M = \frac{[12]^4}{[12][23][31]}, \quad (46)$$

which agrees with the usual spinor-helicity result.

2. $1^-2^-3^+$

In this case

$$\begin{aligned} M &= b_{13}c_{23} - b_{23}c_{13} = \frac{\langle 1+ \rangle [3-] \langle 32 \rangle [3-]}{\langle 3+ \rangle [1-] [2-]} - \frac{\langle 2+ \rangle [3-] \langle 31 \rangle [3-]}{\langle 3+ \rangle [2-] [1-]} \\ &= \frac{[3-]^2}{\langle 3+ \rangle [1-] [2-]} (\langle 1+ \rangle \langle 32 \rangle - \langle 2+ \rangle \langle 31 \rangle) = \frac{[3-]^2 \langle 12 \rangle \langle 3+ \rangle}{\langle 3+ \rangle [1-] [2-]} = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \end{aligned} \quad (47)$$

which agrees with the usual spinor-helicity result.

VIII. $n = 4$

We shall compare calculations of various choices of $\lambda, \nu, +, -$ in the helicity configuration $1^+2^+3^-4^-$, with the constant lines taken to be $r, s, t = 1, 2, 3$ and the color taken to be $\alpha = (1234)$. In that case the amplitude (1) is

$$M = \frac{-1}{2\pi i} \oint_{\Gamma} \frac{d\sigma_4}{f_4} \frac{\sigma_{(123)}^2 \text{Pf}'\Psi}{\sigma_{(1234)}}. \quad (48)$$

The possible crystals are $S = \{34\}$ and $S = \{41\}$. Terms in $\text{Pf}'\Psi$ proportional to $1/\sigma_{34}$ gives rise to $1/s_{34}$, and terms proportional to $1/\sigma_{41}$ gives rise to $1/s_{41}$.

A. $1_{\lambda}^+ 2^+ 3^- 4_{\nu}^-$

The allowed permutations are

$$[1234], [1324], [124](3), [134](2), [14](23), [14](2)(3). \quad (49)$$

Depending on the choice of the reference lines \pm , some of these permutations are zeros. In what follows we will consider several assignments of $(-, +)$.

We will also check the shift invariance (28) for each of the three cases below. Since $\lambda = 1$ and $\nu = 4$, we should have $\delta_2(\text{Pf}'\Psi) = \delta_3(\text{Pf}'\Psi) = \delta'_2(\text{Pf}'\Psi) = \delta'_3(\text{Pf}'\Psi) = 0$, provided these changes do not affect (31), (32), and (33).

1. $1_{\lambda^-} 2^+ 3^- 4_{\nu^+}$

The only non-zero b is b_{23}^{+-} , so the open cycle must contain both 2 and 3, which leaves only [1234] and [1324] as the allowed permutations. In this case,

$$\begin{aligned} \frac{1}{2}\text{Pf}'(\Psi) &= \frac{W_{[1234]}}{\sigma_{(1234)}} + \frac{W_{[1324]}}{\sigma_{(1324)}} = b_{23}^{+-} \left(\frac{c_{12}^+ c_{43}^-}{\sigma_{(1234)}} + \frac{c_{13}^+ c_{42}^-}{\sigma_{(1324)}} \right), \\ c_{12}^+ c_{43}^- &= [12]\langle 43 \rangle \frac{\langle 24 \rangle [31]}{\langle 14 \rangle [41]}, \\ c_{13}^+ c_{42}^- &= [13]\langle 42 \rangle \frac{\langle 34 \rangle [21]}{\langle 14 \rangle [41]} = c_{12}^+ c_{43}^-, \\ \frac{1}{\sigma_{(1234)}} + \frac{1}{\sigma_{(1324)}} &= \frac{1}{\sigma_{23}\sigma_{41}} \left(\frac{1}{\sigma_{12}\sigma_{34}} - \frac{1}{\sigma_{13}\sigma_{24}} \right) = -\frac{1}{\sigma_{(1243)}}, \\ \frac{1}{2}\text{Pf}'\Psi &= -\frac{b_{23}^{+-} c_{12}^+ c_{43}^-}{\sigma_{(1243)}}. \end{aligned} \quad (50)$$

In this gauge $0 = c_{14} = c_{34} = c_{31} = c_{41} = b_{13} = b_{14} = b_{24}$. These conditions must not be violated in checking shift invariance. This means that we should not apply shifts δ'_i of the second kind. The only thing left is δ_2 and δ_3 . Shift invariance is trivial under δ_2 because $b_{12} = 0$, and is equally trivial under δ_3 because $b_{43} = 0$.

The integral to be carried out is just that of a double-color scalar amplitude, with $\alpha = [1234]$ and $\beta = [1243]$. This integral gives rise to $1/2k_1 \cdot k_2 = 1/2k_3 \cdot k_4$, hence

$$M = -\frac{b_{23}^{+-} c_{12}^+ c_{43}^-}{k_3 \cdot k_4} = -\frac{\langle 34 \rangle [21] [12] \langle 43 \rangle \langle 24 \rangle [31]}{\langle 24 \rangle [31] \langle 14 \rangle [41] \langle 34 \rangle [43]} = \frac{[12]^4}{[12][23][34][41]}, \quad (51)$$

which is the Parke-Taylor formula. Momentum conservation has been used to convert $\langle 43 \rangle / \langle 14 \rangle$ to $[12] / [32]$.

2. $\mathbf{1}_\lambda^+ \mathbf{2}_-^+ \mathbf{3}_+^- \mathbf{4}_\nu^-$

The only non-zero b is b_{14}^{+-} , thus the only allowed permutations are $[14](23)$ and $[14](2)(3)$. With $2 = -$ and $3 = +$, $c_{32} = c_{42} = 0 = c_{13} = c_{23}$, resulting in $(23)=0$. Hence we are left with only $[14](2)(3)$, which gives

$$\frac{1}{2} \text{Pf}' \Psi = \frac{W_{[14]}}{\sigma_{(14)}} C_{22} C_{33}, \quad (52)$$

$$W_{[14]} = b_{14}^{+-},$$

$$C_{22} = -\frac{c_{21}}{\sigma_{21}} - \frac{c_{24}}{\sigma_{24}} = c_{24} \left(\frac{1}{\sigma_{21}} - \frac{1}{\sigma_{24}} \right) = c_{24} \frac{\sigma_{14}}{\sigma_{21}\sigma_{24}},$$

$$C_{33} = -\frac{c_{31}}{\sigma_{31}} - \frac{c_{34}}{\sigma_{34}} = c_{34} \left(\frac{1}{\sigma_{31}} - \frac{1}{\sigma_{34}} \right) = c_{34} \frac{\sigma_{14}}{\sigma_{31}\sigma_{34}},$$

$$\frac{1}{2} \text{Pf}' \Psi = \frac{b_{14}^{+-} c_{24}^+ c_{34}^-}{\sigma_{(1243)}},$$

$$M = -\frac{b_{14}^{+-} c_{24}^+ c_{34}^-}{k_3 \cdot k_4} = -\frac{\langle 43 \rangle [12] [42] \langle 43 \rangle \langle 43 \rangle [42]}{\langle 13 \rangle [42] \langle 23 \rangle [32] \langle 34 \rangle [43]} \frac{1}{[12][23][34][41]} = \frac{[12]^4}{[12][23][34][41]}, \quad (53)$$

which is the Parke-Taylor result.

Like the situation under item 1, gauge constraints of the vanishing b_{ij} forbid us to apply δ'_i . Shift invariance under δ_2, δ_3 is trivial because $W_{[14]} = b_{14}$ is not shifted by any of them.

This calculation uses momentum conservation to combine the σ -factors into the standard form, after which rules developed for double-color scalar amplitudes can be used to integrate. For $n > 5$, it is no longer possible to convert the σ -dependence into the standard form this way. Although there are other methods to do so, they are fairly complicated and in any case do not seem to work well with 1-cycles. In those cases it may be easier to do the integrations without combining the σ -factors. Let us illustrate how that goes with the present example.

3. $\mathbf{1}_{\lambda^-}^+ \mathbf{2}^+ \mathbf{3}_+^- \mathbf{4}_\nu^-$

The only non-zero b is b_{24}^{+-} , thus the only allowed permutations are $[124](3)$ and $[1324]$. With $+ = 3$ and $- = 1$, $c_{13} = c_{23} = 0 = c_{31} = c_{41}$. Therefore

$$\frac{1}{2} \text{Pf}' \Psi = -\frac{W_{[124]}}{\sigma_{(124)}} C_{33} + \frac{W_{[1324]}}{\sigma_{(1324)}} = -\frac{W_{[124]}}{\sigma_{(124)}} C_{33},$$

$$C_{33} = -\frac{c_{32}}{\sigma_{32}} - \frac{c_{34}}{\sigma_{34}} = c_{34} \left(\frac{1}{\sigma_{32}} - \frac{1}{\sigma_{34}} \right) = c_{34} \frac{\sigma_{24}}{\sigma_{32}\sigma_{34}},$$

$$\frac{1}{2} \text{Pf}' \Psi = \frac{c_{12}^+ b_{24}^{+-} c_{34}^-}{\sigma_{(1234)}},$$

$$\begin{aligned}
M &= c_{12}^+ b_{24}^{+-} c_{34}^- \left(\frac{1}{k_2 \cdot k_3} + \frac{1}{k_1 \cdot k_2} \right) = -c_{12}^+ b_{24}^{+-} c_{34}^- \frac{k_1 \cdot k_3}{(k_1 \cdot k_2)(k_2 \cdot k_3)} \\
&= -\frac{[21]\langle 23\rangle\langle 43\rangle[21]\langle 43\rangle[41]}{\langle 13\rangle\langle 23\rangle[41]}\frac{\langle 13\rangle[31]}{[31]\langle 34\rangle[43]\langle 23\rangle[32]} = \frac{[12]^4}{[12][23][34][41]}. \tag{54}
\end{aligned}$$

Momentum conservation identities $\langle 34\rangle[41] + \langle 32\rangle[21] = 0$ has been used to get the final result. Shift invariance is trivial under δ_2 and δ_3 because none of the factors of $\text{Pf}'\Psi$ get shifted. Shifts under δ'_i should not be applied because of the gauge constraints.

Note that although the different choices of $(-, +)$ in Sec. VIIIA.1, 2, 3 all give the same final result, they give rise to different cycle structures and different permutations, and also different σ -structure before momentum conservation was used.

B. $1_{\lambda^-}^+ 2_{\nu^+}^+ 3^- 4_+^-$

Since the only non-vanishing b is still b_{23}^{+-} , the allowed permutations are $[1432]$ and $[132](4)$. With $4 = +$, $c_{14}^+ = c_{24}^+ = 0$. With $1 = -$, $c_{31} = c_{41} = 0$. Hence

$$\begin{aligned}
\frac{1}{2}\text{Pf}'\Psi &= \frac{W_{[1432]}}{\sigma_{(1432)}} - \frac{W_{[132]}}{\sigma_{(132)}} C_{44} = b_{23}^{+-} \left(\frac{c_{14}^+ c_{43}^-}{\sigma_{(1432)}} - \frac{c_{13}^+}{\sigma_{(132)}} C_{44} \right) = -\frac{b_{23}^{+-} c_{13}^+}{\sigma_{(132)}} C_{44}, \\
C_{44} &= -\frac{c_{41}^-}{\sigma_{41}} - \frac{c_{42}^-}{\sigma_{42}} - \frac{c_{43}^-}{\sigma_{43}} = -\frac{c_{42}^-}{\sigma_{42}} - \frac{c_{43}^-}{\sigma_{43}} = c_{42}^- \left(\frac{1}{\sigma_{43}} - \frac{1}{\sigma_{42}} \right) = c_{42}^- \frac{\sigma_{32}}{\sigma_{43}\sigma_{42}}, \\
\frac{1}{2}\text{Pf}'\Psi &= \frac{b_{23}^{+-} c_{13}^+ c_{42}^-}{\sigma_{(1243)}}, \\
M &= -\frac{2b_{23}^{+-} c_{13}^+ c_{42}^-}{s_{34}} = -\frac{\langle 34\rangle[21][31]\langle 34\rangle\langle 24\rangle[21]}{\langle 24\rangle[31]\langle 14\rangle[41]\langle 34\rangle[43]} \frac{1}{[12]^4} \\
&= -\frac{\langle 34\rangle[12]^2}{\langle 14\rangle[41][43]} = \frac{[12]^4}{[12][23][34][41]}. \tag{55}
\end{aligned}$$

C. $1_{\lambda^-}^+ 2^+ 3_{\nu^-}^- 4_+^-$

The only non-zero b is b_{23}^{+-} , so the only allowed permutations are $[123](4)$ and $[1423]$. Since $4 = +$ and $1 = -$, $c_{14} = c_{24} = 0 = c_{31} = c_{41}$. Thus

$$\begin{aligned}
\frac{1}{2}\text{Pf}'\Psi &= -\frac{W_{[123]}}{\sigma_{(123)}} C_{44} + \frac{W_{[1423]}}{\sigma_{(1423)}}, \\
C_{44} &= -\frac{c_{42}}{\sigma_{42}} - \frac{c_{43}}{\sigma_{43}} = c_{42} \left(\frac{1}{\sigma_{43}} - \frac{1}{\sigma_{42}} \right) = c_{42} \frac{\sigma_{32}}{\sigma_{42}\sigma_{43}}, \\
W_{[123]} &= c_{12}^+ b_{23}^{+-}, \\
W_{[1423]} &= c_{14}^+ c_{42}^- b_{23}^{+-} = 0, \quad \Rightarrow
\end{aligned}$$

$$\frac{1}{2}\text{Pf}'\Psi = -\frac{c_{12}^+ b_{23}^{+-} c_{42}^-}{\sigma_{(1243)}} = \frac{c_{13}^+ b_{23}^{+-} c_{42}^-}{\sigma_{(1243)}}, \quad (56)$$

which is the same as (55).

D. $1_\lambda^+ 2_-^+ 3_{\nu+}^- 4^-$

In this case $c_{32} = c_{42} = 0 = c_{13} = c_{23}$, and the only non-zero b 's are b_{14} . The allowed permutations are $A = [143](2)$. Choose the constant lines to be 1,3,4 so that the variable line is 2. The amplitude is then

$$\begin{aligned} M &= \frac{-1}{2\pi i} \oint_{\Gamma} \frac{2d\sigma_2}{f_2} \frac{\sigma_{(134)}^2 b_{14} c_{34} C_{22}}{\sigma_{(1234)} \sigma_{(143)}} = -\frac{2}{s_{21}} b_{14} c_{34} c_{21} \\ &= -\frac{2}{s_{34}} \frac{\langle 43 \rangle [12] \langle 43 \rangle [42] [12] \langle 13 \rangle}{\langle 13 \rangle [42] [32] \langle 23 \rangle} = -\frac{\langle 43 \rangle [12]^2}{\langle 23 \rangle [43] [32]} = \frac{[12]^4}{[12][23][34][41]}. \end{aligned} \quad (57)$$

E. Summary for $n = 4$

Table 2 presents a summary of the situations studied above. In spinor helicity language, they all reduce to the Parke-Taylor formula.

	configuration	permutation	$M = [12]^4/[12][23][34][41]$
A1	$1_\lambda^+ 2^+ 3^- 4_{\nu+}^-$	$[1234], [1324]$	$-b_{23} c_{12} c_{43} / k_3 \cdot k_4$
A2	$1_\lambda^+ 2_+^+ 3_-^- 4_\nu^-$	$[14](2)(3)$	$-b_{14} c_{24} c_{34} / k_3 \cdot k_4$
A3	$1_\lambda^+ 2^+ 3_+^- 4_\nu^-$	$[1324], [124](3)$	$b_{24} c_{12} c_{34} (1/k_2 \cdot k_3 + 1/k_3 \cdot k_4)$
B	$1_\lambda^+ 2_\nu^+ 3^- 4_+^-$	$[1432], [132](4)$	$-b_{23} c_{14} c_{42} / k_3 \cdot k_4$
C	$1_\lambda^+ 2^+ 3_\nu^- 4_+^-$	$[1423], [123](4)$	$-b_{23} c_{14} c_{42} / k_3 \cdot k_4$
D	$1_\lambda^+ 2_-^+ 3_{\nu+}^- 4^-$	$[143](2)$	$-b_{14} c_{34} c_{21} / k_3 \cdot k_4$

Table 2. Summary of the $n = 4$ amplitude $M = M^{(1234)}$

IX. $n = 5: 1_\lambda^+ 2_-^+ 3_{\nu+}^- 4^- 5^-$

In this case $c_{32} = c_{42} = c_{52} = 0 = c_{13} = c_{23}$, and the only non-zero b 's are b_{14} and b_{15} . Therefore the non-zero permutations contributing to $\text{Pf}'\Psi$ are $A = [143](2)(5)$, $B =$

$[1453](2), C = [153](2)(4), D = [1543](2)$. Choosing the constant lines to be 1,3,5 and the variable lines to be 2 and 4 yields the amplitude

$$\begin{aligned}
M &= \left(\frac{-1}{2\pi i}\right)^2 \oint_{\Gamma} \frac{d\sigma_2 d\sigma_4}{f_2 f_4} \frac{\sigma_{(135)}^2 \text{Pf}'\Psi}{\sigma_{(12345)}}, \\
\text{Pf}'\Psi &= 2^2 C_{22} \left(\frac{\alpha}{\sigma_{(143)}} C_{55} + \frac{\beta}{\sigma_{(1453)}} + \frac{\gamma}{\sigma_{(153)}} C_{44} + \frac{\delta}{\sigma_{(1543)}} \right), \\
\alpha &= b_{14} c_{34}, \quad \beta = b_{14} c_{54} c_{35}, \quad \gamma = b_{15} c_{35}, \quad \delta = b_{15} c_{45} c_{34}.
\end{aligned} \tag{58}$$

All four terms in $\text{Pf}'\Psi$ have a positive sign. Actually, these plus signs are all of the form $(-1)^2$. To start with, there is an overall minus sign in $\text{Pf}'\Psi$. The permutation signature of the first term is $+$, but the surviving matrix element in $\epsilon_1 U_4 \epsilon_3$ has a minus sign. The permutation signature of the second term is $-$, but the surviving matrix element in $\epsilon_1 U_4 U_5 \epsilon_3$ has a $+$ sign. The third term is like the first term and the fourth term is like the second term. In this way, all the minus signs pile up to give a plus sign for all four terms.

Let us check the shift invariance of (58). Since $\lambda = 1$ and $\nu = 3$, we should check it only for $\delta_2, \delta_4, \delta_5$. The invariance under δ_2 is trivial because none of $\alpha, \beta, \gamma, \delta$ get shifted. (58) also remains unchanged under δ_4 and δ_5 , because the only non-zero b 's are b_{14} and b_{15} , and neither c_{14} nor c_{15} appears in α, β, γ , or δ . The shifts δ'_i should not be applied because of the gauge constraints.

We shall evaluate this integral in two ways. In subsection A, momentum conservation is used to combine the multi-cycle σ -dependences into single 5-cycles, thus allowing rules for double-color scalar amplitudes to be applied to carry out the integrations. This method will not work for $n > 5$, in which case it is better to carry out direct integrations with the multi-cycle structure present. We shall illustrate how this can be done in subsection B. Finally in subsection C, these results will be expressed in spinor-helicity language to show that they agree with the Parke-Taylor formula.

A. Conversion of multi-cycles into single 5-cycles

Using $c_{23} = 0 = c_{42} = c_{52}$ and the momentum conservation relation (39), we can write

$$\begin{aligned}
C_{22} &= -\frac{c_{21}}{\sigma_{21}} - \frac{c_{24}}{\sigma_{24}} - \frac{c_{25}}{\sigma_{25}} = c_{21} \frac{\sigma_{51}}{\sigma_{25}\sigma_{21}} + c_{24} \frac{\sigma_{54}}{\sigma_{25}\sigma_{24}} = c_{21} \frac{\sigma_{41}}{\sigma_{24}\sigma_{21}} + c_{25} \frac{\sigma_{45}}{\sigma_{24}\sigma_{25}}, \\
C_{44} &= -\frac{c_{41}}{\sigma_{41}} - \frac{c_{43}}{\sigma_{43}} - \frac{c_{45}}{\sigma_{45}} = c_{41} \frac{\sigma_{51}}{\sigma_{45}\sigma_{41}} + c_{43} \frac{\sigma_{53}}{\sigma_{45}\sigma_{43}},
\end{aligned}$$

$$C_{55} = -\frac{c_{51}}{\sigma_{51}} - \frac{c_{53}}{\sigma_{53}} - \frac{c_{54}}{\sigma_{54}} = c_{51} \frac{\sigma_{41}}{\sigma_{54}\sigma_{51}} + c_{53} \frac{\sigma_{43}}{\sigma_{54}\sigma_{53}}. \quad (59)$$

These relations allow us to write

$$\begin{aligned} \frac{C_{22}C_{55}}{\sigma_{(143)}} &= +\frac{c_{51}c_{21}}{\sigma_{(12543)}} - \frac{c_{51}c_{24}}{\sigma_{(15243)}} - \frac{c_{53}c_{21}}{\sigma_{(12453)}} + \frac{c_{53}c_{25}}{\sigma_{(14253)}}, \\ \frac{C_{22}}{\sigma_{(1453)}} &= \frac{c_{21}}{\sigma_{(12453)}} - \frac{c_{25}}{\sigma_{(14253)}}, \\ \frac{C_{22}C_{44}}{\sigma_{(153)}} &= -\frac{c_{43}c_{21}}{\sigma_{(12543)}} + \frac{c_{43}c_{24}}{\sigma_{(15243)}} + \frac{c_{41}c_{21}}{\sigma_{(12453)}} - \frac{c_{41}c_{25}}{\sigma_{(14253)}}, \\ \frac{C_{22}}{\sigma_{(1543)}} &= +\frac{c_{21}}{\sigma_{(12543)}} - \frac{c_{24}}{\sigma_{(15243)}}. \end{aligned} \quad (60)$$

Now that all the σ -dependences are of the double-color scalar type, we can use the trivalent [3], polygon [11], or pairing [8] rule to evaluate the integral to get

$$\begin{aligned} \frac{1}{4}M^\alpha &= \frac{1}{s_{12}} \left(\frac{1}{s_{34}} + \frac{1}{s_{45}} \right) (\alpha c_{51}c_{21} - \gamma c_{43}c_{21} + \delta c_{21}) - \frac{1}{s_{15}s_{43}} (-\alpha c_{51}c_{24} + \gamma c_{43}c_{24} - \delta c_{24}) \\ &\quad - \frac{1}{s_{12}s_{45}} (-\alpha c_{53}c_{21} + \beta c_{21} + \gamma c_{41}c_{21}) \\ &= \frac{c_{21}}{s_{12}s_{45}} (-\alpha c_{54} + \gamma c_{45} - \beta + \delta) + \frac{c_{21}}{s_{12}s_{34}} (\alpha c_{51} - \gamma c_{43} + \delta) \\ &\quad - \frac{c_{24}}{s_{15}s_{43}} (-\alpha c_{51} + \gamma c_{43} - \delta). \end{aligned} \quad (61)$$

This method to convert multi-cycles into a single 5-cycle no longer works for $n > 5$ for the following reason. Each C_{ii} consists of $(n-1)$ terms of the form c_{ij}/σ_{ij} , but with $c_{ik} = 0$ where k is the reference momentum of the appropriate helicity, there remains only $(n-2)$ terms. For $n = 5$, one of the three terms is eliminated using momentum conservation. If $c_{i\ell}$ is eliminated, then the factor $\sigma_{\ell j}$ ($j \neq i, k, \ell$) will appear in the numerator, as shown in (59). To get the single 5-cycle structure shown in (60). ℓ must be chosen so that every $\sigma_{\ell j}$ get cancelled out. Since in any cycle ℓ has only two nearest neighbors, this can be achieved with two j 's, as shown in (60), but not with three or more j 's, which is the case for $n > 5$. For that reason this method of converting multi-cycle structures into single a n -cycle structure cannot work for $n > 5$. In that case we have to do the integrations without such a conversion. How this can be done for $n = 5$ is illustrated in the next subsection.

B. Direct integrations

As discussed in Sec. VI, to evaluate (58) directly, we must first determine the allowed crystal sets. These sets must contain consecutive lines, including one constant line. With the constant lines chosen to be 1,3,5, these sets are $S = \{21\}, \{23\}, \{234\}, \{43\}, \{45\}$. There are three terms resulting from the σ_2 -integration, coming from the first three S . The first two sets involve only $(2) = C_{22} = -c_{21}/\sigma_{21} - c_{23}/\sigma_{23} - \dots$, but since $c_{23} = 0$, the pole $1/\sigma_{23}$ is absent, so that leaves only sets $\{21\}$ and $\{234\}$. If we denote the contribution of these two sets to M^α to be respectively M' and M'' , then

$$\begin{aligned} \frac{1}{4}M' &= \frac{-1}{2\pi i} \oint_{\Gamma_4} \frac{c_{21} d\sigma_4}{s_{21} f'_4} \frac{\sigma_{(135)}^2}{\sigma_{(1345)}} \left(\frac{\alpha}{\sigma_{(143)}} C_{55} + \frac{\beta}{\sigma_{(1453)}} + \frac{\gamma}{\sigma_{(153)}} C_{44} + \frac{\delta}{\sigma_{(1543)}} \right), \\ f'_4 &= \frac{s_{41} + s_{42}}{\sigma_{41}} + \frac{s_{43}}{\sigma_{43}} + \frac{s_{45}}{\sigma_{45}}. \end{aligned} \quad (62)$$

To carry out the σ_4 -integration, move the contour Γ_4 away from $f'_4 = 0$ to enclose the poles at $\sigma_{43} = 0$ and $\sigma_{45} = 0$. In this way we get

$$\begin{aligned} \frac{1}{4}M' &= \frac{c_{21}\sigma_{(135)}}{s_{21}} \left[\alpha \left(\frac{c_{54}}{s_{45}\sigma_{(153)}} - \frac{C'_{55}}{s_{34}\sigma_{(13)}} \right) + \frac{\beta}{s_{45}\sigma_{(153)}} + \frac{\gamma}{\sigma_{(153)}} \left(\frac{c_{43}}{s_{34}} - \frac{c_{45}}{s_{45}} \right) \right. \\ &\quad \left. - \frac{\delta}{\sigma_{(153)}} \left(\frac{1}{s_{34}} + \frac{1}{s_{45}} \right) \right] \\ &= -\frac{c_{21}}{s_{21}} \left[\alpha \left(\frac{c_{54}}{s_{45}} - \frac{C'_{55}\sigma_{(153)}}{s_{34}\sigma_{(13)}} \right) + \frac{\beta}{s_{45}} + \gamma \left(\frac{c_{43}}{s_{34}} - \frac{c_{45}}{s_{45}} \right) - \delta \left(\frac{1}{s_{34}} + \frac{1}{s_{45}} \right) \right] \\ &= -\frac{c_{21}}{s_{21}} \left[\alpha \left(\frac{c_{54}}{s_{45}} - \frac{c_{51}}{s_{34}} \right) + \frac{\beta}{s_{45}} + \gamma \left(\frac{c_{43}}{s_{34}} - \frac{c_{45}}{s_{45}} \right) - \delta \left(\frac{1}{s_{34}} + \frac{1}{s_{45}} \right) \right] \\ &= \frac{c_{21}}{s_{21}s_{34}} (\alpha c_{51} - \gamma c_{43} + \delta) - \frac{c_{21}}{s_{21}s_{45}} (\alpha c_{54} + \beta - \gamma c_{45} - \delta), \end{aligned} \quad (63)$$

where C'_{55} is C_{55} evaluated at $\sigma_{43} = 0$, which gives $c_{51}\sigma_{13}/\sigma_{15}\sigma_{53}$.

To evaluate the contribution of the set $\{234\}$ to M^α , make a scaling change $\sigma_{23} = s\sigma'_{23} = s, \sigma_{34} = s\sigma'_{34}$, and look for poles at $s = 0$. As $s \rightarrow 0$,

$$\begin{aligned} f_2 &\rightarrow \frac{1}{s} \left(s_{23} + \frac{s_{24}}{\sigma'_{24}} \right) := \frac{1}{s} f''_2, \\ f_4 &\rightarrow \frac{1}{s} \left(\frac{s_{42}}{\sigma'_{42}} + \frac{s_{43}}{\sigma'_{43}} \right) := \frac{1}{s} f''_4, \\ \sigma_{(12345)} &\rightarrow s\sigma_{(135)}\sigma'_{34}, \\ C_{22} &\rightarrow -\frac{c_{24}}{s\sigma'_{24}}, \\ C_{55} &\rightarrow -\left(\frac{c_{51}}{\sigma_{51}} + \frac{c_{52} + c_{53} + c_{54}}{\sigma_{53}} \right) = c_{51} \frac{\sigma_{31}}{\sigma_{51}\sigma_{53}}, \end{aligned}$$

$$C_{44} \rightarrow -\frac{1}{s} \frac{c_{43}}{\sigma'_{43}}. \quad (64)$$

After doing the s -integration, we obtain the contribution of $\{234\}$ to the amplitude to be

$$\begin{aligned} \frac{1}{4} M'' &= \frac{-1}{2\pi i} \oint_{\Gamma_4} \frac{c_{24} d\sigma'_4 \sigma_{(135)}}{\sigma'_{24} \sigma'_{34} f_2'' f_4''} \left(\frac{-\alpha c_{51}}{\sigma_{(13)} \sigma'_{43}} \frac{\sigma_{31}}{\sigma_{51} \sigma_{53}} + \frac{\gamma c_{43}}{\sigma_{(153)} \sigma'_{43}} + \frac{\delta}{\sigma_{(153)} \sigma'_{43}} \right) \\ &= \frac{-1}{2\pi i} \oint_{\Gamma_4} \frac{c_{24} d\sigma'_4}{\sigma'_{24} \sigma'_{34} f_2'' f_4''} (-\alpha c_{51} + \gamma c_{43} + \delta). \end{aligned} \quad (65)$$

The contour Γ_4 surrounds $f_4'' = 0$ counter clockwise. The pole for $f_4'' = 0$ is located at $\sigma'_{43} = s_{43}/(s_{42} + s_{43})$. This means $\sigma'_{42} = \sigma'_{43} + \sigma'_{32} = -s_{42}/(s_{42} + s_{43})$, $f_2'' = s_{23} + s_{34} + s_{24} = s_{51}$. Hence

$$M'' = 4 \frac{c_{24}}{s_{34} s_{51}} (\alpha c_{51} - \gamma c_{43} + \delta). \quad (66)$$

Combining (63) and (66), we get finally

$$\begin{aligned} M^\alpha = M' + M'' &= \frac{4c_{21}}{s_{21} s_{34}} (\alpha c_{51} - \gamma c_{43} + \delta) + \frac{4c_{21}}{s_{21} s_{45}} (-\alpha c_{54} - \beta + \gamma c_{45} + \delta) \\ &\quad + \frac{4c_{24}}{s_{34} s_{51}} (\alpha c_{51} - \gamma c_{43} + \delta), \end{aligned} \quad (67)$$

which is identical to the result obtained in (61) by multi-cycle conversion.

C. Spinor helicity expression

Recall from (58) that

$$\alpha = b_{14} c_{34}, \quad \beta = b_{14} c_{54} c_{35}, \quad \gamma = b_{15} c_{35}, \quad \delta = b_{15} c_{45} c_{34}.$$

The three terms in (67) can then be written as

$$\begin{aligned} X &= \frac{4c_{21}}{s_{21} s_{34}} (\alpha c_{51} - \gamma c_{43} + \delta) = \frac{4c_{21}}{s_{21} s_{34}} (b_{14} c_{34} c_{51} + b_{15} (-c_{35} c_{43} + c_{45} c_{34})), \\ Y &= \frac{4c_{21}}{s_{21} s_{45}} (-\alpha c_{54} - \beta + \gamma c_{45} + \delta) = \frac{4c_{21}}{s_{21} s_{45}} (b_{14} (-c_{34} c_{54} - c_{54} c_{35}) + b_{15} (c_{35} c_{45} + c_{45} c_{34})) \\ &= \frac{4c_{21}}{s_{21} s_{45}} (-b_{14} c_{54} + b_{15} c_{45}) (c_{34} + c_{35}), \\ Z &= \frac{4c_{24}}{s_{34} s_{51}} (\alpha c_{51} - \gamma c_{43} + \delta) = \frac{4c_{24}}{s_{34} s_{51}} (b_{14} c_{34} c_{51} + b_{15} (-c_{35} c_{43} + c_{45} c_{34})). \end{aligned} \quad (68)$$

The spinor-helicity expression for these quantities are

$$c_{34} + c_{35} = \frac{\langle 43 \rangle [42]}{[32]} + \frac{\langle 53 \rangle [52]}{[32]} = -\frac{\langle 13 \rangle [12]}{[32]},$$

$$\begin{aligned}
c_{35}c_{43} - c_{45}c_{34} &= \frac{\langle 53 \rangle [52] \langle 34 \rangle [32]}{[32] [42]} - \frac{\langle 54 \rangle [52] \langle 43 \rangle [42]}{[42] [32]} = \frac{\langle 34 \rangle [52]}{[32][42]} (\langle 53 \rangle [32] + \langle 54 \rangle [42]) \\
&= \frac{\langle 34 \rangle [52] \langle 15 \rangle [12]}{[32][42]}, \\
X &= \frac{4c_{21}}{s_{21}s_{34}} \left(\frac{\langle 43 \rangle [12] \langle 43 \rangle [42] \langle 15 \rangle [12]}{\langle 13 \rangle [42] [32] [52]} - \frac{\langle 53 \rangle [12] \langle 34 \rangle [52] \langle 15 \rangle [12]}{\langle 13 \rangle [52] [32][42]} \right) \\
&= \frac{4c_{21}}{s_{21}s_{34}} \frac{\langle 43 \rangle \langle 15 \rangle [12]^3}{[32][42][52]} = \frac{4}{s_{21}s_{34}} \frac{[12] \langle 13 \rangle \langle 43 \rangle \langle 15 \rangle [12]^3}{\langle 23 \rangle [32][42][52]} = -\frac{\langle 13 \rangle \langle 15 \rangle [12]^3}{\langle 23 \rangle \langle 21 \rangle [32][42][52][43]}, \\
Y &= \frac{4c_{21}}{s_{21}s_{45}} \left(\frac{\langle 43 \rangle [12] \langle 45 \rangle [42]}{\langle 13 \rangle [42] [52]} - \frac{\langle 53 \rangle [12] \langle 54 \rangle [52]}{\langle 13 \rangle [52] [42]} \right) \frac{\langle 13 \rangle [12]}{[32]} \\
&= -\frac{4c_{21}}{s_{21}s_{45}} \frac{\langle 13 \rangle \langle 45 \rangle [12]^3}{[32][42][52]} = -\frac{4}{s_{21}s_{45}} \frac{[12] \langle 13 \rangle \langle 13 \rangle \langle 45 \rangle [12]^3}{\langle 23 \rangle [32][42][52]} = -\frac{\langle 13 \rangle^2 [12]^3}{\langle 23 \rangle \langle 21 \rangle [32][42][52][54]}, \\
Z &= \frac{4c_{24}}{s_{34}s_{51}} \left(\frac{\langle 43 \rangle [12] \langle 43 \rangle [42] \langle 15 \rangle [12]}{\langle 13 \rangle [42] [32] [52]} - \frac{\langle 53 \rangle [12] \langle 34 \rangle [52] \langle 15 \rangle [12]}{\langle 13 \rangle [52] [32][42]} \right) \\
&= -\frac{4c_{24}}{s_{34}s_{51}} \frac{\langle 15 \rangle \langle 43 \rangle [12]^3}{[32][42][52]} = -\frac{4}{s_{34}s_{51}} \frac{[42] \langle 43 \rangle \langle 15 \rangle \langle 43 \rangle [12]^3}{\langle 23 \rangle [32][42][52]} = -\frac{\langle 43 \rangle [12]^3}{\langle 23 \rangle [32][52][43][15]}, \\
X + Y &= -\frac{\langle 13 \rangle [12]^3}{\langle 23 \rangle \langle 21 \rangle [32][42][52]} \left(\frac{\langle 15 \rangle}{[43]} + \frac{\langle 13 \rangle}{[54]} \right) = \frac{\langle 13 \rangle [12]^3}{\langle 23 \rangle [32][43][52][54]}. \tag{69}
\end{aligned}$$

putting all these together, we finally have

$$M^\alpha = X + Y + Z = \frac{[12]^3}{\langle 23 \rangle [32][43][52]} \left(\frac{\langle 13 \rangle}{[54]} + \frac{\langle 43 \rangle}{[51]} \right) = \frac{[12]^4}{[12][23][34][45][51]}, \tag{70}$$

which agrees with the Parke-Taylor formula.

X. CRYSTAL GRAPH, σ -TABLE, AND C -TABLE

In Sec. IX B, integrations of a five-point amplitude were carried out directly, without first using algebraic manipulation to bring its σ -dependence into the form of a double-color scalar amplitude. In this section, we discuss how to generalize that to an n -point amplitude, using the method of Sec. VI to get the denominator and the numerator factors of the Feynman diagrams. The numerator is essentially $\text{Pf}'\Psi$, after replacing the σ_{ij} and the C_{ii} in it by some appropriate values. The value for σ_{ij} is either $\pm 1, \pm \sigma_{rs}, \pm \sigma_{st}$, or $\pm \sigma_{tr}$, depending on what the Feynman diagram is, and how the integration is done. The precise value can be read off from a ‘crystal graph’ and tabulated in a ‘ σ -table’. How C_{ii} turns out to be is tabulated in a ‘ C -table’.

The basis of the replacement stems from the following observations. Let S_1 and S_2 be two non-intersecting crystals, with defects r_1 and r_2 , triggers p_1 and p_2 , and integration

variables s_1 and s_2 , respectively. Denote the lines in $S_1 \setminus \{r_1, p_1\}$ by a_1, b_1, \dots , and $S_2 \setminus \{r_2, p_2\}$ by a_2, b_2, \dots , and the new σ -variables after the scaling change in S_1 and S_2 by σ' and σ'' . Then since the integral is evaluated in the vicinity of $s_1 = 0$ and $s_2 = 0$, the σ_{ij} variables after the integrations turn into

$$\begin{aligned} \sigma_{a_1 b_1} &\rightarrow \sigma'_{a_1 b_1}, & \sigma_{a_2 b_2} &\rightarrow \sigma''_{a_2 b_2}, & \sigma_{a_1 a_2} &\rightarrow \sigma_{r_1 r_2}, \\ \sigma'_{p_1 r_1} &= 1, & \sigma''_{p_2 r_2} &= 1, \\ \sigma_{A a_1} &\rightarrow \sigma_{A r_1}, & \sigma_{A a_2} &\rightarrow \sigma_{A r_2}, & (A \notin S_1, S_2). \end{aligned} \quad (71)$$

If $S_2 \subset S_1$, then let a_2, b_2, \dots denote lines in S_2 as before, but let a_1, b_1, \dots denote lines in $S_1 \setminus S_2$. Since s_1 is integrated before s_2 , after the integrations, we have

$$\begin{aligned} \sigma_{a_1 b_1} &\rightarrow \sigma'_{a_1 b_1}, & \sigma_{a_2 b_2} &\rightarrow \sigma''_{a_2 b_2}, & \sigma_{a_1 a_2} &\rightarrow \sigma'_{r_1 a_2}, \\ \sigma'_{p_1 r_1} &= 1, & \sigma''_{p_2 r_2} &= 1, \\ \sigma_{A a_1} &\rightarrow \sigma_{A r_1}, & \sigma_{A a_2} &\rightarrow \sigma_{A r_1}, & (A \notin S_1, S_2). \end{aligned} \quad (72)$$

Note that the defect r_2 for the smaller crystal S_2 must be either r_1 or p_1 .

A. Crystal graph and σ -table

From the largest crystals we can proceed to create smaller and smaller crystals, using as a defect either a Möbius constant line, or a previous trigger. Each integration converts a variable line into a trigger line, until every one of the $(n-3)$ variable lines has been turned into a trigger. The information of how integrations are carried out for a Feynman diagram, and the resulting relations using (71) and (72), can be summarized in a *crystal graph*. Fig. 1 shows the Feynman diagram of an 8-point gauge amplitude and its crystal graph, with the constant lines indicated dashed in the Feynman diagram, and underlined in the crystal graph.

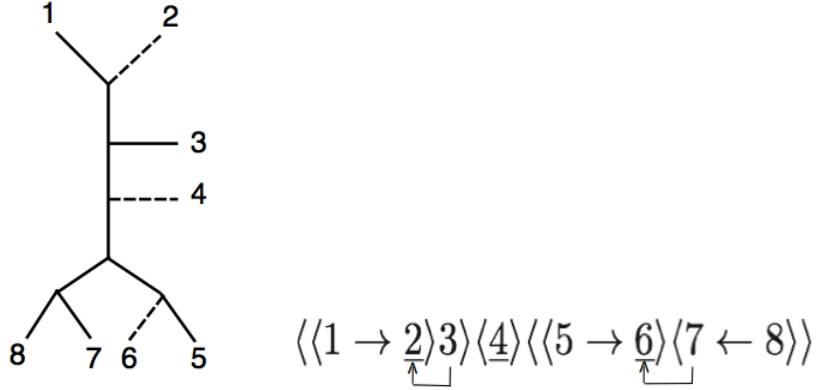


Fig. 1. An 8-point Feynman diagram and its crystal graph

If we forget about the arrows, then the crystal graph is simply an algebraic way to represent the Feynman diagram [8], with external lines forming a propagator grouped inside an angular bracket $\langle \dots \rangle$. The arrows indicate the triggering relations $\sigma_{pr} = 1$, pointing from the trigger p to the defect r . The manner integrations are carried out can be read out from the graph. In the case of Fig. 1, it tells us that starting from two non-intersecting crystal $A = \{1\underline{2}3\}$ and $B = \{5\underline{6}78\}$, with defects $r = 2$ and $r = 6$ respectively, an integration is carried out in each crystal with respective triggers $p = 3$ and $p = 7$. After that, the triggers may be used as defects for smaller crystals to be formed. Subsequent integrations are carried out in the crystal $\{12\} \subset A$, with 2 as the defect and 1 as the trigger, and in the crystals $\{56\} \subset B$ and $\{78\} \subset B$, with 6 and 7 as the respective defects and 5 and 8 as the respective triggers.

Using the rules of (71) and (72), the final expressions for σ_{ij} can be read out from the crystal graph. For Fig. 1, we have $1 = \sigma_{12} = \sigma_{32} = \sigma_{56} = \sigma_{87} = \sigma_{76}$ because each pair is connected by an arrow. For the remaining σ_{ij} , move i and j along the arrows, until the two are connected directly by an arrow, or both end up at some constant lines. Then read off the final σ value at the end. If there are two conflicting ways of doing so, then the movement in the smaller crystal takes precedence. This is so because in a multi-crystal structure, as the scaling variables s become small, the σ_{ij} within a smaller crystal is much smaller than that in the larger crystal. In this way the σ -table of Fig. 1 is obtained and shown in Table 3.

$i \setminus j$	1	2	3	4	5	6	7	8
1	0	+1	-1	σ_{24}	σ_{26}	σ_{26}	σ_{26}	σ_{26}
2	-1	0	-1	σ_{24}	σ_{26}	σ_{26}	σ_{26}	σ_{26}
3	+1	+1	0	σ_{24}	σ_{26}	σ_{26}	σ_{26}	σ_{26}
4	$-\sigma_{24}$	$-\sigma_{24}$	$-\sigma_{24}$	0	σ_{46}	σ_{46}	σ_{46}	σ_{46}
5	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{46}$	0	+1	-1	-1
6	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{46}$	-1	0	-1	-1
7	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{46}$	+1	+1	0	-1
8	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{26}$	$-\sigma_{46}$	+1	+1	+1	0

Table 3. The σ -table for Fig. 1

B. C -table

The C -table can be similarly obtained. For reasons mentioned at the end of last paragraph, the j -summation in $C_{ii} = -\sum_{j \neq i} c_{ij}/\sigma_{ij}$ can be truncated into a sum over the smallest crystal S_i containing i . Moreover, if S_i contains an even smaller crystal S_x with a defect r_x , then all σ_{ij} for $j \in S_x \subset S_i$ can be replaced by σ_{ir_x} . After this replacement and truncation, the remaining σ_{ij} in the sum should be replaced by those in the σ -table to get the C -table.

For example, the C -table for the eight-point amplitude is

$-C_{11}$	c_{12}
$-C_{22}$	$-c_{21}$
$-C_{33}$	$c_{31} + c_{32}$
$-C_{44}$	$(c_{41} + c_{42} + c_{43})/\sigma_{42} +$ $(c_{45} + c_{46} + c_{47} + c_{48})/\sigma_{46}$
$-C_{55}$	c_{56}
$-C_{66}$	$-c_{65}$
$-C_{77}$	$-c_{78}$
$-C_{88}$	c_{87}

Table 4. The C -table for Fig. 1

Using (22), the gauge amplitude in (1) is given by the formula

$$M^{(123\dots n)} = -2^{n-3} \sigma_{(rst)}^2 \sum (\text{Pf}'\Psi) \left(\prod_S \frac{1}{s_S} \right), \quad (73)$$

where the product is taken over a compatible set of $(n-3)$ crystals S , and the sum is over all compatible crystal sets. Each compatible set gives rise to a crystal graph, a σ -table, a C -table, and a product of $(n-3)$ propagators $1/s_S$ that fixes the Feynman diagram involved. The numerator of that Feynman diagram is obtained by substituting the relations of σ_{ij} and C_{ii} contained in the two tables into the expression for $\text{Pf}'\Psi$. To be consistent, the $\sigma_{(rst)}^2$ factor must be cancelled by the same factor contained in $\text{Pf}'\Psi$.

C. Five-point amplitude

As shown in (61), the five-point amplitude computed in Sec. IX consists of three Feynman diagrams. We will illustrate the method outlined above by using it to compute the term with the propagator $1/s_{12}s_{45}$, taking as before the constant lines to be 1, 3, 5.

The Feynman diagram for this term and its crystal graph are shown in Fig. 2, from which we can obtain its σ -table shown in Table 5 and its C -table shown in Table 6.

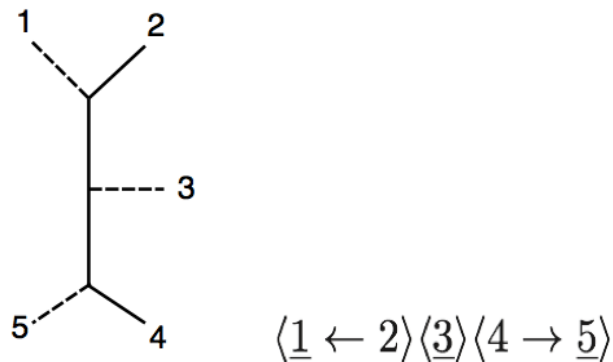


Fig. 2. A Feynman diagram and its crystal graph for a five-point amplitude

$i \setminus j$	1	2	3	4	5
1	0	-1	σ_{13}	σ_{15}	σ_{15}
2	+1	0	σ_{13}	σ_{15}	σ_{15}
3	$-\sigma_{13}$	$-\sigma_{13}$	0	σ_{35}	σ_{35}
4	$-\sigma_{15}$	$-\sigma_{15}$	$-\sigma_{35}$	0	+1
5	$-\sigma_{15}$	$-\sigma_{15}$	$-\sigma_{35}$	-1	0

Table 5. The σ -table for the amplitude in Fig. 2

$-C_{11}$	$-c_{12}$
$-C_{22}$	c_{21}
$-C_{33}$	$(c_{31} + c_{32})/\sigma_{31} + (c_{34} + c_{35})/\sigma_{35}$
$-C_{44}$	c_{45}
$-C_{55}$	$-c_{54}$

Table 6. The C -table for the amplitude in Fig. 2

According to (58), the reduced Pfaffian of the amplitude is given by

$$\text{Pf}'\Psi = 2^2 C_{22} \left(\frac{\alpha}{\sigma_{(143)}} C_{55} + \frac{\beta}{\sigma_{(1453)}} + \frac{\gamma}{\sigma_{(153)}} C_{44} + \frac{\delta}{\sigma_{(1543)}} \right),$$

$$\alpha = b_{14} c_{34}, \quad \beta = b_{14} c_{54} c_{35}, \quad \gamma = b_{15} c_{35}, \quad \delta = b_{15} c_{45} c_{34}.$$

To get the part of the amplitude proportional to $1/s_{12}s_{45}$, all we need is to use Fig. 2, or equivalently Tables 5 and 6, to substitute in the expressions for σ_{ij} and C_{ii} in $\text{Pf}'\Psi$. Then

$$\text{Pf}'\Psi = -4c_{21} \left(\frac{-\alpha}{\sigma_{(135)}} c_{54} - \frac{\beta}{\sigma_{(135)}} + \frac{\gamma}{\sigma_{(135)}} c_{45} + \frac{\delta}{\sigma_{(135)}} \right). \quad (74)$$

Thus, according to (73), this part of the amplitude is

$$4c_{21} (-\alpha c_{54} - \beta_\gamma c_{45} + \delta), \quad (75)$$

agreeing with the result (61) obtained previously.

XI. CHY, FEYNMAN AMPLITUDES, AND THE FOUR-GLUON VERTEX

In the usual field theory approach, a color-stripped Feynman tree amplitude is given by a sum of Feynman diagrams, constructed by using propagators to link up triple-gluon and four-gluon vertices in all possible ways, while keeping the cyclic order of external lines fixed. In a covariant gauge, the numerator of each Feynman diagram consists of sums of products of the form $p \cdot p'$, $\epsilon \cdot p$, $\epsilon \cdot \epsilon'$, where ϵ , ϵ' are the polarization vectors of the external lines, and p , p' are internal or external momenta. To get the final result for each Feynman diagram, we must use momentum conservation to express all internal momenta as sums of external momenta, then assemble and simplify the terms. The algebra involved is very complicated, the result varies from diagram to diagram, generally without any discernible pattern.

In contrast, the CHY amplitude has a very regular numerator no matter what n is. It contains $(n - 1)!$ terms given by (22), each with a distinct σ -structure. The corresponding denominators are obtained by carrying out integrations in σ . Neither the triple nor the four gluon vertex appears explicitly.

A Feynman amplitude is built up from local vertices. It has a complicated numerator but simple denominators in each Feynman diagram. A CHY amplitude has a regular numerator and relatively complicated denominators that must be obtained by integration. It is global; local structures can be extracted only after the amplitude is computed.

In that connection, one puzzling feature about the CHY gauge amplitude is that it is closely related to the CHY scalar amplitude, which corresponds to a ϕ^3 coupling in field theory. Where is the four-gluon vertex coming from? Since the CHY gauge amplitude is gauge invariant, if the triple-gluon vertex is contained in the CHY amplitude, which we know that it does, then the four-gluon vertex must be present. To find out what it is we must carry out a computation of the $n = 4$ CHY amplitude, with general polarization vectors. Recall from Sec. VII that our normalization for $n = 3$ corresponds to a coupling constant of $g = \frac{1}{2}$ for the triple-gluon vertex. It turns out that in order to compare with the Feynman amplitude, the appropriate normalization for $n = 4$ turns out to be $\frac{1}{2}$ of that used in (22). In the rest of this section, we will carry out that calculation to find the four-gluon vertex from the CHY amplitude.

The $n = 4$ amplitude is given by (1) to be

$$M = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{d\sigma_4}{f_4} \frac{\sigma_{(123)}^2 \text{Pf}'\Psi}{\sigma_{(1234)}},$$

$$f_4 = \frac{s_{41}}{\sigma_{41}} + \frac{s_{42}}{\sigma_{42}} + \frac{s_{43}}{\sigma_{43}}, \quad (76)$$

where the contour Γ encircles $f_4 = 0$ counter-clockwise.

The solution of $f_4 = 0$ is $\sigma_4 = \sigma_4^0$, with

$$\sigma_4^0 = \frac{1}{x} (s_{41}\sigma_2\sigma_{13} + s_{41}\sigma_2\sigma_{23}), \quad x = s_{41}\sigma_{13} + s_{42}\sigma_{23}. \quad (77)$$

This implies

$$\sigma_{41}^0 = \frac{1}{x} (s_{41}\sigma_{31}\sigma_{12}), \quad \sigma_{42}^0 = \frac{1}{x} (s_{42}\sigma_{12}\sigma_{23}), \quad \sigma_{43}^0 = \frac{1}{x} (s_{43}\sigma_{23}\sigma_{31}), \quad (78)$$

and

$$\frac{1}{f_4} = \frac{\sigma_{41}\sigma_{42}\sigma_{43}}{x(\sigma_4 - \sigma_4^0)}. \quad (79)$$

The amplitude M is therefore given by the residue at $\sigma_4 = \sigma_4^0$ to be

$$M = L(\text{Pf}'\Psi)_0, \quad L = - \left(\frac{\sigma_{41}\sigma_{42}\sigma_{43}\sigma_{(123)}^2}{x\sigma_{(1234)}} \right)_0 = \frac{s_{42}\sigma_{(123)}^2}{x^2}, \quad (80)$$

where the subscript 0 indicates that σ_4 should be evaluated at σ_4^0 given in (77) and (78).

The reduced Pfaffian $\text{Pf}'\Psi$ contains $3! = 6$ terms with the following cycle structure:

$$a = [142](3), \quad b = [132](4), \quad c = [12](3)(4), \quad d = [12](34), \quad e = [1342], \quad f = [1432]. \quad (81)$$

The reduced Pfaffian is given by (22) to be -2 times the sum of the following six quantities,

$$\begin{aligned} (-)^a \Psi_a &= \frac{\epsilon_1 U_4 \epsilon_2}{\sigma_{(142)}} C_{33}, \\ (-)^b \Psi_b &= \frac{\epsilon_1 U_3 \epsilon_2}{\sigma_{(132)}} C_{44}, \\ (-)^c \Psi_c &= -\frac{\epsilon_1 \cdot \epsilon_2}{\sigma_{(12)}} C_{33} C_{44}, \\ (-)^d \Psi_d &= \frac{\epsilon_1 \cdot \epsilon_2}{\sigma_{(12)}} \frac{\frac{1}{2} \text{Tr}(U_3 U_4)}{\sigma_{(34)}}, \\ (-)^e \Psi_e &= -\frac{\epsilon_1 U_3 U_4 \epsilon_2}{\sigma_{(1342)}}, \\ (-)^f \Psi_f &= -\frac{\epsilon_1 U_4 U_3 \epsilon_2}{\sigma_{(1432)}}, \\ (C_{33})_0 &= \left(-\frac{c_{31}}{\sigma_{31}} - \frac{c_{32}}{\sigma_{32}} - \frac{c_{34}}{\sigma_{34}} \right)_0 = \frac{c_{32}s_{43} - c_{34}s_{41}}{s_{43}} \frac{\sigma_{12}}{\sigma_{13}\sigma_{23}}, \\ (C_{44})_0 &= \left(-\frac{c_{41}}{\sigma_{41}} - \frac{c_{42}}{\sigma_{42}} - \frac{c_{43}}{\sigma_{43}} \right)_0 = \frac{(s_{41}\sigma_{13} + s_{42}\sigma_{23})(s_{41}\sigma_{12} - s_{43}\sigma_{23})(s_{43}c_{41} - s_{41}c_{43})}{s_{41}s_{42}s_{43}\sigma_{12}\sigma_{13}\sigma_{23}}. \end{aligned} \quad (82)$$

Their contributions to M , denoted by $M_i = -2n_i m_i$, $i = a, b, c, d, e, f$, are listed in Table 7. The first column under m_i gives the coefficients of $1/s_{43}$, the second column gives the coefficients of $1/s_{41}$, the third column displays other contributions.

i	n_i		m_i		
			s_{43}^{-1}	s_{41}^{-1}	others
a	$\epsilon_1 U_4 \epsilon_2$	$-b_{14}c_{24} + c_{14}b_{24}$	c_{34}	$-c_{32}$	0
b	$\epsilon_1 U_3 \epsilon_2$	$-b_{13}c_{23} + c_{13}b_{23}$	$-c_{43}$	c_{41}	0
c	$\epsilon_1 \cdot \epsilon_2$	b_{12}	$-c_{32}c_{43} - c_{34}c_{41}$	$c_{32}c_{41}$	$c_{34}c_{43}s_{41}/s_{43}^2$
d	$\epsilon_1 \cdot \epsilon_2 \frac{1}{2} \text{Tr}(U_3 U_4)$	$b_{12}(c_{34}c_{43} - a_{34}b_{43})$	0	0	s_{42}/s_{43}^2
e	$\epsilon_1 U_3 U_4 \epsilon_2$	$b_{13}c_{43}c_{24} + c_{13}c_{34}b_{24}$	1	0	0
f	$\epsilon_1 U_4 U_3 \epsilon_2$	$-b_{13}a_{34}b_{24} - c_{13}b_{34}c_{24}$ $b_{14}c_{34}c_{23} + c_{14}c_{43}b_{23}$ $-b_{14}a_{43}b_{23} - c_{14}b_{43}c_{23}$	-1	-1	0

Table 7. The result of (76) given by $M = -2 \sum_{i=a,b,c,d,e,f} n_i m_i$

Since this calculation is relatively complicated, it is useful to check it by checking its shift invariance under (28). The reduced Pfaffian is proportional to the sum of the terms in (82). Since C_{ii} is not to be shifted, and the different line in (82) have difference σ -dependence, $\text{Pf}'\Psi$ is shift invariant if and only if each of the factors n_i in Table 7 is invariant under the shifts $\delta_3, \delta_4, \delta'_3$, and δ'_4 . The computation in the equation below shows that this is indeed the case.

$$\begin{aligned}
\delta_3 n_a &= 0, & \delta_4 n_a &= -b_{14}b_{24} + b_{14}b_{24} = 0, \\
\delta_3 n_b &= -b_{13}b_{23} + b_{13}b_{23} = 0, & \delta_4 n_b &= 0, \\
\delta_3 n_c &= 0, & \delta_4 n_c &= 0, \\
\delta_3 n_d &= b_{12}(b_{34}b_{43} - b_{34}b_{43}) = 0, & \delta_4 n_d &= b_{12}(b_{34}c_{43} - c_{43}b_{43}) = 0, \\
\delta_3 n_e &= b_{13}b_{43}c_{24} + b_{13}c_{34}b_{24} - b_{13}c_{34}b_{24} - b_{13}b_{34}c_{24} = 0, \\
\delta_4 n_e &= b_{13}c_{43}b_{24} + c_{13}b_{34}b_{24} - b_{13}c_{43}b_{24} - c_{13}b_{34}b_{42} = 0, \\
\delta_3 n_f &= b_{14}c_{34}b_{23} + c_{14}b_{43}b_{23} - b_{14}c_{34}b_{23} - c_{14}b_{43}b_{23} = 0, \\
\delta_4 n_f &= b_{14}b_{34}c_{23} + b_{14}c_{43}b_{23} - b_{14}c_{43}b_{23} - b_{14}b_{43}c_{23} = 0; \\
\delta'_3 n_a &= 0, & \delta'_4 n_a &= -c_{14}c_{24} + c_{14}c_{24} = 0, \\
\delta'_3 n_b &= -c_{13}c_{23} + c_{13}c_{23} = 0, & \delta'_4 n_b &= 0, \\
\delta'_3 n_c &= 0, & \delta'_4 n_c &= 0,
\end{aligned}$$

$$\begin{aligned}
\delta'_3 n_d &= b_{12}(a_{34}c_{43} - a_{34}c_{43}) = 0, & \delta'_4 n_d &= b_{12}(c_{34}a_{43} - a_{34}c_{34}) = 0, \\
\delta'_3 n_e &= c_{13}c_{43}c_{24} + c_{13}a_{34}b_{24} - c_{13}a_{34}b_{24} - c_{13}c_{43}c_{24} = 0, \\
\delta'_4 n_e &= b_{13}a_{43}c_{24} + c_{13}c_{34}c_{24} - b_{13}a_{34}c_{24} - c_{13}c_{34}c_{24} = 0, \\
\delta'_3 n_f &= b_{14}a_{34}c_{23} + c_{14}c_{43}c_{23} - b_{14}a_{43}c_{23} - c_{14}c_{43}c_{23} = 0, \\
\delta'_4 n_f &= c_{14}c_{34}c_{23} + c_{14}a_{43}b_{23} - c_{14}a_{43}b_{23} - c_{14}c_{34}c_{23} = 0.
\end{aligned} \tag{83}$$

The third column of m_i is non-zero only for $i = c$ and d . The reason why these two rows are different from the others can be seen from (82). We have evaluated the integral (76) at the $f_4 = 0$ pole, but we could have evaluated it in a different way, by distorting the contour Γ away from $f_4 = 0$ to surround the $\sigma_{4i} = 0$ poles. If we do so, then we can see that rows c and d are different from the others because both Ψ_c and Ψ_d have a double pole at $\sigma_{43} = 0$ while every other term contains only simple poles. In Ψ_d this is so because $\sigma_{(34)} = -\sigma_{43}^2$, and in Ψ_c , this is because C_{33} and C_{44} can each contribute a term proportional to $1/\sigma_{43}$. It is these double poles that contribute to the expression shown in the third row of m_i . Direct evaluation of double poles will be discussed in the next section.

These strange terms never appeared in our previous calculations because they are both proportional to b_{12} . In our previous calculations, we took particles 1 and 2 to have the same helicity, so in the helicity gauge $b_{12} = 0$.

At first sight these strange terms seem to spell trouble because Feynman propagators, at least in the Feynman gauge, could only give rise to terms proportional to $1/s_{43} = 1/s_{12}$ and $1/s_{41} = 1/s_{23}$, but not $1/s_{43}^2$. However, on closer examination, we see that all the troublesome terms get cancelled out:

$$\begin{aligned}
(n_c m_c + n_d m_d)_{others} &= \frac{b_{12}}{s_{43}^2} [c_{34}c_{43}s_{41} + (c_{34}c_{43} - a_{34}b_{43})s_{42}] \\
&= -\frac{b_{12}}{s_{43}} \left(c_{34}c_{43} + \frac{1}{2}b_{43}s_{42} \right).
\end{aligned} \tag{84}$$

Another way of saying this is that the contribution from the double poles effectively cancels out.

Since we already know the triple-gluon vertex from (43), we can extract the four-gluon vertex from Table 7 by subtracting out the s_{43} - and s_{41} -channel diagrams with two triple-gluon vertices. In a covariant gauge whose propagator is parametrized by ξ , the s_{43} -channel

diagram is

$$\begin{aligned} \mathcal{S} &= g^2 (\epsilon_1 \cdot \epsilon_2 k_{1\mu} - \epsilon_1 \cdot \epsilon_2 k_{2\mu} - 2\epsilon_2 \cdot k_1 \epsilon_{1\mu} + 2\epsilon_1 \cdot k_2 \epsilon_{2\mu}) (g^{\mu\nu} + \xi(k_3 + k_4)^\mu (k_3 + k_4)^\nu / s_{43}) \frac{1}{s_{43}} \\ & (\epsilon_3 \cdot \epsilon_4 k_{3\nu} - \epsilon_3 \cdot \epsilon_4 k_{4\nu} - 2\epsilon_4 \cdot k_3 \epsilon_{3\nu} + 2\epsilon_3 \cdot k_4 \epsilon_{4\nu}) := \frac{4g^2}{s_{43}} \mathcal{S}_0 = \frac{\mathcal{S}_0}{s_{43}}, \end{aligned} \quad (85)$$

independent of ξ , where

$$\mathcal{S}_0 = \frac{1}{4} (b_{12}, -b_{12}, -2c_{21}, 2c_{12}) \begin{pmatrix} a_{13} & a_{14} & c_{31} & c_{41} \\ a_{23} & a_{24} & c_{32} & c_{42} \\ c_{13} & c_{14} & b_{13} & b_{14} \\ c_{23} & c_{24} & b_{23} & b_{24} \end{pmatrix} \begin{pmatrix} b_{34} \\ -b_{34} \\ -2c_{43} \\ 2c_{34} \end{pmatrix}, \quad (86)$$

and the last equality in (85) assumes $g = \frac{1}{2}$. Table 8 shows the difference between $M/2$ obtained from Table 7 and \mathcal{S}_0/s_{43} , for terms proportional to $s_{43}^{-1} = s_{12}^{-1}$. Location $(i|j)$ indicates that the expression is proportional to the (i, j) element of the square matrix in (86).

term	$-\sum_i n_i m_i$ in Table 7		\mathcal{S}_0/s_{43}		$A - B$
	location	value = A	location	value = B	
b_{12}	c, d	$-c_{31}c_{43} + c_{34}c_{41} + \frac{1}{2}b_{34}s_{24}$	$(1, 2 3, 4)$	$-c_{31}c_{43} + c_{34}c_{41} + \frac{1}{4}b_{34}(s_{24} - s_{23})$	$0 - \frac{1}{4}b_{34}s_{12}$
b_{13}	b, e	$c_{21}c_{43} + \frac{1}{2}b_{24}s_{34}$	$(3 3)$	$c_{21}c_{43}$	$\frac{1}{2}b_{24}s_{34}$
b_{14}	a, f	$-c_{34}c_{21} - \frac{1}{2}b_{23}s_{34}$	$(3 4)$	$-c_{21}c_{34}$	$-\frac{1}{2}b_{23}s_{34}$
b_{23}	b, f	$-c_{12}c_{43}$	$(4 3)$	$-c_{12}c_{43}$	0
b_{24}	a, e	$c_{12}c_{34}$	$(4 4)$	$c_{12}c_{34}$	0
b_{34}	e, f	$c_{13}c_{24} - c_{14}c_{23}$	$(3, 4 1, 2)$	$c_{13}c_{24} - c_{14}c_{23}$	0

Table 8. Terms proportional to s_{43}^{-1}

Similarly, the s_{41} -channel diagram is

$$\mathcal{T} = \frac{4g^2}{s_{41}} \mathcal{T}_0 = \frac{\mathcal{T}_0}{s_{41}}, \quad (87)$$

where

$$\mathcal{T}_0 = \frac{1}{4} (b_{23}, -b_{23}, -2c_{32}, 2c_{23}) \begin{pmatrix} a_{24} & a_{21} & c_{42} & c_{12} \\ a_{34} & a_{31} & c_{43} & c_{13} \\ c_{24} & c_{21} & b_{24} & b_{21} \\ c_{34} & c_{31} & b_{34} & b_{31} \end{pmatrix} \begin{pmatrix} b_{41} \\ -b_{41} \\ -2c_{14} \\ 2c_{41} \end{pmatrix}. \quad (88)$$

Table 9 shows the difference between $M/2$ obtained from Table 7 and \mathcal{T}_0/s_{41} , for terms proportional to $s_{41}^{-1} = s_{23}^{-1}$.

term s_{41}^{-1}	$-\sum_i n_i m_i$ in Table 7		\mathcal{T}_0/s_{41}		$A - B$
	location	value = A	location	value = B	
b_{12}	c, d	$-c_{32}c_{41}$	(3 4)	$-c_{32}c_{41}$	0
b_{13}	b, e	$c_{23}c_{41}$	(4 4)	$c_{23}c_{41}$	0
b_{14}	a, f	$-c_{24}c_{32} + c_{34}c_{23}$ $-\frac{1}{2}b_{23}s_{34}$	(3, 4 1, 2)	$-c_{24}c_{32} + c_{34}c_{23}$ $-\frac{1}{4}b_{23}(s_{23} + 2s_{12})$	0 $\frac{1}{4}b_{23}s_{23}$
b_{23}	b, f	$-c_{13}c_{41} + c_{14}c_{43}$	(1, 2 3, 4)	$-c_{13}c_{41} + c_{14}c_{43}$	0
b_{24}	a, e	$c_{14}c_{32}$	(3 3)	$c_{14}c_{32}$	0
b_{34}	e, f	$-c_{14}c_{23}$	(4 3)	$-c_{14}c_{23}$	0

Table 9. Terms proportional to s_{41}^{-1}

The total difference between the CHY amplitude and the sum of the two triple-vertex diagrams is given by the sum of the last column of Table 8, divided by s_{34} , and the last column of Table 9, divided by s_{41} . It is equal to

$$-\frac{1}{4}b_{12}b_{34} - \frac{1}{4}b_{23}b_{41} + \frac{1}{2}b_{13}b_{24}, \quad (89)$$

which is just the four-gluon vertex for the color-stripped amplitude [12],

$$D_{\alpha\beta\gamma\delta} = g^2(-g_{\alpha\beta}g_{\gamma\delta} + 2g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \quad (90)$$

with $\alpha, \beta, \gamma, \delta$ being the Lorentz indices for particles 1, 2, 3, 4 and $g = \frac{1}{2}$.

XII. DOUBLE POLE

As discussed in the last section, between equations (82) and (84), a double pole is present at $\sigma_{43} = 0$ for the $n = 4$ amplitude. This double pole was absent in Sec. VIII where the $n = 4$ amplitude was computed in the helicity gauge, which illustrates that with a suitable choice of the gauge parameters $r, s, t, \lambda, \nu, +, -$, and in suitable helicity configurations, it may also be absent for larger- n amplitudes. In that case the multi-crystal technique discussed in previous sections can be used to evaluate the gauge amplitude for any n . However, for large

n , there are situations when its presence cannot be avoided. In that case we must find a way to calculate the residue of those terms containing a double pole.

There is of course no difficulty in principle to compute the residue of a double pole. It is equal to the derivative of the rest of the integrand evaluated at the pole. Since the derivative must act on the reduced Pfaffian, and on the scattering functions, the computation is very tedious even for small n . The question is whether a simpler way can be found to calculate the residue without differentiation. The rest of this section is devoted to a discussion of this point. This is what we refer to as the ‘double pole problem’. It is not a problem that prevents us from getting the final result of the gauge amplitude, because residues at double poles can always be computed. It is just a question of whether such computations can be simplified.

We will show in Sec. XIIA that there is a simple way to calculate the $n = 4$ double pole encountered in the last section. This method is generalized to any n in Sec. XIIB, provided the double pole comes from a 2-cycle and the corresponding product of two 1-cycles. The discussion in Sec. XIIC shows that the double pole problem of a k -cycle can be reduced to a double pole problem of a $(k - 1)$ -cycle, thus all double pole problems can be solved by induction.

A. $n = 4$

The amplitude (76) in the last section was evaluated at the simple pole $f_4 = 0$. As mentioned there, if the contour is distorted to have the integral evaluated at the poles $\sigma_{4i} = 0$ instead, then a double pole at $\sigma_{43} = 0$ is encountered for the terms Ψ_c and Ψ_d of $\text{Pf}'\Psi$. In what follows we will compute these two terms with the distorted contour, to illustrate how a double pole can be evaluated by changing it into a product of two simple poles, with the help of the scattering equation.

These two terms are given by (76) and (82) to be

$$\begin{aligned}
 M_{cd} &= -\frac{1}{2\pi i} \oint_{\Gamma} \frac{d\sigma_4}{f_4} \frac{\sigma_{(123)}^2 \text{Pf}'\Psi_{cd}}{\sigma_{(1234)}}, \\
 \text{Pf}'\Psi_{cd} &= \frac{\epsilon_1 \cdot \epsilon_2}{\sigma_{12}^2} \left(C_{33} C_{44} + \frac{\frac{1}{2} \text{Tr}(U_3 U_4)}{\sigma_{43}^2} \right), \tag{91}
 \end{aligned}$$

with

$$\begin{aligned}
\epsilon_1 \cdot \epsilon_2 &= b_{12}, \\
C_{33} &= -\frac{c_{31}}{\sigma_{31}} - \frac{c_{32}}{\sigma_{32}} - \frac{c_{34}}{\sigma_{34}}, \\
C_{44} &= -\frac{c_{41}}{\sigma_{41}} - \frac{c_{42}}{\sigma_{42}} - \frac{c_{43}}{\sigma_{43}}, \\
\frac{1}{2} \text{Tr}(U_3 U_4) &= c_{34} c_{43} - b_{34} a_{43}.
\end{aligned} \tag{92}$$

The integrand consists of simple poles at $\sigma_{41} = 0$ and at $\sigma_{43} = 0$, and a double pole at $\sigma_{43} = 0$. The simple poles can be evaluated in the usual way resulting in

$$M_{cd}^{simple} = \frac{b_{12} \sigma_{(123)}}{\sigma_{12}^2} \left[\frac{c_{41}}{s_{41}} \left(\frac{c_{31} + c_{34}}{\sigma_{31}} + \frac{c_{32}}{\sigma_{32}} \right) - \left(\frac{c_{31}}{\sigma_{31}} + \frac{c_{32}}{\sigma_{32}} \right) \frac{c_{43}}{s_{43}} + \left(\frac{c_{41}}{\sigma_{31}} + \frac{c_{42}}{\sigma_{32}} \right) \frac{c_{34}}{s_{43}} \right] \tag{93}$$

Using momentum conservation which implies $c_{31} = -c_{32} - c_{34}$ and $c_{42} = -c_{41} - c_{43}$, this can be reduced to

$$M_{cd}^{simple} = b_{12} \left(\frac{c_{41} c_{32}}{s_{41}} - \frac{c_{32} c_{43} + c_{41} c_{34} + c_{34} c_{43}}{s_{43}} \right). \tag{94}$$

The double-pole contribution to $\text{Pf}'\Psi$ is

$$\frac{b_{12}}{\sigma_{12}^2} [-c_{34} c_{43} + (c_{34} c_{43} - b_{34} a_{43})] \frac{1}{\sigma_{43}^2} = -\frac{1}{2} \frac{b_{12} b_{34} s_{43}}{\sigma_{12}^2 \sigma_{43}^2}. \tag{95}$$

Since it is proportional to s_{43}/σ_{43} , the double pole can be transformed into the product of two simple poles using the scattering equation $f_4 = 0$ to get

$$\frac{1}{2} \frac{b_{12} b_{34}}{\sigma_{12}^2 \sigma_{43}} \left(\frac{s_{41}}{\sigma_{41}} + \frac{s_{42}}{\sigma_{42}} \right). \tag{96}$$

Of course this has to be done before distorting the contour Γ away from $f_4 = 0$. After this change, we can distort the contour and evaluate the resulting simple poles at $\sigma_{43} = 0$ and $\sigma_{41} = 0$ in the usual way to get

$$M_{cd}^{double} = \frac{1}{2} \frac{b_{12} b_{34} \sigma_{(123)}}{\sigma_{12}^2} \left[-\frac{1}{s_{43}} \left(\frac{s_{41}}{\sigma_{31}} + \frac{s_{42}}{\sigma_{32}} \right) + \frac{1}{\sigma_{13}} \right] = -\frac{1}{2} \frac{b_{12} b_{34} s_{42}}{s_{43}}, \tag{97}$$

where the last expression is obtained by using momentum conservation to replace s_{41} with $-s_{42} - s_{43}$.

The final result

$$M_{cd} = M_{cd}^{simple} + M_{cd}^{double} = b_{12} \left[\frac{c_{41} c_{32}}{s_{41}} - \frac{1}{s_{43}} \left(c_{32} c_{43} + c_{41} c_{34} + c_{34} c_{43} + \frac{1}{2} b_{34} s_{42} \right) \right] \tag{98}$$

agrees with the result of rows c and d of Table 7, after taking into account equation (84).

B. 2-cycles

Assuming as usual that $\sigma_\alpha = \sigma_{(12\dots n)}$ in (1). Let i, j be two of the n external lines, then both the 2-cycle (ij) and the product of two 1-cycles $(i)(j)$ contain $1/\sigma_{ji}^2$. With this σ_α , the factor $1/\sigma_{ij}^2$ yields a double pole in s only when i and j are adjacent, *viz.*, when $j = i \pm 1$. We will also assume that neither i nor j is at the two ends of an open cycle, a condition that can be satisfied by a choice of λ and ν .

Consider any term in $\text{Pf}'\Psi$ of the form $X(ij)$ and $X(i)(j)$, where X represents the rest of the cycle structure in $\text{Pf}'\Psi$. Then their contribution to $\text{Pf}'\Psi$ is similar to that in (91) and (92), and can be written as

$$\begin{aligned}\Psi_{ij} &= \Psi_X (\Psi_{(i)(j)} - \Psi_{(ij)}) = \Psi_X \left(C_{ii}C_{jj} + \frac{c_{ij}c_{ji} - b_{ij}a_{ji}}{\sigma_{ij}^2} \right), \\ C_{ii} &= \frac{c_{ij}}{\sigma_{ij}} + \sum_{k \neq i, j} \frac{c_{ik}}{\sigma_{ik}}, \\ C_{jj} &= \frac{c_{ji}}{\sigma_{ji}} + \sum_{k \neq i, j} \frac{c_{jk}}{\sigma_{jk}}.\end{aligned}\tag{99}$$

As in the last subsection, the contribution of $\text{Pf}'\Psi_{ij}$ to (1) is made up of simple and double poles. Simple poles are evaluated in the way discussed in earlier sections, so let us concentrate on the double pole. It is

$$\text{Pf}'\Psi_{ij} = \frac{\Psi_X}{\sigma_{ij}^2} [-c_{ij}c_{ji} + (c_{ij}c_{ji} - b_{ij}a_{ji})] = \frac{1}{2}\Psi_X \frac{b_{ij}}{\sigma_{ij}} \frac{s_{ij}}{\sigma_{ij}}.\tag{100}$$

Once again, scattering equation can be used to replace s_{ij}/σ_{ij} with $-\sum_{h \neq i, j} s_{ik}/\sigma_{ik}$, thereby converting the double to a product of two simple poles. The rest can be evaluated as before, using a method similar to that of Secs. VI, X, and XIII.

C. k -cycles

The crucial step that allows the double-pole problem in a 2-cycle to be solved is equation (95), in which terms quadratic in c disappear, leaving behind an expression proportional to s_{43}/σ_{43}^2 . Then using the scattering equation, the double-pole can be transformed into product of two simple poles as is done in (96). If the quadratic term in c did not vanish, then of course the scattering equation can still be used to replace one $1/\sigma_{43} = (s_{43}/\sigma_{43})/s_{43}$, but then we would end up with a term proportional to $1/s_{43}^2$ which could not be the whole

story, as Feynman diagrams only allow a simple-pole propagator $1/s_{43}$ but not a double pole term $1/s_{43}^2$.

For $k \geq 3$, it turns out that this crucial step still holds: terms of $\text{Pf}'\Psi$ of degree- k in c add up to zero. To see more explicitly how that happens, let us consider $k = 3$, with the three consecutive lines in the 3-cycle to be 1,2,3. In the crystal $\{12\bar{3}\}$ with 3 being the defect, double-pole appear in $\text{Pf}'\Psi$ from the cycles (123), (132), (12)(3), (31)(2), (23)(1), and (1)(2)(3). Together with the signature factor, their respective contributions to $\text{Pf}'\Psi$ are

$$\begin{aligned}
\Psi_{(123)} &= \frac{1}{2\sigma_{12}\sigma_{23}\sigma_{31}} (-c_{12}c_{23}c_{31} + c_{12}b_{23}a_{31} - c_{21}a_{23}b_{31} + c_{21}c_{32}c_{13}) \\
&\quad + \frac{1}{2\sigma_{12}\sigma_{23}\sigma_{31}} (-b_{12}c_{32}a_{31} + a_{12}c_{23}b_{31} + b_{12}a_{23}c_{31} - a_{12}b_{23}c_{13}), \\
\Psi_{(132)} &= \frac{1}{2\sigma_{13}\sigma_{32}\sigma_{21}} (-c_{13}c_{32}c_{21} + c_{13}b_{32}a_{21} - c_{31}a_{32}b_{21} + c_{31}c_{23}c_{12}) \\
&\quad + \frac{1}{2\sigma_{13}\sigma_{32}\sigma_{21}} (-b_{13}c_{23}a_{21} + a_{13}c_{32}b_{21} + b_{13}a_{32}c_{21} - a_{13}b_{32}c_{12}), \\
-\Psi_{(12)(3)} &= -\frac{c_{12}c_{21} - b_{12}a_{21}}{\sigma_{12}\sigma_{21}} C_{33}, \\
-\Psi_{(31)(2)} &= -\frac{c_{31}c_{13} - b_{31}a_{13}}{\sigma_{31}\sigma_{13}} C_{22}, \\
-\Psi_{(23)(1)} &= -\frac{c_{23}c_{32} - b_{23}a_{32}}{\sigma_{23}\sigma_{32}} C_{11}, \\
\Psi_{(1)(2)(3)} &= C_{11}C_{22}C_{33}, \tag{101}
\end{aligned}$$

with

$$\begin{aligned}
C_{11} &= -\frac{c_{12}}{\sigma_{12}} - \frac{c_{13}}{\sigma_{13}} - \dots, \\
C_{22} &= -\frac{c_{21}}{\sigma_{21}} - \frac{c_{23}}{\sigma_{23}} - \dots, \\
C_{33} &= -\frac{c_{31}}{\sigma_{31}} - \frac{c_{32}}{\sigma_{32}} - \dots. \tag{102}
\end{aligned}$$

All these terms in $\text{Pf}'\Psi$ scale like $1/s^3 = 1/s^{m+1}$ ($m+1 = k$), which produces a double pole $1/s^2$ in the integrand of (1). The ellipses in (59) represent terms that do not contribute a double pole so they will be ignored.

It can be verified by direct algebraic computation that the c^3 terms in (101) add up to zero. As stated before, the c^k terms in a k -cycle also add up to zero for any k , and there is a simple reason for that. If we let $x_{ij} = c_{ij}/\sigma_{ij}$ for $i \neq j$, and $x_{ii} = C_{ii}$. The products of k c 's add up to zero because the sum is proportional to the determinant of the $k \times k$ matrix (x_{ij}) . This determinant vanishes because its row sums are zero.

With the c^3 terms gone, a factor s_{12}/σ_{12} , s_{23}/σ_{23} , or s_{31}/σ_{31} is always present in every remaining term of $\text{Pf}'\Psi$. If we could use the three scattering equations $f_1 = f_2 = f_3 = 0$ to solve them in terms of s_{ir}/σ_{ir} , with $1 \leq i \leq 3$ and $r > 3$, then we would have transformed every double pole in $\text{Pf}'\Psi$ into a product of simple poles, and the problem is solved.

Unfortunately this cannot be done because the matrix to be inverted for the linear equation is singular, so the three scattering equations allows only two $x_{ij} = s_{ij}/\sigma_{ij}$ ($1 \leq i, j \leq 3$) to be solved, but not 3. Moreover, the solution of these two depends on the value of the third, so the double pole cannot be gotten rid of this way, for it still resides in the pole of the third quantity. For example, x_{23} and x_{13} can be solved in terms of x_{12} and x_{ir} with $1 \leq i \leq 3$ and $r > 3$. As a result, the double pole involving s_{12}/σ_{12}^2 is still present.

However, this remaining double pole involves only lines 1 and 2, and can be solved by the method of Sec. XIIB. For example, consider the 8-point amplitude whose Feynman diagram and crystal graph are shown in Fig. 1. The relevant part of the crystal graph is $\langle\langle 1 \rightarrow 2 \rangle\rangle 3$, with $p = 3$ being the trigger and $r = 2$ being the defect of the larger crystal $\{123\}$. With this consideration, the double pole problem of the 3-cycle (123) in the larger crystal becomes the double pole problem in the smaller crystal $\{12\}$ for the 2-cycle (12) .

Note that this procedure relies on the fact that the σ_{ij} in a smaller crystal is much smaller than that in a larger crystal. For that reason, when $f_1 = 0$ is used to solve for x_{12} , the term x_{13} can be ignored. Otherwise the problem is not solved because the solutions would have become circular.

Similarly, we know that in the presence of a k -cycle involving lines $\{123 \cdots k\}$, the c^k terms in $\text{Pf}'\Psi$ together with those from all the other relevant product cycles add up to zero. We will use the letters i, j to denote numbers between 1 and k , and the letter r to denote a numbers larger than k . In every remaining term in $\text{Pf}'\Psi$, at least one factor $x_{ij} = s_{ij}/\sigma_{ij}$ is present, and altogether there are $k(k-1)/2$ such factors. Using the the scattering equations $f_i = 0$, which is of rank $k-1$, we can solve $k-1$ of the x_{ij} in terms of the rest. Let us choose those $k-1$ factors to be x_{ak} , with $1 \leq a \leq k-1$. After that, all the double pole terms become double pole terms involving x_{ab} , where $1 \leq a, b \leq k-1$, thereby reducing the double pole problem of the k -cycle into a double pole problem for $(k-1)$ -cycle. Since σ_{ij} in a smaller crystal is much smaller than that in a larger crystal, by induction, all double pole problems can be solved this way.

XIII. SUMMARY

The evaluation of the CHY n -point gauge tree amplitude in (1) is discussed. There are two basic difficulties: how to handle the $(2n-3)!!$ terms in the reduced Pfaffian, and how to carry out the $(n-3)$ integrations in the presence of a σ dependence much more complicated than the Parke-Taylor form found in the CHY double-color scalar amplitude.

We found a way to solve the first difficulty by grouping together terms of the reduced Pfaffian into open and closed cycles of the permutation group S_n . We discovered a shift invariance for individual cycles which can be used to check explicit calculations. The use of helicity gauge to further simplify calculations has also been amply discussed.

Integrations are carried out by dividing the dominant integration regions into $(n-3)$ compatible ‘crystals’, each possessing a pole that can be evaluated in one of the $(n-3)$ integrations using residue calculus. A complete recipe is provided when only simple poles are present to extract the $(n-3)$ Feynman diagram propagators, with different compatible crystals giving rise to different Feynman diagrams. A recipe making use of the ‘crystal graph’ is also given to convert the σ_{ij} factors in the reduced Pfaffian into constants, after which the reduced Pfaffian can be identified as the numerator of the Feynman diagram. Double poles are rarely present in small- n amplitudes, but when they appear in large- n amplitudes, there will be more contribution to the amplitude in addition to what is given by the recipes above. These double poles can also be handled.

Calculations depend on a set of gauge parameters: the constant lines r, s, t , the Pfaffian lines λ, ν , and the helicity reference lines $+, -$ in the helicity gauge. Although the final answer must not depend on the choice, they have to be fixed in a calculation, and the structure and complexity of the calculation depend on their choice. Many examples are given for the $n = 3, n = 4, n = 5$ amplitudes to illustrate these calculations.

Unlike the Feynman amplitude, the CHY amplitude is global without the built-in local parts in the form of vertices and propagators. Propagators turn out to emerge from the scattering functions f_i , but the vertices are harder to extract. We have been successful in extracting the triple-gluon vertex from the $n = 3$ amplitude, and the four-gluon vertex from the $n = 4$ amplitude. The calculation for the latter is somewhat lengthy.

Both the CHY formula and the Feynman-diagram technique give the same scattering amplitude, but they group their terms differently. Roughly speaking, the CHY amplitude

groups by numerator, and the Feynman amplitude groups by denominator. The numerator factors in CHY amplitudes are universal for all n , being the cycles of the reduced Pfaffian. Its denominators, however, have to be computed by integration, and a single numerator cycle may involve propagators from several Feynman diagrams. In contrast, Feynman amplitudes are arranged according to Feynman diagrams whose propagators are easy to write down, but its numerator factors have to be built up from products of triple and four gluon vertices. Detailed algebraic manipulation which depends on the topology of the Feynman diagram is needed to expand into its final form as a function of $\epsilon_i \cdot \epsilon_j$, $\epsilon_i \cdot k_j$, and $k_i \cdot k_j$.

This comparison of the two approaches is based on a general helicity configuration. For special helicity configurations, it is often simpler to calculate in the helicity gauge, for either formalism.

Since complication of a gluon amplitude comes largely from its numerator factors, especially for large n , there is a definite advantage in computing it in the numerator-grouping approach. Another advantage is that the CHY formula has many invariances that can be used to check the calculations. They all stem from the basic Möbius invariance of the CHY amplitude, including the shift invariance, and the independence of the amplitude on the choice of the gauge parameters $r, s, t, \sigma_r, \sigma_s, \sigma_t, \lambda, \nu, +, -$. The disadvantage of the CHY formalism is the lack of visible local structures in terms of propagators and vertices, to enable the underlying physical process to be easily understood.

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